SEQUENTIAL DECISIONS IN THE CONTROL OF A SPACESHIP

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1. Introduction and summary

Imagine a spaceship travelling towards a certain planet with predetermined speed, in a direction which will bring it close to the target after a known period of time. Observations on the position of the target, relative to the present course, are made continuously and lead to a gradually improving prediction of the eventual miss distance. On the other hand, the fuel available in the spaceship for making minor changes in the direction of motion, is gradually losing its effectiveness. This is because the final change of position caused by a small velocity imposed perpendicular to the present motion, is roughly proportional to the remaining time. Thus we have a control problem which is essentially one of compromise between the extremes of using the fuel early and perhaps in the wrong way, because of poor information; or waiting too long for more precise information, so that the fuel becomes ineffective.

The statistical decision problem considered here actually arises from a simplified formulation of the above question, but one which contains its main features. We suppose first that the motion of the spaceship relative to its target is confined to a fixed plane with the target as origin. The horizontal component of velocity is fixed as unity, so that the time coordinate $\tau \leq 0$, also represents the horizontal distance to be travelled before the target is passed. It is enough to represent the vertical components of position and velocity together by μ, the height at which the present line of motion meets the axis $\tau = 0$. However, μ is unknown, and must be estimated continuously by observing a certain stochastic process $\{W(\tau); \tau \leq 0\}$ whose mean drift is μ per unit time.

A second fiction, which will be maintained throughout the present paper, is that an infinite quantity of fuel is available for adjusting the vertical velocity, and hence μ, at a fixed price c per unit change of velocity. Thus at any time τ, an instantaneous velocity increment Δ, costing $c\Delta$, will change the unknown quantity μ by a known amount $\Delta|\tau|$. The problem is to find a control procedure which minimizes the sum of all fuel costs together with a cost associated with the final miss. For the most part, we shall assume that this terminal cost is given by $\frac{1}{2}k\mu^2$. Because of the symmetry of this function, the direction of any control

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applied is always determined by the sign of the current estimate of μ. The important question is how large this estimate must be before it is advantageous to take some action.

Section 2 is concerned with finding a convenient description of the state of the system. Under the assumption that the information process $\{W(\tau)\}$ is Gaussian with suitable initial conditions, the posterior distribution of μ has the normal form $\mathfrak{N}(y, s)$, where the variance s is a strictly increasing function of $-\tau$: approximately linear. Then by transforming the time parameter, we can conveniently regard s as the time to go. The corresponding mean y is determined, after observing a related stochastic process $\{Y(s')\}$ during $s' \geq s$, simply by $y = Y(s)$. It turns out that $\{Y(s); s \geq 0\}$ is a standard Wiener process in the $(-s)$ scale. In other words, given any position (y, s), if no controlling action is taken before s falls to $s - \delta$, then a new position $(y + \delta Y, s - \delta)$ arises in which $\delta Y = Y(s - \delta) - Y(s)$ has the conditional distribution $\mathfrak{N}(0, \delta)$.

Both actions and costs are easily translated into the (y, s) coordinate system. An action is represented by instantaneous change in the value of y, and a shift Δy in either direction incurs a cost $D(s)\,|\Delta y|$, where $D(s)$ is a specified positive function. The terminal cost, incurred when any position $(y, 0)$ is reached, is given in general by a function $R(y, 0)$ which is easily evaluated since, at this final stage $\mu = y$ is known. In the special case mentioned previously, suitable changes of scale lead to a standard form in which $R(y, 0) = \frac{1}{2}y^2$ and $D(s) = 1/s$.

Since the pair (y, s) always provides a complete description of the state of the system, it follows that in seeking an optimal control procedure to minimize costs, we may restrict attention to policies which depend only on these coordinates. In effect, we must classify each point (y, s) in the half-plane $s > 0$, according to whether or not some action is involved when that position arises.

For a discrete time variation of this problem where fuel could be used only at certain specified times, C. T. Striebel and F. Tung [9] used dynamic programming techniques to show that an optimal procedure can be expressed in terms of a boundary $\tilde{y}(s)$ as follows: if s corresponds to an allowable action time and $|y| > \tilde{y}(s)$, use fuel to go to $(\operatorname{sgn} y)\tilde{y}(s)$. Otherwise, do not use fuel.

This result clearly indicates that the solution of the continuous time version of the problem has the same form. It is now also clear that variations of the problem with different and possibly asymmetric terminal costs $R(y, 0)$ would lead to similar solutions. That is, the optimal policy corresponds to an action region $\mathfrak{A}$. If (y, s) is inside the action region, fuel must be instantaneously applied to bring (y, s) to an appropriate point of the waiting region Ω, which is the complement of $\mathfrak{A}$. This characterization is also suggested by results of R. T. Orford [7] on a related problem.

The optimal boundary curves $\pm\tilde{y}(s)$ can be determined, in principle, in terms of the Bayes risk function $R(y, s)$, which represents the minimum expected cost incurred when one starts at the point (y, s). The properties of $R(y, s)$ and the fact that the optimal procedure corresponds to the solution of a Free Boundary Problem are discussed in section 3. C. T. Striebel [8] independently and previously derived the "necessity" of these free boundary conditions in the sense that conditions are presented under which there is an optimal procedure that satisfies the free boundary problem. The effect of a policy determined by boundary curves $\pm\tilde{y}(s)$ is to constrain the process $\{Y(s)\}$ so that its trajectory always lies within or on the boundary of the region Ω. To see how this works, let us consider only the modifications imposed at the upper boundary. Conditional on $Y(s_0) = y_0$, for any initial position (y_0, s_0) we define the process

$$(1.1) \qquad M(s_1) = \max\,[0, \sup_{s_0 \geq s \geq s_1} (Y(s) - \tilde{y}(s))], \qquad (s_1 < s_0).$$

This represents the cumulative effect of suppressing the original path below the curve $\tilde{y}(s)$ and would lead to a position (y_1, s_1), where $y_1 = Y(s_1) - M(s_1)$.

It is instructive to regard the policy as the limit of a sequence of restrictions to discrete time when the corresponding time intervals approach zero. Here each member of the sequence is defined by a discrete set of values of s and actions, determined by the critical levels $\pm\tilde{y}(s)$, are allowed only at the specified instants. Of course, when actions are restricted in this way, the results are suboptimal, but any discrete time formulation can be considered in its own right. The relation between the two approaches is illustrated by the fact that sequences of optimal discrete time policies and the associated risk functions, converge in a natural way to their continuous counterparts.

The last assertion can be justified by reference to similar results which have been established for sequential tests of a normal mean [5]: a problem which is in a certain sense, equivalent to ours. The relevant characteristics of the testing problem, originally described in [3], are illustrated in the following stopping problem.

This rather artificial version of the testing problem, where the s-axis forms one boundary, will be associated with our present control problem. We restrict attention to the quadrant $y, s \geq 0$ of the plane. Changes of position within this region occur exactly as before, according to the process $\{Y(s)\}$, but termination may occur in any one of three ways: (1) if a position $(0, s)$ is reached, the process stops automatically without cost; (2) similarly, the process must stop at any position $(y, 0)$, and a cost R_y, $(y, 0)$ is incurred; and (3) in any other position (y, s), not on either axis, it may be elected to stop and pay a specified amount $D(s)$.

In this problem, the optimal risk function $V(y, s)$ determines the optimal policy in a very simple way. Any point (y, s) must be assigned to the stopping or action region if $V(y, s) = D(s)$ and to the continuation region if $V(y, s) < D(s)$. In particular, the strict inequality indicates that there is a policy which achieves a definite advantage over the given stopping cost, at the particular position (y, s).

It will be shown in section 4 that $V(y, s)$ and the derivative $R_y(y, s)$ of the original risk function, restricted to $y, s \geq 0$, are determined by precisely the same properties and may be identified. Thus, the same curve $\tilde{y}(s)$ defines the optimal policy for both problems.

With this interpretation of $R_y(y, s)$, the general techniques discussed in [1], can be applied to find approximations to the unknown boundary. In spite of their formal similarity, there is a practical difference between the two problems. For example, the restriction to $y \geq 0$ is unnatural for the original testing problem and such cost functions as $D(s) = 1/s$ and $R_y(y, 0) = y$ would be inappropriate. It is perhaps more accurate to describe the minimization of $V(y, s)$ as a stopping problem. Then it is natural to refer to the upper halves of $\mathfrak{A}$ and Ω as the optimal stopping and continuation regions.

Section 5 is concerned with obtaining specific inner and outer approximations to the optimal boundary curve $\tilde{y}(s)$, for the above special case. As a consequence, $\tilde{y}(s)$ is determined within fairly narrow limits for all values of s. In particular, it is deduced that

$$\tilde{y}(s) = \frac{1}{s} + O(s^2), \qquad (s \to 0), \tag{1.2}$$

$$\tilde{y}(s) = s^{1/2} + O(s^{-1/2}), \qquad (s \to \infty). \tag{1.3}$$

The main object of the remaining sections is to indicate techniques which can be used to refine these asymptotic bounds. One such refinement shows that $\tilde{y}(s) = \frac{1}{2}s^2 + o(s^2)$, $(s \to 0)$, and this is supplemented by a formal expansion which gives $\tilde{y}(s) = 1/s + \frac{1}{2}s^2 - \frac{1}{2}s^5 + \frac{7}{2}s^8 + \cdots$. For $s \to \infty$, we establish that $\tilde{y}(s)s^{-1/2} - 1$ is roughly of the order $s^{-\eta_0}$, where $\eta_0 = 1.61005$.

Our approach throughout is based on comparisons between the given problem and certain auxiliary stopping problems for which the solution is known. For example, the treatment of the case $s \to 0$ is closely related to the solution as $s \to \infty$ of another optimal stopping problem, defined by the stopping cost $d(y, s) = -s$ for $s > 0$, and with terminal cost on the s-axis, $d(y, 0) = \min(y, 0)$. In section 6 it is shown that the optimal boundary for this auxiliary problem is $\tilde{z}(s) = -s + \frac{1}{2} + o(1)$ as $s \to \infty$ and the corresponding minimum risk is $v(y, s) = y - \frac{1}{2}\exp(2y + 2s - 1) + o(1)$ in the continuation region. This result is applied to give $\tilde{y}(s) = s^{-1} + \frac{1}{2}s^2 + o(s^2)$ as $s \to 0$. The same ideas motivate the formal expansion for $\tilde{y}(s)$ in section 7.

The refined inner and outer bounds for $s \to \infty$ are presented in section 8. We rely on two facts. In the first place $u(y, s) = s^{-\lambda/2}\alpha F\{(\lambda + 1)/2, \frac{3}{2}, -\alpha^2/2\}$ with $\alpha = ys^{-1/2}$, is a solution of the basic diffusion equation satisfied by $V(y, s)$. Here F is the confluent hypergeometric function. Second, there is a number $\lambda_0 = 2\eta_0 + 2$, which is the smallest $\lambda \geq 1$ such that u vanishes when $\alpha = 1$.

In the final section, assumptions slightly different from those of section 2 lead to a stopping risk of the form $D(y, s) = s^{-1} + a$ for $y > 0$, $s > 0$. In this case, a formal expansion of the type described in [4], [6], is initiated to show that $\tilde{y}(s)s^{-1/2} \sim (2 \log s)^{1/2}$ as $s \to \infty$.

