# STRONG MIXING PROPERTIES OF MARKOV CHAINS WITH INFINITE INVARIANT MEASURE

KLAUS KRICKEBERG
UNIVERSITY OF HEIDELBERG

## 1. Introduction

It is common knowledge that the existence of weakly wandering sets for ergodic transformations $T$ in spaces of infinite measure excludes the existence of strongly mixing transformations in such spaces if this concept is defined by requiring the "mixing equation" (2.2) below to be true, with some sequence $\rho_n$, for all measurable sets $A$ and $B$ of finite measure. However, Hopf's famous example of a transformation in the plane ([19], p. 67) shows that (2.2) may hold for all bounded sets $A$ and $B$ whose boundary has measure 0. This and the desire to put the so-called "individual" or "strong" ratio limit property of Markov chains into the general framework of measure preserving transformations might motivate the present attempt to treat mixing in topological measure spaces, and to restrict the attention to almost everywhere continuous transformations.

The second section describes the measure spaces and the class of their admissible isomorphisms. These isomorphisms, too, are required to be almost everywhere continuous, and thus leave invariant the concept of a mixing and, more generally, of a quasi-mixing transformation as defined here. What is involved is, essentially, a particular kind of weak convergence of functions of two sets which are sigma-finite measures in each variable.

One of the basic tools is the construction of mixing transformations in Euclidean spaces from the shift in the sample space of Markov chains via an isomorphism between these two measure spaces; this isomorphism is given in section 3. The next section describes the relation between the strong ratio limit property and the quasi-mixing and mixing property of the shift. Section 5 treats some examples, and the final one contains, without proofs, category theorems which can be obtained by exploiting these methods systematically.

The present paper owes much to oral and written discussions with F. Papanghelou, much more, indeed, than will be apparent from the references given below. In particular, he drew my attention to the papers [14], [16], [18], and [21].

This work was done while the author was a visiting professor at Columbia University.

## 2. Mixing transformations

Let $X$ be a completely regular topological space and $\mathfrak{F}$ the sigma-algebra of its Borel sets, that is, the sigma-algebra generated by its open sets. A sigma-finite measure $\mu$ defined on $\mathfrak{F}$ is called *tight* if every Borel set $A$ can be written as $A = \bigcup_{k=1}^{\infty} C_k \cup N$ where each set $C_k$ is compact, $N \in \mathfrak{F}$, and $\mu(N) = 0$. We may assume that $\mu(C_k) < +\infty$. In fact, there are Borel sets $D_{kl}$ with finite measure such that $C_k = \bigcup_{l=1}^{\infty} D_{kl}$, and compact sets $C_{klm}$ and Borel null sets $N_{kl}$ such that $D_{kl} = \bigcup_{m=1}^{\infty} C_{klm} \cup N_{kl}$.

If $\mu$ is tight and $\nu$ is $\mu$-continuous, $\nu$ is also tight. In the case of a finite measure, tightness as defined here means the same as "regularity and tightness" as studied by Le Cam [15], Prohorov [17], Varadarajan [20], Hildenbrand [8], and others; Bochner [2] and Kappos [12] use the term "strictly topological measure." In the general case, $\mu$ being sigma-finite, it can be represented as $\mu = \sum_{l=1}^{\infty} \mu_l$ where each $\mu_l$ is a finite measure on $\mathfrak{F}$, and since $\mu_l$ is $\mu$-continuous, it is tight if $\mu$ is. Conversely, the sum of countably many tight measures $\mu_l$, if it is sigma-finite, is tight, because $A = \bigcup_k C_{kl} \cup N_l$ with $\mu_l(N_l) = 0$, $l = 1, 2, \cdots$ implies $A = \bigcup_{kl} C_{kl} \cup N$ with $N = \bigcap_l N_l$, hence $\mu(N) = 0$.

Consider another completely regular space $X'$ and the sigma-algebra $\mathfrak{F}'$ of its Borel sets. Let $\mathfrak{F}^*$ stand for the domain of the completion of $\mu$; for the sake of simplicity, this completion will again be denoted by $\mu$. Let $S$ be a mapping from $X$ into $X'$ which is *$\mu$-almost everywhere continuous.* By this we mean that the set $M$ of the points of $X$ where $S$ is discontinuous belongs to $\mathfrak{F}^*$ and $\mu(M) = 0$. Then $S$ is measurable between $\mathfrak{F}^*$ and $\mathfrak{F}'$, that is, $S^{-1}\mathfrak{F}' \subseteq \mathfrak{F}^*$. To prove this, note that we may write $X = D \cup L$ where $D, L \in \mathfrak{F}$, $D \cap L = \emptyset$, $\mu(L) = 0$, and $M \subseteq L$; in particular, $S$ is continuous on $D$. If $A' \in \mathfrak{F}'$, we have $D \cap S^{-1}A' \in \mathfrak{F}$ and $L \cap S^{-1}A' \in \mathfrak{F}^*$ with $\mu(L \cap S^{-1}A') = 0$; therefore, $S^{-1}A' \in \mathfrak{F}^*$ and $\mu(S^{-1}A') = \mu(D \cap S^{-1}A')$.

The *image* $\mu'$ of $\mu$ under $S$ is, then, the measure given on $\mathfrak{F}'$ by $\mu'(A') = \mu(D \cap S^{-1}A')$ where $D \cap S^{-1}A' \in \mathfrak{F}$ for every $A' \in \mathfrak{F}'$. We are going to show that, if $\mu$ is tight, so is $\mu'$. In fact, given a set $A' \in \mathfrak{F}'$, write $D \cap S^{-1}A' = \bigcup_k C_k \cup N$ with compact sets $C_k$ and $\mu(N) = 0$. Since $S$ is continuous on $D$ and $C_k \subseteq D$, the sets $C'_k = SC_k$ are compact, and $C_k \subseteq S^{-1}A'$ implies $C'_k \subseteq SS^{-1}A' \subseteq A'$. Set $N' = A' - \bigcup_k C'_k$. Then we have $C_k \subseteq S^{-1}SC_k = S^{-1}C'_k$, hence $S^{-1}N'$ is equal to $S^{-1}A' - \bigcup_k S^{-1}C'_k \subseteq S^{-1}A' - \bigcup_k C_k$ which gives $\mu'(N') = \mu(D \cap S^{-1}N') \leq \mu(D \cap (S^{-1}A' - \bigcup_k C_k)) \leq \mu(N) = 0$.

Given any two measure spaces $(X, \mathfrak{F}, \mu)$ and $(X', \mathfrak{F}', \mu')$ of the type considered here, a mapping $S$ from $X$ into $X'$ is called a *homomorphism* of the former into the latter if it is $\mu$-almost everywhere continuous, and $\mu'$ is the image of $\mu$ under $S$. Thus, tightness is preserved by homomorphisms. If $S$ is such a homomorphism, and if $S'$ is one from $(X', \mathfrak{F}', \mu')$ into a third measure space $(X'', \mathfrak{F}'', \mu'')$, the composition $S' \circ S$ is a homomorphism of $(X, \mathfrak{F}, \mu)$ into $(X'', \mathfrak{F}'', \mu'')$. In particular, if $f'$ is a $\mu'$-almost everywhere continuous real-valued function, the function $f' \circ S$ is $\mu$-almost everywhere continuous. A subset $A$ of $X$

is called *$\mu$-almost clopen* if its indicator function $1_A$ is $\mu$-almost everywhere continuous, that is, if the boundary of $A$ has measure 0; this implies $A \in \mathfrak{F}^*$. Applying the preceding remark, we see that the inverse image $S^{-1}A'$ of a $\mu'$-almost clopen set $A'$ under a homomorphism $S$ is $\mu$-almost clopen, since $1_{S^{-1}A'} = 1_{A'} \circ S$.

