RANDOM PACKING DENSITY

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1. Introduction

Random packing and random space-filling problems have received attention in recent articles in the literature. Investigation of seemingly disparate topics has produced the analyses and the results that are published, and there may be other settings in which related work has been underway that is not available in print. Writings on this subject appear in a wide variety of journals, and it is not difficult to miss pertinent published papers.

In one dimension, this topic usually bears the label "parking problem," and this has been treated by several authors. Here, the packing, or space filling, is achieved by automobiles of unit length which are parked at random, one at a time, in the unfilled intervals remaining on the line until space for one car is no longer available. The center of the interval representing the length of the automobile is assumed to follow a uniform distribution over the unfilled portions of the line. The mean, variance, and distribution of the proportion of the line filled by cars in this manner, as the length of the line becomes infinite, is the object of the analysis. Rényi's paper [13] is the first on this subject and it is followed by Ney [10], and most recently by Mannion [9] and Dvoretzky and Robbins [7]. We will return to these in later sections.

Related one-dimensional problems for the discrete situation are analyzed by Jackson and Montroll [8] and Page [11]. Page considers pairs of adjacent points selected at random from n points on a straight line, one pair at a time, such that neither point of a selected pair is allowed then to form a pair with its other neighbor. The process of selection is repeated until the only points remaining are isolated from one another by intervening pairs already selected. The proportion of isolated points as n approaches infinity is a parameter of interest in some physical models and we shall return to this study. Jackson and Montroll investigated the number of possible configurations for n points on a line, the total number of vacancies in all these configurations, and then the average proportion of vacancies in a configuration. In both papers, the actual physical problem is two-dimensional, but this is very difficult to manage; therefore, resort is made to one-dimensional analogues which are interesting in their own right and may suggest approaches and approximations for the two-dimensional model. Both sets of studies, namely the discrete and continuous models, differ from a large class of studies on counter problems and renewal theory models because there is a dependence of later random fillings on those previously registered.

Whereas both the continuous random space-filling model and the discrete random selection of pairs of points model are one-dimensional versions of some studies in geometrical probability, the geometrical aspects do not loom as large as they do in higher dimensional analogues where, together with the dependent probability aspects, they make a formal analysis most unmanageable. Each one-dimensional situation relates specifically to a physical model--Rényi was injected into what is labeled the "parking problem" by examining the one-dimensional analogue of a three-dimensional model for an ideal liquid. For the discrete case, Page is motivated by a model in physical chemistry which investigates the proportion of hydrogen atoms trapped in a film of hydrogen which is later adsorbed by mercury. Jackson and Montroll consider the same physical problem.

In two and three dimensions, mathematical analysis of the models becomes more intractable, despite the ingenuity demonstrated by those who contributed to the resolution of a large piece of the problem of random space filling in one dimension. The geometry of the region to be filled and the geometry of the units to be placed at random cause additional problems which do not emerge in the one-dimensional case where intervals on a line present the only way to "fill" or "pack" the line. However, there is interest and effort in three-dimensional random packing since some proposed molecular models for liquids require a study of this representation.

Work along these lines is indicated in the papers of several writers, notably J. D. Bernal [1]–[4], who are concerned with geometrical models of liquid structure. For example, a random packing of equal spheres in three dimensions may provide a useful model for an ideal liquid, ideal in the sense that all the distances between closest neighbors are equal. Naturally, random and nonrandom packing in two, three, and higher dimensions have an intellectual and mathematical appeal which have engaged the attention of a number of scholars over the years.

Apart from questions of random packing, there are geometrical exercises in packing which extend back in time to Newton's era. We shall return to questions of random packing and the results of our investigations, but some information on the geometry of the situation from a deterministic point of view is pertinent and will be useful in examining random packing. In fact, attempts at random packing may be useful because there are difficulties inherent in obtaining maximum or minimum packing solely from geometrical approaches. Coxeter [5] has suggested the use of random packing attempts to learn whether a random configuration can provide a packing density higher than the best bound which is attained from lattice packing. This paper and Coxeter's book [6] also provide interesting links between the geometry of polytopes, many of which occur in nature as crystals, and questions of packing density. It is the relationship between structure in crystallography and liquid structure that engaged Bernal's attention in his papers.