From the practical point of view, our assumption that there is an infinite quantity of fuel available in the spaceship is unsatisfactory. The assumption can be relaxed at the expense of dealing with a third variable. We have found that the present approach can still be applied in a fairly straightforward manner. It is hoped that these developments will be discussed later.

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2. Preliminaries

In studying the information process, we may imagine that μ is a fixed parameter. Any controls are easily taken into account, since they lead to corresponding translations of μ. We suppose that initially, μ has a normal prior distribution denoted by $\mathfrak{N}(y_0, s_0)$, and that the observed process $\{W(\tau); \tau_0 \leq \tau \leq 0\}$ has independent normal increments with mean μ and known variance $\sigma^2(\tau)$ per unit time. The situation is analogous to one in which a succession of independent normal observations have a common unknown mean and arbitrary known variances. In that case, it is not difficult to verify that the posterior distribution is normal at every stage, and a similar result holds here, where observation takes place continuously. Thus the information accumulated up to the instant τ, can be summarized in a posterior distribution of the form $\mathfrak{N}(y, s)$. Let us consider briefly how the parameters $y(\tau)$ and $s(\tau)$ behave.

It is enough to represent the results $W(\tau'); \tau < \tau' \leq \tau + \delta\tau$, of a very short period of observation by the final value or equivalently by the total increment δW. For example, this would certainly be valid for a constant variance function $\sigma^2(\tau')$ since δW would then constitute a sufficient statistic from the new observations. We must investigate the joint distribution of μ and δW, where the marginal for μ is given by the pair (y, s). Then by finding the conditional distribution of μ given δW, which is the new posterior distribution at $\tau + \delta\tau$, the increments δy and δs can be evaluated. This calculation leads to the following equations.

$$\frac{\delta s}{s^2} = -\frac{\delta\tau}{\sigma^2(\tau)} + o(\delta\tau), \tag{2.1}$$

$$\delta y = \frac{s}{\sigma^2(\tau)}(\delta W - y\delta\tau) + o(\delta\tau). \tag{2.2}$$

The first corresponds to a simple differential equation for $s(\tau)$, and the solution is

$$\frac{1}{s} = \frac{1}{s_0} + \int_{\tau_0}^{\tau} \frac{du}{\sigma^2(u)}. \tag{2.3}$$

This determines s as a monotone strictly decreasing function of τ. On the other hand, y changes stochastically. We replace δy by δY, in order to represent the increment as a random variable, conditional on the information available at time τ. With this conditioning, δW can be expressed as

$$\begin{aligned} \delta W &= \mu\delta\tau + \sigma\epsilon_1(\delta\tau)^{1/2} \\ &= (y + s^{1/2}\epsilon_2)\delta\tau + \sigma\epsilon_1(\delta\tau)^{1/2}, \end{aligned}$$

where ϵ_1, ϵ_2 are independent standard normal variates and it follows from (2.2) and (2.1) that δY has the distribution $\mathfrak{N}(0, -\delta s)$. Thus we have a representation of the original information process by a derived process $\{Y(s)\}$, for which the decreasing quantity s is a natural index.

Our treatment of the information does not depend on any special characteristics of the variance function $\sigma^2(\tau)$, but the intended application suggests a particular function, for which the determination of the Wiener process can be made more explicit. We are regarding $-\tau$ as the horizontal distance which remains, and it is reasonable to suppose that any errors in observing the target will have standard deviations proportional to this distance. In view of this, it is important to consider the special case where

$$\sigma^2(\tau) = a\tau^2, \qquad (\tau \leq 0). \tag{2.4}$$

Then, relation (2.3) reduces to

$$s = \frac{-a\tau}{1 - b\tau}, \qquad (\tau_0 \leq \tau \leq 0), \tag{2.5}$$

where the constant $b = a/s_0 + 1/\tau_0$. The function $s(\tau)$ is approximately linear, and exactly so if $s_0 = -a\tau_0$. It was assumed earlier that the initial information is in a convenient normal form, and we might pretend further that the corresponding variance s_0 is such that $b = 0$. Alternatively, it is not unrealistic to suppose that both s_0 and $-\tau_0$ are very large and in the limit, again b vanishes. Then we have

$$s = -a\tau, \tag{2.6}$$

and $s \geq 0$ can be interpreted as the time to go.

The function $y(\tau)$, and hence the observed path of the process $\{Y(s); s_0 \geq s \geq 0\}$, can be evaluated by solving the stochastic differential equation (2.2). On substituting the special forms (2.4) and (2.6), it becomes $\delta y - (y/\tau)\,\delta\tau = -(1/\tau)\,\delta W$. Here there is no difficulty, since $W(\tau)$ is a.s. continuous and the solution is

$$\frac{y}{\tau} = \frac{y_0}{\tau_0} - \frac{W(\tau)}{\tau^2} - \int_{\tau_0}^{\tau} \frac{2}{u^3}\, W(u)\, du. \tag{2.7}$$

From now on, we shall treat the control problem entirely in terms of the (y, s) coordinate system. In general, let $D(s)$ denote the cost of an optional unit change in the value of y. Any such action results in a corresponding translation of the whole future path $\{Y(s'); s \geq s' \geq 0\}$. The terminal cost incurred on arrival at any point on the y axis is given by $R(y, 0)$.

One further preliminary task remains and this is to normalize the specification of our main application. When equation (2.6) holds and the price of fuel is c, it follows from the discussion in section 1, that $D(s) = -c/\tau = ac/s$. Also, we have $R(y, 0) = \frac{1}{2}ky^2$. Now consider the transformation

$$y^* = \beta y, \qquad s^* = \beta^2 s. \tag{2.8}$$

By relating the scale changes in this way, we have ensured that the transformed process $\{Y^*(s^*)\}$ is still a standard Wiener process. For example,

$$\operatorname{var}(\delta Y^*) = \beta^2 \operatorname{var}(\delta Y) = \beta^2(-\delta s) = -\delta s^*. \tag{2.9}$$

We can also change the unit of cost by a factor γ, so that in the new system the cost of a shift Δy^* is given by

$$D^*(s^*)\,\Delta y^* = \gamma\,\frac{ac}{s}\,\Delta y = \beta\gamma\,ac\,\frac{1}{s^*}\,\Delta y^*. \tag{2.10}$$

The new terminal cost is

$$R^*(y^*, 0) = \gamma\,\tfrac{1}{2}k\,y^2 = \frac{\gamma k}{\beta^2}\,\tfrac{1}{2}y^{*2}. \tag{2.11}$$

It follows by examining these final coefficients, that if

$$\beta = a^{-1/3}c^{-1/3}k^{1/3}, \qquad \gamma = a^{-2/3}c^{-2/3}k^{-1/3}, \tag{2.12}$$

then $D^*(s^*) = 1/s^*$ and $R^*(y^*, 0) = \frac{1}{2}y^{*2}$.

In the later sections it will be assumed without loss of generality, that $ac = k = 1$, but before we discard the present notation, it is worth mentioning a consequence of the scale changes. It will be shown that the optimal policy for the normalized version is determined by a curve $\tilde{y}^*(s^*)$, such that $\tilde{\alpha}^*(s^*) \to 1$ as $s^* \to \infty$, where $\tilde{\alpha}^*(s^*) = \tilde{y}^*(s^*)/s^{*1/2}$. From the practical point of view, it is reasonable to consider values for the constants with $k/c \gg 1$, since it may be relatively important to 'hit' the target. In this case, the optimal boundary $\tilde{y}(s)$ in the original specification is given approximately by $\tilde{y}(s) \approx s^{1/2}$ for every $s > 0$. More precisely, for any fixed value of s,

$$\tilde{\alpha}(s) = \tilde{\alpha}^*(sa^{-2/3}(k/c)^{2/3}), \tag{2.13}$$

which converges to 1 as $k/c \to \infty$, by the asymptotic result just quoted.

3. Properties of the risk function

In this section, we consider the characteristic properties of the Bayes risk function $R(y, s)$. The actual costs of any procedure must be calculated according to specified continuous functions $D(s)$ and $R(y, 0)$, where the latter is symmetric and convex. In general for $s > 0$, $R(y, s)$ can be defined as the infimum for all possible control procedures of the total expected cost incurred after starting in the position (y, s). However, this local definition is not completely satisfactory, since we are assuming that there is a control procedure, determined by curves $y = \pm\tilde{y}(s)$, which is uniformly optimal for every position. Hence, $R(y, s)$ can be used alternatively to denote the risk function for a particular (well-behaved) policy. It will be assumed further, that $R(y, s)$ possesses continuous partial derivatives R_{yy} and R_s, except perhaps at points along the boundary curves. A slightly deficient justification for the heuristic argument will emerge later. Its proper foundation depends, as we shall see, on the existence of a solution with the appropriate formal properties, but no such proof will be attempted. Even for the application we have in mind, it is no easy task to find an explicit solution. However, the technique of comparing the central problem with similar cases for which the formal solution is known, and therefore justified, should leave little room for doubt.

It is clear that the optimal risk $R(y, s)$ must be symmetric in y and for the most part, we shall restrict attention to positions with $y, s \geq 0$. Consider first, the value of an action which changes y by an amount Δy. The original risk, expressed in terms of the new position, is simply $D(s)\,\Delta y + R(y + \Delta y, s)$, and since the action may or may not be profitable, we have $R(y, s) \leq D(s)\,|\Delta y| + R(y + \Delta y, s)$. It follows by letting Δy approach zero through positive and negative values that in general

$$|R_y(y, s)| \leq D(s). \tag{3.1}$$

Again, when $y > \tilde{y}(s)$ and the optimal policy prescribes a shift $\Delta y = \tilde{y}(s) - y$, we obtain

$$R(y, s) = R(\tilde{y}(s), s) + D(s)(y - \tilde{y}(s)). \tag{3.2}$$

The risk function is linear in y throughout the action region, and $R_y(y, s) = \pm D(s)$ according to the sign of y. In view of this, the determination of $R(y, s)$ depends largely on its properties within the continuation region Ω and at the boundary.

We now show that $R(y, s)$ is a solution of the diffusion equation

$$\tfrac{1}{2}R_{yy} = R_s, \qquad (|y| < \tilde{y}(s), s > 0). \tag{3.3}$$

Let $(y, s)\epsilon\Omega$ and consider the transition to $(y + \delta Y, s - \delta)$ after a short period of length δ. It follows that $R(y, s) = E[R(y + \delta Y, s - \delta)]$, apart from the possibility that some action is necessary during the period. Here, we can rely on the fact that δY is distributed as $\mathfrak{N}(0, \delta)$. Only terms of order δ will be needed in evaluating the expectation. By making use of the differentiability assumptions, it is not difficult to establish the following expansion:

$$R(y, s) = E[R(y, s) + R_y(y, s)\delta Y + \tfrac{1}{2}R_{yy}(y, s)\delta Y^2 - R_s(y, s)\delta] + o(\delta), \tag{3.4}$$

or equivalently,

$$0 = R_y(y, s)E(\delta Y) + \tfrac{1}{2}R_{yy}(y, s)E(\delta Y^2) - R_s(y, s)\delta + o(\delta). \tag{3.5}$$

But $E(\delta Y) = 0$, $E(\delta Y^2) = \delta$, and since the coefficient of δ must vanish, the result is equation (3.3).