By an *isomorphism* of $(X, \mathfrak{F}, \mu)$ onto $(X', \mathfrak{F}', \mu')$, we mean a homomorphism $S$ of $(X, \mathfrak{F}, \mu)$ into $(X', \mathfrak{F}', \mu')$ with the following property: there exists a homomorphism $S'$ of $(X', \mathfrak{F}', \mu')$ into $(X, \mathfrak{F}, \mu)$ such that $S'Sx = x$ for $\mu$-almost all $x \in X$ and $SS'x' = x'$ for $\mu'$-almost all $x' \in X'$. It will be assumed in the remainder of the paper that all the measures occurring there are tight.

Consider now a homomorphism $T$ of $(X, \mathfrak{F}, \mu)$ into itself, or *endomorphism*, in short. In other words, this is a measure preserving transformation which is defined and continuous almost everywhere. The homomorphism $T$ is called *strongly quasi-mixing* if there exists a sequence of almost clopen sets $H_k$, $k = 1, 2, \cdots$, a sequence of positive numbers $\rho_n$, $n = 1, 2, \cdots$, and a function $\varphi$ on $\mathfrak{F} \times \mathfrak{F}$ such that the following is true: $0 < \varphi(H_k) < +\infty$; $H_k \subseteq H_{k+1}$; $\mu(X - \bigcup_k H_k) = 0$; $\varphi$ is a sigma-finite tight measure in each variable if the other one is fixed; $\varphi(A, B) > 0$ whenever $A$ and $B$ are almost clopen, and $\mu(A) > 0$ and $\mu(B) > 0$; and finally

$$\lim_{n\to\infty} \rho_n \mu(A \cap T^{-n}B) = \varphi(A, B) < +\infty \tag{2.1}$$

for any two almost clopen sets $A$ and $B$ included in some $H_k$. If for some choice of $H_k$ and $\rho_n$ the function $\varphi$ has the particular form $\varphi(A, B) = \mu(A)\mu(B)$, that is,

$$\lim_{n\to\infty} \rho_n \mu(A \cap T^{-n}B) = \mu(A)\mu(B), \tag{2.2}$$

we say that $T$ is *strongly mixing*.

For continuous, but not necessarily measure preserving transformations in a space of finite measure, strong mixing was defined by Jacobs [10]. It would now be easy to define strong mixing for general positive contractions in $L_1(\mu)$ which preserve almost everywhere continuous functions so as to embrace both the situation studied by Jacobs and that considered here.

Weaker concepts could be obtained by understanding (2.1) or (2.2) in the sense of Cesàro convergence or strong Cesàro convergence. In the latter case we would have to speak about weak quasi-mixing and weak mixing, but we will not use these concepts here, and we will abandon the adverb "strongly" in the sequel.

In a sense, $\rho_n$ and $\varphi$ are independent of the choice of the sequence $H_k$, and $\varphi$ is, up to a positive factor, determined by its values (2.1) for almost clopen sets $A$ and $B$ included in some $H_k$. In fact, suppose that we have other sequences $H'_k$ and $\rho'_n$ and another function $\varphi'$ with analogous properties. Then there are indices $k$ and $l$ such that $\mu(H_k \cap H'_l) > 0$. Upon applying the "mixing equations" (2.1) to $A_0 = B_0 = H_k \cap H'_l$, we find that the finite and positive limit

$$\alpha = \lim_{n\to\infty} \frac{\rho'_n}{\rho_n} = \frac{\varphi'(A_0, B_0)}{\varphi(A_0, B_0)} \tag{2.3}$$

exists; that is, $\rho_n$ and $\rho'_n$ are asymptotically equal up to a positive factor. It follows that $\varphi'(A, B) = \alpha\varphi(A, B)$ whenever $A$ and $B$ are almost clopen and included in some $H_k$ as well as in some $H'_l$. Since $\varphi$ and $\varphi'$ are tight in each variable, an argument similar to those employed in [1] and [8] shows that $\varphi'(A, B) = \alpha\varphi(A, B)$ if $A, B \in \mathfrak{F}$ and $A, B \subseteq H_k \cap H'_l$ for some $k$ and $l$, and this finally implies that $\varphi' = \alpha\varphi$. In the case of a mixing transformation, upon normalizing $\varphi$ and $\varphi'$ by (2.2), $\rho_n$ and $\rho'_n$ become asymptotically equal.

As an immediate consequence of the definitions, quasi-mixing and mixing are invariant under isomorphisms: if $T'$ is an endomorphism of $(X', \mathfrak{F}', \mu')$ and $S$ an isomorphism of $(X, \mathfrak{F}, \mu)$ onto $(X', \mathfrak{F}', \mu')$ such that $T' \circ S = S \circ T$ holds $\mu$-almost everywhere, then $T$ is quasi-mixing if and only if $T'$ is, and the same sequence $\rho_n$ serves both for $T$ and $T'$.

Let us look first at the case $\mu(X) < +\infty$, and suppose that $T$ is quasi-mixing. Then there is a $k$ such that $\mu(H_k) > \frac{2}{3}\mu(X)$; hence, $\mu(H_k \cap T^{-n}H_k) > \frac{1}{3}\mu(X)$ for all $n$, and therefore $\limsup_{n\to\infty} \rho_n \leq 3\varphi(H_k, H_k)/\mu(X)$. Thus the sequence $\rho_n$ is bounded by some constant $\alpha$. By (2.1) this implies $0 \leq \varphi(A, B) \leq \alpha \min(\mu(A), \mu(B))$ for almost clopen sets $A$ and $B$ contained in an $H_k$, and hence for all $A, B \in \mathfrak{F}$. On account of the complete regularity of $X$, the class of all almost clopen sets included in some $H_k$ is dense in $\mathfrak{F}$ for the metric $\mu(\cdot + \cdot)$ where $+$ stands for the symmetric difference. The usual approximation argument shows now that (2.1) holds for all $A, B \in \mathfrak{F}$. It follows that $\lim_{n\to\infty} \rho_n = \varphi(X, X)/\mu(X)$; hence,

$$\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \varphi(A, B)\mu(X)/\varphi(X, X). \tag{2.4}$$

If $T$ is mixing, we find that $T$ is mixing in the classical sense ([9], p. 36 and [6], p. 37). Upon normalizing $\varphi$ by $\varphi(X, X) = \mu(X)^2$, we have $\lim_{n\to\infty} \rho_n = \mu(X)$ and (2.2), and we can satisfy (2.2) with $\rho_n = \mu(X)$ for all $n$.

Suppose next that $\mu(X) = +\infty$, and $T$ is mixing. Then

$$\mu(H_k)^2 = \lim_{n\to\infty} \rho_n\mu(H_k \cap T^{-n}H_k) \leq \liminf_{n\to\infty} \rho_n\mu(H_k), \tag{2.5}$$

and therefore, $\mu(H_k) \leq \liminf_{n\to\infty} \rho_n$ for all $k$, hence $\lim_{n\to\infty} \rho_n = +\infty$.