2. Previous results

Sphere packing is said to have density ρ, if the ratio of the volume of that part of a cube covered by the spheres, where no two spheres have any inner point in common, to the volume of the whole cube, tends to the limit ρ as the side of the cube tends to infinity. Let ρ_n denote the upper bound of this density in n-dimensional space. Rogers [14], [15] then proves the following theorem.

THEOREM. *Consider the regular simplex in n dimensions of side 2, and the system of $(n + 1)$ spheres of unit radius with their centers at the vertices of the simplex. Let σ_n denote the ratio of the volume of that part of the simplex covered by the spheres to the volume of the whole simplex. Then $\rho_n \leq \sigma_n$, and $\sigma_n < [(n+2)/2](1/\sqrt{2})^n$.*

This bound had been known before, but not in the sharp way developed by Rogers.

An asymptotic formula due to H. E. Daniels and given in Rogers [14], [15] provides an upper bound for the packing density, ρ_n, of n-dimensional spheres in an n-dimensional cube as n approaches infinity. Daniel's result as $n \to \infty$ is

$$\sigma_n \sim \frac{(n+1)!e^{(n/2)-1}}{\sqrt{2}\Gamma\left(1+\frac{n}{2}\right)(4n)^{n/2}} \sim \frac{n}{e}\left(\frac{1}{\sqrt{2}}\right)^n. \tag{1}$$

In fact,

$$\sigma_n = \frac{n}{e}\left(\frac{1}{\sqrt{2}}\right)^n\left[1 + 0\left(\frac{1}{n}\right)\right] \qquad \text{as} \quad n \to \infty. \tag{2}$$

This suggests that as n increases, and consequently the upper bound decreases, a random packing may yield a packing density greater than an arrangement arrived at through the best lattice packing. However, the mean packing densities and variances through $n = 5$ in table I, which are obtained by random packing

TABLE I

RANDOM PACKING OF SPHERES

Results of placing an n-dimensional sphere with radius equal .5 at random into a larger n-dimensional sphere with radius equal 5.0.

Dimension	2	3	4	5
Sample Size (Replications)	20	10	4	2
Mean Packing Ratio	.4756	.280	.146	.075
St. Dev. of Packing Ratio	.01964	.0049	.00408	.000453

of a large sphere with unit spheres, suggest that this is very much an open problem when they are compared with the closest lattice packing densities for each dimensionality.

In table II, densities are given for random packing of spheres in cubes. Our work indicates that present computer and programming technology both require some forward advance before we can try economically for random packing densi-

TABLE II

DENSITY OF PACKING OF SPHERES IN A CUBE

Source \ Dimension	2	3	4	5
Closest Lattice Packing	$\frac{\pi}{\sqrt{12}} = .9069$	$\frac{\pi}{\sqrt{18}} = .7404$	$\frac{\pi^2}{16} = .6168$	$\frac{\pi^2}{\sqrt{450}} = .4652$
Uniform Distribution	.468	.282	.146*	.075*
Worst Lattice Packing (Best Lattice Covering) $\left(\frac{1}{2^n}\right)$	$\left(\frac{2\pi}{\sqrt{27}}\right)\frac{1}{4} =$ .3023	$\left(\frac{5\sqrt{5}\pi}{24}\right)\frac{1}{8} =$ .1829	$\left(\frac{3\pi^2}{5\sqrt{5}}\right)\frac{1}{16} =$ .1103**	

* Packing spheres in a sphere.
** Less than value in cell, greater than .1036.

TABLE III

n	$\sigma_n = \frac{n+2}{n}\left(\frac{1}{\sqrt{2}}\right)^n$	Best Lattice Packing Value	Best Lattice Packing Decimal Form	$\frac{(n+1)!e^{(n/2)-1}}{\sqrt{2}\Gamma\left(1+\frac{n}{2}\right)(4n)^{n/2}}$
2	1.0000	$\frac{\pi}{2\sqrt{3}}$	.9069	.3671
3	.8839	$\frac{\pi}{3\sqrt{2}}$	.7404	.3902
4	.7500	$\frac{\pi^2}{16}$	.6168	.3679
5	.6187	$\frac{\pi^2}{15\sqrt{2}}$	.4652	.3252
6	.5000	$\frac{\pi^3}{48\sqrt{3}}$	.3729	.2759
7	.3977	$\frac{\pi^3}{105}$	.2952	.2276
8	.3125	$\frac{\pi^4}{384}$	.2536	.1839
9	.2431	$\geq \frac{2\pi^4}{945\sqrt{2}}$	$\geq .1457$	.1463
10	.1875	$\geq \frac{\pi^5}{1920\sqrt{3}}$	$\geq .0920$	.1150
11	.1436	$\geq \frac{64\pi^5}{187{,}110\sqrt{3}}$	$\geq .0604$	.0894
12	.1094	$\geq \frac{\pi^6}{19{,}440}$	$\geq .0494$	.0690

ties through $n = 12$—the largest value of n where a lattice packing solution is available for comparison at present.