Thus, in order to determine the risk function, we must solve a differential equation; but the boundary is unknown, and it is not immediately clear what conditions should be imposed there. Let us assume a priori that $R(y, s)$ itself must be continuous at the boundary but that its derivatives may have simple discontinuities. The previous analysis can be adapted for positions on the boundary curves, but a more sensitive treatment is needed.

Consider a fixed point (y_0, s_0) on the upper boundary curve. We suppose that $\tilde{y}(s) = y_0 + o(s_0 - s)^{1/2})$ as $s \to s_0-$, which is slightly stronger than our original continuity assumption. After a short period δ, the new position is $(Y(s_0 - \delta) - M(s_0 - \delta), s_0 - \delta)$, where $Y(s_0) = y_0$, $Y(s_0 - \delta) = y_0 + \delta Y$, and $M(s_0 - \delta)$ represents the total action which occurs. It will be enough to retain terms of order $\delta^{1/2}$ in relating the corresponding risks, and hence we can approximate the distribution of $M(s_0 - \delta)$.

$$(3.6)\qquad \begin{aligned} M(s_0 - \delta) &= \max\,[0, \sup_{s_0 \geq s \geq s_0 - \delta} \{Y(s) - \tilde{y}(s)\}] \\ &= \sup_{s_0 \geq s \geq s_0 - \delta} [\{Y(s) - Y(s_0)\} - \{\tilde{y}(s) - \tilde{y}(s_0)\}] \\ &= \sup_{s_0 \geq s \geq s_0 - \delta} \{Y(s) - Y(s_0)\} + o(\delta^{1/2}). \end{aligned}$$

It can be shown by symmetry considerations that the main term here has precisely the same distribution as $|\delta Y|$. In particular,

$$(3.7)\qquad E\{M(s_0 - \delta)\} = E\{|\delta Y|\} + o(\delta^{1/2}) = 2^{1/2}\pi^{-1/2}\delta^{1/2} + o(\delta^{1/2}).$$

Now let $R_y^- = \lim_{y \uparrow y_0} R_y(y, s_0)$. By (3.2), the corresponding right limit is $R_y^+ = D(s_0)$, since it is approached through $\mathcal{A}$. We note that all the costs incurred during the period of interest can be evaluated according to $D(s_0)$, whereas the final position must lie in Ω, so that the corresponding risk involves R_y^-. Then

(3.8)

$$\begin{aligned} R(y_0, s_0) &= E[M(s_0 - \delta)D(s_0) + R(Y(s_0 - \delta) - M(s_0 - \delta), s_0 - \delta)] + o(\delta^{1/2}) \\ &= D(s_0)E[M(s_0 - \delta)] + E[R(y_0, s_0) + R_y^- \{\delta Y - M(s_0 - \delta)\}] + o(\delta^{1/2}). \end{aligned}$$

On collecting the terms of order $\delta^{1/2}$, we are left with

$$(R_y^- - D(s_0))E[M(s_0 - \delta)] = o(\delta^{1/2}),$$

and it follows that $R_y^- = D(s_0)$. In general, R_y is continuous at the boundary, and hence R_s must be also. In particular,

$$(3.9)\qquad R_y = D, \qquad (y = \tilde{y}(s)).$$

It is important to recognize that relations (3.2), (3.3), and (3.8) do not depend in any way on the optimality of the policy. In fact, all three can be applied to the risk function for an arbitrary policy with specified boundaries. For the optimal policy, there is an extra condition which will be useful in locating the curve $\tilde{y}(s)$: the second derivative R_{yy} must be continuous at the boundary of Ω. This means that

$$(3.10)\qquad R_{yy} = 0, \qquad (y = \pm\tilde{y}(s)).$$

To verify this necessary condition for optimality, we again consider a starting point (y_0, s_0) with $y_0 = \tilde{y}(s_0)$. But now, let us modify the optimal policy locally. No action is permitted during the period $s_0 \geq s > s_0 - \delta$, but the original procedure must be resumed at $s_0 - \delta$. The initial risk for the modified policy is $R^{(\delta)}(y_0, s_0) = E[R(y_0 + \delta Y, s_0 - \delta)]$.

In view of (3.9), there will be no terms of order $\delta^{1/2}$ here, and the expectation can be dealt with almost as before, in the derivation of equation (3.3). However, in this case, the term involving δY^2 needs special treatment. Let R_{yy}^+ and R_{yy}^- denote the one-sided second derivatives evaluated at (y_0, s_0) by suitable limiting operations. We obtain

$$(3.11)\qquad \begin{aligned} R^{(\delta)}(y_0, s_0) &= R(y_0, s_0) + R_y(y_0, s_0)E(\delta Y) + \tfrac{1}{2}R_{yy}^- E(\delta Y^2) - R_s(y_0, s_0)\delta \\ &\quad + \tfrac{1}{2}(R_{yy}^+ - R_{yy}^-)E(\delta Y^2; \delta Y > \tilde{y}(s_0 - \delta) - \tilde{y}(s_0)) + o(\delta). \end{aligned}$$

Equation (3.3) holds as (y, s_0) approaches (y_0, s_0) from below, and continuity ensures that $\frac{1}{2}R_{yy}^- = R_s(y_0, s_0)$. Also, the restricted expectation can be replaced by $E(\delta Y^2; \delta Y > 0) + o(\delta)$ and what remains of the expansion is

$$R^{(\delta)}(y_0, s_0) = R(y_0, s_0) + \tfrac{1}{4}(R_{yy}^+ - R_{yy}^-)\delta + o(\delta). \tag{3.12}$$

The modified policy is suboptimal, and hence $R^{(\delta)}(y_0, s_0) \geq R(y_0, s_0)$ no matter what the value of δ. It follows that $R_{yy}^+ \geq R_{yy}^-$. On the other hand, relation (3.2) indicates that $R_{yy}^+ = 0$, but conditions (3.1) and (3.9) together imply that $R_{yy}^- \geq 0$. The result is a contradiction unless $R_{yy}^+ = R_{yy}^-$.

For a given boundary, equations (3.3) and (3.9) determine a solution of the diffusion equation. In our problem, the boundary is not known, and, hopefully, the extra boundary condition (3.10) determines the boundary. Equations (3.3), (3.9), and (3.10) are said to define a free boundary problem, and we have shown that a well-behaved solution of the optimization problem is a solution of the free boundary problem. Essentially this result was independently and previously derived by C. T. Striebel [8]. Somewhat more crucial to our applications are the conditions discussed in section 4 by which a solution of the free boundary problem is a solution of the optimality problem.

4. An associated problem

Having established the most useful properties of $R(y, s)$, we shall employ them in a rather indirect way. In section 1, we introduced another problem with minimum risk function $V(y, s)$ for positions in the positive quadrant. This problem is conceptually simpler, because a decision to act is necessarily final and the corresponding instantaneous cost $D(s)$ can be compared with the expected cost of continuing. Hence, the appropriate boundary curve $\tilde{y}(s)$ and the two axes can be treated as absorbing barriers. There is no need to consider the process $\{Y(s)\}$ after absorption takes place. Nevertheless, we shall see that the two problems are equivalent in the sense that the same continuation region is optimal for both. For the moment, let $\tilde{y}(s)$ represent the optimal policy corresponding to $V(y, s)$.

The formal properties of $V(y, s)$ are listed below.

$$\tfrac{1}{2}V_{yy} = V_s, \qquad (0 < y < \tilde{y}(s), \quad s > 0), \tag{4.1}$$

$$V = D, \qquad (y \geq \tilde{y}(s)), \tag{4.2}$$

$$V_y = 0, \qquad (y = \tilde{y}(s)). \tag{4.3}$$

These conditions are analogous to (3.3), (3.9), and (3.10) respectively and can be derived similarly. Notice, however, that (4.2) is intuitively obvious. As before, only the last is an optimality condition. In addition, the automatic termination of the path when either axis is reached, leads to the specified costs,

$$V(0, s) = 0, \qquad (s > 0), \tag{4.4}$$

$$V(y, 0) = R_y(y, 0), \qquad (y \geq 0). \tag{4.5}$$

We now observe that all these properties are satisfied by the function $R_y(y, s)$; $y, s \geq 0$. Equation (4.1) follows by differentiating (3.3), and the rest are directly applicable.

From a more fundamental point of view, it is clear that a solution to the associated problem must provide a solution to the original. If $V(y, s)$ represents the local minimum expected cost for every position (y, s) in the positive quadrant, then the function $R(y, s)$ obtained by setting $R_y(y, s) = V(y, s)$ must be minimal. More precisely,

$$R(y, s) = \int_0^{|y|} R_y(y', s)\, dy' + \int_0^s R_s(0, s')\, ds' + R(0, 0), \tag{4.6}$$

$$R(y, s) = \int_0^{|y|} V(y', s)\, dy' + \int_0^s \tfrac{1}{2} V_y(0, s')\, ds' + R(0, 0). \tag{4.7}$$

Here we use the fact that

$$R_s(0, s') = \tfrac{1}{2} R_{yy}(0, s') = \tfrac{1}{2} V_y(0, s'). \tag{4.8}$$

Since $V_y(0, s') = \lim_{h \downarrow 0} (1/h)\, V(h, s')$, each of the above integrands is everywhere minimal and the conclusion follows.

It remains to make sure whether we have specified enough conditions to determine the function $V(y, s)$. In general, the properties (4.1)–(4.5) are not sufficient, because (4.3) does not fully represent the optimality of the policy. We shall impose two further optimality conditions in order to ensure that there is at most one solution. It was remarked earlier that the optimal policy can be defined by the inequality

$$V < D, \qquad (0 < y < \tilde{y}(s)). \tag{4.9}$$

Again, for any position (y, s) and $0 < \delta < s$, consider the suboptimal modified procedure: continue for a period of length δ and then resume the optimal policy. In particular, for points (y, s) in the optimal stopping region, we have

$$E[V(y + \delta Y, s - \delta)] \geq D(s), \qquad (y \geq \tilde{y}(s)). \tag{4.10}$$

Strictly speaking, the path should terminate if it crosses the s-axis, but we can alternatively and equivalently treat the integrand as an odd function of $y + \delta Y$. The boundary condition (4.3) is a convenient, but very special form of (4.10). The general condition is needed in order to complete the characterization of $V(y, s)$.

We have regarded the function $V(y, s)$ as the minimum risk for each separate position. But it has been assumed that there is a single policy which attains this minimum everywhere and determines $V(y, s)$. More important; we have assumed in deriving its formal properties that $V(y, s)$ is suitably differentiable. In order to justify the approach, let us distinguish temporarily between the extremal and the formal properties of $V(y, s)$.

In what follows, let $V(y, s)$ denote a risk function which satisfies the formal conditions (4.1)–(4.5), (4.9)–(4.10), assuming that such a solution exists. Let $V^*(y, s)$ be the infimum over all control procedures, of the risk at the point (y, s).

