SPORADIC RANDOM FUNCTIONS AND CONDITIONAL SPECTRAL ANALYSIS: SELF-SIMILAR EXAMPLES AND LIMITS

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1. Introduction

The definition of sporadic random functions has arisen out of a pessimistic evaluation of the suitability of ordinary random functions as models of certain "turbulencelike" chance phenomena. Now that one has evaluated a number of spectra of turbulence, one must indeed admit that their sampling behavior is often not at all as predicted by the developments of the Wiener-Khinchin second order theory. True, given two frequency intervals (λ_1, λ_2) and (λ_3, λ_4), the ratio between the energies in these intervals rapidly tends to a limit, as was expected. But the energies within each of the intervals (λ_1, λ_2) or (λ_3, λ_4) continue to fluctuate wildly, however large the sample may be. Another puzzling fact: some turbulencelike phenomena appear to have an infinite energy in low frequencies, a syndrome often colorfully referred to as an "infrared catastrophe."

The reason the concept of stationary random function must be generalized may be further explained as follows. Many physical time series X are "intermittent," that is, alternate between periods of activity, and quiescent "intermissions" during which X is constant and may even vanish. Moreover, and this is the crucial point, some series are quiescent *most* of the time. Using the intuitive *physical* meaning of the phrase, "almost sure event," such a series would appear to satisfy the following properties, which will entitle it to be called "sporadic": X is almost surely constant in any prescribed finite span (t', t''), but X "almost surely" varies sometime. It would be very convenient if each of the above occurrences of the term "almost sure" could also be interpreted in the usual mathematical terms. Unfortunately, one has so rigged the theory of random functions, that the above two requirements are incompatible ([8], pp. 51 and 70).

There is, however, a simple way of generalizing the concept of a stationary random function, so as to accommodate the sporadically varying series $X(t)$. It suffices to amend in two ways the classical Kolmogorov's probability space triplet $(\Omega, \mathcal{A}, \mu)$: (a) the measure μ is assumed unbounded though sigma-finite; and (b) a family $\mathcal{B}$ of conditioning events B is added, where $0 < \mu(B) < \infty$. It is further agreed that, henceforth, the only well-posed questions will be those relative to conditional probabilities $\Pr\{A|B\}$ where $B \in \mathcal{B}$.

The first part of the present paper is devoted to the preliminary task of defining mathematically the concept of "sporadic behavior." Sporadic renewal sets will be stressed.

The second part of this paper is devoted to its main purpose, which is to investigate the strange correlation and spectral properties of sporadic processes. Three types of results can be distinguished here. The first relates to the definition of conditional correlations, given a fixed conditioning event. The second relates to the limit behavior of conditional correlations, as the conditioning event is steadily weakened; a central role will be played here by functions of "regular variation." The third type of result relates to the limit behavior of sample values of correlation. A central role will be played here by the little known "Mittag-Leffler" distribution, and by some self-similar renewal sets and processes.

It should be noted that the "generalized random functions" due to Gelfand and Itô [13] can be characterized as using Schwartz distributions to solve problems associated with an excessive amount of high frequency energy. The present generalization is on the contrary directed towards low frequency problems. (It has, however, suggested a natural solution to a long standing problem of hard analysis concerning high frequencies; see [15].)

The very convenient notation for g.r.f., $(\Omega, \mathcal{A}, \mathcal{B}, \mu)$, comes from Rényi's general axiomatic of conditional probability [24]. But measure theory as such is not essential to the present enterprise, whose spirit is very different from Rényi's: we shall not strive for generality for its own sake, nor for irreducible axiomatics. Our sole object is to select new mathematical objects adapted to the needs of physics, and to prove a few basic results justifying our choice. Examples of applications can be found in [20], [21], [22], each of which includes a more detailed discussion of the motivating empirical findings. For the intermittency of turbulence in fluids, see also [25] and the latter portions of [23]. The author will not be surprised if some details of his work turn out to require refinement and correction.

2. Sample spaces, events, and generalized random functions based upon sigma-finite unbounded measures

It is useful to begin by restating the general definition of a random function (r.f.) $X(t, \omega)$. It begins with the following *measurable space* $(\Omega, \mathcal{A})$, where Ω is the *sample space* of *elementary events* ω, each of which is a *sample function* $x(t)$ of $t \in R$ (that is, $-\infty < t < \infty$). *Simple events* are Ω, $\varnothing$, and the finite unions, finite intersections, or complements of ω sets of the form $\{\omega: x'_i < X(t_i, \omega) \leqq x''_i\}$, where the x'_i, t_i, and x''_i are rational. The smallest Borel field containing those simple events will be $\mathcal{A}$.

The next step is to form a *measure space* $(\Omega, \mathcal{A}, \mu)$, where the *measure* is a nonnegative completely additive set function defined for $A \in \mathcal{A}$.

The final step in the definition is to assume that $\mu(\Omega) < \infty$; then a probability $\Pr\{A\}$ can be defined for all $A \in \mathcal{A}$ by $\Pr\{A\} = \mu(A)/\mu(\Omega)$. This is the point where the usual definition has proven inadequate.

When $\mu(A)$ is unbounded, the triplet $(\Omega, \mathcal{A}, \mu)$ specifies $X(t, \omega)$ through measures that cannot be interpreted as nontrivial absolute probabilities. Therefore, in order to specify $X(t, \omega)$ through some kind of probability, one must complete the triplet $(\Omega, \mathcal{A}, \mu)$ by also specifying an appropriate collection $\mathcal{B}$ of admissible conditioning events B, such that $B \in \mathcal{A}$ and $0 < \mu(B) < \infty$. A *generalized random function* (g.r.f.) will be defined as a quadruplet $(\Omega, \mathcal{A}, \mathcal{B}, \mu)$ and *well-set probabilistic results* will be defined as those relative to the conditional probabilities of the form $\Pr \{A|B\} = \mu(A \cap B)/\mu(B)$.

An alternative approach to some g.r.f. In important cases, the measure μ is "sigma-finite," meaning that there exists a denumerable family of events Ω_i with $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$ and $0 < \mu(\Omega_i) < \infty$. Moreover, every B satisfies $B \subset \bigcup_{i=1}^{I(B)} \Omega_i$, with $I(B) < \infty$. Under these two conditions, the g.r.f. $(\Omega, \mathcal{A}, \mathcal{B}, \mu)$ is equivalent to the indexed family of ordinary r.f. $(\bigcup_{i=1}^{I} \Omega_i, \mathcal{A}_I, \mu_I)$ where I ranges from 1 to ∞, the elements of $\mathcal{A}_I$ are the sets of the form $A \cap (\bigcup_{i=1}^{I} \Omega_i)$, and μ_I is a probability conditioned by $\bigcup_{i=1}^{I} \Omega_i$. These r.f. are related to each other by hosts of conditions of compatibility; $\lim_{I\to\infty} (\bigcup_{i=1}^{I} \Omega_i, \mathcal{A}_I, \mu_I)$ is not an ordinary r.f.

Concrete problems cannot involve infinite values of I; therefore, one can always avoid g.r.f.'s by choosing some large but finite "external scale" J, and working with the r.f. $(\bigcup_{i=1}^{J} \Omega_i, \mathcal{A}_J, \mu_J)$. This J will be nonintrinsic, however, and the solution of concrete problems will require the study of "transient small sample behavior" rather than the simpler study of limit theorems. Thus, even from the viewpoint of concrete problems, it is usually simpler to work with the g.r.f. $(\Omega, \mathcal{A}, \mathcal{B}, \mu)$ directly.

3. Definition of sporadically varying generalized random functions and related concepts

3.1. *Sporadically varying g.r.f.* We must first define some sets of Ω. Let

$$(3.1)\qquad \begin{aligned} A^* &= \{\omega\colon X(t, \omega) \text{ is constant over } -\infty < t < \infty\} \\ &= \{\omega\colon -\infty < t < \infty, \text{ g.l.b. } X(t, \omega) = \text{l.u.b. } X(t, \omega)\} \end{aligned}$$

and, for every open interval (t', t'') such that $-\infty < t' < t'' < \infty$, let

$$(3.2)\qquad \begin{aligned} B(t', t'') &= [\omega\colon X(t, \omega) \text{ is not constant over } t \in (t', t'')] \\ &= \{\omega\colon t' < t < t'' \text{ g.l.b. } X(t, \omega) < \text{l.u.b. } X(t, \omega)\} \end{aligned}$$

It will be necessary that $A^* \in \mathcal{A}$ and $B(t', t'') \in \mathcal{A}$. For that, it is sufficient that time be restricted to some denumerable subset of R. When time is continuous, problems of "separability" may arise. But a discussion of separability is not needed for the examples to be examined in this paper, and will therefore be postponed.

The family of all the events $B(t', t'')$ will be designated by $\mathcal{B}^*$.

DEFINITION 3.1. *A sporadically varying g.r.f. is a quadruplet* $(\Omega, \mathcal{A}, \mathcal{B}, \mu)$, *where* $(\Omega, \mathcal{A})$ *is a measurable space, and where the following conditions are imposed upon the measure* μ *and upon the subfamily* $\mathcal{B}$ *of* $\mathcal{A}$:

The measure μ is assumed to satisfy two conditions

$$\mu(\Omega) = \infty, \qquad \mu(\varnothing) = 0; \tag{3.3}$$
$$\mu(A^*) = 0.$$

Two measures μ_1 and μ_2 will not be distinguished if $\mu_1(A)/\mu_2(A)$ is a constant independent of $A \in \mathcal{A}$. The family $\mathcal{B}$ of conditioning events B is assumed to satisfy two conditions

$$0 < \mu(B) < \infty, \qquad \text{if} \quad B \in \mathcal{B}; \tag{3.4}$$
$$\mathcal{B}^* \subset \mathcal{B}.$$

PROPOSITION 3.1. *Conditions (3.3) and (3.4) imply that μ is sigma-finite.*

PROOF. Write $\Omega = A^* \cup [\cup_{i=-\infty}^{\infty} B(i, i + 1 + \epsilon)]$, where $\epsilon > 0$.