There are known results for best lattice packing of n-dimensional spheres in n-dimensional space for $n = 2, 3, \cdots, 11, 12$. These are summarized and tabled in Rogers [15]. These values are listed below in table III together with the approximations obtained by Rogers and by Daniels.

For $n = 2, 3, 4, 5$ they are given in table VIII to serve as anchors for the other entries. In one dimension, the whole line can be filled and thus nonrandom packing is a trivial question. For $n = 2$, the maximum lattice packing density of $\pi/\sqrt{12}$ is attained for a hexagonal array of equal circles, each circle tangent to six other circles of the array. For $n = 3$, the maximum lattice packing density of $\pi/\sqrt{18}$ is attained by tangency of each sphere to twelve other spheres.

3. Random packing attempts

These geometrical results are of interest and most useful as bench marks in examining the density due to random packing in both two and three dimensions. In fact, we employ similar results on lattice covering (see Rogers [15]) of n-dimensional space by n-dimensional spheres to obtain results of worst lattice packing, which are given in table VIII. A strict probabilistic analysis for $n \geq 2$ appears to be intractable at present and so we are compelled to investigate and employ monte carlo or machine simulation techniques to obtain results.

Ordinarily one would assume that the centroids of the space-filling units follow a uniform distribution over the unfilled regions until packing is accomplished. We have done this for $n = 2, 3, 4, 5$ for spheres packed in a larger sphere. In addition we have considered a "restricted" randomness situation for $n = 1, 2$ which was motivated by a physical model in three dimensions. This has produced more packing density values and has provided some new theoretical results to check on some simulation results for the modified or "restricted" random parking problem. In fact, the original "parking problem" is imbedded in this revised one-dimensional model which produces solutions to additional problems of interest. This is discussed in a later section.

These analytical results in one dimension, simulated results in two dimensions, and physical results in three dimensions add to the mosaic of values for packing density given in table VIII.

This paper had its beginnings in a query by Peter Ney. In attempting to extend the parking problem results to higher dimensions, Ilona Palasti [12] conjectured that if in two dimensions, a rectangle is filled at random by unit squares with sides parallel to the sides of the rectangle, then the ratio of the area filled to the total area of the rectangle approaches (as the rectangle becomes infinite) a mean value which is the square of the mean value (packing density) obtained in the parking problem. In one dimension, as the line becomes infinite in length, the packing density (mean value) is

(3) $$c = \int_0^\infty \exp\left\{-2\int_0^t \frac{1-e^{-u}}{u}\,du\right\}dt = .74759.$$

See the papers by Rényi, Ney, Mannion, Dvoretzky and Robbins. Mannion also obtained the variance of the occupied portion of the line segment as the line becomes infinite—it is proportional to c_2x where x is the length of the line segment and $c_2 = 0.035672$.

Thus the Palasti conjecture, which is based on some mathematical development in her paper, states that the random packing density of unit squares in two-dimensional space is:

(4) $$c^2 = (.74759)^2 = .56,$$

and this value was produced empirically in four experiments reported by Palasti in her paper.

The conjecture is extended by Palasti to n-dimensional unit cubes filling n-dimensional space at random where the sides of the cubes are always oriented appropriately; namely, that the packing density (mean value) is c^n. Ney was unconvinced mainly because the four sampling experiments reported always produced a value of .56. A thorough check on this conjecture in two dimensions by simulation requires good approximations of the uniform distribution over a rectangle, and so we first examined a discrete analogue in two dimensions which also is of practical interest.

