ERGODIC THEORY OF SHIFT TRANSFORMATIONS

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1. Introduction

The main purpose of this note is to discuss the ergodic properties of a certain class of strictly ergodic dynamical systems which appear as subsystems of the shift dynamical system defined on the power space $X = A^Z$, where Z is the set of all integers. We discuss only the cases when the base space A is a finite set. We are particularly interested in two examples of strictly ergodic dynamical systems which are constructed by using certain number-theoretic functions. Among other things it will be shown that there exist a continuum number of strictly ergodic dynamical systems, no two of which are spectrally isomorphic.

2. Strictly ergodic dynamical systems

Let $X = \{x\}$ be a nonempty compact metrizable space, and let φ be a homeomorphism of X onto itself. The pair (X, φ) is called a *dynamical system.* A subset X_0 of X is said to be φ-invariant if $\varphi(X_0) = X_0$. If X_0 is a nonempty closed φ-invariant subset of X, then (X_0, φ) may be considered as a dynamical system, and is called a *dynamical subsystem* of (X, φ). A dynamical system (X, φ) is said to be *minimal* if there is no dynamical subsystem of (X, φ) except (X, φ) itself, that is if there is no nonempty closed φ-invariant subset of X except X itself.

Let

$$Z = \{n | n = 0, \pm 1, \pm 2, \cdots\} \tag{1}$$

be the set of all integers. For any point $x_0 \in X$, the set

$$\text{Orb}\,(x_0) = \{\varphi^n(x) | n \in Z\} \tag{2}$$

is called the *orbit* of x_0, and its closure $\overline{\text{Orb}}\,(x_0)$ is called the *orbit closure* of x_0. Obviously, $\overline{\text{Orb}}\,(x_0)$ is a closed φ-invariant subset of X, and hence $(\overline{\text{Orb}}\,(x_0), \varphi)$ is a dynamical subsystem of (X, φ). It is clear that a dynamical system (X, φ) is minimal if and only if Orb (x_0) is dense in X for any $x_0 \in X$.

Let $\mathcal{B} = \{B\}$ be the σ-field of all Borel subsets B of X. It was proved by N. Kryloff and N. Bogoliouboff [6] that, for any dynamical system (X, φ), there exists a normalized, countably additive, nonnegative measure μ defined on $\mathcal{B}$ which is invariant under φ; that is, $\mu(\varphi(B)) = \mu(B)$ for any $B \in \mathcal{B}$. Such a measure μ is not necessarily unique. A dynamical system (X, φ) is said to be *uniquely ergodic* if such a measure μ is unique. A dynamical system is said to be *strictly ergodic* if it is mininal and uniquely ergodic at the same time.

Let (X, φ) be a dynamical system, and let x_0 be a point of X. It was proved by W. H. Gottschalk [1] that $(\overline{\text{Orb}}\,(x_0), \varphi)$ is minimal if and only if, for any neighborhood W of x_0, there exists a positive integer n such that, for any integer $m \in Z$, at least one of the points $\varphi^k(x_0)$, $k = m + 1, \cdots, m + n$, belongs to W. On the other hand, it was proved by J. C. Oxtoby [10] that $(\overline{\text{Orb}}\,(x_0), \varphi)$ is uniquely ergodic if and only if the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=m+1}^{m+n} f(\varphi^k(x_0)) = \bar{f}(x_0) \tag{3}$$

exists uniformly in $m \in Z$ for any real-valued continuous function f defined on X. By combining these two results, we see that $(\overline{\text{Orb}}\,(x_0), \varphi)$ is strictly ergodic if and only if (i) the limit (3) exists uniformly in $m \in Z$ for any real-valued continuous function f defined on X, and if (ii) $\bar{f}(x_0) > 0$ for any nonnegative continuous function defined on X such that $f(x_0) > 0$.

3. Shift dynamical systems

Let $A = \{a\}$ be a finite set containing more than one element. Let

$$X = A^Z = \prod_{n \in Z} A_n; \qquad A_n = A \text{ for all } n \in Z, \tag{4}$$

be the set of all A-valued functions x defined on Z, or equivalently, the set of all two-sided infinite sequences

$$x = \{a_n | n \in Z\}; \qquad a_n \in A \text{ for all } n \in Z. \tag{5}$$

The mapping

$$\pi_n\colon x \to a_n = \pi_n(x) \tag{6}$$

is called the n-th *projection* of the *power space* $X = A^Z$ onto the *base space* A, and a_n is called the n-th *coordinate* of x.

The space X is a totally disconnected compact metrizable space with respect to the usual direct product topology in which a defining neighborhood of a point x_0 of X is given by

$$W_{n_1,\cdots,n_\ell}(x_0) = \{x | \pi_{n_i}(x) = \pi_{n_i}(x_0),\, i = 1, \cdots, \ell\}, \tag{7}$$

where $\{n_1, \cdots, n_\ell\}$ is a finite subset of Z.