As intended, a sporadically varying $X(t, \omega)$ almost surely does not vary in any *prescribed* bounded interval (t', t''), but almost surely varies *somewhere* along R. These characteristics would be mutually incompatible if $X(t, \omega)$ were an ordinary random function.

3.2. *Intermissions and set of variation.* Given a left-continuous g.r.f. $X(t, \omega)$ and x, the interior of the set $\{t: X(t, \omega) = x\}$ either is empty or is the denumerable union of open intervals. The latter needs a distinguishing name.

DEFINITION 3.2. *An intermission of the g.r.f. $X(t, \omega)$ is a maximal open interval contained in the interior of a set of the form $\{t: X(t, \omega) = x\}$.*

For each ω, the intermissions of $X(t, \omega)$ are denumerable, and can be designated as $(t'_h(\omega), t''_h(\omega))$, where $1 \leqq h < \infty$; the ω will usually be omitted.

DEFINITION 3.3. *The set of variation of the g.r.f. $X(t, \omega)$ will be the closed set $S(\omega) = R - \cup_{h=1}^{\infty}(t'_h, t''_h)$.*

If $X(t, \omega)$ is sporadically varying, so that $\mu(\Omega) = \infty$, there is a vanishing absolute probability that $S(\omega)$ and (t', t'') intersect. The set $S(\omega)$ will be said to be "sporadically distributed" or "sporadic."

(Some sporadic random sets can be treated directly, by generalizing to infinite $\mu(\Omega)$ the definition of random compact set quoted in [3], p. 309. In this way, one may define functions that satisfy sporadically a property other than the property of being nonconstant.)

An extreme example of sporadic g.r.f. is a function with a single randomly located step: $\Omega = R$, where ω is distributed with Lebesgue measure; for $t > \omega$, $X(t, \omega) = X'$ with $|X'| < \infty$; for $t < \omega$, $X(t, \omega) = X'' \neq X'$ with $|X''| < \infty$. Thus, $S(\omega) = \omega$, and there are two intermissions, both of infinite duration. Well-set probabilistic problems relate to step functions with a step randomly located over (t', t'') with the corresponding (finite) Lebesgue measure.

4. Measure preservation and conditional stationarity

4.1. *Shift invariant indecomposable measures and conditions.* The shift transformation φ_τ is defined as usual. If ω represents the function $X(t)$, then $\varphi_\tau\omega$ represents $X(t + \tau)$, and one assumes $\varphi_\tau\omega \in \Omega$. Similarly, for every $A \in \mathcal{A}$, one defines $\varphi_\tau A$, and one assumes $\varphi_\tau A \in \mathcal{A}$.

If μ is a measure on $(\Omega, \mathcal{A})$, possibly unbounded, the *shift invariance* of μ (that is, the *measure preserving* character of φ_τ) is defined by the usual condition:

(a) for every $A \in \mathcal{A}$, and for every τ, one has $\mu(\varphi_\tau A) = \mu(A)$.

The conditioning events $B(t', t'')$, members of $\mathcal{B}^*$, have the following property: for every $B \in \mathcal{B}^*$ and for every τ, $\varphi_\tau B \in \mathcal{B}^*$. Therefore, $\mathcal{B}^*$ will be said to be *shift invariant*. More generally, $\mathcal{B}$ will be said to be *shift invariant*, if it fulfills the following condition:

(b) for every $B \in \mathcal{B}$, and for every τ, one has $\varphi_\tau B \in \mathcal{B}$.

(If $\mathcal{B} - \mathcal{B}^*$ is nonvoid, this is no longer a necessary consequence of (a).)

A, a set $\in \mathcal{A}$, will be said *shift invariant* if it fulfills the usual condition:

(c) for every τ, $\mu[\varphi_\tau A \cup A - \varphi_\tau A \cap A] = 0$.

Such A form a Borel field, designated as $\mathfrak{I}$. The least interesting case occurs when every $A \in \mathfrak{I}$ is the union of sets in $\mathfrak{I}$ having a finite μ measure. The most interesting case is when every set A in $\mathfrak{I}$ satisfies either $\mu(A) = 0$ or $\mu(\Omega - A) = 0$; as usual, such a φ_τ will be called indecomposable.

4.2. *Conditional restricted stationarity.* The usual concept of stationarity, being relative to absolute probabilities, is degenerate in the case of g.r.f.'s such that $\mu(\Omega) = \infty$. Partial conditional equivalents are available, however.

DEFINITION 4.1. *The g.r.f.* $(\Omega, \mathcal{A}, \mathcal{B}, \mu)$ *will be said to be* conditionally stationary *if, for every* $B \in \mathcal{B}$, *there exists a nonvanishing open interval* $(t', t'')_B$ *with the following property. Let* t_i *be I time instants,* $I < \infty$, *and* τ *a time span such that* $[\bigcup \{t_i\}] \cup [\bigcup \{t_i + \tau\}] \subset (t', t'')$, *and let* $A_R^{(i)}$ *be I Borel sets of R. Then*

$$\Pr\{\forall i, X(t_i, \omega) \in A_R^{(i)} | B\} = \Pr\{\forall i, X(t_i + \tau, \omega) \in A_R^{(i)} | B\}. \tag{4.1}$$

The expression

$$\Pr\{X(t, \omega) \in A_R | B\} = \mu\{\omega: [X(t, \omega) \in A_R] \cap [\omega \in B]\}/\mu(B), \tag{4.2}$$

which is independent of t as long as $t \in (t', t'')_B$, is a conditional marginal distribution for $X(t, \omega)$, given B.

Returning to the shift invariance of μ and $\mathcal{B}$, note that it has the following obvious consequences

$$\Pr\{\varphi_\tau A | \varphi_\tau B\} = \Pr\{A|B\}; \tag{4.3}$$

$$\text{if } A \cup \varphi_\tau A \subset B, \quad \text{then} \quad \Pr\{\varphi_\tau A | B\} = \Pr\{A|B\}. \tag{4.4}$$

This (4.4) follows from

$$\begin{aligned} \Pr\{\varphi_\tau A|B\} &= \mu(\varphi_\tau A \cap B)/\mu(B) = \mu(\varphi_\tau A)/\mu(B) = \mu(A)/\mu(B) \\ &= \mu(A \cap B)/\mu(B) = \Pr\{A|B\}. \end{aligned} \tag{4.5}$$

EXAMPLE. Let $\mathcal{B} = \mathcal{B}^*$ and let $X(t, \omega)$ be a sporadically varying g.r.f. For the relation $\varphi_\tau A \cup A \subset B(t', t'')$ to hold, it is sufficient that A be of the form $\{\omega: X(t_1, \omega) \in A_R^{(1)}, X(t_2, \omega) \in A_R^{(2)}\}$, where $A_R^{(1)}$, $A_R^{(2)}$ are nonintersecting Borel sets on $R(A_R^{(1)} \cap A_R^{(2)} = \varnothing)$, and where t_1, t_2, $t_1 + \tau$, $t_2 + \tau$ all $\in (t', t'')$. The relation $\varphi_\tau A \cup A \subset B(t', t'')$ also holds for events A involving a larger number of instants t_i, as long as $[\bigcup \{t_i\}] \subset [\bigcup \{t_i + \tau\}] \subset (t', t'')$ and $A \subset B(t', t'')$.

First example of conditional stationarity. Now, make the stronger assumptions that $\mathcal{B} = \mathcal{B}^*$ and that $X(t, \omega)$ vanishes during its intermissions. Consider the event $A = \{\omega: X(t, \omega) \in A_R\}$. If $0 \notin A_R$ and $t \in (t', t'')$, one has $A \subset B(t', t'')$, so that $\mu(A) < \infty$. In particular, $\mu[\omega: X(t, \omega) \neq 0] < \infty$, so that X may be integrable without vanishing identically. For example, if $\mathcal{B} = \mathcal{B}^*$ and $X(t, \omega)$ is bounded and vanishes during the intermissions, it is integrable.

Now consider two instants t and $t + \tau$, both belonging to (t', t''). If $0 \notin A_R$, then $A \cup \varphi_\tau A \subset B(t', t'')$, so that $\Pr\{\varphi_\tau A | B(t', t'')\} = \Pr\{A | B(t', t'')\}$ by (4.4). If $0 \in A_R$, then $0 \notin R - A_R$ and

$$\Pr\{\varphi_\tau(\Omega - A) | B(t', t'')\} = \Pr\{(\Omega - A) | B(t', t'')\}.$$

Since $\Pr\{A | B(t', t'')\} = 1 - \Pr\{(\Omega - A) | B(t', t'')\}$, the relation

$$\Pr\{\varphi_\tau A | B(t', t'')\} = \Pr\{A | B(t', t'')\}$$

is valid for every A of the form $\{\omega: X(t, \omega) \in A_R\}$. In fact:

PROPOSITION 4.1. *If $\mathcal{B} = \mathcal{B}^*$, if μ is shift invariant, and if $X(t, \omega)$ vanishes during its intermissions, then $[X(t, \omega) | B(t', t'')]$ is conditionally stationary over some interval $(t', t'')_B$ satisfying $(t', t'')_B \supseteq (t', t'')$.*

PROPOSITION 4.2. (Converse of 4.1.) *If $\mathcal{B} = \mathcal{B}^*$, X vanishes during its intermissions, and if, for every (t', t''), $[X(t, \omega) | B(t', t'')]$ is conditionally stationary over (t', t''), then μ is shift invariant.*

Second example of conditional stationarity. A second important class of g.r.f.'s is one for which the conditional stationarity is *not* a consequence of the shift invariance of μ, and must be postulated separately. This class is defined as having the following property.

For almost all t (that is, except if t belongs to a set of R of vanishing Lebesgue measure), the conditioned marginal distribution $\Pr\{X(t, \omega) \in A_R | B\}$ is the same for all conditions B such that $t \in (t', t'')_B$. One can, therefore, speak in this case of a *nonconditional* "pseudomarginal" distribution for $X(t, \omega)$. The ordinary r.v., whose distribution is $\Pr\{X(t, \omega) < x | B\}$, will be called the "pseudomargin" of X.