4. Two-dimensional "parking" problem

Consider the simple discrete process developed in the paper by E. S. Page [11] in connection with the following problem in physical chemistry. A surface received a film of hydrogen which was later adsorbed by mercury. If the surface is regarded as a rectangular lattice of sites available for the single hydrogen atoms, a possible model for the formation of the film is that a hydrogen molecule consisting of two atoms comes on to the surface and occupies a pair of adjacent sites. It is supposed that these atoms stay in position until one or both are later adsorbed by the mercury. The hydrogen molecules occupy the pairs by sites in sequence until all the pairs of *adjacent* sites are filled. At this stage, the average proportion of the sites which have not been filled by hydrogen atoms is a quantity of interest.

In the adsorption stage of the experiment a pair of adjacent hydrogen atoms, but *not necessarily* a pair, which originally formed a molecule when occupying the surface, is replaced by a pair of atoms of mercury. The selection of pairs of adjacent hydrogen atoms continues as long as possible, and eventually some isolated hydrogen atoms remain trapped in the film. The proportion of hydrogen trapped is then desired.

Because this two-dimensional problem seemed too difficult for straightforward mathematical analysis, Page considered a reduced version in one dimension. We now wish to employ Page's asymptotic results for the one-dimensional version for comparison with our computer simulation results for the same model. Then we will simulate a two-dimensional analogue on the computer which relates to the rectangular lattice model above for the proportion of trapped hydrogen. In effect, this is a discrete analogue of the parking problem in two dimensions. Our computer simulation results for the proportion in two dimensions will be compared then with the square of Page's proportion for one dimension.

Consider the points $1, 2, \cdots, n$ on the real line. Pick an adjacent pair of points $(X, X + 1)$ at random by choosing $X = i, i = 1, \cdots, n - 1$, with probability $1/(n - 1)$. Then choose a second pair at random from the remaining points, and continue in this way until no adjacent pairs remain there. Let $N(n)$ denote the number of pairs so chosen. This process has been considered by Page [11], from whose work one can conclude after some modifications that

$$\lim_{n\to\infty} \frac{N(n)}{n} = .4323. \tag{5}$$

A two-dimensional analogue is defined in the obvious way by starting with an $n \times n$ array of points and randomly choosing an adjacent pair of points situated along the top row of the array $(X, X + 1)$ with probability $1/(n - 1)$; and an adjacent pair of points situated along the first column of the array $(Y, Y + 1)$ with probability $1/(n - 1)$. This identifies the four vertices of a "unit square" in the $n \times n$ array of points which cannot be selected again. Continue in this way until no "unit squares" remain. Let $N(s)$ denote the number of "unit squares" so chosen, and we conjecture in the fashion of Palasti that

$$\lim_{n\to\infty} \frac{N(s)}{n^2} = (.4323)^2 = .1869. \tag{6}$$

This procedure was carried out on a computer for $n = 10, 20, 40, 50, 100$ in the one-dimensional case with 10, 25, 50, 100 replications; and for $n \times n = 10 \times 10$, 20×20, 40×40, 50×50, 100×100 in the two-dimensional case with 10 and 25 replications except for the 100×100 array which was replicated 10 times. The results are given in table IV and are convincingly close to the postulated value in one dimension and the conjectured value for two dimensions.

The spectacular results listed in table IV suggest why Palasti may have had the phenomenal occurrences for the two-dimensional continuous model displayed in her article. In four experiments of random space filling of rectangles with unit squares, she achieved the space-filling ratio .56. The rectangular dimensions were 5×15, 10×15, 15×15 and 20×15. Moreover, this stability occurred around an asymptotic ratio for unit squares in rather finite rectangles. For one thing, the conjecture is reinforced since the arithmetic means in table IV increase as n increases and as they converge to the asymptotic ratio, and the variances are small and decrease as n increases.

On the other hand, the physical situation for the discrete model inhibits variability more than one should expect for the continuous version examined by Palasti—for example, tangency of unit squares is possible only in the continuous model. For this reason, and because the samplings were not replicated, the