Thus, given any position (y_0, s_0) and any $\epsilon_0 > 0$, there exists a policy with risk function $V^{(0)}(y, s)$ such that

$$V^*(y_0, s_0) \leq V^{(0)}(y_0, s_0) \leq V^*(y_0, s_0) + \epsilon_0. \tag{4.11}$$

We now prove that $V(y, s) = V^*(y, s)$, which means that the minimization problem will be properly treated provided that we can find the formal solution. Although the existence of $V(y, s)$ is in general open to question, it can sometimes be found explicitly, and we shall rely on this later for certain special cases connected with our main application. The equivalence of the definitions can be established from the fact that $V(y, 0) = V^*(y, 0)$, but since the cost function $D(s)$: continuous for $s > 0$, may be unbounded as $s \to 0$, it is convenient to assume further that

$$\sup_y \,[V(y, s) - V^*(y, s)] \to 0 \qquad \text{as} \quad s \to 0. \tag{4.12}$$

In practice, this is not difficult to verify, by finding a crude approximation to $V^*(y, s)$. The following argument is based essentially on that given in [2], section 6.

LEMMA 4.1. *If $V(y, s)$ is a risk function which satisfies conditions* (4.1)–(4.5), (4.9)–(4.10) *and* (4.12), *and if $V^*(y, s)$ is the minimum risk function, then*

$$V(y, s) = V^*(y, s), \qquad (y, s \geq 0).$$

PROOF. Consider any fixed position (y_0, s_0) with $y_0, s_0 > 0$ and let $\epsilon_0, \epsilon_1, \epsilon_2$ be arbitrary positive numbers. By (4.11), we can find a policy such that its risk function satisfies

$$V^{(0)}(y_0, s_0) \leq V^*(y_0, s_0) + \epsilon_0. \tag{4.13}$$

Now choose $s_1 < s_0$, using assumption (4.12), so that for every $y \geq 0$,

$$V(y, s_1) \leq V^*(y, s_1) + \epsilon_1 \leq V^{(0)}(y, s_1) + \epsilon_1. \tag{4.14}$$

The function $D(s)$ is uniformly continuous on the closed interval $[s_1, s_0]$. Hence there is a $\delta = (s_0 - s_1)/n$ for some integer $n > 0$, which ensures that

$$\sup_{|s-s'|<\delta} |D(s) - D(s')| \leq \epsilon_2,$$

within the interval.

We now restrict attention to the period $s_0 \geq s \geq s_1$ and consider two procedures in which stopping is permitted only at the instants $s_0 = s_1 + n\delta$, $s_1 + (n - 1)\delta, \cdots, s_1$. Automatic stops on the s-axis can be included in this restriction by extending the appropriate risks as odd functions of y. Let $V_\delta(y, s)$ represent the optimal discrete procedure determined from the final cost $V_\delta(y, s_1) = V(y, s_1)$. Similarly, let $V_\delta^{(0)}(y, s)$ be the minimum risk when $V_\delta^{(0)}(y, s_1) = V^{(0)}(y, s_1)$. Related to these is the function $V_+^{(0)}(y, s)$ defined as follows. We make a slight modification of the stopping cost $D(s)$, but not the continuous policy associated with $V^{(0)}(y, s)$. Whenever any optional stop occurs, the cost is determined according to the next discrete instant. Thus $D(s)$ is replaced by $D(s')$, where $s' = s_1 + k\delta$ for some integer k and $s \geq s' > s - \delta$. It follows from our choice of δ that the extra cost can never exceed ϵ_2, and hence

$$V_{+}^{(0)}(y_0, s_0) \leq V^{(0)}(y_0, s_0) + \epsilon_2. \tag{4.15}$$

On the other hand, $V_{+}^{(0)}(y, s)$ can be regarded as the risk function for a certain discrete procedure. The same continuous procedure as before can be applied, with the provision that stopping actually takes place always at the next discrete instant. This means that part of the path will be disregarded. Since $V_\delta^{(0)}(y, s)$ represents the minimum risk for the restriction to discrete time, we have

$$V_\delta^{(0)}(y_0, s_0) \leq V_{+}^{(0)}(y_0, s_0). \tag{4.16}$$

The relation between $V_\delta^{(0)}(y, s)$ and $V_\delta(y, s)$ is a consequence of our choice of s_1. We have $V_\delta(y, s_1) \leq V_\delta^{(0)}(y, s_1) + \epsilon_1$, $(y \geq 0)$. Then $V_\delta(y, s_1 + \delta)$ and $V_\delta^{(0)}(y, s_1 + \delta)$ can be evaluated in terms of these quantities and the inequality is preserved. After n repetitions of this technique, we obtain

$$V_\delta(y_0, s_0) \leq V_\delta^{(0)}(y_0, s_0) + \epsilon_1. \tag{4.17}$$

Finally, we must make use of the properties of $V(y, s)$ to show that

$$V(y_0, s_0) \leq V_\delta(y_0, s_0). \tag{4.18}$$

Assume inductively that for some $k \geq 0$, $V(y, s_1 + k\delta) \leq V_\delta(y, s_1 + k\delta)$, $(y \geq 0)$. When $k = 0$, we have equality. In order to extend the result to the case $(k + 1)$, we note first that

$$\begin{aligned} V_\delta(y, s_1 + (k + 1)\delta) \\ = \min [D(s_1 + (k + 1)\delta), E\{V_\delta(y + \delta Y, s_1 + k\delta)\}]. \end{aligned} \tag{4.19}$$

If $y \geq \tilde{y}(s_1 + (k + 1)\delta)$, then

$$\begin{aligned} E\{V_\delta(y + \delta Y, s_1 + k\delta)\} \\ \geq E\{V(y + \delta Y, s_1 + k\delta)\} \geq D(s_1 + (k + 1)\delta) \end{aligned} \tag{4.20}$$

by (4.10) and hence $V(y, s_1 + (k + 1)\delta) \leq V_\delta(y, s_1 + (k + 1)\delta)$. A similar argument shows that

$$V(y', s') \leq V_\delta(y', s'), \qquad (y' = \tilde{y}(s');\quad s_1 + (k + 1)\delta > s' > s_1 + k\delta). \tag{4.21}$$

To obtain the corresponding inequality for positions $(y, s_1 + (k + 1)\delta)$ with $0 < y < \tilde{y}(s_1 + (k + 1)\delta)$, it is enough to show that

$$V(y, s_1 + (k + 1)\delta) \leq E\{V_\delta(y + \delta Y, s_1 + k\delta)\},$$

since condition (4.9) applies. The right-hand side can be evaluated as a conditional expectation as follows: $E\{V_\delta(y + \delta Y, s_1 + k\delta)\} = E\{V_\delta(Y', S')\}$, where (Y', S') is the point where the Wiener process through $(y, s_1 + (k + 1)\delta)$ first hits the barrier consisting of the curve $\tilde{y}(s)$, the line $s = s_1 + k\delta$, and the s-axis. But $V(y, s)$ is a solution of the diffusion equation in the region lying between these curves and is sufficiently well behaved (see [2], section 6) to justify a similar expression, $V(y, s_1 + (k + 1)\delta) = E\{V(Y', S')\}$. Then the required inequality is valid if $V(y', s') \leq V_\delta(y', s')$ at every point of the barrier. But this has already been verified for the case $y' = \tilde{y}(s')$, and it certainly holds along the two linear sections. The induction is now complete and (4.18) follows.

We conclude from the inequalities (4.13), (4.15)–(4.18) that

$$V(y_0, s_0) \leq V^*(y_0, s_0) + \epsilon_0 + \epsilon_1 + \epsilon_2. \tag{4.23}$$

Then, since $\epsilon_0, \epsilon_1, \epsilon_2$ were chosen arbitrarily, we obtain $V(y_0, s_0) \leq V^*(y_0, s_0)$, and this can only mean equality.

5. Approximations to the optimal boundary

In practice, we hope that conditions (4.1)–(4.5) and (4.9) will be sufficient to determine $V(y, s)$ and $\tilde{y}(s)$ from the given cost functions $D(s)$ and $R_y(y, 0)$. However, the methods used here to find approximations to the curve $\tilde{y}(s)$ involve a reversal of the natural direction of inference. We shall consider various solutions of the diffusion equation and investigate the variations of our minimization problem, determined from them in such a way that the appropriate properties hold by definition. The cost functions of these variations might seem irrelevant, but it is possible to arrange useful comparisons with the given $D(s)$ and $R_y(y, 0)$, so that a relation between the artificial policy and $\tilde{y}(s)$ can be inferred. In this connection, it is advantageous to think in terms of $V(y, s)$ rather than the original risk $R(y, s)$.

We confine our application of the above techniques to the case when

$$D(s) = \frac{1}{s}, \qquad R_y(y, 0) = y. \tag{5.1}$$

It is interesting that because $R_{yy}(y, 0) = 1$ here, the function $R_{yy}(y, s)$ has a special interpretation. Conditions (4.1) and (4.3) suggest that for $|y| \leq \tilde{y}(s)$, this second derivative represents the probability that the Wiener process escapes to the y-axis without hitting either boundary curve. Thus, we are attempting to select $\tilde{y}(s)$ so as to minimize the escape probability for every initial position, subject to an integral constraint given by (4.2). Much of the analysis which follows could also be developed from this point of view.

The solutions of the diffusion equation discussed here are all generated by the relation

$$V^{(\alpha)}(y, s) = E[V^{(\alpha)}(Y(0), 0)|Y(s) = y], \qquad (y, s \geq 0), \tag{5.2}$$

with suitably selected functions $V^{(\alpha)}(y, 0)$. The obvious choice indicated by (5.1), leads to $V^{(1)}(y, s) = y$. This satisfies equation (4.1) trivially.

If we now set $\tilde{y}^{(1)}(s) = 1/s$ and make the modification $V^{(1)}(y, s) = 1/s$ when $y > 1/s$, then conditions (4.2), (4.4), and (4.5) are satisfied. It can be shown that $V^{(1)}(y, s)$ is uniquely determined by these four properties, given the boundary curve $\tilde{y}^{(1)}(s)$, (see [1], section. 4]). Then we may conclude that $V^{(1)}(y, s)$ is the risk function which corresponds to this boundary. Condition (4.3) does not hold, so the policy is suboptimal. On the other hand, (4.9) is satisfied, and whenever $y < 1/s$, we have $V(y, s) \leq V^{(1)}(y, s) < D(s)$. It follows that

$$\tilde{y}(s) \geq \tilde{y}^{(1)}(s) = \frac{1}{s}. \tag{5.3}$$

We remark that the same risk function $V^{(1)}(y, s)$ also represents a quite different discrete procedure. This other policy is optimal when it must be decided to stop immediately at (y, s), or not at all until one of the axes is reached.

We now consider another solution of the diffusion equation, given by

$$V^{(2)}(y, s) = A\varphi(ys^{-1/2})ys^{-3/2} \qquad \text{where} \quad \varphi(u) = (2\pi)^{-1/2}e^{u^2/2}. \tag{5.4}$$

Let the constant $A = 1/\varphi(1)$, so that $V^{(2)}(y, s) = s^{-1}$ along the curve $\tilde{y}^{(2)}(s) = s^{1/2}$. Further, we have

$$V_y^{(2)}(y, s) = A\varphi(ys^{-1/2})s^{-3/2}(1 - y^2s^{-1}), \tag{5.5}$$

which vanishes when $y = s^{1/2}$. Hence, if we redefine the risk outside this curve by $V^{(2)}(y, s) = s^{-1}$, conditions (4.1)–(4.4) and (4.9) are all satisfied. The procedure specified by the curve $\tilde{y}^{(2)}(s)$ is not optimal for the cost function $R_y(y, 0) = y$. However, it is a useful policy whenever $s > 0$, since every path will be stopped before it reaches the y-axis. The corresponding risk function satisfies (4.9) and we can obtain an inner approximation just as before:

$$\tilde{y}(s) \geq \tilde{y}^{(2)}(s) = s^{1/2}. \tag{5.6}$$

Finally, we note that the special policy determined by $\tilde{y}^{(2)}(s)$ would be optimal if the terminal cost $R_y(y, 0)$ had been infinite for all $y < 0$. This reinforces the remarks made at the end of section 2.