Given an endomorphism $T$, a sequence of numbers $\rho_n$, and a fixed set $A \in \mathfrak{F}^*$ with $\mu(A) < +\infty$, the class $\mathfrak{B}$ of all sets $B \in \mathfrak{F}^*$ of finite measure such that the limit (2.1) exists and is finite, is obviously stable against unions of finitely many disjoint sets, and differences $B_2 - B_1$ with $B_1 \subseteq B_2$. If $\varphi(A, \cdot)$ is defined on $\mathfrak{B}$ by (2.1), $\mathfrak{B}$ is also stable against completion relative to $\varphi(A, \cdot)$ in the following sense: if $B \in \mathfrak{F}^*$ has the property that, for every positive $\epsilon$, there are sets $B_1$ and $B_2$ in $\mathfrak{B}$ such that $B_1 \subseteq B \subseteq B_2$ and $\varphi(A, B_2) - \varphi(A, B_1) < \epsilon$, then $B \in \mathfrak{B}$. The same remarks apply to the class of all $B$ which satisfy (2.2); in this case, completion relative to $\varphi(A, \cdot)$ means the same as completion relative to $\mu$ if $\varphi(A) > 0$.

Various sufficient criteria for quasi-mixing or mixing follow easily. In view of a later application, we mention an example. Suppose that to a certain sequence $\rho_n$ there exists a class $\mathfrak{C}$ of sets with the following properties: every set of $\mathfrak{C}$ is almost clopen and has finite measure; $X$ is, up to a null set, the union of countably many sets of $\mathcal{C}$; the class $\mathcal{C}$ is stable against finite intersections; $\mathcal{C}$ is a basis for the topology of $X$, that is, given any open set $G$ and any $x \in G$ there is a set $E$ of $\mathcal{C}$ such that $E \subseteq G$ and $x$ is an interior point of $E$; the equation (2.2) holds for all $A, B \in \mathcal{C}$. Then $T$ is mixing with the given sequence $(\rho_n)$. In fact, (2.2) holds for all $A$ and $B$ in the smallest class $\mathfrak{D}$ over $\mathcal{C}$ which is stable against finite unions of disjoint sets, and against differences of sets where one of them includes the other one. As $\mathcal{C}$ is stable against finite intersections, $\mathfrak{D}$ turns out to be the ring generated by $\mathcal{C}$ ([5], p. 27), and since the class of all almost clopen sets with finite measure is a ring, every set in $\mathfrak{D}$ is almost clopen with finite measure. We can then write $X = \bigcup_k H_k \cup N$ with $H_k \in \mathfrak{D}$, $\mu(N) = 0$, and $H_k \subseteq H_{k+1}$. Let $A$ be an almost clopen subset of $H_k$ and $A_0$ its interior. The measure $\mu$ being tight, we have $\mu(A) = \sup \{\mu(C);\ C \subseteq A_0, C \text{ compact}\}$, and since $\mathcal{C}$ is a basis for the topology of $X$, there exists for each compact subset $C$ of $A_0$ a finite union $D$ of sets of $\mathcal{C}$ such that $C \subseteq D \subseteq A_0$; note that $D \in \mathfrak{D}$. Applying the same argument to $H_k - A$ instead of $A$, we find that $A$ belongs to the completion of $\mathfrak{D}$ relative to $\mu$, which finishes the proof.

There is, of course, a corresponding sufficient criterion for quasi-mixing if the function $\varphi$ defined by (2.1) for $A, B \in \mathcal{C}$ admits an extension to $\mathfrak{F} \times \mathfrak{F}$ which is a tight measure in each variable, the other one being fixed, and such that, for almost clopen sets $A$ and $B$, the relation $\varphi(A, B) = 0$ implies $\mu(A) = 0$ or $\mu(B) = 0$.

Conversely, we remark that, given a sequence $H_k$ as in the definition of quasi-mixing, the class of all clopen sets included in some $H_k$ has the properties required of the class $\mathcal{C}$ of the preceding criteria, since $X$ is completely regular.

It may often be convenient to restate the definition (2.1) or (2.2) in a "functional" form. We show that, given the sequences $H_k$ and $\rho_n$, the equation (2.2) holds for all almost clopen sets $A$ and $B$ included in some $H_k$ if and only if the relation

$$\lim_{n\to\infty} \rho_n \mu(f(g \circ T^n)) = \mu(f)\mu(g), \tag{2.6}$$

where $\mu(f) = \int_X f\, d\mu$, is true for any two bounded and almost everywhere continuous function $f$ and $g$ whose carrier is part of some $H_k$. In fact, (2.6) implies (2.2) upon taking indicator functions. Conversely, (2.2) entails (2.6) for finite linear combinations $f$ and $g$ of indicator functions of almost clopen sets contained in an $H_k$. Suppose now that $f$ is such a linear combination with $f \geq 0$, whereas $g$ is of the general type described above. As shown in [1] and [8], given $\epsilon > 0$, there is a finite linear combination $g^*$ of indicator functions of almost clopen sets included in some $H_k$ such that $g \leq g^*$ and $\mu(g^*) \leq \mu(g) + \epsilon$; hence

$$\begin{aligned} \limsup_{n\to\infty} \rho_n \mu(f(g \circ T^n)) &\leq \limsup_{n\to\infty} \rho_n \mu(f(g^* \circ T^n)) \\ &= \mu(f)\mu(g^*) \leq (f)(\mu(g) + \epsilon), \end{aligned} \tag{2.7}$$

and therefore $\limsup_{n\to\infty} \rho_n \mu(f(g \circ T^n)) \leq \mu(f)\mu(g)$. In the same way we obtain

$\mu(f)\mu(g) \leq \liminf_{n\to\infty} \rho_n\mu(f(g \circ T^n))$, that is, (2.6). The extension to a "step" function $f$ of arbitrary sign, and from there to the case where $f$ is also of the general type, is now immediate.

As with the criterion before, quasi-mixing may be treated in a similar manner. Given $H_k$, $\rho_n$, and $\varphi$ with the properties listed in the definition, equation (2.1) holds for all $A$ and $B$ admitted there if and only if

$$\lim_{n\to\infty} \rho_n\mu(f(g \circ T^n)) = \int_X \int_X f(x)g(y)\varphi(dx, dy) \tag{2.8}$$

is true with all bounded and almost everywhere continuous functions $f$ and $g$ which are carried by some $H_k$.

Let $r$ be a natural number, $X^r = Xx \cdots xX$ the product space with $r$ factors provided with the product topology, $\mathfrak{F}^{(r)}$ the class of the Borel sets in $X^r$ which is also the product sigma-algebra of the sigma-algebra $\mathfrak{F}$ in each factor, $\mu^{(r)}$ the product measure of $\mu$ on each factor, and $T^{(r)}$ the endomorphism defined by $T^{(r)}(x_1, \cdots, x_r) = (Tx_1, \cdots, Tx_r)$ for $(x_1, \cdots, x_r) \in X^r$. It is easy to see that, if $T$ is quasi-mixing, $T^{(r)}$ also is, and that the corresponding sequence $\rho_n^{(r)}$ and function $\varphi^{(r)}$ are determined by $\rho_n^{(r)} = \rho_n^r$ and $\varphi^{(r)}(A_1x \cdots xA_r;\ B_1x \cdots xB_r) = \varphi(A_1, B_1) \cdots \varphi(A_r, B_r)$. A similar remark applies to a product of different spaces.