Unless X vanishes identically, it is not integrable, but its pseudomarginal variable can be integrable.

4.3. *The structure of $\mu[B(u^*)]$ when μ is shift invariant.* If μ is shift invariant, it will often be unnecessary to specify the value of t' in $B(t', t' + u^*)$, because t' is indifferent or obvious from the context. We shall write then $B(t', t' + u^*)$ as $B(u^*)$.

PROPOSITION 4.3. *If μ is shift invariant, then the function $\mu[B(u)]$, defined for $u > 0$, is concave, continuous and right and left differentiable.*

PROOF. This will follow classically after it is proved that for every $u^* > 0$ and $h > 0$, one has

$$(4.6)\quad \mu[B(t^*, t^* + u^* + 2h)] + \mu[B(t^*, t^* + u^*)] - 2\mu[B(t^*, t^* + u^* + h)] \leqq 0.$$

Note that $B(t^*, t^* + u^* + 2h) \supseteq B(t^*, t^* + u^* + h) \supseteq B(t^*, t^* + u^*)$. Therefore, (4.6) holds if

(4.7) $\mu[B(t^*, t^* + u^* + 2h) - B(t^*, t^* + u^* + h)]$
$$\leqq \mu[B(t^*, t^* + u^* + h) - B(t^*, t^* + u^*)].$$

By stationarity, this holds if

(4.8) $\mu[B(t^* - h, t^* + u^* + h) - B(t^* - h, t^* + u^*)]$
$$\leqq \mu[B(t^*, t^* + u^* + h) - B(t^*, t^* + u^*)]$$

which in turn holds if

(4.9) $B(t^* - h, t^* + u^* + h) - B(t^* - h, t^* + u^*)$
$$\subseteq B(t^*, t^* + u^* + h) - B(t^*, t^* + u^*).$$

This last statement is true because the event on the left of $\subseteq$ differs from that on the right by the set of ω such that X varies for $t \in [t^* + u^*, t^* + u^* + h)$ and $t \in (t^* - h, t^*]$ but not for $t \in (t^*, t^* + u^*)$.

PROPOSITION 4.4. *One can write* μ *as*

$$\mu[B(u^*)] = Q + \int_0^{u^*} P(s)\, ds, \tag{4.10}$$

where $Q \geqq 0$ *and* $P(u)$ *is positive nonincreasing, and such that* $\int_0^{\epsilon} P(s)\, ds < \infty$ *for all* $\epsilon > 0$. $P(0)$ *is defined as* $\lim_{s\to 0} P(s)$, *and may be infinite.*

COROLLARY 4.1. *Let* $T_0(\omega) = g.l.b.\ \{t: t^* + t \in S(\omega) | S(\omega) \cap (t^*, t^* + u^*) \neq \varnothing\}$. *It follows from the above proposition that, given the interval* $(t^*, t^* + u^*)$,

$$\Pr\{0 < T_0 \leqq t_0 | 0 \leqq T_0 < u^*\} = \int_0^{t_0} P(s)\, ds \Big[Q + \int_0^{u^*} P(s)\, ds\Big]^{-1}, \tag{4.11}$$

which has a probability density proportional to $P(t_0)$. *Moreover,*

$$\Pr\{T_0 = 0 | 0 \leqq T_0 < u^*\} = Q\Big[Q + \int_0^{u^*} P(s)\, ds\Big]^{-1}, \tag{4.12}$$

which is positive if $Q > 0$ *and vanishes if* $Q = 0$.

COROLLARY 4.2. *If* $Q = 0$, *and* $S(\omega) \cap (t', t'') \neq \varnothing$, $S(\omega) \cap (t', t'')$ *has zero Lebesgue measure, except for a set of* ω *of zero* μ *measure.*

PROOF. Given $S(\omega)$, choose T^* with Lebesgue measure over some bounded (t', t'') such that $S(\omega) \cap (t', t'') \neq \varnothing$. Then $\Pr\{T^* \in S(\omega) \cap (t', t'') \neq \varnothing\} = 0$ is a consequence of $Q = 0$, and it implies that $S(\omega) \cap (t', t'')$ has zero Lebesgue measure with a conditional probability equal to 1.

If $Q > 0$ and $S(\omega) \cap (t', t'') \neq \varnothing$, $\mu[\omega: S(\omega) \cap (t', t'')$ has positive Lebesgue measure$] > 0$. But $\mu[\omega: S(\omega) \cap (t', t'')$ has zero Lebesgue measure$]$ may be either positive or zero; examples of both kinds will be given in the sequel.

4.4. *Degrees of intermittency.* The behavior of $P(u)$ allows an important classification of certain ordinary and generalized random processes.

The *nonintermittent case* will be defined by $P \equiv 0$. One must have $Q > 0$ in order that $\mu[B(u)] > 0$. Then $\mu[B(u)] \equiv \mu(\Omega)$ and $\Pr\{T_0 > 0 | 0 \leqq T_0 < u^*\} \equiv 0$. Such functions $X(t, \omega)$ have almost surely no open interval of constancy.

The *intermittent cases* will be defined by $P \not\equiv 0$. The constant Q may be $\geqq 0$. Then $\Pr\{T_0 > 0 | 0 \leqq T_0 < u^*\} > 0$ and $X(t, \omega)$ almost surely possesses open intervals of constancy.

A finer classification of intermittent r.f. will now be considered.

Finitely intermittent r.f. will be those corresponding to $\int_0^\infty P(s)\,ds < \infty$. They are ordinary random functions that "flip" between a "quiescent" state where X is constant and an "active" state where X may vary.

Infinitely intermittent r.f. will be those corresponding to $\int_0^\infty P(s)\,ds = \infty$. They *are identical to the sporadically varying generalized random functions.* Note that for these functions, it is not excluded that $P(\infty) > 0$.

5. Examples of sporadically varying g.r.f.'s and sporadic sets constructed through generalized renewal processes

The simplest sporadic g.r.f. are those whose structure is wholly determined by the function $\mu[B(u^*)]$ implied in the definition of "sporadic." If $Q = 0$, $P(0) < \infty$, and $P(\infty) = 0$, the function $P(u)$ fully determines a classical renewal process, as we shall now demonstrate.

5.1. *Synchronized classical renewals.* This is a sequence $S^0 = \{T_k^0\}$ with k integer, $-\infty < k < \infty$, such that $T_0^0 = 0 \in S^0$, and such that the intermissions $U_k = T_{k+1}^0 - T_k^0$ are independent and identically distributed nonnegative random variables with $\Pr\{U \geqq u\} = P(u)/P(0)$. The Ω space of the S^0 is that of all sequences containing the origin and such that $S^0 \cap (t', t'')$ is finite or denumerable. By well-known rules, a measure is attached to the events of this sample space, and $S^0 \cap (t', t'')$ is almost surely finite (surely finite if $P(\epsilon) = P(0)$ for some $\epsilon > 0$) [6].

5.2. *Stationary classical renewals and generalization to* $E(U) = \int_0^\infty P(s)\,ds = \infty$. Starting from the set S^0 of section 5.1, we shall construct a generalized random set S as follows. The first step is to replace $T_0^0 = 0$ by a T_0 distributed over $R^+(t \geqq 0)$, with the measure of *density* $P(u_0)$; this measure may be unbounded. The second step is to translate the set $\{T_k^0, k > 0\}$ to the right by the amount T^0; thus, the measure of T_k is the convolution of the measures of T_0 and T_k^0. The third step is to choose T_{-1} on R^- $(t < 0)$ with the conditional probability measure $\Pr\{T_{-1} \leqq -u | T^0 = t^0\} = P(u + t_0)/P(t_0)$; it is easy to see that the unconditioned distribution of T_{-1} has a measure of *density* $P(-u)$. The final step is to translate the set $\{T_k^0, k < -1\}$ to the left by the amount $T_{-1} - T_{-1}^0$; thus, the measure of T_k is the convolution of the measures of T_k^0 and $T_{-1} - T_{-1}^0$.

PROPOSITION 5.1. *The measure of the generalized random set S is shift invariant and indecomposable.*

For example, $\mu[\omega: S(\omega) \cap (t', t'') \neq \varnothing]$ is shift invariant. This result is well-known if $E(U) < \infty$; then, S is an ordinary random set (intermittent) and one can normalize $P(s)$ into the probability density $P(u)\left[\int_0^\infty P(s)\,ds\right]^{-1}$ [7]. Most of the classical proofs of stationarity can also be extended to the case $E(U) = \infty$, in which case S is sporadic.

PROPOSITION 5.2. *Let $E(U) = \infty$. Over the time span $(0, u^*)$, the following processes are identical in law*: (1) *the stationary sporadic process S constructed above;* (2) *the nonstationary ordinary process obtained from S^0 by making T_0 random over $(0, u^*)$ with the density $P(t_0) \Big/ \left[\int_0^{u^*} P(s)\, ds\right]$*; (3) *the stationary ordinary random process constructed, as in section 5.2, using the truncated $P^*(u, u^*)$ defined by $P^*(u, u^*) = P(u)$ if $u < u^*$, $P^*(u, u^*) = 0$ if $u > u^*$.*

This last proposition shows how to replace the generalized random set $S(\omega)$ by an indexed family of ordinary random sets (see end of section 2).

5.3. *Sporadically varying g.r.f.'s having S as set of variation.* The function $V(t, \omega)$, defined by $V(t, \omega) = 1$, if $t \in S(\omega)$ and $V(t, \omega) = 0$, if $t \notin S(\omega)$, is a conditionally stationary g.r.f. with the required property, and belonging to the second class of section 4.2. The integral $K(t, \omega)$ of $V(t, \omega)$ is a nonstationary g.r.f. with the required property.