The inequalities (5.3) and (5.6) together provide a fairly accurate inner approximation to the whole curve $\tilde{y}(s)$. But in order to show this, we must find suitable outer approximations. Here, the investigation of special suboptimal policies is no longer enough. Roughly speaking, we need to consider procedures which are optimal in situations where the decision maker is encouraged to continue by a reduction of future costs.

The last example can be modified to produce such a procedure. Let

$$V^{(3)}(y, s) = A\varphi(y(s + h)^{-1/2})y(s + h)^{-3/2}, \qquad (y, s \geq 0), \tag{5.7}$$

where $A, h > 0$ are parameters of the solution. For any fixed value of h, let us choose A in such a way that $V^{(3)}(y, 0) \leq y$, whenever $y \geq 0$. Since

$$V_y^{(3)}(y, s) = A\varphi(y(s + h)^{-1/2})(s + h)^{-3/2}\{1 - y^2(s + h)^{-1}\}, \tag{5.8}$$

which does not exceed $A\varphi(0)/h^{3/2}$ along the line $s = 0$, this can be achieved by setting $A = h^{3/2}/\varphi(0)$.

Consider the procedure determined by the curve $\tilde{y}_h(s) = (s + h)^{1/2}$; $s \geq 0$. If we imagine that the cost functions (5.1) are replaced by

$$V^{(3)}((s + h)^{1/2}, s) = A\varphi(1)(s + h)^{-1}, \qquad V^{(3)}(y, 0) = A\varphi(yh^{-1/2})yh^{-3/2} \tag{5.9}$$

respectively, then we have an optimal policy by reference to the lemma of section 4. The risk function $V^{(3)}(y, s)$ can be modified according to (4.2) outside the continuation region, and then all the conditions (4.1)–(4.5), (4.9) and (4.12) are satisfied.

It only remains to compare this auxiliary problem with the original. We observe that the stopping cost $A\varphi(1)/(s + h) \leq 1/s$, provided that $0 < s \leq s_h$, with equality at the instant $s_h = h/(A\varphi(1) - 1)$.

Now consider the boundary position $(\tilde{y}_h, s_h)$ for the optimal policy indexed by h. According to this constructed policy, the minimum risk attainable under the specifications (5.9) is $V^{(3)}(\tilde{y}_h, s_h) = 1/s_h$. But the actual cost incurred by stopping at any future time s: $s_h > s > 0$, or at $s = 0$, is less than that prescribed by (5.1). It follows that $V(\tilde{y}_h, s_h) \geq V^{(3)}(\tilde{y}_h, s_h) = D(s_h)$, and hence $\tilde{y}_h \geq \tilde{y}(s_h)$. Thus, as h varies, the point $(\tilde{y}_h, s_h)$ describes a curve $\tilde{y}^{(3)}(s)$ say, which is an outer approximation to the required optimal boundary. The form of $\tilde{y}^{(3)}(s)$ is implicit in our special choice of A, $\tilde{y}_h(s)$ and s_h:

$$(5.10) \qquad \left.\begin{aligned} s &= e^{1/2}h(h^{3/2} - e^{1/2})^{-1} \\ y &= (s + h)^{1/2} \end{aligned}\right\}, \qquad (h > e^{1/3}).$$

It is easily verified that s increases through every positive value as h decreases from ∞ to $e^{1/3}$. In particular, the following asymptotic formulae can be deduced.

$$(5.11) \qquad \tilde{y}(s) \leq \tilde{y}^{(3)}(s) = s^{1/2}\left\{1 + \tfrac{1}{2}e^{1/3}s^{-1} + O(s^{-2})\right\}, \qquad (s \to \infty),$$

$$(5.12) \qquad \tilde{y}(s) \leq \tilde{y}^{(3)}(s) = e^{1/2}s^{-1}\{1 + O(s^3)\}, \qquad (s \to 0).$$

The first of these, taken with (5.6), shows that

$$(5.13) \qquad \tilde{y}(s) = s^{1/2}\{1 + O(s^{-1})\}, \qquad (s \to \infty).$$

However, the second does not match so well with (5.3). Another outer approximation will be constructed to give a more precise description of the optimal boundary when s is small.

The following solution of the diffusion equation generates a useful auxiliary problem.

$$(5.14) \qquad V^{(4)}(y, s) = y - Be^{\beta^2 s/2}\sinh(\beta y), \qquad (y, s \geq 0).$$

We aim to choose the parameters $B, \beta > 0$ for a particular instant s and by making the proper comparison, find the level $\tilde{y}^{(4)}(s) \geq \tilde{y}(s)$, which provides a best local approximation. But it is more convenient to study the auxiliary problem first in its general form and then try to pick out a special position where the comparison is most relevant. We note that

$$(5.15) \qquad V_y^{(4)}(y, s) = 1 - B\beta e^{\beta^2 s/2}\cosh(\beta y),$$

$$(5.16) \qquad V_{yy}^{(4)}(y, s) = -B\beta^2 e^{\beta^2 s/2}\sinh(\beta y).$$

The optimal boundary for the auxiliary problem is determined simply by setting $V_y^{(4)}(y, s) = 0$, so that $B\beta\cosh(\beta y) = e^{-\beta^2 s/2}$.

Provided that B is not too large, the corresponding continuation region is bounded. Let us denote the boundary curve by $\tilde{y}_\beta(s)$ and restrict attention to positions (y, s) with $\tilde{y}_\beta(s) > 0$. As before, the risk outside the continuation region must be determined by applying condition (4.2) with D replaced by an appropriate $D^{(4)}(s)$. In spite of this, the relation $V^{(4)}(y, 0) \leq y$ remains valid, and it is enough to examine the auxiliary stopping cost along the boundary with reference to (5.1).

Consider the difference $V^{(4)}(\tilde{y}_\beta(s),s) - s^{-1}$, as s increases from zero as far as the instant when $\tilde{y}_\beta(s) = 0$. The quantity is negative at both ends of the curve, but it is clear that we can adjust its maximum value by choosing B. Suppose now that B is selected to make this maximum value zero. A necessary condition is that the differential along the curve should vanish, and by applying the diffusion equation, it can be expressed as

$$0 = V_y^{(4)}(\tilde{y}_\beta(s), s)\, dy + \{V_s^{(4)}(\tilde{y}_\beta(s), s) + s^{-2}\}\, ds, \tag{5.17}$$

$$0 = \left\{\tfrac{1}{2} V_{yy}^{(4)}(\tilde{y}_\beta(s), s) + s^{-2}\right\} ds. \tag{5.18}$$

Let $(\tilde{y}_\beta, s_\beta)$ be a point on the curve, at which the maximum is attained. For this position, the future costs associated with the auxiliary problem are uniformly less than those given by (5.1), but the present stopping cost is the same. Consequently we have $\tilde{y}_\beta \geq \tilde{y}(s_\beta)$.

The above construction implies that the following equations must hold simultaneously at the special position $(y, s) = (\tilde{y}_\beta, s_\beta)$:

$$V^{(4)}(y, s) = s^{-1}, \qquad V_y^{(4)}(y, s) = 0, \qquad V_{yy}^{(4)}(y, s) = -2s^{-2}. \tag{5.19}$$

Fortunately, these three properties are sufficient to define the construction. In fact, there is no difficulty in eliminating the parameters B and β. We have

$$\begin{cases} Be^{\beta^2 s/2} \sinh(\beta y) = y - s^{-1}, \\ B\beta e^{\beta^2 s/2} \cosh(\beta y) = 1, \\ B\beta^2 e^{\beta^2 s/2} \sinh(\beta y) = -2s^{-2}. \end{cases} \tag{5.20}$$

The elimination leads to a relation between y and s, which defines the required outer approximation $\tilde{y}^{(4)}(s)$ for all values of s:

$$2^{1/2}s^{-1}(y - s^{-1})^{1/2} - \tanh\{2^{1/2}ys^{-1}(y - s^{-1})^{-1/2}\} = 0. \tag{5.21}$$

It is a straightforward matter to verify that the expression on the left is a strictly increasing function of y in the range $y > s^{-1}$ and deduce the existence of a unique zero at $y = \tilde{y}^{(4)}(s)$.

The effectiveness of the approximation is suggested by simpler formulae which can be derived from (5.21) for extreme values of s. A limited expansion of the hyperbolic tangent can be used to show that

$$\tilde{y}^{(4)}(s) = (\tfrac{3}{2})^{1/2} s^{1/2}\{1 + O(s^{-3/2})\}, \qquad (s \to \infty). \tag{5.22}$$

In this case, according to (5.13), the approximation is too large by a factor $(\frac{3}{2})^{1/2}$. But for small values of s, the result is much more satisfactory. Since $\tanh(u) \leq 1$ always, equation (5.21) yields the inequality $\tilde{y}^{(4)}(s) \leq 1/s + \frac{1}{2}s^2$. Then by making use of $\tilde{y}^{(1)}(s)$, we obtain

$$\frac{1}{s} \leq \tilde{y}(s) \leq \frac{1}{s} + \tfrac{1}{2}s^2, \qquad (s > 0). \tag{5.23}$$

In particular, the difference between these two bounds approaches zero rapidly as $s \to 0$. The above inequality for $\tilde{y}^{(4)}(s)$ leads to a more robust formula by substituting for y in the second term of (5.21):

$$\tilde{y}^{(4)}(s) \approx \frac{1}{s} + \tfrac{1}{2}s^2 \tanh^2 \{1 + 2s^{-3}\}, \tag{5.24}$$

which is fairly accurate when $s < 1$.

The table below contains a summary of our results so far, giving each of the four approximations for several values of s. The first two functions tabulated are lower bounds and the others are upper bounds.

TABLE I

s	0.1	0.2	0.5	1.0	1.5	2.0	4.0	6.0	8.0	10.0
$\tilde{y}^{(1)}$	10.0	5.0	2.0	1.0	0.67	0.50	0.25	0.17	0.13	0.10
$\tilde{y}^{(2)}$	0.32	0.45	0.71	1.0	1.22	1.41	2.00	2.45	2.83	3.16
$\tilde{y}^{(3)}$	16.5	8.27	3.43	2.27	2.08	2.08	2.40	2.73	3.07	3.38
$\tilde{y}^{(4)}$	10.0	5.02	2.13	1.49	1.61	1.80	2.48	3.02	3.48	3.88

Bounds on $\tilde{y}(s)$

6. An auxiliary problem

The inner and outer bounds on $\tilde{y}(s)$ obtained in the last section, leave the unknown boundary curve covered by a narrow strip. The table indicates that we already have a reasonably accurate determination of the optimal policy, for all values of $s > 0$. Nevertheless, it is of interest to investigate whether the techniques can be developed further. In what follows, we seek more precise asymptotic descriptions of $\tilde{y}(s)$, first as $s \to 0$ and later as $s \to \infty$.