Let $T$ be a quasi-mixing endomorphism of $(X, \mathfrak{F}, \mu)$ and $A_0$ an invariant almost clopen set. Then $\mu(A_0) = 0$ or $\mu(X - A_0) = 0$. In fact, in the opposite case, setting $B_0 = X - A_0$, we could choose $k$ in such a way that $A = A_0 \cap H_k$ and $B = B_0 \cap H_k$ have positive measure. Since $\mu(A \cap T^{-n}B) = 0$ for all $n$, we would have $\varphi(A, B) = 0$ by (2.1); hence, $\mu(A) = \mu(B) = 0$ which would contradict the choice of $A$ and $B$.

However, a mixing transformation is, in general, not ergodic if $\mu(X) = +\infty$ as we will see in section 5. For a later reference we recall here the definition of the *ergodic index* of $T$. We note first that the ergodicity of $T^{(r+1)}$ implies that of $T^{(r)}$. The ergodic index of $T$ is, then, the largest integer $r$ such that $T^{(r)}$ is ergodic; it is 0 if $T$ is not ergodic and $\infty$ if every $T^{(r)}$ is. The ergodic index is, of course, invariant under isomorphisms.

## 3. An isomorphism between Markovian and Lebesguian measure

Let I stand for the set of all integers, and let $Z$, the set of "states," be some nonempty subset of $I$. We consider a *stochastic matrix* $\pi = (\pi(i,j))_{i,j\in Z}$, thus $\pi(i,j) \geq 0$ and $\sum_j \pi(i,j) = 1$; for all facts about Markov chains used here we refer to [3]. We assume that $\pi$ admits a nonvanishing *invariant measure*, that is, a vector $\lambda = (\lambda(j))_{j\in Z}$ such that $\lambda \neq 0$, $\lambda \geq 0$, and $\lambda = \lambda\pi$; more explicitly, $\lambda(j) \geq 0$ and $\lambda(j) = \sum_i \lambda(i)\pi(i,j)$ for all $j \in Z$. We pick one such $\lambda$ and keep it fixed. We form the set $X = Z^I$ of the possible sample paths of bilateral Markov chains with transition matrix $\pi$; its elements are sequences $x = (x_n)_{n\in I}$ where $x_n \in Z$ for every $n$. The topology in $X$ will be the product of the discrete topologies in the factor spaces $Z$; a basis is given by the class of the *elementary cylinders*

$$A = \{x\colon x_{n_1} = i_1, \cdots, x_{n_k} = i_k\} \tag{3.1}$$

with $n_1 < n_2 < \cdots < n_k$ and $i_l \in Z$, $l = 1, \cdots, k$. This basis, together with the empty set, is stable against finite intersections and consists of clopen sets. Since it is also countable, it generates the same sigma-algebra $\mathfrak{F}$ as the open sets. It is easy to provide $X$ with a metric under which it becomes complete; thus $X$, being separable, is a *Polish space.*

Let $\pi = (\pi^n(i, j))_{i,j \in I}$ be the matrix of the $n$-step transition probabilities. A measure $\mu$ will be defined for cylinders of the type (3.1) in the usual way [7], [11], by

$$\mu(A) = \lambda(i_1)\pi^{n_2-n_1}(i_1, i_2) \cdots \pi^{n_k-n_{k-1}}(i_{k-1}, i_k) \tag{3.2}$$

and then extended to all of $\mathfrak{F}$. It is tight because $X$ is a Polish space ([5], p. 40, (10)). Setting $X_i = \{x\colon x_0 = i\}$ we find that $X_i \cap X_j = \emptyset$ for $i \neq j$, $\bigcup_i X_i = X$, and $\mu(X_i) = \lambda(i) < +\infty$; hence $\mu$ is sigma-finite with $\mu(X) = \sum_i \lambda(i)$. If $\lambda(i) = 0$ for some $i$, then $\mu\{x\colon x_n = i \text{ for at least one } n\} = 0$; thus by discarding such states $i$, we may, and will, assume that $\lambda(i) > 0$ for every $i$.

With the measure $\mu$ and the product topology, the *shift* $T((x_n)_{n\in I}) = (x_{n+1})_{n \in I}$ is a measure preserving homeomorphism of $X$ onto $X$ with a measurable inverse; in particular, it is an automorphism in our sense.

The topological space $X$ is, of course, not locally compact. Assuming that for every $x = (x_n)_{n\in I}$ we have $\mu\{y\colon y_n = x_n \text{ for } n \geq 0\} = \mu\{y\colon y_n = x_n \text{ for } n < 0\} = 0$, we will now construct an isomorphism $S$ of $(X, \mathfrak{F}, \mu)$ onto a measure space $(X', \mathfrak{F}', \mu')$ where $X'$ is an elementary closed subset of the plane, $\mathfrak{F}'$ the sigma-algebra of the Borel subsets of $X'$, and $\mu'$ the two-dimensional Lebesgue measure on $\mathfrak{F}'$. To do so, let us take a sequence of mutually disjoint closed rectangles $X_i'$ of height $\lambda(i)$ and width 1, $i \in Z$, and set $X' = \bigcup_i X_i'$. Let $x = (x_n)_{n\in I}$ be a point of $X$. Its image $Sx$ is determined in the following way. To begin with, we stipulate that $Sx \in X'_{x_0}$. Next we divide $X'_{x_0}$ into closed vertical strips of width $\pi(x_0, i)$, $i \in Z$, arranged according to the natural ordering of the states $i$ as integers, and let $Sx$ fall into the strip indexed by $i = x_1$. In the third step we subdivide the strip into vertical closed strips of width $\pi(x_0, x_1)\pi(x_1, i)$, $i \in Z$, and select the strip with $i = x_2$ as the one to contain $Sx$, and so on. Our assumptions imply that

$$\begin{aligned}\lambda(x_0)\pi(x_0, x_1) &\cdots \pi(x_{k-1}, x_k) \\ &= \mu\{y\colon y_0 = x_0, \cdots, y_k = x_k\} \to \mu\{y\colon y_n = x_n \text{ for } n \geq 0\} \\ &= 0\end{aligned} \tag{3.3}$$

for $k \to \infty$; thus the intersection of the closed strips just constructed is a vertical segment which determines the abscissa of $Sx$. In a similar way, we obtain its ordinate: we divide $X'_{x_0}$ into horizontal strips of height $\lambda(i)\pi(i, x_0)$, making use of the fact that these numbers add up to $\lambda(x_0)$, and require $Sx$ to fall into the strip indexed by $i = x_{-1}$. This strip will then be subdivided into horizontal strips of height $\lambda(i)\pi(i, x_{-1})\pi(x_{-1}, x_0)$ and the strip with $i = x_{-2}$ selected, and so on.

