SOME THEOREMS ON STANDBYS

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1. Introductory remarks

The investigation of the effect of having standbys, that is, of the insertion of redundant elements, modules, blocks, or even whole units, occupies a leading position in modern reliability theory. One can become acquainted with the principles of the theory of standbys from many monographs [1], [2], [3], which have recently appeared, as well as from the excellent paper of A. D. Soloviev [4].

Henceforth, our use of the word "unit" will be associated with each specific problem. If elements are in standby, then the unit will be an element; if a whole machine is in standby, then the unit will be this machine. We designate the operating unit and its attached standbys as a standby system.

Hot, warm, and cold standbys are differentiated according to the state in which the standby is placed. For hot standbys the units are loaded in exactly the same way as the operating units; for warm standbys they have a diminished load, and cold standbys are completely unloaded. The probability of loss of operational ability is the same for the hot standby as for the operating regime. Warm standby units may fail, but the failure rate is less than for the operational units. It is customary to assume that cold standby units may not lose their operational ability. This assumption distorts the true situation somewhat; however, it may be considered a good approximation to reality for a majority of items.

Standbys are differentiated as being repairable or not repairable. In repairable standbys each unit which has failed can be repaired and after recovery can be put into standby. In standbys without repair, the unit which has failed is eliminated and it no longer takes part in the system's operation. We shall henceforth deal only with repairable systems. We shall therefore pay particular attention to the case of greatest practical interest, when the time to repair is short compared to the time of faultless operation of the individual unit.

The distribution of the time of faultless operation of a standby system consisting either of operating units and one standby, or of one basic unit and some number of standby units, is studied here.

2. The case of n operating units and one standby

In this section we start from the following assumptions.

1. There are n operating units and one standby; the time of faultless operation of the system is random and distributed according to the law $F(x) = 1 - e^{-\lambda x}$

($\lambda > 0$ is a constant) for the operating unit, and the law $F_1(x) = 1 - e^{-\lambda_1 x}$ ($\lambda_1 \geq 0$ is a constant) for the standby unit. In particular, if the standby is hot, $\lambda_1 = \lambda$, and if the standby is cold, then $\lambda_1 = 0$.

2. When the basic unit fails, the standby quickly takes over the load borne by the unit which has failed.

3. Repair completely restores the properties of the unit. The repair time is a random variable with the distribution function $G(x)$.

4. Repair of the unit which has failed starts immediately after failure.

Let $R(x)$ denote the probability that the time of faultless operation of the standby system will be at least x. It is easy to see that $R(x)$ will satisfy the following integral equation:

$$(1) \qquad R(x) = e^{-n\lambda x} + n\lambda e^{-n\lambda x} \int_0^x e^{-\lambda_1 z}[1 - G(x - z)]\, dz + n\lambda \iint_{y+z<x} e^{-n\lambda(y+z)-\lambda_1 z} R(x - y - z)\, dz\, dG(y).$$

In the Laplace transform formulation,

$$(2) \qquad \varphi(s) = -\int_0^\infty e^{-sx}\, dR(x); \qquad g(s) = \int_0^\infty e^{-sx}\, dG(x),$$

the solution of this equation has the following form

$$(3) \qquad \varphi(s) = \frac{n\lambda(n\lambda + \lambda_1)[1 - g(n\lambda + s)]}{(n\lambda + s)[s + (n\lambda + \lambda_1)(1 - g(n\lambda + s))]}.$$

Benjamin Epstein and J. Hosford [5] first found this formula for the case where $n = 1$ and $G(x) = 1 - e^{-\nu x}$. Yu. K. Belyaev [6] derived this formula for an arbitrary distribution $G(x)$, in the hot standby case and for $n = 1$, as a corollary of the general theory of linear Markov processes. D. Gaver [7] and B. V. Gnedenko [8] also obtained this formula for the case of an arbitrary standby with $n = 1$.

We find the formula

$$(4) \qquad a = -\left[\frac{d\varphi(s)}{ds}\right]_{s=0} = \frac{n\lambda + (n\lambda + \lambda_1)[1 - g(n\lambda)]}{n\lambda(n\lambda + \lambda_1)[1 - g(n\lambda)]}$$

for the mean time of faultless operation of the standby system.

Let us point out the fact that the mean time of faultless operation of a standby system depends only on the quantity $g(n\lambda)$ and not on the whole distribution $G(x)$. Hence, if there are two distributions $\bar{G}(x)$ and $G(x)$, for which $\bar{g}(n\lambda) = g(n\lambda)$, the associated quantities a also coincide.