TABLE IV

ONE DIMENSION

No. of points \ Sample Size	Entries in cells are the average no. of pairs of points chosen and the variance within the sample				Entries in cells are the average ratio of no. of pairs of points chosen divided by the total no. of points and variance within the sample			
	10	25	50	100	10	25	50	100
10	4.20 .4000	4.20 .1667	4.20 .2041	4.26 .2752	.4200 .004000	.4200 .001667	.4200 .002041	.4260 .002752
20	8.60 .2667	8.56 .3400	8.54 .2943	8.53 .4132	.4300 .000667	.4280 .000850	.4270 .000736	.4265 .001033
40	17.40 .2667	17.28 .9600	17.20 .5306	17.18 .6743	.4350 .000167	.4320 .000600	.4300 .000332	.4295 .000421
50	21.20 .4000	21.20 .7500	21.40 .8980	21.51 .8989	.4240 .000160	.4240 .000300	.4280 .000359	.4302 .000359
100	43.20 1.0667	43.32 1.6434	42.90 1.8061	43.15 2.2096	.4320 .000107	.4332 .000164	.4290 .000181	.4315 .000221
					Ratio from asymptotic development in E. S. Page's paper: .4323			

TWO DIMENSIONS

	Entries in cells are the average no. of "unit squares" chosen and the variance within the sample		Entries in cells are the average ratio of no. of "unit squares" chosen divided by n^2 and variance within the sample	
10 by 10	16.80 .8444	17.36 1.5733	.1680 .000084	.1736 .000157
20 by 20	73.00 3.3333	73.24 6.3567	.1825 .000002	.1831 .00040
40 by 40	294.20 67.0668	295.68 17.3112	.1839 .000026	.1848 .000007
50 by 50	460.10 23.2118	461.16 20.2917	.1840 .000004	.1844 .000003
100 by 100	1866.60 118.8		.1867 .000001	
			Conjectured ratio in asymptotic case: $(.4323)^2 = .1869$	

results from the Palasti experiment bear re-examination. A computer simulation of Palasti's four experiments produced the following results which are listed together with her sampling yields. Each case was replicated ten times on the computer. The mean, variance, and standard deviation in table V are based on this sampling.

TABLE V

Number of unit squares placed					Proportion of area covered by unit squares			
		Computer				Computer		
Rectangle	Palasti	Mean	Variance	Strd. Dev.	Palasti	Mean	Variance	Strd. Dev.
5 × 5	42	37.7	2.455	1.567	.56	.50270	.0004365	.02089
10 × 15	84	80.9	9.307	3.282	.56	.53933	.0004785	.02188
15 × 15	126	120.7	5.346	2.311	.56	.53644	.0001056	.01027
20 × 15	167	161.1	14.319	3.873	.56	.53700	.0001591	.01261
20 × 30		329*				.54833*		

* One trial.

These results indicate some fortuitous findings by Palasti. It is highly conceivable that as the area of the rectangle approaches infinity, the mean ratio of filled area approaches .56, and the computer simulation data are not inconsistent with this conjecture. Perhaps some new theoretical developments or additional computer simulations will provide closure to this almost resolved problem. We have added a 20 × 30 rectangle which produced a ratio of .54833. This was the result of only one trial, and no larger rectangles were attempted on the computer because the expense and the efficiency of the computer dictated this.

Simulation of this random space filling in two dimensions led naturally to random space filling in higher dimensions and for different geometric figures. In table I there are listed the results of placing n-dimensional spheres with unit diameter at random into a larger n-dimensional sphere with diameter equal to 10 units.

5. Restricted random packing in three dimensions

The value of the random-packing density in three dimensions arrived at by sampling, namely .27, provides a bound of interest in physical chemistry in connection with models of ideal simple liquids. It is in this setting that our attention was turned to questions of packing density where randomness is restricted through some structure imposed by the realities of the situation. Several experiments on the packing of equal spheres in three dimensions are reported in the literature. These experiments were attempts to produce estimates of packing density through physical simulation by pouring spherical balls into a rigid container.

Two kinds of random packing are noted in these papers and are labelled "dense random packing" and "loose random packing." For dense packing, the balls in the vessel are gently shaken down for several minutes; for loose packing, the balls fill the vessel by rolling down a slope of random-packed balls. Experiments were reported by Scott [16], and the results were plotted to provide extrapolations for infinite containers so as to remove "peripheral" or "edge" effects. Values of packing density equal to 0.64 for dense random packing and equal to 0.60 for loose random packing were obtained.