The mapping $S$ of $X$ into $X'$ thus defined is everywhere continuous. In fact, if $y_i = x_i$, $i = -k, \cdots, k$, then the abscissas of $Sx$ and $Sy$ differ by at most $\lambda(x_0)\pi(x_0, x_1) \cdots \pi(x_{k-1}, x_k)$ and their ordinates by at most $\lambda(x_{-k})\pi(x_{-k}, x_{-k+1}) \cdots \pi(x_{-1}, x_0)$, and both quantities go to 0 for $k \to \infty$. The construction of $S$ shows that the equation $\mu'(A') = \mu(S^{-1}A')$ holds whenever $A'$ is the intersection of a vertical and a horizontal strip of the type described above; hence, it holds for every set $A' \in \mathfrak{F}'$; that is, $S$ is a homomorphism. Since every vertical or horizontal segment in one of the rectangles $X'_i$ is the intersection of at least one sequence of our vertical or horizontal strips, respectively, $S$ maps $X$ onto $X'$. Therefore, a mapping $S'$ from $X'$ into $X$ satisfying $S(S'(x')) = x'$ for all $x' \in X'$ exists and shall be kept fixed in the sequel. If $Sx = Sy$ but $x \neq y$, there are either two adjacent vertical or two adjacent horizontal strips which both contain $Sx$. Denoting the union of boundary segments of strips by $N'$, we have $\mu(N') = 0$ since there are only countably many such segments. Let $N = S^{-1}N'$, thus $\mu(N) = 0$. The restriction of $S$ to $X - N$ is a bijective mapping of $X - N$ onto $X' - N'$, and $S'Sx = x$ for every $x \in X - N$. Moreover, $S'$ is continuous in every point of $X - N'$, and is therefore a homomorphism of $(X', \mathfrak{F}', \mu')$ into $(X, \mathfrak{F}, \mu)$; hence $S$ is an isomorphism of $(X, \mathfrak{F}, \mu)$ onto $(X', \mathfrak{F}', \mu')$.

In view of a later application, we remark that if $\mu(X) = +\infty$, every bounded subset of $X'$ is included in the union of a finite number of rectangles $X'_i$, that is, in the image of a finite number of elementary cylinders $X_i$.

It is pretty obvious how the space $X'$, the sigma-algebra $\mathfrak{F}'$ and the measure $\mu'$ have to be modified when points $x$ with, say, $\alpha = \mu\{y\colon y_n = x_n \text{ for } n \geq 0\} > 0$ occur. Roughly speaking, if $\mu\{y\colon y_n = x_n \text{ for } n < 0\} = 0$ still holds, the set $\{y\colon y_n = x_n \text{ for } n \geq 0\}$ will be mapped onto a horizontal segment of length $\alpha$, endowed with one-dimensional Lebesgue measure instead of a two-dimensional set, whereas in the case where $\mu\{y\colon y_n = x_n \text{ for } n < 0\}$ is positive, too, that is, where $\mu\{x\} > 0$, we have to incorporate an atom into our measure space $(X', \mathfrak{F}', \mu')$ which is then going to correspond to the point $x$.

The mapping $T' = S \circ T \circ S^{-1}$ represents an automorphism of $(X', \mathfrak{F}', \mu')$ which is continuous everywhere on $X' - N'$. It is of a fairly elementary nature. For example, the image of a rectangle $X'_i$ is the union of "horizontal" strips of width 1 and height $\lambda(i)\pi(i, j)$, $j \in Z$, where the strip indexed by $j$ is the one included in $X'_j$ as constructed above. Similarly, the inverse image of $X'_i$ is the union of vertical strips of height $\lambda(j)$ and width $\pi(j, i)$, the one indexed by $j$ being the one included in $X'_j$. A vertical strip of width $\pi(i, j)$ and height $\lambda(i)$ included in $X'_i$ is mapped by $T'$ onto the horizontal strip of width 1 and height $\lambda(i)\pi(i, j)$ included in $X'_j$. The transformation $T'$ becomes especially simple if each row and each column of $\pi$ contains only a finite number of elements different from 0. Examples will be given in section 5.

We remark that if we consider the unilateral sequence space $Z^{I^+}$ with $I^+ = \{0, 1, 2, \cdots\}$ instead of $Z^I$, an isomorphism $S$ of the kind described here will be, of course, onto a union of intervals instead of rectangles, endowed with one-dimensional Lebesgue measure, and mappings of the type $S \circ T \circ S^{-1}$ will furnish a large class of noninvertible endomorphisms of subsets of the line.

Finally, examples of this kind as well as the invertible transformations $T'$ in the plane treated before yield examples of transformations in unions of $k$-dimensional intervals. In fact, the following familiar mapping $S_k$ of the closed unit interval onto the closed $k$-dimensional cube, based on dyadic expansions, is an isomorphism of the one-dimensional Lebesgue measure space onto the $k$-dimensional one:

$$\sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}} \to \left( \sum_{n=0}^{\infty} \frac{x_{nk}}{2^{n+1}}, \sum_{n=0}^{\infty} \frac{x_{nk+1}}{2^{n+1}}, \cdots, \sum_{n=0}^{\infty} \frac{x_{nk+k-1}}{2^{n+1}} \right) \tag{3.4}$$

where $x_n = 0, 1$, and $x_n = 1$ for infinitely many $n$, or $x_n = 0$ for all $n$. This mapping is continuous at every nondyadic point of $[0, 1]$ and has an inverse $S'_k$ which is continuous at every point with nondyadic coordinates.

## 4. Mixing of Markov chains

Given the stochastic matrix $\pi$ and the invariant measure $\lambda$ as in the preceding section, we consider two elementary cylinders

$$A = \{x \colon x_{n_1} = i_1, \cdots, x_{n_k} = i_k\}, \qquad B = \{x \colon x_{m_1} = j_1, \cdots, x_{m_l} = j_l\} \tag{4.1}$$

where $n_1 < \cdots < n_k$ and $m_1 < \cdots < m_l$. Then

$$T^{-n}B = \{x \colon x_{m_1+n} = j_1, \cdots, x_{m_l+n} = j_l\}. \tag{4.2}$$

For $n > n_k - m_1$, it follows from (3.2) that

$$\mu(A \cap T^{-n}B) = \mu(A)\mu(B) \frac{\pi^{m_1-n_k+n}(i_k, j_1)}{\lambda(j_1)}. \tag{4.3}$$

In particular, we find that

$$\mu(X_i \cap T^{-n}X_j) = \mu(X_i)\mu(X_j) \frac{\pi^n(i, j)}{\lambda(j)} \tag{4.4}$$

for all $n > 0$. Hence if there exists a sequence $\rho_n$ such that

$$\liminf_{n\to\infty} \rho_n \mu(X_i \cap T^{-n}X_j) > 0 \tag{4.5}$$

for all $i, j \in Z$, any Markov chain with transition matrix $\pi$ is irreducible and aperiodic; these two assumptions will always be made in the sequel.

The matrix $\pi$ is said to have the *strong ratio limit property* in the sense of Pruitt [18] if there exist positive numbers $\gamma$, $\tau(i)$, and $\kappa(i)$, $i \in Z$, such that

$$\lim_{n\to\infty} \frac{\pi^{n+m}(i, j)}{\pi^n(i_0, j_0)} = \gamma^m \frac{\tau(i)\kappa(j)}{\tau(i_0)\kappa(j_0)} \tag{4.6}$$

for all $m \in I$ and $i, j, i_0, j_0 \in Z$. Assuming this to be true, we select two states, $i_0$ and $j_0$, to be kept fixed and set

$$\rho_n = \frac{\tau(i_0)\kappa(j_0)}{\pi^n(i_0, j_0)}. \tag{4.7}$$

Then (4.6) imespli

(4.8) $$\lim_{n\to\infty} \rho_n \pi^{n+1}(i, j) = \gamma\tau(i)\kappa(j)$$

and

(4.9) $$\lim_{n\to\infty} \rho_n \pi^{n}(i, j) = \tau(i)\kappa(j)$$

for all states $i$ and $j$, and it follows that

(4.10) $$\sum_{i\in Z} \kappa(i)\pi(i, j) \leq \gamma\kappa(j), \qquad \sum_{j\in Z} \pi(i, j)\tau(j) \leq \gamma\tau(i)$$

for all $i, j \in Z$. We will now also assume that we have equality in these inequalities, that is,

(4.11) $$\sum_{i\in Z} \kappa(i)\pi(i, j) = \gamma\kappa(j),$$

(4.12) $$\sum_{j\in Z} \pi(i, j)\tau(j) = \gamma\tau(i).$$

By (4.8) and (4.9), a sufficient condition for (4.11) is that for every $j$ there is only a finite number of states $i$ with $\pi(i, j) > 0$, and similarly for (4.12); a deeper condition will be discussed below.