It will be convenient to change the notation slightly and denote the given stopping and terminal costs together by

$$\begin{aligned} D(y, s) &= \frac{1}{s}, && (y > 0, s > 0), \\ D(y, 0) &= y, && (y \geq 0), \\ D(0, s) &= 0, && (s > 0). \end{aligned} \tag{6.1}$$

Consider the auxiliary stopping problem for the Wiener process $\{Y(s), s \geq 0\}$ in the half plane $s \geq 0$, with the following stopping and terminal cost function:

$$\begin{aligned} d(y, s) &= -s, && (s \geq 0), \\ d(y, 0) &= 0, && (y \geq 0), \\ d(y, 0) &= y, && (y < 0). \end{aligned} \tag{6.2}$$

LEMMA 6.1. *As $s \to \infty$, the minimum risk*

$$v(y, s) = y - \tfrac{1}{2}e^{2y+2s-1} + o(1), \tag{6.3}$$

where $o(1)$ is positive and applies uniformly in y. The optimal policy consists of stopping whenever $y \geq \tilde{z}(s)$, where $-s + \frac{1}{2} + o(1) \leq \tilde{z}(s) \leq -s + \frac{1}{2}$.

PROOF. We note that $v'(y, s) = y - \frac{1}{2}e^{2y+2s-1}$ satisfies the diffusion equation with boundary conditions

$$\begin{aligned} v'(y, s) &= -s, && \text{for } y = -s + \tfrac{1}{2}, && s > 0, \\ v'_y(y, s) &= 0, && \text{for } y = -s + \tfrac{1}{2}, && s > 0, \\ v'(y, 0) &= y - \tfrac{1}{2}e^{2y-1}, && \text{for } y \leq \tfrac{1}{2}, && s = 0. \end{aligned} \tag{6.4}$$

Thus $v'(y, s)$ and $\tilde{z}'(s) = -s + \frac{1}{2}$, represent the solution of a more favorable problem, since $v'(y, 0) \leq d(y, 0)$, where it applies. Hence, $v(y, s) \geq v'(y, s)$ and (y, s) is a stopping position if $y \geq -s + \frac{1}{2}$.

The procedure defined by $\tilde{z}'(s)$ is suboptimal for the original problem. For that problem it yields a risk

$$v''(y, s) = v'(y, s) + E' \{\tfrac{1}{2}e^{2Y(0)-1} + \min(-Y(0), 0) | Y(s) = y\}, \tag{6.5}$$

where E' represents the expectation restricted to those paths which are not stopped until $s = 0$. It is easily shown that this term approaches zero uniformly in y as $s \to \infty$. Then (6.3) follows, since $v(y, s) \leq v''(y, s)$. It is also clear that for any $\epsilon > 0$, when s is sufficiently large, $y \leq -s + \frac{1}{2} - \epsilon$ implies that $v(y, s) < -s$ and (y, s) is a continuation point.

It remains to show that the optimal stopping set S consists of an interval $[\tilde{z}(s), \infty]$ for each s. But if (y, s) is in the continuation region, we can easily show that $v(y - \Delta, s) \leq v(y, s) < -s$ from a consideration of the policy obtained by translating S downwards an amount Δ. The resulting procedure is suboptimal, but since $d(y, 0)$ is monotone in y, it leads to the desired inequality.

We remark that the lemma was prompted by the fact that a standard Wiener process $\{W(t); t \geq 0\}$ starting at the origin, intersects the line $w = a + mt$; $a > 0, m > 0$, with probability e^{-2am}.

In section 5, we showed that the boundary $\tilde{y}(s)$ of the optimal stopping region for the spaceship control problem specified by (6.1), satisfies $\tilde{y}(s) \leq s^{-1} + \frac{1}{2}s^2$ in general. We can now prove the following theorem.

THEOREM 6.2. *If $(s \to 0)$, then $\tilde{y}(s) = s^{-1} + \frac{1}{2}s^2 + o(s^2)$.*

PROOF. If no stopping is permitted when $0 < s < s_0$, the problem becomes one with minimum risk $V^*(y, s) \geq V(y, s)$, where $V^*(y, s_0) = \min(y, s_0^{-1})$ for $y \geq 0$. Again, if $\{Y(s)\}$ is a Wiener process in the $(-s)$ scale, then $\{aY(s)\}$ is a Wiener process in the $(-a^2s)$ scale. Thus, the above constrained problem may be transformed by setting

$$\begin{aligned} y^* &= s_0^{-2}(y - s_0^{-1}), \\ s^* &= s_0^{-4}(s - s_0), \\ v^* &= s_0^{-2}(V - s_0^{-1}), \end{aligned} \tag{6.6}$$

to the problem with stopping and terminal risk

$$\begin{aligned} d^*(y^*, s^*) &= s_0^{-2}\{(s_0 + s_0^4 s^*)^{-1} - s_0^{-1}\} \\ &= -s^* + s_0^3 s^{*2} - \cdots, && (y^* > -s_0^{-3},\ s^* > 0), \\ d^*(y^*, 0) &= \min (y^*, 0), && (y^* \geq -s_0^{-3}), \\ d^*(-s_0^{-3}, s^*) &= -s_0^{-3}, && (s^* > 0). \end{aligned} \tag{6.7}$$

This is approximately the auxiliary problem of lemma 6.1, when s_0 is small.

Consider the procedure with risk $v^{*\prime\prime}(y^*, s^*)$, which consists of stopping whenever $y^* \geq -s^* + \frac{1}{2}$. By comparing (6.7) with (6.2), the difference $v^{*\prime\prime}(y^*, s^*) - v''(y^*, s^*)$ can be expressed as a sum of two contributions. One is due to paths which stop along $y^* = -s^* + \frac{1}{2}$ for some $s^* > 0$, and the other is due to paths which stop along $y^* = -s_0^{-3}$. We take the initial $s^* = s_0^{-1}$. Clearly,

$$\sup_{0 < s' \leq s_0^{-1}} |d^* (-s' + \tfrac{1}{2}, s') - d\,(-s' + \tfrac{1}{2}, s')| = O(s_0), \tag{6.8}$$

$$\sup_{0 < s' \leq s_0^{-1}} |d^*(-s_0^{-3}, s') - v''(-s_0^{-3}, s')| = o(1). \tag{6.9}$$

It follows that

$$v^{*\prime\prime}(y^*, s_0^{-1}) = v''(y^*, s_0^{-1}) + o(1) = v'(y^*, s_0^{-1}) + o(1), \tag{6.10}$$

and the optimal risk for (6.7),

$$v^*(y^*, s_0^{-1}) \leq v^{*\prime\prime}(y^*, s_0^{-1}) < d^*(y^*, s_0^{-1}), \tag{6.11}$$

when $y^* \leq -s_0^{-1} + \frac{1}{2} - \epsilon$ and s_0 is sufficiently small. Then, on substituting back into (6.4), with $s = s_0 + s_0^4 s^* = s_0 + s_0^3$, we find that $V(y, s) \leq V^*(y, s) < s^{-1}$, whenever

$$y \leq s_0^{-1} + s_0^2\,(-s_0^{-1} + \tfrac{1}{2} - \epsilon) = s^{-1} + (\tfrac{1}{2} - \epsilon)\, s^2 + o(s^2). \tag{6.12}$$

In other words,

$$\tilde{y}(s) \geq s^{-1} + \tfrac{1}{2}s^2 + o(s^2), \qquad (s \to 0), \tag{6.13}$$

which concludes the proof.

The above derivation provides a close upper bound for the optimal risk $V(y, s)$, near the boundary. Since $s^* = s_0^{-1}$ corresponds to $s = s_0 + s_0^3$, this indicates that for small values of s_0, the effect of forbidding any stops between 0 and s_0, is small and diminishes rapidly as s increases from s_0.

7. Formal expansions for small s

In this section we derive the formal expansions for $s \to 0$;

$$\tilde{y}(s) = s^{-1} + \tfrac{1}{2}s^2 - \tfrac{1}{2}s^5 + \tfrac{7}{2}s^8 + \cdots, \tag{7.1}$$

$$V(y, s) = y - s^2 e^{2\delta'} \{\tfrac{1}{2} + s^3(\delta' - \delta'^2) + s^6 (\tfrac{1}{4} - 7\delta' + 6\delta'^2 - 3\delta'^3 + \delta'^4) + \cdots\}, \tag{7.2}$$

where $\delta' = s^{-2}(y - s^{-1}) - \frac{1}{2}$, and $y < \tilde{y}(s)$.

The expansions are motivated by the preceding results, which indicated the major importance of the boundary $y = -s + \frac{1}{2}$ for large s in the auxiliary problem (6.2). This in turn points to the relevance of the distribution and moments of the time T when the Wiener process $\{W(t), t \geq 0\}$, starting at the origin, first intersects the line $w = a + mt$; $a > 0$, $m > 0$. The distribution is known to have moment generating function $E\{e^{\lambda T}\} = \exp \{-a(m + \sqrt{m^2 - 2\lambda})\}$. Thus, in the notation of section 6, the moments of the time required for a path from (y^*, s^*) to hit the line $L: y^* = -s^* + \frac{1}{2}$, can be expressed in the form $e^{2\delta}P(\delta)$, where $P(\delta)$ is a polynomial in δ and where

$$\delta = -a = y^* + s^* - \tfrac{1}{2}. \tag{7.3}$$

One is led to consider solutions of the diffusion equation $\frac{1}{2}H_{y^*y^*} = H_{s^*}$, of the form $H_n(y^*, s^*) = e^{2\delta}u_n(\delta + s^*, s^*)$, where the functions $u_n(x, t)$ are polynomial solutions of the corresponding equation $\frac{1}{2}u_{xx} = u_t$:

$$\begin{aligned} u_0(x, t) &= 1, \\ u_1(x, t) &= x, \\ u_2(x, t) &= \tfrac{1}{2}(x^2 + t), \\ u_3(x, t) &= \tfrac{1}{6}!(x^3 + 3xt), \\ u_4(x, t) &= \tfrac{1}{24}!(x^4 + 6x^2t + 3t^2), \end{aligned} \tag{7.4}$$

and so on. We replace these for convenience by the linear combinations $w_n(x, t)$, selected so that $w_n(t, t) = t^n$. Hence,

$$\begin{aligned} w_0(x, t) &= u_0(x, t) = 1, \\ w_1(x, t) &= u_1(x, t) = x, \\ w_2(x, t) &= 2u_2 - w_1 = x^2 + t - x, \\ w_3(x, t) &= 6u_3 - 3w_2 = x^3 + 3xt - 3x^2 - 3t + 3x, \\ w_4(x, t) &= 24u_4 - 6w_3 - 3w_2 \\ &= x^4 - 6x^3 + 15x^2 - 15x + 15t - 18xt + 6x^2t + 3t^2. \end{aligned} \tag{7.5}$$