Bernal and Mason [3], who were following up some earlier work of Bernal [1], [2], examined the mutual coordination (number of nearest neighbors) of spheres arranged at random in three dimensions and more or less closely packed. They report a packing density of 0.62 for an approximation to Scott's dense packing, and 0.60 for a situation approximating Scott's loose packing. Also, they estimate that in any physical random packing, each sphere should be in close contact with at least four others as a necessary condition of stability, and at most twelve. They suggest that the most probable average is six in that each sphere, in general, would rest on three others and support another three. Bernal and Mason [3] also conjecture that the dense random-packing density of 0.64 must be capable of mathematical determination, since it appears that it should occur for minimum unoccupied volume given this restricted random situation (the maximum density for lattice packing is .7404, and this occurs when each sphere touches twelve others). The joint authors suggest that random close packing is probably related to the configuration of atoms in liquids, such as those of the rare gases, and the difference between random close packing (.64) and regular lattice close packing (.74) is nearly the same as the percentage increase in volume which occurs in the melting of argon. Random loose packing may represent the approximation of the liquid at higher temperatures.

Scott [17] examined empirically the radial distribution of random close packing of equal spheres in a rigid container. The number of nearest neighbors is estimated at 9.3 ± 0.8. This is consistent with the estimate of the co-ordination number made by Bernal and Mason [3], namely a modal co-ordination between 8 and 9 contacts and a mean number of total contacts equal to 8.5. Scott reports that from the radial distributions of the atoms in liquid helium, neon, and argon, the number of nearest neighbors reported in the literature is 8.5–9.7 for helium, 8.8 for neon, and 8.0–8.5 for argon. Thus the figure of 9.3 ± 0.8 obtained from random close packing of spherical balls can be considered in agreement with experimental results for the liquefied rare gases.

As mentioned above, Bernal and Mason suggest that the fractional change of volume in the melting of argon is the same as the fractional difference between the densities of regular close packing and random close packing. To reinforce this conjecture, Scott [17] refers to some data in the literature which gives the ratio of solid and liquid densities for neon, argon, krypton, and xenon as 1.15, 1.15, 1.15, 1.14 respectively, whereas the ratio of regular to random packing densities of spherical balls is found to be 1.16.

All this suggests both the use of random packing as a statistical index for the classification of liquids and the need to explore methods of obtaining this packing density without resorting to physical experiment. The latter clause is especially pertinent when one desires packing densities in higher dimensional spaces. Mathematical determination of this phenomenon is too difficult; therefore, one is led to computer simulation. It would be good to program what goes on physically in loose or dense random packing, and then have a computer simulate this, thus producing a packing density. Putting this structure into random space filling in three dimensions poses problems on a computer which require resolution.

6. Restricted random packing in one and two dimensions

Consideration of these problems led to attempts in one and two dimensions where simulation can be accomplished easily with and without the use of a computer. Results from one-dimensional simulations motivated some new theoretical results which add to the results previously obtained by those who have worked on the "parking problem."

For one and two dimensions, structured (loose or dense) random packing may be thought of in the following ways.

(a) *One dimension.* The center of a unit interval is placed at random along a line of length ℓ according to the uniform distribution over the unfilled portions of the line. If the interval does not overlap any previously placed intervals, it remains where it is placed; if it does overlap, it may move either to the right or to the left (whichever is the shortest distance). It may then be placed so that its endpoint coincides with the endpoint of the interval it has overlapped, provided that after this is done, the interval has not overlapped any previously placed interval. If it still overlaps, it is rejected. This is done sequentially until it is impossible for any unit interval to fit on the line. We desire the ratio of the line filled in this manner as ℓ approaches infinity.

(b) *Two dimensions.* Consider a rectangle with dimension w for open top and closed bottom, and dimension ℓ for the two sides. Circles of unit diameter will be placed at random within it in the following way. The center of a circle is a point positioned at random according to the uniform distribution over the interval $(w - 1)$ centered at the top of the rectangle. When a circle is selected it moves vertically down an imagined line ℓ, perpendicular to the sides, which passes through the point selected at random in the interval of length $(w - 1)$. It falls in this way to the bottom of the rectangle unless its descent is blocked by a previously placed circle. If the latter occurs, the circle moves along a line perpendicular to the imagined line to the right or the left, whichever distance is smaller for it to clear the blocking circle, but always less than a radius length, and keeps falling until it hits bottom, or if the bottom is filled, until it hits an upper layer which does not permit any additional fall across the width of the rectangle in the manner just described. This process continues until the rectangle is filled and no additional circles can be accommodated. This is an attempt at simulating in two dimensions what has been labelled "loose" random packing in three dimensions.