5.4. *The infinitely divisible process technique and the function N^0.* It is convenient to consider the synchronized renewal set S^0 as the set of values of an auxiliary r.f. $T^0(n)$ of a variable n, not necessarily integer [26]. One will have $T^0(n_0) = t_0$; T^0 will be left-continuous, it will have independent infinitely divisible increments. That is, given n' and n'', $-\infty < n' < n'' < \infty$, $T^0(n') - T^0(n'')$ will be an infinitely divisible positive r.v. whose Lévy's jump function is $1 - P(u)$; its characteristic function [10], [14], [18], [19] is

$$\varphi(\zeta) = \exp\left\{-(n'' - n') \int_0^\infty (e^{i\zeta s} - 1)[-dP(s)]\right\}. \tag{5.1}$$

The number of jumps of T^0, located in the n-span (n', n'') and having a size in the range $(s, s + ds)$, is a Poisson random variable of mean $|(n'' - n')\, dP(s)|$. The intervals, between successive values of n on which T^0 varies, are exponential random variables of unit expectation.

The function $N^0(t)$ will conversely be defined as being the largest n such that $T^0(n) \leqq t$.

5.5. *The fundamental sporadically varying function $N(t)$.* Suppose now that $T_0 \in R^+$ has the density $P(t_0)$ (see section 5.2 concerning T_{-1}). Thus, by shifting randomly the origin of $N^0(t)$, we construct a function $N(t, \omega)$. If $E(U) = \infty$, this g.r.f. (dependent on n_0) constitutes a basic example of sporadically varying g.r.f. Its set of variation is $S(\omega)$, independent of n_0.

5.6. *Several generalizations of renewal processes.* The possibilities excluded so far, $Q > 0$, $P(0) = \infty$, and $P(\infty) > 0$, are readily introduced by generalizing the renewal processes.

The inequality $P(\infty) > 0$ implies that the process is *not* recurrent with probability one. In the infinitely divisible technique, $T(n)$ is not defined for all values of n, but only over a span whose duration is an exponential random variable of mean $1/P(\infty)$. S is then *almost surely a finite set.*

Suppose in particular that $P(u) \equiv P(\infty)$ so that $\mu[B(u)] = uP(\infty)$. In that case, S is *almost surely reduced to a single point* as in section 3.2.

The most natural way of making $Q > 0$ is to consider the closure of the set of values of the function $T_Q(n) = T(n) + Qn$. Renewals are no longer instantaneous, but their duration is an exponential random variable of expectation Q. The set S^0 is a union of closed intervals, whose total number is almost surely denumerable if $P(\infty) = 0$, and infinite and finite if $P(\infty) > 0$. If $S(\omega) \cap (t', t'') \neq \emptyset$, this set has a positive Lebesgue measure, except for a set of ω that has vanishing μ measure. Another way of making $Q > 0$ is to have an "act" of duration $Q[P(s') - P(s'')]^{-1}$ follow every intermission of duration $u \in (s', s'')$ [where $P(s') > P(s'')$]. Then, the ω sets, over which $S(\omega) \cap (t', t'')$ has zero or positive Lebesgue measure, are both of positive μ measure.

The infinitely divisible technique is well known to generalize with no difficulty to the case $P(0) = \infty$ with $\int_0^\epsilon P(s)\,ds < \infty$. The expected number of jumps of $T(n)$ such that $n' < n < n''$ is almost surely infinite. Moreover, if $S \cap (t, t'') \equiv \emptyset$, the number of points of $S \cap (t', t'')$ is almost surely infinite, the Lebesgue measure of $S \cap (t', t'')$ is almost surely zero; the function $N(t)$ is almost surely a singular continuous function (it is the counterpart of the classical Lebesgue function of Cantor's triadic set).

To combine $P(0) = \infty$ with $Q > 0$, consider again the closure of the values of the r.f. $T_Q(n) = T(n) + Qn$. It is then almost sure that S has a nonvanishing Lebesgue measure but contains no closed interval.

The most interesting examples of sporadic sets (those introduced through limit theorems) will be characterized by $Q = 0$ and $P(\infty) = 0$, but $P(0) = \infty$ with $\int_0^\epsilon P(s)\,ds < \infty$. They present both the low frequency problems associated with sporadic processes, and the high frequency problems of which an example appears in [15].

5.7. *A generalization that is not a renewal process.* Consider finally a process $T_0(n, \omega)$, whose increments are stationary but are not independent, and whose jumps have a marginal distribution $P(u)$ such that $\int_0^\epsilon P(s)\,ds < \infty$ and $\int_0^\infty P(s)\,ds = \infty$. Let T_0 have the measure of density $P(s)$ (see section 5.2 concerning T_{-1}). The closure of the set of values of $T(n, \omega)$ is then a sporadically distributed generalized random sequence, but is not a renewal sequence.

6. The concepts of conditioned deltavariance and covariance, of weak stationarity and of core function

If the stationary ordinary random function $X(t, \omega)$ is such that $E\{[X(t)]^2\} < \infty$, one defines its *covariance* $C(\tau)$ as equal to $E[X(t)X(t + \tau)]$. It will be convenient to deduce $C(\tau)$ from the function

$$D(\tau) = C(0) - C(\tau) = \frac{1}{2}E\{[X(t) - X(t + \tau)]^2\}. \tag{6.1}$$

This $D(\tau)$ is meaningful for all X for which $C(\tau)$ is defined, and also for many others (such as the Brownian notion of Bachelier-Wiener-Lévy). Lacking a generally agreed upon term for this important concept, we propose to call it *deltavariance.* If X is a sporadically varying g.r.f., $D \equiv 0$.

6.1. *The conditioned deltavariance and deltavariance stationarity.* Given a conditioning event $B \in \mathcal{B}$, consider the function

$$D[\tau, B, t] = \frac{1}{2} \int_B [X(t, \omega) - X(t + \tau, \omega)]^2 \mu(d\omega)/\mu[B]. \tag{6.2}$$

Whenever this expression is finite, it may be used to define the *conditioned deltavariance* of X, given B.

DEFINITION 6.1. $(\Omega, \mathcal{A}, \mathcal{B}, \mu)$ *will be said to be* conditionally deltavariance stationary, *if, for every* $B \in \mathcal{B}$, *there exists a time span* $(t', t'')_B$ *of positive duration such that* $D(\tau, B, t)$ *is independent of* t *as long as* $(t, t + \tau) \subset (t', t'')_B$. *One will write* D *as* $D(\tau, B)$.

PROPOSITION 6.1. *Let* $\mathcal{B} = \mathcal{B}^*$ *and let* μ *be shift invariant. Then,* $D[\tau, B(t', t''), t]$ *is conditionally deltavariance stationary for some* $(t', t'')_B \supseteq (t', t'')$.

PROOF. The ω such that $X(t, \omega) - X(t + \tau, \omega) = 0$ contribute nothing to D. If A_R is a Borel set of R such that $0 \notin A_R$, the ω set

$$\{\omega: [X(t, \omega) - X(t + \tau, \omega)] \in A_R\}$$

belongs to $B(t', t'')$ and its μ measure is independent of t. If $\tau < u^*$, the numerator of $D[\tau, B(u^*), t]$ is an integral carried out over $B(u^*)$. However, if $\omega \in B(u^*) - B(\tau) \subset B(u^*)$, then ω contributes nothing to the integral. That is, one can write

$$D[\tau, B(u^*)] = D^*(\tau)/\mu[B(u^*)], \tag{6.3}$$

where

$$D^*(\tau) = \frac{1}{2} \int_{B(\tau)} [X(0, \omega) - X(\tau, \omega)]^2 \mu(d\omega). \tag{6.4}$$

This $D^*(\tau)$ will be referred to as the *unweighted deltavariance* of X; it is defined only up to multiplication by an arbitrary finite positive number (as is also the case for $P(s)$ and μ). The function $D^*(\tau)$ is nonnegative definite.

If X is an ordinary process of covariance $C(\tau)$, then μ can be so chosen that $\mu(\Omega) = 1$, and D boils down to

$$D[\tau, B(u^*)] = D(\tau)/\mu[B(u^*)] = [C(0) - C(\tau)]/\mu[B(u^*)]. \tag{6.5}$$

Fixing τ and varying u^*, we see that $D[\tau, B(u^*)]$ is a decreasing function of u^*, defined for $u^* > \tau$.

Varying τ, we see that one can eliminate u^* by forming the *relative deltavariance,* defined as equal to

$$\frac{D[\tau', B(u^*)]}{D[\tau'', B(u^*)]} = \frac{D^*(\tau')}{D^*(\tau'')}, \qquad 0 < \tau', \quad \tau'' < u^*. \tag{6.6}$$

(By considering this ratio, one circumvents the difficulties mentioned in the introduction that were encountered in the empirical estimation of the covariance. See section 8, also.)

6.2. *The conditional covariance and covariance stationarity.* Again, given $B \in \mathcal{B}$, consider the function

$$C(\tau, B, t) = \int_B X(t, \omega)X(t + \tau, \omega)\mu(d\omega)/\mu(B). \tag{6.7}$$

Whenever this expression is finite, it will define the *conditioned covariance* of X, given B.

DEFINITION 6.2. *X will be said to be* conditionally covariance stationary *if, for every $B \in \mathcal{B}$, there exists a time span $(t', t'')_B$ of positive duration, such that $C(\tau, B, t)$ is independent of t as long as $(t, t + \tau) \subset (t', t'')_B$. Then, $C(\tau, B, t)$ will be designated by $C(\tau, B)$.*

PROPOSITION 6.2. *Let $\mathcal{B} = \mathcal{B}^*$, let μ be shift invariant, and, for every $B \in \mathcal{B}^*$ and $t \in (t', t'')$, let $C(0, B, t) = C(0, B) < \infty$. Then $C[\tau, B(t', t''), t]$ is conditionally covariance stationary for some interval $(t', t'')_B \supseteq (t', t'')$, and one has $|C(\tau, B)| \leqq C(0, B)$.*

Examine two examples that slightly generalize those singled out in section 4.2.