Now let

$$J_n(\delta, s^*) = e^{2\delta}w_n(\delta + s^*, s^*). \tag{7.6}$$

Then we have

$$\begin{aligned} w_0(\delta + s^*, s^*) &= 1, \\ w_1(\delta + s^*, s^*) &= s^* + \delta, \\ w_2(\delta + s^*, s^*) &= s^{*2} + \delta(2s^* - 1) + \delta^2, \\ w_3(\delta + s^*, s^*) &= s^{*3} + \delta(3s^{*2} - 3s^* + 3) + \delta^2(3s^* - 3) + \delta^3, \\ w_4(\delta + s^*, s^*) &= s^{*4} + \delta(4s^{*3} - 6s^{*2} + 12s^* - 15) \\ &\quad + \delta^2(6s^{*2} - 12s^* + 15) + \delta^3(4s^* - 6) + \delta^4; \end{aligned} \tag{7.7}$$

(7.8)

$$J_0 = e^{2\delta} = 1 + 2\delta + 2\delta^2 + \tfrac{4}{3}\delta^3 + \tfrac{2}{3}\delta^4 + \cdots,$$
$$J_1 = s^* + \delta(2s^* + 1) + \delta^2(2s^* + 2) + \delta^3(\tfrac{4}{3}s^* + 2) + \delta^4(\tfrac{2}{3}s^* + \tfrac{4}{3}) + \cdots,$$
$$J_2 = s^{*2} + \delta(2s^{*2} + 2s^* - 1) + \delta^2(2s^{*2} + 4s^* - 1) + \delta^3(\tfrac{4}{3}s^{*2} + 4s^*) + \cdots,$$
$$J_3 = s^{*3} + \delta(2s^{*3} + 3s^{*2} - 3s^* + 3) + \delta^2(2s^{*3} + 6s^{*2} - 3s^* + 3) + \cdots,$$
$$J_4 = s^{*4} + \delta(2s^{*4} + 4s^{*3} - 6s^{*2} + 12s^* - 15) + \cdots,$$

and so on. We can easily expand $\partial J_n/\partial\delta$ from (7.8).

For the problem specified by (6.7), we shall consider solutions of the following form, for the optimal risk and boundary curve.

$$v^*(y^*, s^*) = y^* - \tfrac{1}{2}e^{2\delta} + c_0(s_0)J_0 + c_1(s_0)J_1 + c_2(s_0)J_2 + \cdots, \tag{7.9}$$

$$\begin{aligned} \tilde{y}^*(s^*) &= -s^* + \tfrac{1}{2} + \tilde{\delta}(s^*), \\ \tilde{\delta}(s^*) &= \delta_1(s^*)s_0^3 + \delta_2(s^*)s_0^6 + \delta_3(s^*)s_0^9 + \cdots. \end{aligned} \tag{7.10}$$

The coefficients $c_n(s_0)$ and $\delta_n(s^*)$ can be selected to approximate the boundary conditions $v^*(\tilde{y}^*, s^*) = d^*(\tilde{y}^*, s^*)$ and $v^*_{y^*}(\tilde{y}^*, s^*) = 0$. We have

$$\begin{aligned} \sum c_n(s_0)J_n(\tilde{\delta}, s^*) &= s_0^3s^{*2} - s_0^6s^{*3} + s_0^9s^{*4} - \cdots \\ &\quad + \tilde{\delta}^2 + \tfrac{2}{3}\tilde{\delta}^3 + \tfrac{1}{3}\tilde{\delta}^4 + \cdots, \end{aligned} \tag{7.11}$$

$$\sum c_n(s_0)\frac{\partial J_n}{\partial\delta}(\tilde{\delta}, s^*) = 2\tilde{\delta} + 2\tilde{\delta}^2 + \tfrac{4}{3}\tilde{\delta}^3 + \cdots. \tag{7.12}$$

We match the coefficients, taking s_0 small and treating s^* as $0(1)$. The dominant term on the right of (7.11) is $s_0^3s^{*2}$, which calls for $c_2 = s_0^3$. Then the left side of (7.12) begins with $s_0^3(2s^{*2} + 2s^* - 1)$, and the right side with $s_0^3 2\delta_1(s^*)$, which implies that $\delta_1(s^*) = s^{*2} + s^* - \tfrac{1}{2}$. By substituting again in (7.11) and carrying terms of order s_0^6, we find that

$$c_4 = -s_0^6, \qquad c_3 = -3s_0^6, \qquad c_1 = s_0^6, \qquad c_0 = -\tfrac{1}{4}s_0^6. \tag{7.13}$$

Next, a similar comparison for (7.12) shows that

$$\delta_2(s^*) = -s^{*3} + \tfrac{1}{2}s^{*2} - \tfrac{5}{2}s^* + \tfrac{7}{2}, \tag{7.14}$$

and so on. The formal, and so far unjustified, substitution of $s^* = 0$, with the transformation (6.6), gives

$$s = s_0, \qquad \tilde{y}(s) = s_0^{-1} + s_0^2\tilde{y}^*(0), \qquad V(y, s) = s_0^{-1} + s_0^2v^*(y^*, 0), \tag{7.15}$$

and finally yields the desired terms of (7.1) and (7.2).

The formal expansions (7.9) and (7.10) seem to be justifiable for the auxiliary problem, when s^* is large. Although the authors have not carried out the necessary details for a proof, it seems to be fairly straightforward. We have treated s^* as $0(1)$, but clearly the operations would be meaningful for large s^*, provided that $s_0^3s^{*2} \to 0$. However, it cannot be expected that the substitution $s^* = 0$ will yield the solution of the auxiliary problem (6.7). In fact, this substitution does not yield values for $v^*(y^*, 0)$ which coincide with $d^*(y^*, 0) = \min(y^*, 0)$. On the other hand, if the expansions (7.9) and (7.10) are meaningful, they should be valid for $s^* = s_0^{-1}$ and $s^* = 2s_0^{-1}$ say. Thus, one would expect the expansions to be self-consistent in the sense that different pairs (s_0, s^*) yield approximately the same results, whenever the corresponding values of $s = s_0 + s_0^4 s^*$ coincide. Indeed, the initial terms have this consistency property exactly, for all $s^* \geq 0$. This fact indicates that there is no effect due to the computationally convenient substitution $s^* = 0$ and justifies the results (7.1) and (7.2).

The above consistency provides a useful check on the calculations of $\delta_1(s^*)$ and $\delta_2(s^*)$. For example, the condition that $\tilde{y} = s_0^{-1} + s_0^2\tilde{y}^*$ remains constant when $s = s_0 + s_0^4 s^*$ is constant, can be seen to imply that $\delta_1(s^*) = s^{*2} + s^* + K_1$, and given $K_1 = -\frac{1}{2}$, that $\delta_2(s^*) = -s^{*3} + \frac{1}{2}s^{*2} - \frac{5}{2}s^* + K_2$.

8. Refined bounds for $s \to \infty$

In this section, we shall establish the following upper and lower bounds for $\tilde{\alpha}(s) = \tilde{y}(s)s^{-1/2}$ as $s \to \infty$, to improve the results obtained in section 5.

THEOREM 8.1. (i) *There is a constant* $K_0 > 0$, *such that* $\tilde{\alpha}(s) \geq 1 + K_0 s^{-\eta_0}$, $(s \to \infty)$, *where* $\eta_0 = 1.61005$.

(ii) *If* $\eta < \eta_0$, *there is a constant* $K_\eta > 0$, *such that* $\tilde{\alpha}(s) \leq 1 + K_\eta s^{-\eta}$, $(s \to \infty)$.

We review a few relevant facts before proceeding to the main argument. Certain important solutions of the diffusion equation have the form, $u(y, s) = s^{-\lambda/2}A_\lambda(\alpha)$, $\alpha = ys^{-1/2}$, where $A_\lambda(\alpha)$ satisfies

$$A_\lambda''(\alpha) + \alpha A_\lambda'(\alpha) + \lambda A_\lambda(\alpha) = 0. \tag{8.1}$$

We observe that $A_\lambda'(\alpha)$ is a candidate for $A_{\lambda+1}(\alpha)$. One example for $\lambda = 2$, which will be useful, is $\alpha\varphi(\alpha)$. The odd solutions of (8.1) are of special interest. A power series expansion shows that these can be expressed in terms of the confluent hypergeometric function as

$$A_\lambda(\alpha) = \alpha F\left(\frac{\lambda + 1}{2}, \frac{3}{2}; -\frac{\alpha^2}{2}\right), \tag{8.2}$$

where

$$F(\beta, \gamma; w) = 1 + \frac{\beta}{\gamma} w + \frac{\beta(\beta + 1)}{\gamma(\gamma + 1)} \frac{w^2}{2!} + \cdots . \tag{8.3}$$

With this definition, the function $A_\lambda(\alpha)$ is continuous in λ and the smallest value of $\lambda \geq 1$, for which $A_\lambda(1) = 0$ is $\lambda_0 = 5.22010$. We note also that $A_\lambda(\alpha) > 0$ for $1 \leq \lambda \leq \lambda_0$, $0 < \alpha < 1$, and $A_{\lambda_0}'(1) < 0$. Finally, $A_\lambda(\alpha)$ is a bounded function of α.

The proof of (i) involves considering a solution of the diffusion equation of the form

$$B\left\{s^{-1}\frac{\alpha\varphi(\alpha)}{\varphi(1)} - bs^{\lambda_0/2}A_{\lambda_0}(\alpha)\right\}, \tag{8.4}$$

which corresponds to a "less favorable" problem than ours. As a preliminary, we note that if $c > 0$ is small enough, the function

$$G_0(c,\alpha) = \frac{\alpha\varphi(\alpha)}{\varphi(1)} - cA_{\lambda_0}(\alpha), \tag{8.5}$$

which is equal to 1 at $\alpha = 1$, attains a local maximum $M_0(c)$ near $\alpha = 1$. In fact, as $c \to 0+$, $M_0(c) \approx 1 + \frac{1}{4}c^2A_{\lambda_0}'^2(1)$. This maximum is increasing in c for small $c > 0$ and is attained at $\alpha_0(c) \approx 1 - \frac{1}{2}cA_{\lambda_0}'(1) > 1$.

Now take any $s_0 > \frac{1}{2}$ and consider

$$V_0(y,s) = \{M_0(cs_0^{-\eta_0})s\}^{-1}\left\{\frac{\alpha\varphi(\alpha)}{\varphi(1)} - cs^{-\eta_0}A_{\lambda_0}(\alpha)\right\}, \tag{8.6}$$

where $\eta_0 = \frac{1}{2}\lambda_0 - 1 = 1.61005$. If $c > 0$ is sufficiently small (not depending on s_0), it is easy to see that $D_0(y, \frac{1}{2}) = V_0(y, \frac{1}{2}) \geq V(y, \frac{1}{2})$, at least when $0 \leq y \leq 2^{-1/2}\alpha_0(cs_0^{-\eta_0})$. Since $G_0(cs^{-\eta_0}, 1) = 1$ and $M_0(cs^{-\eta_0})$ is decreasing in s for $\frac{1}{2} \leq s \leq s_0$, the equation $V_0(y,s) = s^{-1}$ has a solution $\tilde{y}_0(s) = s^{1/2}\tilde{\alpha}_0(s)$. Further, $\tilde{\alpha}_0(s)$ always lies between 1 and $\alpha_0(cs_0^{-\eta_0})$, and $\tilde{\alpha}_0(s_0)$ coincides with $\alpha_0(cs_0^{-\eta_0})$.