This two-dimensional version of loose random packing was attempted in three computer runs in which circles with unit diameters were placed in a square measuring 50 units to a side. This led to packing densities of .798, .802, and .806. In some computer runs for unrestricted random packing of circles in a rectangle, packing density values hovered around .60. Thus the summary given in table VIII lists values of .80 and .60 in the appropriate cells for the two-dimensional column. It would be good to simulate three-dimensional "loose packing" on a computer to see if values of packing density center around .63, the value obtained from physical simulation by several authors. This has not yet been done, but we are looking into the size and complexity of the programming effort.

Simulations of restricted random packing in two and three dimensions led us back to the one-dimensional case which we described above. Computer simulation for restricted random packing (loose packing) in one dimension produced the following (random unit intervals).

TABLE VI

Line Length	Replications	Mean	Variance	Standard Deviation
20	10	.8000	.00222	.04714
50	10	.7880	.00108	.03293
100	5	.8060	.00073	.02702

Computer simulation of the unrestricted random packing case—the "parking problem"—provided the following results to serve as an anchor to compare known results with computer simulation (random unit intervals).

TABLE VII

Line Length	Replications	Mean	Variance	Standard Deviation
20	10	.7000	.00167	.04082
50	10	.7480	.00046	.02150
100	5	.7500	.00040	.02000

Thus the computer simulation mean values are consistent with the theoretical mean value of .74759 for the infinite line in the parking problem. Also, the computer simulation variances are consistent with the theoretical variance of the number of cars necessary to fill the line. Mannion [9] gives for this variance the value $0.035672x$ when x, the length of the line, approaches infinity. For example, when the length is 100, we note the estimated variance based on a sample of size 5 is .000400 as compared with .000357.

This suggests that our computer simulation for the restricted random situation is satisfactory. However, it leads to a gnawing doubt and uneasy feeling since the mean value of .806 is so close to the packing density arrived at by simulation for circles in the two-dimensional restricted random (loose packing) situation, namely .80. Table VIII suggests that one or both packing densities may be in error.

TABLE VIII

SUMMARY

Type of Packing \ Dimensions of Sphere	2	3	4	5
Best Lattice Packing	$\frac{\pi}{\sqrt{12}} = .9069$	$\frac{\pi}{\sqrt{18}} = .7404$	$\frac{\pi^2}{16} = .6168$	$\frac{\pi^2}{\sqrt{450}} = .4652$
Restricted Randomness	.80	.63		
Cubic Packing	$\frac{\pi}{4} = .7854$	$\frac{\pi}{6} = .5236$	$\frac{\pi^2}{32} = .3084$	$\frac{\pi^2}{60} = .1645$
Uniform Distribution	.60	.27	.148*	.075*
Worst Lattice Packing	$\frac{2\pi}{\sqrt{27}} \cdot \frac{1}{4} =$ .3023	$\frac{5\sqrt{5}\pi}{24} \cdot \frac{1}{8} =$ .1829	$\frac{3\pi^2}{5\sqrt{5}} \cdot \frac{1}{16} =$ .1103**	

* Packing spheres in a sphere.

** Less than value in cell, greater than .1036.

The following argument suggested by Herman Rubin proved decisive in demonstrating the validity of our simulation results in one dimension. Thus, it indicates some re-examination of the simulation results for the two-dimensional case. Consider a street of length x with intervals of length $\alpha \leq 1$ and $\beta \leq 1$ at each end of the street. Let $\alpha + \beta = c \leq 2$. These parameters are chosen to represent the loose packing of cars of unit length in one dimension. For $x \geq 0$, let $[t, t + 1]$ be the random interval occupied by the first car parked on the street, unless it overlaps the interval α or the interval β, in which case it is moved the appropriate distance so that it touches the end of the interval it overlapped. Continue this process until no additional cars can be parked. This should permit more space filling than the original parking problem since cars are permitted more leeway to park bumper to bumper.