The relations (4.3)–(4.7) entail

(4.13) $$\lim_{n\to\infty} \rho_n \mu(A \cap T^{-n}B) = \varphi(A, B)$$

with

(4.14) $$\varphi(A, B) = \mu(A)\mu(B)\gamma^{m_1 - n_k}\tau(i_k)\frac{\kappa(j_1)}{\lambda(j_1)}$$

if $A$ and $B$ are given by (4.1). Let us now keep $B$ fixed, and consider the sigma-algebra $\mathfrak{F}_{n_1\cdots n_k}$ of the sets of $\mathfrak{F}$ determined by conditions on $x_{n_1}, \cdots, x_{n_k}$, that is, the atomic sigma-algebra whose atoms are the elementary cylinders $A$ given by (4.1). The function $\varphi(\cdot, B)$ defined for such elementary cylinders $A$ by (4.14) admits a unique sigma-additive extension to $\mathfrak{F}_{n_1\cdots n_k}$. A simple computation shows that the relations (4.12) are equivalent to the compatibility of these measures on the various sigma-algebras $\mathfrak{F}_{n_1\cdots n_k}$, thus giving rise to a nonnegative function $\varphi(\cdot, B)$ defined on $\mathcal{G} = \bigcup_{n_1<\cdots<n_k} \mathfrak{F}_{n_1\cdots n_k}$ whose restriction to every $\mathfrak{F}_{n_1\cdots n_k}$ is sigma-additive and sigma-finite. Similarly, the relations (4.11) ensure the compatibility of the measures obtained by extending $\varphi(A, \cdot)$ to the sigma-algebras $\mathfrak{F}_{m_1\cdots m_l}$ if $A$ has the form (4.1), and these compatibility conditions are then still satisfied by the corresponding extensions of $\varphi(A, \cdot)$ to the various $\mathfrak{F}_{m_1\cdots m_l}$ if $A \in \mathcal{G}$; in fact, this amounts to interchanging the order of summation in a doubly infinite series with positive terms.

Next, an adaptation of some proof of Kolmogorov's extension theorem ([5], p. 212) shows that $\varphi$ admits one and only one extension to $\mathfrak{F} \times \mathfrak{F}$ which is a sigma-finite measure in each variable, the other one being fixed. Since $X$ is a Polish space, these measures are then tight. We keep the letter $\varphi$ to denote the extension to $\mathfrak{F} \times \mathfrak{F}$. If $A_0$ and $B_0$ are almost clopen sets with positive measure, there exist elementary cylinders $A$ and $B$ such that $A \subseteq A_0$ and $B \subseteq B_0$; hence $\varphi(A, B) > 0$ by (4.14), and therefore $\varphi(A_0, B_0) > 0$. Upon applying the criterion for quasi-mixing deduced in the first section, where $\mathcal{C}$ consists of the empty set and the elementary cylinders, we find that the shift $T$ is quasi-mixing, with $\rho_n$ and $\varphi$ given by (4.7) and (4.14).

In some cases the sequence $\rho_n$ can be found directly. The formula (4.7) gives then, up to a factor, the asymptotic behavior of $\pi^n(i_0, j_0)$, and the criteria established by Kakutani and Parry [11] lead to a way of determining the ergodic index of $T$: the index is $r$ if $\sum_{n=1}^{\infty} \rho_n^{-r} = +\infty$, but $\sum_{n=1}^{\infty} \rho_n^{-r-1} < +\infty$; it is $\infty$ if $\sum_{n=1}^{\infty} \rho_n^{-r} = +\infty$ for every $r$. Roughly speaking, a high ergodic index corresponds to a sequence $\rho_n$ which increases slowly, that is, a slowly quasi-mixing transformation. Also, $T$ is ergodic if and only if $\sum_{n=1}^{\infty} \rho_n^{-1} = +\infty$.

Let us now look at the case of a so-called *R-recurrent* chain. According to Vere-Jones [21], $\pi$ is called $R$-recurrent if $R$ is the radius of convergence of the power series $\sum_{n=1}^{\infty} \pi^n(i, j)z^n$, and $\sum_{n=1}^{\infty} \pi^n(i, j)R^n = +\infty$; both statements do not depend on $i$ and $j$. We assume $\pi$ to be $R$-recurrent, but for a while it need not have the strong ratio limit property. We set $\gamma = R^{-1}$; thus $0 < \gamma \leq 1$. It was shown in [21] that the equations (4.11) and (4.12) have nontrivial positive solutions $\bar{\kappa} = (\bar{\kappa}(i))_{i \in Z}$ and $\bar{\tau} = (\bar{\tau}(i))_{i \in Z}$, respectively, which are unique up to a factor, and

$$\lim_{n \to \infty} \frac{\sum_{m=0}^{n} \pi^m(i, j)R^m}{\sum_{m=0}^{n} \pi^m(i_0, j_0)R^m} = \frac{\bar{\tau}(i)\bar{\kappa}(j)}{\bar{\tau}(i_0)\bar{\kappa}(j_0)} \tag{4.15}$$

for all $i, j, i_0, j_0 \in Z$.

On the other hand, let us assume that $\pi$ has the strong ratio limit property and that (4.6) holds. Upon setting $i = i_0$, $j = j_0$, and $m = 1$, we find that $R = \gamma^{-1}$ is the radius of convergence of the power series $\sum_{n=1}^{\infty} \pi^n(i, j)z^n$ and that $0 < \gamma \leq 1$. If, in addition, $\pi$ is $R$-current and $\bar{\kappa}$ and $\bar{\tau}$ are determined as before, it follows from (4.6), (4.13), and $\sum_{n=1}^{\infty} \pi^n(i_0, j_0)R^n = +\infty$ that $\tau = \alpha\bar{\tau}$ and $\kappa = \beta\bar{\kappa}$ with some positive constants $\alpha$ and $\beta$.

The chain $\pi$ is recurrent, that is, $\sum_{n=1}^{\infty} \pi^n(i, j) = +\infty$, if and only if it is 1-recurrent. In this case the vector $\bar{\tau}$ described above as the solution of (4.12) with $\gamma = 1$ is constant, that is, $\bar{\tau}(i) = \bar{\tau}(i_0)$ for all $i$ and $i_0$, and $\bar{\kappa}$ is an invariant measure $\lambda$, which is unique up to a positive factor. Hence, if $\pi$ is recurrent and has the strong ratio limit property, we obtain

$$\lim_{n \to \infty} \frac{\pi^{n+m}(i, j)}{\pi^n(i_0, j_0)} = \frac{\lambda(j)}{\lambda(j_0)} \tag{4.16}$$

for every $m \in I$ and all $i, j, i_0, j_0 \in Z$.