In the first special class: $\mathcal{B} = \mathcal{B}^*$, μ is shift invariant, and X is such that, if $t \in (t', t'')$ and $0 \notin A_R$, one has $\{\omega: X(t, \omega) \in A_R\} \subset B(t', t'')$. This is, for example, the case if X is a g.r.f. that vanishes during its intermissions. Then the numerator of C becomes independent of B, as long as $(t, t + \tau) \subset (t', t'')$. The existence of C is, thus, reduced to the usual conditions that X must be square integrable, and C can be factored out in the form

$$C[\tau, B(u^*)] = C^*(\tau)/\mu[B(u^*)], \tag{6.8}$$

which serves to define the *unweighted covariance* $C^*(\tau)$.

In the second special class: $\mathcal{B} = \mathcal{B}^*$, μ is shift invariant, and X has a pseudomarginal distribution (independent of B). Here, $C[0, B(u^*)]$ is independent of u^*; it is finite if and only if the pseudomarginal r.v. of $X(t, \omega)$ is square integrable. In that case, one will designate $C[0, B(u^*)]$ by the same notation $C(0)$ as an ordinary variance, and $C[\tau, B(u^*)]$ takes the form

$$C[\tau, B(u^*)] = C(0) - D^*(\tau)/\mu[B(u^*)]. \tag{6.9}$$

6.3. *The concept of the core function of a set of zero Lebesgue measure.* The r.f. with independent values is the simplest of all r.f., and the only one to be defined fully by its marginal distribution. It is, however, not measurable if the allowed values of t are all the points of R.

No such problem arises if time is restricted to a discrete set S of R such as the integers k, so that X is a sequence of independent r.v. One can then extend X to a left-continuous r.f. of continuous time, $W(t, \omega)$, constant over the intervals of the form $k < t \leqq k + 1$; if the marginal distribution of W is continuous, its set of variation is almost surely S.

Now consider a general set S, having a vanishing Lebesgue measure and expressible in the form $S = R - \bigcup_{h=1}^{\infty} (t_h', t_h'')$, where the (t_h', t_h'') are nonoverlapping. Consider also a r.v. W_M (where M stands for marginal), continuously distributed, of mean zero and of unit variance, and let W_h be a discrete sequence of independent r.v. having the distribution of W_M.

DEFINITION 6.3. *The independent core function of S, with margin W_M, is defined as follows. If $t_h' < t \leq t_h''$, then $W(t, \omega) = W(t_h'', \omega) = W_h$. If $\{t_k\}$ are K points ($K < \infty$) of S, the $W(t_k, \omega)$ will be independent r.v. with the distribution of W_M.*

6.4. *Covariance properties of a core function.* If $\omega \notin B(t, t + \tau)$, then $E\{[W(t) - W(t + \tau)]^2\}/2 = 0$. If $\omega \in B(t, t + \tau)$, then $E\{[W(t) - W(t + \tau)]^2\}/2$ is one half of the variance of the difference $W_{h'} - W_{h''}$, where $W_{h'}$ and $W_{h''}$ are independent r.v.'s of variance one. Therefore,

$$D[\tau, B(t', t''), t] = \Pr\{\omega \in B(t, t + \tau) | \omega \in B(t', t'')\}. \tag{6.10}$$

If, moreover, μ is shift invariant, $(t', t'') = (t^*, t^* + u^*)$ and $(t, t + \tau) \subset (t^*, t^* + u^*)$, one has

$$D[\tau, B(u^*)] = \Pr\{\omega \in B(t, t + \tau) | \omega \in B(t^*, t^* + u^*)\} = \frac{\mu[B(\tau)]}{\mu[B(u^*)]} = \frac{Q + \int_0^\tau P(s)\, ds}{Q + \int_0^{u^*} P(s)\, ds}. \tag{6.11}$$

The relative deltavariance and the unweighted deltavariance, respectively, take the forms

$$\begin{aligned} \mu[B(\tau')]/\mu[B(\tau'')] &= \left[Q + \int_0^{\tau'} P(s)\, ds\right] \Big/ \left[Q + \int_0^{\tau''} P(s)\, ds\right] \\ \mu[B(\tau)] &= Q + \int_0^\tau P(s)\, ds. \end{aligned} \tag{6.12}$$

REMARK. The conditioned deltavariance properties derived above apply irrespectively of whether $\mu(\Omega) = \infty$ or $\mu(\Omega) < \infty$. But they are not very useful in the classical case $\mu(\Omega) < \infty$ where W is not sporadic.

6.5. *Generalization; the orthogonal core function.* From the viewpoint of the present study of second order properties, the independence of the r.v. W_h is an unnecessarily strong assumption. If W_M is non-Gaussian, the same covariance is obtained if the W_h are only orthogonal. It is not obvious how the function W should be defined for $t \in S - \bigcup_{h=1}^{\infty} t_h''$; but this set has vanishing Lebesgue measure, so that the corresponding values of W do not matter.

7. Conditional self-similarity and asymptotic self-similarity

7.1. *Self-similarity in time, in the sense of conditional deltavariance.*

DEFINITION 7.1. *The process $X(t, \omega)$ is said to be self-similar in time, in the sense of conditional deltavariance, if one has $D[hu^*, B(u^*)] = D_L(h)$, where $D_L(h)$, with $0 \leq h \leq 1$, is finite and is not identically zero.*

The following theorem expresses the intimate relation between self-similarity and core functions.

THEOREM 7.1. *The definition of self-similarity requires that* $X(t, \omega)K^{-1/2}$, *where* $K > 0$, *be an orthogonal core g.r.f. such that* $\mu[B(u)] = u^{1-\theta}$, *with* $0 \leqq \theta \leqq 1$.

PROOF. The proof proceeds in several steps.

(a) Since

$$D_L(1) = K = D^*(u^*)/\mu[B(u^*)] = D^*(\tau)/\mu[B(\tau)], \tag{7.1}$$

$D[\tau, (B(u^*)]$ can be written as $K\mu[B(\tau)]/\mu[B(u^*)]$, where the function $\mu[B(u)]$ remains to be specified.

(b) The statement that $D^*(\tau) = K\mu[B(\tau)]$ can be rewritten

$$E\{[X(t)K^{-1/2} - X(t + \tau)K^{-1/2}]^2|\omega \in B(t, t + \tau)\}/2 = \mu[B(\tau)], \tag{7.2}$$

which, indeed, means that, when $\omega \in B(t, t + \tau)$, the r.v. $X(t)K^{-1/2}$ and $X(t + \tau)K^{-1/2}$ have unit variance and are orthogonal.

(c) Choose any couple (h', h'') such that $0 \leqq h' \leqq 1$ and $0 \leqq h'' \leqq 1$. Self-similarity requires

$$\begin{aligned} \mu[B(h'h''u^*)]/\mu[B(h''u^*)] &= \mu[B(h'u^*)]/\mu[B(u^*)] \\ \mu[B(h'h''u^*)]/\mu[B(u^*)] &= \{\mu[B(h''u^*)]/\mu[B(u^*)]\}\{\mu[B(h'u^*)]/\mu[B(u^*)]\} \end{aligned} \tag{7.3}$$

or, finally,

$$D_L(1)\, D_L(h'\, h'') = D_L(h')D_L(h''). \tag{7.4}$$

This equation has for solution $D_L(h) = Kh^{1-\theta}$, where θ is a constant. $\mu[B(u)]$ is only defined up to a positive multiplier. One can, therefore, choose

$$\mu[B(u)] = u^{1-\theta}. \tag{7.5}$$

The requirement that $0 \leqq \theta \leqq 1$ follows from the convexity of $\mu[B(u)]$, combined with $\int_0^\epsilon P(s)\, ds < \infty$.

The degenerate case $\theta = 0$. If $\theta = 0$, $\mu[B(u)] = u$, the set S is a.s. a single point chosen at random on R, as in section 3.3.

The classical case $\theta = 1$, $S \equiv R$. If $\theta = 1$, then $\mu[B(u)] =$ const., and one obtains the troublesome process of orthogonal values on R.

The sporadic cases $0 < \theta < 1$. In the interesting cases $0 < \theta < 1$, the Lebesgue measure of $S \cap (t', t'')$ almost surely vanishes. Moreover, $\int_1^\infty P(s)\, ds = \infty$, and therefore, the g.r.f. $X(t, \omega)$ is sporadic. $P(u)$ being defined up to a positive multiplier, one can assume $P(u) = u^{-\theta}$.

COROLLARY 7.1. *If X is a g.r.f. that is conditionally deltavariance self-similar in time, it cannot be finitely intermittent.*

Conditional Taylor's scale. G. I. Taylor has proposed that the integral $\int_0^\infty C(s)\, ds$ be used as measure of the temporal scale of a random phenomenon whose covariance is $C(s)$ [12]. If $X(t, \omega)$ is sporadic, the best that one can do is replace $C(s)$ by a conditional covariance. If $X(t, \omega)$ is, moreover, self-similar in time, this "conditional Taylor's scale" turns out to be $u^*(1 - \theta)/(2 - \theta)$.

7.2. *Uniformly self-similar renewal sets.* In the present section, the assumption that $\mu[B(u)] = u^{1-\theta}$ is combined with the assumption that S is a generalized renewal set, as defined in section 5.6. In the renewal case, $S(\omega)$ is fully determined by the requirements of nondegenerate deltavariance self-similarity, namely, $Q = 0$, $P(\infty) = 0$, and $P(u) = u^{-\theta}$ for $0 < u < \infty$, $0 < \theta < 1$. Insert $P = u^{-\theta}$ in Lévy's formula of section 5.3. We obtain S^0 as closure of the set of values of the function $T^0(n)$, such that $T^0(0) = t_0$ and $T^0(n'') - T^0(n')$ has as characteristic function

$$\begin{aligned}(7.6)\qquad \varphi(\zeta) &= \exp\left[-(n'' - n')\int_0^\infty \theta s^{-(\theta+1)}(e^{i\zeta s} - 1)\,ds\right]\\ &= \exp\{-(n'' - n')|\zeta|^\theta\Gamma(1-\theta)\cos(\theta\pi/2)[1 - i|\zeta|\zeta^{-1}\tan(\theta\pi/2)]\}.\end{aligned}$$

Such T^0 are called "stable" and T^0 is a Lévy process of "independent stable increments" [8], [14], [18]. The process $T^0(u)$ varies only by jumps; the positions of the jumps are mutually independent; the number of jumps, whose size is between u and $u + du$ and whose position is between n and $n + dn$, is a Poisson r.v. of expectation equal to $\theta u^{-(\theta+1)}\,dn\,du$; the degree of "thinness" of S can be described by the fact that its Hausdorff dimension is θ([2], p. 267) almost surely. Finally, S is almost surely a "set of multiplicity" [15].