Following the integral equation derivation given by Rényi, Ney, and Dvoretzky and Robbins, we get

$$(7) \qquad (x + c - 1)N(x) = (x + c - 1) + cN(x - 1) + 2 \int_0^{x-1} N(y)\, dy$$

where $N(x)$ is the number of cars parked in the interval $[0, x]$. Putting $c = 0$, we obtain

$$(x-1)N(x) = (x-1) + 2\int_0^{x-1} N(y)\,dy \tag{8}$$

or

$$N(x+1) = 1 + \frac{2}{x}\int_0^{x} N(y)\,dy, \tag{9}$$

the integral equation for the regular parking problem. If we put $c = 1$, then

$$N(x) = 1 + \frac{1}{x}N(x-1) + \frac{2}{x}\int_0^{x-1} N(y)\,dy, \tag{10}$$

and this is for the case where a car which overlaps a previously placed car can move only in one direction for a maximum distance of a car length to find an unobstructed interval.

To find the limiting value for $c = 2$, as $x \to \infty$ we take the Laplace transform of each side of the integral equation and obtain

$$\phi'(t) + \phi(t)\left(-1 + 2e^{-t} + \frac{2}{t}e^{-t}\right) + \left(\frac{1}{t} + \frac{1}{t^2}\right)e^{-t} = 0. \tag{11}$$

Then we may write

$$K = \phi(t)\left\{e^{-t-2e^{-t}-2\int_t^\infty \frac{e^{-u}}{u}du}\right\} - \int_t^\infty e^{-v}\left(\frac{2}{v} + \frac{1}{v^2}\right)e^{-v-2e^{-v}-2\int_v^\infty \frac{e^{-u}}{u}du}\,dv \tag{12}$$

where K is a constant. Since for the limiting situation we have $K = 0$, we obtain

$$\phi(t) = \left[e^{t+2e^{-t}+2\int_t^\infty \frac{e^{-u}}{u}du}\right]\int_t^\infty e^{-v}\left(\frac{2}{v} + \frac{1}{v^2}\right)e^{-v-2e^{-v}-2\int_v^\infty \frac{e^{-u}}{u}du}\,dv, \tag{13}$$

or

$$\phi(t) = \tag{14}$$

$$\left[e^{t+2e^{-t}-2+2\int_t^\infty \frac{e^{-u}}{u}du+2\gamma}\right]\left[\int_0^\infty \left(\frac{2}{v} + \frac{1}{v^2}\right)e^{-2(v+e^{-v}-1)-2\gamma-2\int_t^\infty \frac{e^{-u}}{u}du}\,dv\right],$$

where γ = Euler's constant and

$$\int_t^\infty \frac{e^{-u}}{u}\,du = -\log u - \gamma + \sum_{m=1}^{\infty}\frac{(-1)^{m-1}t^m}{m'm!}. \tag{15}$$

Now $\phi(t)$ behaves as $A/t^2 + B/t + \cdots$, and thus $N(x) \sim Ax + B + \cdots$, and A is the limiting value we seek where

$$A = \int_0^\infty \left(\frac{2}{v} + \frac{1}{v^2}\right)e^{-2(v+e^{-v}-1)-2\gamma-2\int_v^\infty \frac{e^{-u}}{u}du}\,dv. \tag{16}$$

Evaluating this constant by computer we find $.80865 < A < .80866$. This theoretical value for packing density conforms well to the simulated value for a 100 unit line, namely .806, and suggests that our two-dimensional simulated value requires further examination.

The value .80 is entered in the appropriate cell for $n = 2$ in the summary listed in table VIII with the tacit understanding that this value bears watching. In some smaller rectangles, namely 10×10, 10×15, "loose packing" of circles produced filled ratios of .7411 and .8011. The latter is already close to the values obtained in 50×50 rectangles, and both results are consistent for increasing area of rectangle.

The mosaic presented in table VIII is of interest. Cubic packing is obtained by considering an n-dimensional space compartmentalized into n-dimensional cubes of unit edge and n-dimensional spheres with unit diameter inscribed in each cube. Note that for $n = 2, 3$, restricted and unrestricted random packing are neither best nor worst but serve to bound cubic packing. For $n = 4$, one cell is missing, but this phenomenon is indicated here also. For $n = 5$, two cells are empty but this possibility also exists here. A column with results for $n = 1$ is omitted in table VIII since the geometry of the situation is lost in one-dimension, and values are probably not commensurate. In this case, the filled ratio is unity for best lattice packing and cubic packing; it is .8087 and .7476 for restricted and unrestricted randomness respectively; and it is .50 for worst lattice packing.

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