However, even if $\pi$ is not $R$-recurrent, it may happen that for some invariant measure $\lambda$ the equation (4.16) is true. In this case we will say that the couple $(\pi, \lambda)$ has the *strong ratio limit property* in the sense of Orey, although Orey considered only recurrent chains [16]. If $(\pi, \lambda)$ has this property, it follows from our remarks above that $T$ is mixing and satisfies (2.2) with

$$\rho_n = \frac{\lambda(j_0)}{\pi^n(i_0, j_0)}. \tag{4.17}$$

Examples will be given in the next section.

## 5. Examples

We start with a finite measure space: let $Z = \{0, 1\}$ and $\pi(i, j) = \frac{1}{2}$ for all $i$ and $j$. An invariant measure is given by $\lambda(0) = \lambda(1) = \frac{1}{2}$, and $\mu$ is thus the product measure of the measure $\lambda$ on each factor $Z$. The set $X'$, being the union of two rectangles of width 1 and height $\frac{1}{2}$, may be thought of as the unit square $[0, 1] \times [0, 1]$. Then $S$ is the familiar mapping of the bilateral sequence space of zeros and ones onto the unit square ([6], p. 9) which sends the point $(x_n)_{n \in I}$ into the point

$$x' = (\xi, \eta) = \left(\sum_{n=1}^{\infty} \frac{x_n}{2^{n+1}}, \sum_{n=0}^{\infty} \frac{x_n}{2^n}\right). \tag{5.1}$$

Therefore, $T'$ is the baker's transformation: $T'x' = (2\xi - (x_1/2), (\eta/2) + (x_1/2))$ whose mixing property is well known. The mapping $S_2' \circ T' \circ S_2$, where $S_2$ and $S_2'$ were defined at the end of section 3, represents a simple example of an invertible mixing transformation of the unit interval with only countably many discontinuities. (This was suggested by F. Papanghelou. Other examples had been constructed by J. von Neumann and S. Kakutani (oral communication by S. Kakutani).) It would be interesting to know if there is a mixing, or weakly mixing, invertible transformation of the unit interval with a finite number of discontinuities.

Let us now consider irreducible, aperiodic and recurrent chains. Orey [16] and Kingman and Orey [14] showed that each of the following conditions is sufficient for the strong ratio limit property.

(i) The chain is *reversible;* that is, there exists a vector $(\lambda(i))_{i \in Z}$ such that $\lambda(i)\pi(i, j) = \lambda(j)\pi(j, i)$ for all $i, j \in Z$. (It follows immediately that $\lambda$ is an invariant measure.)

(ii) There exists an index $k$ such that $\inf_{i \in Z} \sum_{n=1}^{k} \pi^n(i, i) > 0$.

The symmetric random walk on the set $Z = I^+ = \{0, 1, 2, \cdots\}$ with the reflecting and absorbing barrier 0 of the type $\pi(0, 1) = \pi(i, i + 1) = p$, $\pi(0, 0) = \pi(i, i - 1) = q$ for $i > 0$, where $0 < p, q$ and $p + q = 1$, was studied by Karlin and McGregor [13]. It is reversible with $\lambda(i) = (p/q)^i$. The recurrent random walks are obtained for $p \leq \frac{1}{2}$, and any of the criteria (i) and (ii) implies then that $T$ is mixing. The measure $\lambda$ is infinite for $p \geq \frac{1}{2}$; thus we will look only at the case $p = \frac{1}{2}$ where $\pi$ is also doubly stochastic and $\lambda(i) = 1$ for all $i$. The transformation $T'$ is mixing, and the mixing equation (2.6) with $T'$ holds for all Riemann integrable functions $f$ and $g$ defined on $X'$. Now it turns out that $T'$ is, up to a trivial modification, Hopf's example of a mixing transformation of a set with infinite measure ([9], p. 67). It consists of the baker's transformation performed in each component square of $X'$ plus a simple permutation of the horizontal half-squares making up those components. Hopf proved that we may take

$$\rho_n = \left(\frac{\pi n}{2}\right)^{1/2}. \tag{5.2}$$

It seems that the question of whether $T'$ is ergodic had still been unanswered. However, on account of (5.2), the criterion by Kakutani and Parry in the form given to it in section 4 shows that $T$, and therefore $T'$, has ergodic index 2; in particular, $T$ is ergodic.

Next we discuss a class of examples studied by Kakutani and Parry [11], the centrally biased random walks on the even integers. Let $-1 < \epsilon < 1$, and let $\omega = (\omega(i, j))_{i,j \in I}$ be the following transition matrix: $\omega(0, -1) = \omega(0, 1) = \frac{1}{2}$ and $\omega(i, i-1) = \frac{1}{2}(1 + (\epsilon/i))$, $\omega(i, i+1) = \frac{1}{2}(1 - (\epsilon/i))$ if $i \neq 0$. We consider the state space $Z = \{0, \pm 2, \pm 4, \cdots\}$ and define $\pi$ to be the restriction of $\omega$ to $Z \times Z$; this is again a stochastic matrix. The sequence

$$\nu(0) = 1, \qquad \nu(i) = \nu(-i) = \frac{\Gamma(1+\epsilon) i \Gamma(i-\epsilon)}{\Gamma(1-\epsilon)\Gamma(i+1+\epsilon)} \qquad \text{if} \quad i > 0 \tag{5.3}$$

satisfies $\nu(i)\omega(i, j) = \nu(j)\omega(j, i)$ for all $i$ and $j$, and therefore $\nu(i)\omega^2(i, j) = \nu(j)\omega^2(j, i)$. Hence $\pi$ is reversible, and an invariant measure $\lambda$ which fulfills $\lambda(i)\pi(i, j) = \lambda(j)\pi(j, i)$ for all $i, j \in Z$ is given by the restriction of $\nu$ to $Z$. This measure is infinite if and only if $\epsilon \leq \frac{1}{2}$. Markov chains with transition matrix $\pi$ are irreducible and aperiodic. They are transient if $\epsilon < -\frac{1}{2}$ and recurrent if $-\frac{1}{2} < \epsilon$. We will assume from now on that $-\frac{1}{2} < \epsilon \leq \frac{1}{2}$. Then $T$ is mixing, and the ergodic index of $T$ equals $r$ if

$$\frac{1}{2} - \frac{1}{r} < \epsilon < \frac{1}{2} - \frac{1}{r+1}. \tag{5.4}$$

More precisely, (5.4) implies the existence of positive constants $\alpha$ and $\beta$ and of numbers $r'$ and $r''$ such that

$$r' - 1 < r'' < r < r', \qquad \alpha n^{1/r''+1} \leq \rho_n \leq \beta n^{1/r'} \tag{5.5}$$

where $\rho_n = (\pi^n(0, 0))^{-1}$. Thus for large $r$ we obtain examples of transformations $T$ which, in a sense, mix very slowly, since $\rho_n$ increases slowly.

In the particular case of the classical symmetric random walk on the even integers, that is, $\epsilon = 0$, we have $\pi^n(0, 0) = 2^{-2n}\binom{2n}{n}$ which shows that we may take

$$\rho_n = (\pi n)^{1/2} \tag{5.6}$$

to satisfy the mixing equation (2.2). Therefore $T$ and $T'$ have ergodic index 2. Incidentally, the strong ratio limit property in this case follows also from the results of Chung and Erdös [4] since we are dealing here with a process with independent increments centered at 0.