PROPOSITION 7.1. *After T^0 has been made random, with the unbounded measure of density $(t_0)^{-\theta}$, one obtains a set S that is uniformly self-similar under change of scale, in the following sense. Given that $S \cap (t', t'') \neq \varnothing$, let $H(t', t'')$ be the set of values of $h(0 \leqq h \leqq 1)$ such that $t' + h(t'' - t') \in S \cap (t', t'')$. If $H(t', t'')$ is independent in distribution from t' and t'', then $S(\omega)$ is called uniformly self-similar.*

7.3. *Asymptotic self-similarity in time.*

DEFINITION 7.2. *Let the stationary g.r.f. $X(t, \omega)$ be such that, for $0 \leqq h \leqq 1$,*

$$\lim_{u^*\to\infty} D[hu^*, B(u^*)] = D_L(h)$$

is defined, finite and not identically zero. Then, $X(t, \omega)$ will be said to be asymptotically self-similar in time, in the sense of conditional deltavariance.

THEOREM 7.2. *In order that $\lim_{u^*\to\infty} D[hu^*, B(u^*)] = D_L(h)$ be defined, finite and not identically zero, it is necessary and sufficient that μ and D^* satisfy*

$$\text{(a)}\qquad \lim_{u\to\infty}\mu[B(hu)]/\mu[B(u)] = h^{1-\theta},\qquad 0 \leqq h \leqq 1,\ 0 \leqq \theta \leqq 1,$$

which means that μ "varies regularly" at infinity, in the sense of Karamata, and expresses a kind of asymptotic self-similarity of S;

$$\text{(b)}\qquad \lim_{u\to\infty} D^*(u)/\mu[B(u)] = K > 0,$$

which expresses that the correlation between $X(t', \omega)$ and $X(t'', \omega)$, conditioned by $\omega \in B(t', t'')$, tends to zero with $(t'' - t')^{-1}$.

PROOF. Letting $h = 1$, we see that it is necessary that

$$(7.7)\qquad \lim_{u\to\infty} D^*(u)/\mu[B(u)]$$

exist. It will define $K = D_L(1)$.

Now let $0 < h < 1$, and write $D^*(hu^*)/\mu[B(u^*)]$ as

$$\{D^*(hu^*)/\mu[B(hu^*)]\}/\{\mu[B(hu^*)/\mu[B(u^*)]\}. \tag{7.8}$$

Thus, it is necessary that

$$\lim_{u\to\infty} \mu[B(hu^*)]/\mu[B(u^*)] \tag{7.9}$$

exists.

Well-known arguments ([10], [11]) then show there must exist a constant θ such that

$$\lim_{u\to\infty} \mu[B(hu^*)]/\mu[B(u^*)] = h^{1-\theta} = (\tau/u^*)^{1-\theta}. \tag{7.10}$$

Finally, the condition $0 \leqq \theta \leqq 1$ is a consequence of the convexity of $\mu[B(u)]$. Sufficiency is obvious.

PROPOSITION 7.2. *In order that* $\int_0^\tau P(s)\,ds$ *be monotone nondecreasing, convex and of regular variation with exponent* $0 \leqq 1 - \theta \leqq 1$, *it is necessary and sufficient that* $P(u)$ *be monotone nonincreasing and of regular variation with exponent* θ.

PROOF. For sufficiency, see [11]; for necessity, one can adapt the proof of theorem 2 of [17].

If the limit falls within the degenerate case $\theta = 1$, then $D_L(h)$ is the same as for the process of orthogonal values on R. The intermissions of X are then "few" and/or "short," and they are made effectively negligible by rescaling.

Finally, the interesting limits $0 < \theta < 1$ imply that $P(\infty) = 0$, and that $\mu[B(u)]$ is not identically constant. The limit function D_L is then unaffected by the value of Q, that is, by the Lebesgue measure of R sets of the form $S \cap (t', t'')$.

7.4. *Asymptotically self-similar renewal sets.* They are characterized by a function $P(u)$ such that $\lim_{u\to\infty} P(hu)/P(u) = h^{-\theta}$. This is equivalent to saying that the r.v. $T^0(n'') - T^0(n')$ belongs to the "domain of attraction" of the stable r.v. of exponent θ.

8. Limit theorems relative to the sample deltavariance in the renewal case Uniform self-similarity

8.1. *The "ergodic" problem of the relation between population and sample means of* $X(t, \omega)$; *the conditional ergodic problem.* Let $(\Omega, \mathcal{A}, \mathcal{B}, \mu)$ be such that the shift transformation φ_τ is indecomposable. Birkhoff's individual ergodic theorem applies when X is integrable. We know that such is *not* the case for interesting g.r.f., for example, when X has a pseudomarginal distribution. Even when Birkhoff's theorem applies, it only states that the sample mean tends to zero (or that it is identically zero). Under such circumstances, one becomes interested in the rate at which this limit is achieved, a question that Birkhoff's theorem does not attempt to answer.

The best then is to start anew, and to attack the problem of the behavior of *conditioned* sample moments. We shall see that as $u^* \to \infty$, ratios such as

$$\frac{\int_0^{u^*-\tau} ds\{[X(s, \omega) - X(s + \tau, \omega)]^2|\omega \in B(u^*)\}/2}{(u^* - \tau)E\{[X(t, \omega) - X(t + \tau, \omega)]^2|\omega \in B(u^*)\}/2} \tag{8.1}$$

may tend in distribution to limit r.v. other than unity. This property is weaker than ergodicity.

8.2. *Commentary upon the relations between successive classes of limit theorem concerning* $\sum X(t)$ and $\sum X(t)X(t + \tau)$. As is the case with so many generalizations first suggested by hard facts, the theorems to be described might easily have been first introduced solely to fill in a gap between already existing mathematical theories.

The Wiener-Khinchin theory and the "laws of large numbers." The "second order stationary random processes" of Wiener and of Khinchin can be defined through the requirement that the limit in distribution

$$\lim_{u^* \to \infty} u^{*^{-1}} \sum_{t=t^*}^{t^*+u^*} X(t)X(t + \tau) \tag{8.2}$$

be equal to $E[X(t)X(t + \tau)]$ and be a nonrandom function $C(\tau)$ of τ. This theory is thus parallel in scope to the "laws of large numbers," relative to the case where the first order expression $u^{*^{-1}} \sum_{t=t^*}^{t^*+u^*} X(t)$ tends to the nonrandom limit $E(X)$ as $u^* \to \infty$.

Central limit theorems for normed sums and their counterpart in spectral theory. The next stage of the theory of $\sum X(t)$ was constituted by the central limit theorems relative to the possible behavior of the normed sums $\alpha(u^*) \sum_{t=t^*}^{t^*+u^*} X(t) - \beta(u^*)$. Bochner ([4], p. 295) introduced normed sums of the form $\alpha(u^*) \int X(t)X(t + \tau)\, dt$ in his generalization of Wiener's generalized harmonic analysis of nonrandom functions $X(t)$. There exist stochastic analogs of these theorems, but we have no room to study them here.

Limit theorem of a form analogous to that of Poisson. For sums of r.v.'s, a third stage of the theory was the study of expressions like $\alpha(u^*) \sum_0^{K(u^*)} X(t, u^*) - \beta(u^*)$, where the distribution of the $X(t, u^*)$ is allowed to change as u^* increases.

The theorems of the rest of section 8 and those of section 10 can be cast in this "Poisson" mold, with the restriction that $K(u^*)$ is usually u^* itself, and with the following changes. First, $X(t, u^*)$ is usually to be replaced by $X(t, u^*)X(t + \tau, u^*)$ or $[X(t, u^*) - X(t + \tau, u^*)]^2$. Second, the dependence of these expressions on u^* is assumed to be of a very explicit kind: it is induced by the time lag u^* characteristic of the conditioning event $B(u^*)$. Third, the summands are dependent.

8.3. *The behavior of the function $N(t)$, when $T(n)$ is a process of independent stable increments. The Mittag-Leffler distribution.* It is clear that

$$\begin{aligned} \Pr\{N(t) \geqq n|T(0) = 0\} &= \Pr\{T(n) \leqq t|T(0) = 0\} \\ &= \Lambda_\theta(tn^{-1/\theta}) = \Lambda_\theta[(nt^{-\theta})^{-1/\theta}], \end{aligned} \tag{8.3}$$

where $\Lambda_\theta(y) = \Pr\{T(n + 1) - T(n) \leqq y\}$ is the distribution function of P, Lévy's stable r.v. W. Feller [9] who introduced the r.v. whose distribution function is $\Lambda_\theta(u^{-1/\theta})$, called it a Mittag-Leffler r.v.