Thus, the curve $\tilde{y}_0(s)$ determines a procedure for $\frac{1}{2} \leq s \leq s_0$ with risk $V_0(y,s)$ in the continuation region, for the problem specified by

$$\begin{aligned} D_0(y,s) &= s^{-1}, && (y > 0, \tfrac{1}{2} < s \leq s_0),\\ D_0(0,s) &= 0, && (\tfrac{1}{2} < s \leq s_0),\\ D_0(y,\tfrac{1}{2}) &\geq V(y,\tfrac{1}{2}), && (0 \leq y \leq \tilde{y}_0(\tfrac{1}{2})). \end{aligned} \tag{8.7}$$

This problem is less favorable than the original and furthermore, $V_0(y,s_0) < s_0^{-1}$ when $ys_0^{-1/2} < \tilde{\alpha}_0(s_0) \approx 1 - \frac{1}{2}cs_0^{-\eta_0}A_{\lambda_0}'(1)$. This establishes (i) and also shows that $V(y,s_0) \leq V_0(y,s_0)$.

We shall prove (ii) by comparing the original problem with one for which the minimum risk is given by a solution of the diffusion equation

$$V_\lambda(y,s) = B\left\{s^{-1}\frac{\alpha\varphi(\alpha)}{\varphi(1)} - bs^{-\lambda/2}A_\lambda(\alpha)\right\}, \tag{8.8}$$

with $\lambda = 2\eta + 2 < \lambda_0$. It suffices to consider η close to η_0, and hence, we may assume that $A_\lambda(1) > 0$, $A_\lambda'(1) < 0$. For small values of $c > 0$ and α near 1, the function

$$\begin{aligned} G_\lambda(c,\alpha) &= \frac{\alpha\varphi(\alpha)}{\varphi(1)} - cA_\lambda(\alpha)\\ &\approx 1 - (\alpha-1)^2 - cA_\lambda(1) - c(\alpha-1)A_\lambda'(1). \end{aligned} \tag{8.9}$$

Thus $G_\lambda(c,\alpha)$ has a local maximum at

$$\alpha_\lambda(c) \approx 1 - \tfrac{1}{2}cA_\lambda'(1) > 1, \tag{8.10}$$

and the maximum value

$$M_\lambda(c) \approx 1 - cA_\lambda(1) + \tfrac{1}{4}c^2A_\lambda'^2(1), \tag{8.11}$$

is decreasing in c. Since $A_\lambda(\alpha)$ is bounded, c can be kept small enough to ensure that this is the absolute maximum for $\alpha \geq 0$.

We now choose $B = 1/M_\lambda(bs_0^{-\eta})$ in (8.6) so that

$$V_\lambda(y, s) = \frac{s^{-1}}{M_\lambda(bs_0^{-\eta})} G_\lambda(bs^{-\eta}, \alpha). \tag{8.12}$$

For each s, let $\tilde{y}_\lambda(s) = s^{1/2}\tilde{\alpha}_\lambda(s)$, where $\tilde{\alpha}_\lambda(s) = \alpha_\lambda(bs^{-\eta})$. Thus, provided that $bs^{-\eta}$ is small, $V_\lambda(y, s)$ and the curve $\tilde{y}_\lambda(s)$ satisfy the free boundary conditions and represent the optimal solution of a stopping problem. If we restrict our attention to $\{(y, s), y \geq 0, s_1 \leq s \leq s_0\}$, the appropriate stopping and terminal cost is

$$\begin{aligned} D_\lambda(y, s) &= \frac{M_\lambda(bs^{-\eta})}{M_\lambda(bs_0^{-\eta})} s^{-1} \leq s^{-1}, && (y > 0, s_1 < s \leq s_0), \\ D_\lambda(0, s) &= 0, && (s_1 < s \leq s_0), \\ D_\lambda(y, s_1) &= V_\lambda(y, s_1), && (0 \leq y \leq \tilde{y}_\lambda(s_1)). \end{aligned} \tag{8.13}$$

Assuming for the moment that $V_\lambda(y, s_1) \leq V(y, s_1)$, this problem is more favorable than the original one. Then

$$V(\tilde{y}_\lambda(s_0), s_0) \geq V_\lambda(\tilde{y}_\lambda(s_0), s_0) = s_0^{-1}. \tag{8.14}$$

It follows that $(\tilde{y}_\lambda(s_0), s_0)$ is in the optimal stopping region for the original problem, which gives the required inequality (ii).

It remains to verify that b and s_1 can be selected so that

$$V_\lambda(y, s_1) \leq V(y, s_1), \qquad (0 \leq y \leq s_1^{1/2}\alpha_\lambda(bs_1^{-\eta})), \tag{8.15}$$

whenever s_0 is sufficiently large. One of the results implicit in section 5 is a lower bound on the minimum risk: $V^{(3)}(y, s) \leq V(y, s)$. The function $V^{(3)}(y, s)$ has the following property as $s \to \infty$:

$$sV^{(3)}(\alpha s^{1/2}, s) = \begin{cases} \dfrac{\alpha\varphi(\alpha)}{\varphi(1)} \{1 + o(1)\}, & (0 \leq \alpha \leq 1), \\ 1 + o(1), & (\alpha > 1), \end{cases} \tag{8.16}$$

where $o(1)$ applies uniformly in $\alpha \geq 0$. We now observe that $A_\lambda(\alpha)\varphi(1)/\alpha\varphi(\alpha)$ is bounded away from zero in some fixed interval $0 < \alpha < 1 + \epsilon$. Hence there is a constant $k > 0$, not depending on the value of c, such that

$$G_\lambda(c, \alpha) \leq \frac{\alpha\varphi(\alpha)}{\varphi(1)} (1 - 2ck), \qquad (0 < \alpha < 1 + \epsilon). \tag{8.17}$$

Thus,

$$s_1 V_\lambda(\alpha s_1^{1/2}, s_1) \leq \frac{\alpha\varphi(\alpha)}{\varphi(1)} \frac{(1 - 2bs_1^{-\eta}k)}{M_\lambda(bs_0^{-\eta})}, \qquad (0 < \alpha < 1 + \epsilon). \tag{8.18}$$

We can now select b and s_1 by first choosing a value for $c = bs_1^{-\eta}$ small enough to ensure that (8.10) is applicable, with $\alpha_\lambda(bs_1^{-\eta}) < 1 + \epsilon$. Then s_1 can be chosen large, according to (8.16), so that

$$s_1 V^{(3)}(\alpha s_1^{1/2}, s_1) \geq \frac{\alpha\varphi(\alpha)}{\varphi(1)} (1 - bs_1^{-\eta}k). \tag{8.19}$$

This inequality, together with (8.18) and the general relation between $V^{(3)}(y, s)$ and $V(y, s)$, establishes that

$$V_\lambda(\alpha s_1^{1/2}, s_1) \leq \frac{V(\alpha s_1^{1/2}, s_1)}{M_\lambda(bs_0^{-\eta})} \frac{(1 - 2bs_1^{-\eta}k)}{(1 - bs_1^{-\eta}k)}, \qquad (0 \leq \alpha \leq \alpha_\lambda(bs_1^{-\eta})). \tag{8.20}$$

Finally, since the last ratio here is <1 and since $M_\lambda(bs_0^{-\eta}) \to 1$ as $s_0 \to \infty$, the required inequality (8.15) holds when s_0 is sufficiently large.

9. A formal expansion for a variation of the control problem

The formulation of the spaceship control problem treated so far is based on assumptions, concerning the rate at which information is collected, which led to the function $D(y, s) = s^{-1}$. We shall indicate how the stopping cost

$$D(y, s) = s^{-1} + a, \qquad (a > 0), \tag{9.1}$$

can arise for two distinct variations of these assumptions and then suggest formal expansions as $s \to \infty$ for this modified problem.

First suppose, as in section 2, that information is accumulated at the rate $a_1\tau^{-2}$, where $-\tau$ is the time to go. Then the total information available is $s^{-1} = -a_1\tau^{-1} + a_2$ and $D(y, s) = -a_3\tau^{-1}$. Previously, we assumed that $a_2 = 0$ and made a scale transformation to obtain the stopping cost. If we do not assume $a_2 = 0$, the transformation leads to the form (9.1).

An alternative assumption is that information is collected at a constant rate $1/\sigma^2$. Then if I_0 is the total information to be accumulated by the time $\tau = 0$ when the target is reached, the information at time $\tau < 0$ is $s^{-1} = I_0 + \sigma^{-2}\tau < I_0$. It turns out that

$$D(y, s) = -a_3\tau^{-1} = a_3\sigma^{-2}I_0^{-1} + \frac{a_3\sigma^{-2}I_0^{-2}}{(s - I_0^{-1})}, \tag{9.2}$$

for $s > I_0^{-1}$. Once again, a linear transformation of the y, s and cost coordinates can be found, which produces the form (9.1) without affecting the basic Wiener process.

Apparently, the most substantial effect of changing the stopping cost to $s^{-1} + a$ occurs as $s \to \infty$. We shall initiate a formal expansion of the type described in [4], [6], for the asymptotic behavior of the optimal boundary $\tilde{y}(s) = s^{1/2}\tilde{\alpha}(s)$.

Let

$$V(y, s) = a - 2a\{1 - \Phi(\alpha)\} + E\{f(s^{1/2}(\alpha + Z))\}, \tag{9.3}$$

where $\alpha = ys^{-1/2}$ and Φ is the standard normal distribution function. The expectation is to operate on the Taylor expansion of $f(s^{1/2}(\alpha + Z))$ about $s^{1/2}\alpha$ with Z distributed as $\mathfrak{N}(0, 1)$. Thus, we have the boundary conditions

$$\frac{1}{s} + a = a - 2a\{1 - \Phi(\tilde{\alpha})\} + f(s^{1/2}\tilde{\alpha}) + \frac{s}{2!}f^{(2)}(s^{1/2}\tilde{\alpha}) + \frac{s^2}{4!}f^{(4)}(s^{1/2}\tilde{\alpha}) + \cdots, \tag{9.4}$$

$$\frac{\partial V}{\partial \alpha} = 0 = 2a\varphi(\tilde{\alpha}) + s^{1/2}f^{(1)}(s^{1/2}\tilde{\alpha}) + \frac{s^{3/2}}{2!}f^{(3)}(s^{1/2}\tilde{\alpha}) + \cdots. \tag{9.5}$$

A first approximation to the functions $\tilde{\alpha}$ and f for which the formal expansion applies, is given by

$$\tilde{\alpha}^2(s) = 2 \log s + \log \log s + \log (a^2/\pi), \tag{9.6}$$

$$f(x) = 2x^{-2} \log x^2. \tag{9.7}$$

Further terms can be obtained by the techniques used in [4], [6]. Presumably, the same argument would apply here; that is, to show that the expansion for $\tilde{\alpha}(s)$ yields a valid approximation to the optimal boundary as $s \to \infty$.

In conclusion, we remark that (9.6) indicates the asymptotic form

$$\tilde{\alpha}(s) \sim \sqrt{2 \log s}, \qquad (s \to \infty), \tag{9.8}$$

which is very different from the previous case, where we obtained $\tilde{\alpha}(s) \to 1$ as $s \to \infty$.

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