3. Limit theorems

In the most interesting practical cases the repair time is on the average significantly less than the time of faultless operation. In this connection the following limit theorems are of direct applied value. In place of the function $G(x)$, let us consider a sequence of functions $G_\nu(x)$ dependent on the single parameter ν. Let us assume that for any $\epsilon > 0$ the relation

$$1 - G_\nu(\epsilon) \to 0 \tag{5}$$

holds as $\nu \to \infty$. Relation (5) implies the following: as $\nu \to \infty$, $g_\nu(n\lambda) \to 1$.

Let η_ν denote the time of faultless operation of the standby system under the condition that the repair time has the distribution $G_\nu(x)$, and let us put

$$\alpha_\nu = \left(n + \frac{\lambda_1}{\lambda}\right) [1 - g_\nu(n\lambda)]. \tag{6}$$

THEOREM 1. *If condition* (5) *is satisfied, then for any* $x > 0$ *the relation* $P\{\alpha_\nu \eta_\nu < x\} \to 1 - e^{-\lambda x}$ *holds.*

The proof of this result is a direct consequence of (3).

Now, let us impose an additional condition on the function $G_\nu(x)$. Specifically, let us assume that the $G_\nu(x)$ have finite second moments.

$$\begin{aligned} m_2(\nu) &= \int_0^\infty x^2 \, dG_\nu(x), \\ m_1(\nu) &= \int_0^\infty x \, dG_\nu(x) = \frac{1}{\nu} \end{aligned} \tag{7}$$

and for $\nu \to \infty$,

$$\frac{m_2(\nu)}{m_1(\nu)} \to 0. \tag{8}$$

THEOREM 2. *If condition* (8) *is satisfied in addition to conditions* 1–4, *then for large values of* ν *the mean time of faultless operation of a standby system is asymptotically equal to the mean time of a system under the assumption that* $G_\nu(x) = 1 - e^{-\nu x}$.

The proof of this theorem results from (4) with the aid of the equality

$$1 - g_\nu(n\lambda) = n\lambda \int_0^\infty x \, dG_\nu(x) - \int_0^\infty (e^{-n\lambda x} - 1 + n\lambda x) \, dG_\nu(x) \tag{9}$$

and the inequality

$$\left| \int_0^\infty (e^{-n\lambda x} - 1 + n\lambda x) \, dG_\nu(x) \right| \leq \frac{n^2\lambda^2}{2} m_2(\nu). \tag{10}$$

It is interesting to note that condition (8) is satisfied automatically for an important class of functions $G_\nu(x)$ defined by means of the equalities

$$G_\nu(x) = G_1(\nu x), \qquad \int_0^\infty x \, dG_\nu(x) = \frac{1}{\nu}. \tag{11}$$

The Weibull, gamma, and a number of other distributions used in reliability theory belong to this class of distributions.

Simple computations give the first terms of the asymptotic expansion.

4. Several remarks on the preceding deductions

First, let us note that we may obtain the results of Yu. K. Belyaev and D. Gaver, mentioned earlier, from (3) and (4). However, because of conditions 1–4, the results of D. Gaver allow the derivation of formulas (3) and (4). Indeed, condition 1 of section 2 is such that the n unit system emerges as one which completely recovers its properties with each switching of the standby unit for the one which has lost operational ability.

We illustrate theorem 2 with a small table, which shows how rapidly the practical independence of the mean time of faultless operation of the standby system from the distribution function of the repair time sets in. For definiteness, let us put $n = 4$. We do not indicate the value of a itself in the table, but the value of the ratio a/a_0, where a_0 is the mean time of faultless operation of a standby system without repair:

$$\frac{a}{a_0} = 1 + \frac{n\lambda g(n\lambda)}{(2n\lambda + \lambda_1)(1 - g(n\lambda))}. \tag{12}$$

We limit ourselves here to the cold standby case.