Let t_0 be the first point of variation of $N(t)$, on the right of $t = 0$. If $T(0) = t_0$ does not vanish but is known to be less than u^*, then $[N(u^*) - N(0)]/(u^* - t_0)_\theta$ is a Mittag-Leffler r.v. Let now t_0 be made random with the fundamental density $(1 - \theta)t_0^{-\theta}/u^{*1-\theta}$.

$$[N(u^*) - N(0)|B(u^*)]/E[N(u^*) - N(0)|B(u^*)] \tag{8.4}$$

is a weighted mixture of Mittag-Leffler r.v. of exponent θ. It will be designated by M_θ and called a "modified M.L." r.v. Since [9] showed that

$$E[N(u^*) - N(0)|T(0) = 0] = \sin(\pi\theta)(\pi\theta)^{-1}(u^*)^\theta, \tag{8.5}$$

we have

$$E\{[N(u^*) - N(0)]|B(u^*)\} = (1 - \theta)u^{*\theta}. \tag{8.6}$$

8.4. *Sample deltavariance of W when S is a uniformly self-similar renewal set.* Consider the probability

$$\Pr\left\{Y_D = \frac{\int_0^{u^*-\tau} ds\{[W(s) - W(s + \tau)]^2|B(u^*)\}}{(u^* - \tau)E\{[W(t) - W(t + \tau)]^2|B(u^*)\}} \leqq y\right\}. \tag{8.7}$$

It follows from self-similarity that this expression will be unchanged if u^* is replaced by $qu^*(q > 0$, constant) while $B(u^*)$ is replaced by $B(qu^*)$ and τ by $q\tau$. Thus, to each h, $0 \leqq h \leqq 1$, there corresponds a function $F_D(y, h)$ such that $\Pr\{Y_D \leqq y\} = F_D(y, \tau/u^* = h)$. The form of F_D is readily derived when $h = 1$ and when h is very small.

If $h = 1$, then Y_D is half of the squared difference between the independent r.v. $W(0, \omega)$ and $W(u^*, \omega)$.

Now let $h < 1$. Since $W(t) - W(t + \tau)$ vanishes if $(t, t + \tau)$ is a part of an intermission of S, it suffices to carry the integral

$$\frac{1}{2}\int ds\{[W(s) - W(s + \tau)]^2|B(u^*)\} \tag{8.8}$$

over the set $\Delta^0(u^*, \tau) = (0, u^* - \tau) - \bigcup_\tau [(t_h', t_h'' - \tau) \cap (0, u^* - \tau)]$, where the union $\bigcup_\tau$ is carried out over the values of h such that $t_h'' - t_h' > \tau$.

For small h, we shall presently prove the following results. For two different h', h'', the ratio between the sample values of $D[h'u^*, B(u^*)]$ and $D[h''u^*, B(u^*)]$ tends to a nonrandom limit as $h' \to 0$ and $h'' \to 0$, while h'/h'' remains constant. As $h \to 0$, the ratio between the population and sample value of $D[hu^*, B(u^*)]$ tends to a r.v. M_θ.

To avoid the awkward task of letting $h \to 0$, we shall, instead, let $u^* \to \infty$ with τ fixed. This approach is also advantageous because it increases the generality of the result. Instead of requiring that $P = u^{-\theta}$, it will be necessary and sufficient that the limit in distribution of $[N(u^*) - N(0)|B(u^*)]/E[N(u^*) - N(0)|B(u^*)]$ be a nondegenerate r.v. different from unity. This is equivalent to saying that $P(u)$ is regularly varying at infinity, with an exponent θ such that $0 < \theta < 1$.

8.5. *Limit of the weighted deltavariance of the core function W of a renewal set, when $u^* \to \infty$ with fixed τ.* Define P' and P'' by writing

$$(8.9)\quad \frac{\int_0^{u^*-\tau} [W(s) - W(s+\tau)]^2\, ds}{(u^* - \tau)E\{[W(s) - W(s+\tau)]^2\}} = \frac{\int_{s\in\Delta^0(u^*,\tau)} [W(s) - W(s+\tau)]^2\, ds}{2\,|\Delta^0(u^*,\tau)|} \cdot \frac{2\,|\Delta^0(u^*,\tau)|}{(u^* - \tau)E\{[W(s) - W(s+\tau)]^2\}} = P'P'',$$

where ω is understood to be conditioned by $\omega \in B(u^*)$ and $|\Delta^0(u^*, \tau)|$ is the Lebesgue measure of Δ^0.

The denominator of P' tends almost surely to ∞ as $u^* \to \infty$. Moreover, P' itself is nothing but the deltavariance of lag τ of the core function of an auxiliary ordinary renewal set, based upon the truncated law P^* such that $P^*(u, \tau) = u^{-\theta}$ for $u \leqq \tau$, $P^*(u, \tau) = 0$ for $u > \tau$. This process is τ dependent in the sense that $X(t')$ and $X(t'')$ are independent when $|t' - t''| > \tau$. Therefore, the expression P' is the sample mean of an expression which satisfies the strong law of large numbers. Thus, P' tends almost surely to its population mean, which is one. Therefore, Y_D has the same limit in distribution as P''.

Adapting the argument of [9] that led to the Mittag-Leffler distribution, one obtains the following result:

THEOREM 8.1. *Let W be the core function of a recurrent S, and let $u^* \to \infty$. In order that the ratio between the sample and expectation values of $D[\tau, B(u^*)]$ have a proper limit in distribution, it is necessary and sufficient that $P(u)u^\theta$ be slowly varying for some $\theta \in (0, 1)$. The limit is then a r.v. M_θ.*

COROLLARY 8.1. *$W(t)$ being conditioned by $B(u^*)$, the limit in distribution*

$$(8.10)\quad \lim_{u^*\to\infty} \int_0^{u^*} ds\, [W(t) - W(t+\tau')]^2 \Big/ \int_0^{u^*} ds\, [W(t) - W(t+\tau'')]^2$$

is a nonrandom function of τ' and τ''.

Again, by considering this ratio, one circumvents the difficulties that were encountered in the empirical estimation of the covariance.

9. Alternative spectra or Fourier transforms of alternative deltavariances and the population forms of the infrared catastrophe

For a sporadically varying $X(t, \omega)$, the concept of spectrum is only meaningful in a conditional sense. Moreover, it is ambiguous even when the condition B is fixed, as we shall now see by examples.

9.1. *The expectation of the Schuster periodogram.* Schuster's periodogram will be defined in section 10. Its expectation is the most intrinsic form of spectrum and suggests the following expression for the energy to be found in frequencies above λ,

$$(9.1) \qquad G_S(\lambda, u^*) = \frac{2}{\pi} \int_0^{u^*} (1 - s/u^*) C[s, B(u^*)] \sin (2\pi\lambda s) s^{-1}\, ds.$$

If X were an ordinary r.f., one would have $\lim_{u^* \to \infty} C[s, B(u^*)] = C(s)$, and $\lim_{u^* \to \infty} G_S(\lambda, u^*)$ would be the usual spectrum. In the sporadic case, however, *both terms* of the integrand, $C[s, B(u^*)]$ and $(1 - s/u^*)$, depend on u^*, and neither of them tends to a nondegenerate limit as $u^* \to \infty$.

Suppose, in particular, that the deltavariance is asymptotically self-similar, and $C(0, B) = 1$, so that

$$(9.2) \qquad \lim_{u^* \to \infty} C[hu^*, B(u^*)] = 1 - h^{1-\theta} = 1 - (\tau/u^*)^{1-\theta}.$$

Then,

$$(9.3) \quad G_S(\lambda, u^*) \to \frac{2}{\pi} \int_0^1 (1 - s)(1 - s^{1-\theta}) \sin [2\pi(\lambda u^*)s] s^{-1}\, ds = G_S^*(\lambda u^*).$$

This G_S^* is a bounded function such that $G_S^*(\lambda) \sim \lambda^{\theta-1}$ for $\lambda \to \infty$.

In particular, given that $0 < \lambda', \lambda'' < \infty$,

$$(9.4) \qquad |G_S(\lambda', u^*) - G_S(\lambda'', u^*)|/|\lambda' - \lambda''|$$

is the average spectral density in the frequency span (λ', λ'') that is, an average of the spectral density defined by

$$(9.5) \qquad 2 \int_0^{u^*} (1 - s/u^*) C[s, B(u^*)] \cos (2\pi\lambda s)\, ds.$$

This average tends to zero with $1/u^*$.

On the other hand, the spectral density at $\lambda = 0$ is given by

$$(9.6) \quad 2 \int_0^{u^*} (1 - s/u^*) C[s, B(u^*)]\, ds = 2u^* \int_0^1 (1 - s)(1 - s^{1-\theta})\, ds,$$

which tends to infinity proportionately to u^*.

If one examines the spectrum as a whole, the energy will seem to flow to ever lower frequencies as $u^* \to \infty$.

9.2. *The population infrared catastrophe.* Given the deltavariance

$$D[\tau, B(u^*)] = D^*(\tau)/\mu[B(u^*)],$$

define the function $G^*(\lambda)$ by

$$(9.7) \qquad G^*(\lambda) = \frac{2}{\pi} \int_0^\infty \sin (2\pi\lambda s) s^{-1} D^*(s)\, ds.$$

When $X(t, \omega)$ is an ordinary r.f. normalized so that $\mu(\Omega) = \mu[B(\infty)] = 1$, D^* is a Wiener-Khinchin deltavariance, and $|G^*(\lambda') - G^*(\lambda'')|$ is its energy in the spectral interval (λ', λ'') (we use the convention that the energy of a spectral line at λ_0 is split equally between the intervals (λ', λ_0) and (λ_0, λ''), where $\lambda' < \lambda_0 < \lambda''$). Moreover, $G^*(0) < \infty$.

For sporadically varying g.r.f., on the contrary, $\mu[B(u)]$ is unbounded and $G^*(0) = \infty$. In the asymptotically self-similar case, for example, we have

$$G^*(\lambda) \sim \lambda^{1-\theta}, \qquad \text{as} \quad \lambda \to 0. \tag{9.8}$$

In interpreting empirical spectral measurements, one may be tempted to handle G^* as if it were a Wiener-Khinchin spectrum. But $G^*(0) = \infty$ would then be interpreted as meaning that there is an infinite energy in low frequencies, which is impossible physically and therefore "catastrophic" for the identification of G^* to a Wiener-Khinchin spectrum. To distinguish this difficulty from high frequency divergences, it is called an "infrared catastrophe." As introduced in the theory of sporadically varying g.r.f., G^* is not a spectrum and its divergence is not impossible physically and hence not catastrophic for the theory.