In the following example, let $r$ be a positive integer and $Z^r$ the set of all $r$-tuples $(i_1, \cdots, i_r)$ where each $i_s$ is an even integer. We consider on $Z^r$ the transition matrix $\pi^{(r)}(i_1, \cdots, i_r; j_1, \cdots, j_r) = \pi(i_1, j_1) \cdots \pi(i_r, j_r)$ where $\pi$ stands for the transition matrix of the preceding example, that is, the classical random walk on $Z$. Then the uniform distribution $\lambda^{(r)}(i_1, \cdots, i_r) = 1$ is invariant. The resulting measure space is, up to a trivial isomorphism, the product space $(X^r, \mathfrak{F}^{(r)}, \mu^{(r)})$ considered in the first section, where $(X, \mathfrak{F}, \mu)$ denotes the space of the classical random walk on the even integers; the shift becomes the transformation $T^{(r)}$. The strong ratio limit property of $(\pi^{(r)}, \lambda^{(r)})$ in the sense of Orey follows immediately from that of $(\pi, \lambda)$. Hence, the radius of convergence of the power series $\sum_{n=1}^{\infty} \pi^{(r)n}(i_1, \cdots, i_r; j_1, \cdots, j_r)z^n$ equals one. By (5.6) the shift is mixing with

$$\rho_n = (\pi n)^{r/2}. \tag{5.7}$$

For $r = 2$ we get a recurrent chain, and $T^{(2)}$ has ergodic index one. For $r > 2$, however, $T^{(r)}$ is not ergodic and $\pi^{(r)}$ is transient, and therefore not $R$-recurrent. For large $r$ the transformations $T^{(r)}$ yield simple examples of automorphisms of a union of countably many squares which mix very fast as expressed by (5.7). In this geometrical form they had been proposed by F. Papanghelou. Their mixing property with (5.6) and (5.7) can be proved directly by reasoning as in [9]; the example obtained if $r = 1$ is a little simpler than the one given by Hopf.

The transformation of the next example will be quasi-mixing but not mixing. Let $Z = I$ and $\pi(i, i-1) = \frac{1}{9}$, $\pi(i, i) = \pi(i, i+1) = \frac{4}{9}$. It is easy to verify inductively that

$$\pi^n(i, j) = \frac{2^{n+j-i}}{3^{2n}} \binom{2n}{n-j+i} \sim \left(\frac{8}{9}\right)^n \frac{1}{\sqrt{\pi n}} 2^{j-i} \tag{5.8}$$

for $n \to \infty$. Therefore, $\pi$ has the strong ratio limit property with $\gamma = \frac{8}{9}$, $\tau(i) = 2^{-i}$, and $\kappa(i) = 2^i$. Since $\pi$ is the matrix of processes with independent increments, the uniform distribution $\lambda(i) = 1$ is an invariant measure. Hence, the shift $T$ is quasi-mixing with $\rho_n = (\frac{9}{8})^n \sqrt{\pi n}$ and

$$\varphi(A, B) = \mu(A)\mu(B)(\tfrac{8}{9})^{m_1 - n_k} 2^{j_1 - i_k} \tag{5.9}$$

if $A$ and $B$ are given by (4.1). It also follows from (5.8) that $\pi$ is $R$-recurrent where $R = \frac{9}{8}$. By taking the measure space $(X, \mathfrak{F}, \mu)$ and the shift $T$ of the present example and passing to $X^r$, $\mathfrak{F}^{(r)}$, $\mu^{(r)}$, and $T^{(r)}$ with $r > 2$ as we did before, we obtain examples of quasi-mixing but not mixing transformations where $\pi$ is not $R$-recurrent.

In order to investigate further the set function $\varphi$ given by (4.7), let us assign to each point $x = (x_n)_{n \in I}$ the sequence of sets $D_m(x) = \{y\colon y_l = x_l \text{ for } |l| \leq m\}$, $m = 1, 2, \cdots$. The images $SD_m(x)$ under the isomorphism $S$ constructed in section 3 constitute for fixed $x$ a sequence of rectangles which "converges" to $Sx$ and for fixed $m$ a partition of $X'$ which is a subpartition of the preceding one. Hence, we may use the $D_m(x)$ to compute derivatives of set functions on $\mathfrak{F}$. Keeping $B$ fixed, as given by (4.1), we get

$$\frac{\varphi(D_m(x), B)}{\mu(D_m(x))} = \alpha\left(\frac{9}{8}\right)^m 2^{-x_m} \tag{5.10}$$

with a certain positive number $\alpha$. By the strong law of large numbers, $\lim_{m\to\infty} x_m/m = \frac{1}{3}$ for $\mu$-almost all points $x$. Since $\log \frac{9}{8}/\log 2 < \frac{1}{3}$, we have $\lim_{m\to\infty} (m \log \frac{9}{8} - x_m \log 2) = -\infty$ and therefore, by (5.10),

$$\lim_{m\to\infty} \frac{\varphi(D_m(x), B)}{\mu(D_m(x))} = 0 \tag{5.11}$$

for $\mu$-almost all $x$. Thus the measures $\varphi(\cdot, B)$ and $\mu$ are mutually singular. This is, then, still true if $B \in \mathfrak{G}$, since every set in $\mathfrak{G}$ is the union of countably many elementary cylinders. Finally, the class of all $B \in \mathfrak{F}$ such that $\varphi(\cdot, B)$ and $\mu$ are mutually singular is monotone, and therefore coincides with $\mathfrak{F}$ ([5], p. 27). In the same way, we find that for fixed $A$ in $\mathfrak{F}$ the measures $\varphi(A, \cdot)$ and $\mu$ are mutually singular.

For recent criteria for the strong ratio limit property in the case of chains with independent increments we refer to [19].

## 6. Category theorems

Using the various examples of mixing transformations obtained in the form $T' = S \circ T \circ S^{-1}$ where $T$ is the shift in the sample space of a Markov chain and $S$ the isomorphism described in section 3, we are in a position to derive category theorems similar to those in the case of a finite measure ([6], pp. 77–78); however, the proofs will be published elsewhere.

Let $X$ be a separable manifold and $\mu$ an atomless infinite Radon measure on $X$. We denote by $\mathfrak{I}$ the set of all measure preserving transformations of $X$; thus the elements of $\mathfrak{I}$ need be neither invertible nor almost everywhere continuous. We endow $\mathfrak{I}$ with the usual *weak topology* ([6], p. 62): a net $(T_\alpha)$ in $\mathfrak{I}$ converges weakly to a transformation $T$ from $\mathfrak{I}$ if $\lim_\alpha \mu(|f \circ T_\alpha - f \circ T|) = 0$ for every integrable function $f$. This makes $\mathfrak{I}$ into a Polish space; hence category statements have their customary meaning. Let $\mathfrak{M}$ be the set of all mixing endomorphisms. Then $\mathfrak{M}$ is a set of the first category in $\mathfrak{I}$ under the weak topology. Note that $\mathfrak{M}$ comprises all mixing transformations with all possible sequences $\rho_n$, and not necessarily invertible. On the other hand, given a particular sequence of positive numbers $\rho_n$, let $\mathfrak{M}^0(\rho_n)$ stand for the set of all mixing automorphisms which satisfy the mixing equation (2.2) with just this sequence $\rho_n$. Then the following is true. Given positive numbers $\eta$ and $\eta'$ there exist sequences $\rho_n$ and $\rho_n'$ such that $\mathfrak{M}^0(\rho_n)$ and $\mathfrak{M}^0(\rho_n')$ are weakly dense in $\mathfrak{I}$, and $\rho_n = O(n^\eta)$ and $n^{\eta'} = O(\rho_n')$ for $n \to \infty$.

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