$G_\nu(x)$	$\frac{\lambda}{\nu} = 0.25$	0.50	1.00	2.50
0 for $x \leq 0$, 0.5 for $0 < x \leq \frac{2}{\nu}$, 1 for $x > \frac{2}{\nu}$.	1.66	2.08	3.04	6.02
$1 - e^{-\nu x}$	1.50	2.00	3.00	6.00
$\frac{\nu}{2} x$ for $x \leq \frac{2}{\nu}$, 1 for $x > \frac{2}{\nu}$.	1.38	1.86	2.85	5.84
$\frac{1}{2}(3\nu)^3 \int_0^x z^2 e^{-3\nu z}\, dz$	1.36	1.85	2.84	5.84
0 for $x \leq \frac{1}{\nu}$, 1 for $x > \frac{1}{\nu}$.	1.29	1.77	2.76	5.75

In general, some random time is required to disconnect the unit which has failed. Let $L(x)$ denote its distribution function. Now, let

$$H(x) = \int_0^x L(x - z) n\lambda e^{-n\lambda z}\, dz \tag{13}$$

and

$$\varphi(s) = -\int_0^\infty e^{-sx}\, dR(x); \qquad h(s) = \int_0^\infty e^{-sx}\, dH(x); \tag{14}$$

$$g(s) = \int_0^\infty e^{-sx} G(x)\, dH(x).$$

It can be seen, by the same means as in section 2, that the Laplace transform of the time of faultless operation of a standby system is

$$\varphi(s) = \frac{n\lambda}{n\lambda + s} \frac{h(s) - g(s + \lambda_1)}{1 - g(s + \lambda_1)}. \tag{15}$$

5. Cold redundancy

It has been shown by B. V. Gnedenko [9] that the problem may be solved under broader hypotheses in the cold redundancy case.

Let us assume that the time of faultless operation of the unit has the distribution $F(x)$; the switching time has the distribution $L(x)$, the repair time has the distribution $G(x)$; and that repair completely restores the properties of the unit. Let us use the notation

$$H(t) = \int_0^t L(t - x)\, dF(x); \tag{16}$$

then the following holds.

THEOREM 3. *The Laplace transform of the time of faultless operation of a redundant system equals*

$$\varphi(s) = f(s) \frac{h(s) - g(s)}{1 - g(s)}. \tag{17}$$

If $\alpha_\nu = \int_0^\infty [1 - G_\nu(x)]\, dH(x) \to 0$ *as* $\nu \to \infty$, *then*

$$\varphi(\alpha_\nu s) \to \frac{1}{1 + (a + b)s} \tag{18}$$

where $a = \int_0^\infty x\, dF(x)$ and $b = \int_0^\infty x\, dH(x)$.

This theorem is a generalization of the results of [9] and of A. D. Soloviev [10]. The proof of the theorem is similar to that presented in sections 2 and 3 of the present paper.

Let the switching be accomplished instantaneously. The mean time of faultless operation of the redundant system is $A = 2a + (\alpha a/1 - \alpha)$ where $\alpha = \int_0^\infty G(x)\, dF(x)$, as results from (17).

Evidently, for all functions $F(x)$ and $G(x)$ for which $\alpha = \nu/(\lambda + \nu)$ the mean time of faultless operation is identical and coincides with that obtained for $F(x) = 1 - e^{-\lambda x}$ and $G(x) = 1 - e^{-\nu x}$.

How much greater the role of the function $F(x)$ is may be seen from the following table which was presented in [9]. Values of the ratio $A/2a$ are given in the table.

$F(x)$, $G(x)$	$\frac{\lambda}{\nu} =$ 1	2	4	10
$F(x) = 1 - e^{-\lambda x}$, $G(x) = 1 - e^{-\nu x}$,	1.50	2.00	3.00	6.00
$F(x) = \begin{cases} \frac{\lambda}{2} x, & x < \frac{2}{\lambda}, \\ 1, & x > \frac{2}{\lambda}, \end{cases}$ $G(x) = \begin{cases} \frac{\nu}{2} x \text{ for } x \leq \frac{2}{\nu}, \\ 1 \quad \text{for } x > \frac{2}{\nu}, \end{cases}$	1.50	2.50	4.50	10.50
$F(x) = \begin{cases} 0 \text{ for } x \leq \frac{1}{\lambda}, \\ 1 \text{ for } x > \frac{1}{\lambda}, \end{cases}$ $G(x) = 1 - e^{-\nu x}$	1.86	4.19	27.82	15000

6. The case of one basic unit and n standbys

So far we have considered only a particular kind of standby system, although it is a system often used in practice. Another particular case, in which one basic unit has several similar units in standby, has been considered at my request, by the young mathematician J. Makke from Jena. He successfully showed that if the repair time is short, compared with the time of faultless operation of the individual unit, then the time of faultless operation of the standby system is asymptotically distributed according to an exponential law. Exact formulas have not been derived successfully for this case. The result of J. Makke refers only to cold standbys [11].

Exact formulas are not as important in engineering computations, as are asymptotic results similar to those which have been presented here. As yet only initial results have been obtained and further problems are under study.

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