More reasonable definitions of the spectrum will be proposed presently. They will show that, in order for $|G^*(\lambda'') - G^*(\lambda')|/\mu[B(u^*)]$ to be a rough estimate of the energy in the frequency band (λ', λ''), one must assume that $1/u^* \leqq \lambda' < \lambda'' \leqq \infty$. In particular, the energy in the band $(1/u^*, \infty)$ is roughly $G^*(1/u^*)/\mu[B(u^*)]$. If $G^*(0) = \infty$, then both numerator and denominator increase as $u^* \to \infty$, but their ratio may well tend to a finite limit. The energy will seem to flow into ever lower frequencies, but the total *expected* energy will remain fixed.

9.3. *A case, when one can construct a stationary ordinary r.f., coinciding over* $(0, u^*)$ *with the sporadic* $X(t, \omega)$. Let us now suppose that the conditioned covariance $C[\tau, B(u^*)]$ satisfies $C[u^*, B(u^*)] = 0$, and consider the function $C_L(\tau, u^*)$ such that $C_L(\tau, u^*) = C[\tau, B(u^*)]$ if $|\tau| < u^*$, $C_L(\tau, u^*) = 0$ otherwise. This function is continuous and is easily seen to be positive definite. Therefore, there exists a stationary ordinary r.f. $X_L(t, \omega, u^*)$, of which C_L is the Wiener-Khinchin covariance. In frequencies above λ, it has an energy equal to

$$G_L(\lambda, u^*) = \frac{2}{\pi} \int_0^{u^*} D[s, B(u^*)] \sin (2\pi\lambda s)s^{-1}\, ds + \frac{2}{\pi} D[u^*, B(u^*)] \int_{u^*}^{\infty} \sin (2\pi\lambda s)s^{-1}\, ds. \tag{9.9}$$

This function $G_L(\lambda, u^*)$ is bounded, as it should be and varies little for $\lambda < 1/u^*$. Its behavior for large λ is, however, mostly determined by the behavior of $D[\tau, B(u^*)]$ for small τ: one has $G_L(\lambda, u^*) \sim G^*(\lambda)/\mu[B(u^*)]$. Therefore, an infrared catastrophe would be brought about if this approximation, which is only valid for $\lambda \to \infty$, were applied without justification for $\lambda \to 0$.

9.4. *A periodic function which, over* $(0, u^*)$, *is identical to the conditioned sporadic process* $X(t, \omega)$. The function

$$C_P(\tau, u^*) = (1 - \tau/u^*)C[\tau, B(u^*)] + (\tau/u^*)C[u^* - \tau, B(u^*)] \tag{9.10}$$

is the covariance of a periodic function $X_P(t, \omega, u^*)$. As u^* increases, so does the number of spectral lines whose frequency is between 0 and some fixed λ_0. Energy seems to flow to low frequencies.

10. Limit theorems relative to the sample periodogram and other sample estimators in the renewal case

10.1. *The periodogram of Schuster.* This is the r.v.

$$|Y(\lambda, \omega, u^*)|^2 = (1/u^*)\left|\int_0^{u^*} X(s, \omega)e^{-2\pi is\lambda}\,ds\right|^2. \tag{10.1}$$

It is the squared modulus of the "Fourier transform"

$$Y(\lambda, \omega, u^*) = u^{*^{-1/2}} \int_0^{u^*} X(s, \omega)e^{-2\pi i\lambda s}\,ds. \tag{10.2}$$

Let $X(t, \omega)$ be an ordinary r.f. of zero mean, satisfying appropriate additional conditions; it is then known that, as $u^* \to \infty$, the ratio $|Y|^2/E(|Y|^2)$ tends in limit to an exponential r.v., whose expectation is (naturally) unity. In the present section, the distribution of $|Y|^2/E(|Y|^2)$ will be examined under the assumption that X is the core function of a sporadic renewal set S.

If S is self-similar, the ratio between the sample and expected values of the periodogram is obviously some r.v. having λu^* as only parameter. However, it will only be assumed that X is the core function of any *asymptotically* self-similar renewal sequence. Very low and very high values of the parameter λu^* will be seen to lead to very different limit distributions for $|Y|^2/E[|Y|^2]$.

10.2. *Sample fluctuation of* $Y(0, \omega, u^*)$ *and the sample infrared catastrophe.*

$$|Y(0, \omega, u^*)|^2 = u^*\left[u^{*^{-1}} \int_0^{u^*} W(s)\,ds\right]^2, \tag{10.3}$$

As $u^* \to \infty$, $u^{*\,-1} \int W(s)\,ds$ tends in distribution to a nondegenerate random variable (this can be proved easily by continuing the argument of [5]). Thus, $|Y(0)|^2$ tends to infinity with u^*, in distribution and almost surely, "as if" the function $W(s)$ were an ordinary stationary r.f. whose population mean has not been removed.

10.3. *Sample fluctuation of* $Y(\lambda, \omega, u^*)/(E|Y|^2)^{1/2}$, *at very high frequencies.* As in the study of $D[hu^*, B(u^*)]$ for small τ, the idea is to cut out from $(0, u^*)$ some stretches that contribute nothing to $Y(\lambda, \omega, u^*)$, while leaving a remainder that can be treated by the usual Wiener-Khinchin methods. Note, therefore, that any portion of an intermission of W, whose duration is an integral multiple of $1/\lambda$, contributes nothing to $Y(\lambda, \omega, u^*)$. One can, therefore, carry out the integration of $We^{-2\pi i\lambda s}$ over a subset $\Delta(u^*, \lambda)$ of $(0, u^*)$. That is, one can write Y as the following product of two independent r.v.,

$$Y = \{|\Delta(u^*, \lambda)|/u^*\}^{1/2}\, Y_\Delta \tag{10.4}$$

where Y_Δ is

$$(|\Delta(u^*, \lambda)|)^{-1/2} \int_{s \in \Delta(u^*,\lambda)} W(s)e^{-2\pi i\lambda s}\,ds. \tag{10.5}$$

This is the sample Fourier vector of a portion of length $\Delta(u^*, \lambda)$, cut off from an ordinary r.f. so defined that $X(t')$ and $X(t'')$ are independent r.v. when $|t' - t''| > 1/\lambda$. We know, therefore, that, as $|\Delta| \to \infty$, Y_Δ tends towards an isotropic Gaussian vector of zero mean.

For $\Delta(u^*, \lambda)/u^*$, division by some weight function of u^* and λ again yields a r.v. whose limit in distribution, as $u^* \to \infty$, is a r.v. M_θ.

Combining the two terms of Y, we see that we have proved that the high frequency fluctuations are *unaffected* by the marginal distribution of $W(t, \omega)$, and one has the following theorem.

THEOREM 10.1. *If $u^* \to \infty$, while λ is constant, the limit in distribution*

$$\lim_{u^* \to \infty} \{Y(\lambda, \omega, u^*)[E|Y(\lambda, \omega, u^*)|^2]^{-1/2}\} \tag{10.6}$$

is a compound random variable; an isotropic Gaussian vector whose mean square modulus is a M_θ r.v.

COROLLARY 10.1. *If λ is fixed, the limit in distribution*

$$\lim_{u^* \to \infty} \{|Y(\lambda, \omega, u^*)|^2[E|Y(\lambda, \omega, u^*)|^2]^{-1}\} \tag{10.7}$$

is the product of two independent r.v. EM_θ, where E is an exponential r.v.

10.4. *Joint fluctuations of $Y(\lambda', \omega, u^*)$ and $Y(\lambda'', \omega, u^*)$.* The method used for $Y(\lambda, \omega, u^*)$ remains applicable if there exist two positive *integers* q' and q'' such that $\lambda' q' = \lambda'' q'' = \lambda_0$. For brevity, denote $\Delta(u^*, 1/\lambda_0)$ by Δ, and consider the four-dimensional vector of coordinates

$$\begin{aligned}
&\Delta^{-1/2} \text{ Real part of } \int_{s \in \Delta} W(s) e^{-2\pi i \lambda' s}\, ds, \\
&\Delta^{-1/2} \text{ Imaginary part of } \int_{s \in \Delta} W(s) e^{-2\pi i \lambda' s}\, ds, \\
&\Delta^{-1/2} \text{ Real part of } \int_{s \in \Delta} W(s) e^{-2\pi i \lambda'' s}, \\
&\Delta^{-1/2} \text{ Imaginary part of } \int_{s \in \Delta} W(s) e^{-2\pi i \lambda'' s}\, ds.
\end{aligned} \tag{10.8}$$

As $|\Delta| \to \infty$, this vector tends to an isotropic four-dimensional Gaussian r.v.; therefore, the limits in distribution

$$\lim_{u^* \to \infty} \frac{|Y(\lambda', \omega, u^*)|^2}{\Delta(u^*, \lambda_0)}, \qquad \lim_{u^* \to \infty} \frac{|Y(\lambda'', \omega, u^*)|^2}{\Delta(u^*, \lambda_0)} \tag{10.9}$$

are two independent exponential r.v.

Further, one has the following theorem.

THEOREM 10.2. *If λ' and λ'' are constant and such that λ'/λ'' is rational, the two limits in distribution*

$$\lim_{u^* \to \infty} \frac{|Y(\lambda', \omega, u^*)|^2}{E[|Y(\lambda', \omega, u^*)|^2]}, \qquad \lim_{u^* \to \infty} \frac{|Y(\lambda'', \omega, u^*)|^2}{E[|Y(\lambda'', \omega, u^*)|^2]} \tag{10.10}$$

are, respectively, of the form $M_\theta E'$ and $M_\theta E''$, where M_θ, E', and E'' are mutually independent and E' and E'' are exponential with unit expectation.

10.5. *Weighted spectral estimators.* For functions to which the Wiener-Khinchin spectral analysis applies, reliable estimation of the spectral density is obtained through weighted averages of Schuster's periodograms $E_\lambda M_\theta$. Given a fixed frequency hand (λ', λ'') with $\lambda' > 0$, the factors E_λ are asymptotically eliminated and the ratio, between the sample and population values of the weighted spectral density, tends in distribution to M_θ as $u^* \to \infty$. Given the variable frequency band $(0, 1/u^*)$, the energy that it contains tends in distribution to another limit, not degenerate to zero; this is a sampling form of "infrared catastrophe."

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