A subset P of X is called a *primitive* set if it is of the form

$$P = P^{(n)}(\beta) = \{x |\ \pi_{n+i}(x) = b_i,\, i = 1, \cdots, \ell\}, \tag{8}$$

where $\beta = (b_1, \cdots, b_\ell)$; $b_i \in A$, $i = 1, \cdots, \ell$; and $n \in Z$. In this expression β is called a *block* of *length* ℓ. We do not assume that $b_1, \cdots, b_\ell$ are all different. As a special case, we also consider a block β of length 0. In this case, we put $P^{(n)}(\beta) = X$ for any $n \in Z$.

We observe that a primitive set is a special case of a neighborhood $W_{n_1,\cdots,n}(x_0)$

of the form (7) in which $\{n_1, \cdots, n_\ell\}$ is a consecutive set of integers, namely $n_i = n + i$, $i = 1, \cdots, \ell$.

A subset E of X is called an *elementary* set if it is a union of a finite number of primitive sets. A neighborhood of the form (7) is clearly an elementary set. It is easy to see that a subset of X is open and closed at the same time if and only if it is an elementary set.

The family of all primitive subsets of X is denoted by $\mathcal{P}$, and the family of all elementary subsets of X is denoted by $\mathcal{E}$. Clearly, $\mathcal{E}$ is a field of subsets of X. The σ-field of subsets of X generated by $\mathcal{E}$ is denoted by $\mathcal{B}$. This $\mathcal{B}$ is nothing but the σ-field of all Borel subsets of X.

Let φ be a mapping of X onto itself defined by

$$\pi_n(\varphi(x)) = \pi_{n+1}(x) \qquad \text{for all} \quad n \in Z. \tag{9}$$

It is clear that φ is a homeomorphism of X onto itself. The map φ is called the *shift transformation*, and the dynamical system (X, φ) is called the *shift dynamical system* defined on the power space $X = A^Z$. It is clear that φ maps each of $\mathcal{P}$, $\mathcal{E}$, and $\mathcal{B}$ onto itself.

Let μ be a normalized, φ-invariant, countably additive nonnegative measure defined on $\mathcal{B}$. For any block $\beta = (b_1, \cdots, b_\ell)$, $\mu(P^{(n)}(\beta))$ is independent of $n \in Z$, and hence we may denote it by $D(\beta)$. It is then clear that the following conditions are satisfied:

$$0 \leq D(\beta) \leq 1 \text{ for any block } \beta, \text{ and } D(\beta) = 1 \text{ if } \beta \text{ is a block of length } 0, \tag{10}$$

$$D(\beta) = \sum_{b \in A} D((\beta, b)) = \sum_{b \in A} D((b, \beta)), \tag{11}$$

where $(\beta, b) = (b_1, \cdots, b_\ell, b)$ and $(b, \beta) = (b, b_1, \cdots, b_\ell)$ if $\beta = (b_1, \cdots, b_\ell)$.

Conversely, assume that $D(\beta)$ is defined for any block β and that the conditions (10) and (11) are satisfied. Then it is easy to see that there exists a normalized, φ-invariant, countably additive, nonnegative measure μ defined on $\mathcal{B}$ such that $\mu(P^{(n)}(\beta)) = D(\beta)$ for any block β and $n \in Z$.

Now let x_0 be a point of X. From the result stated at the end of section 2 follows that $(\overline{\text{Orb}}\,(x_0), \varphi)$ is strictly ergodic if and only if (i) the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=m+1}^{m+n} \chi_P(\varphi^k(x_0)) = \mu(P) \tag{12}$$

exists uniformly in $m \in Z$ for any primitive set P, where χ_P is the characteristic function of P, and if (ii) $\mu(P) > 0$ for any primitive set P with $x_0 \in P$.

For any block $\beta = (b_1, \cdots, b_\ell)$, let us put

$$N(\beta, x_0) = \{n \mid \pi_{n+i}(x_0) = b_i,\ i = 1, \cdots, \ell\}. \tag{13}$$

Then the result above can be restated as follows: $(\overline{\text{Orb}}\,(x_0), \varphi)$ is strictly ergodic if and only if (i) the limit

$$\lim_{n\to\infty} \frac{1}{n} \begin{Bmatrix} \text{the number of integers } k \in N(\beta, x_0) \\ \text{such that } m + 1 \leq k \leq m + n \end{Bmatrix} = D(\beta) \tag{14}$$

exists uniformly in $m \in Z$ for any block $\beta = (b_1, \cdots, b_\ell)$ and if (ii) $D(\beta) > 0$ for any block β for which $N(\beta, x_0)$ is not empty.

It is easy to see that the limit $D(\beta)$ satisfies the conditions (10) and (11), and hence there exists a normalized, φ-invariant, countably additive, nonnegative measure μ defined on $\mathcal{B}$ such that $\mu(P^{(n)}(\beta)) = D(\beta)$ for any block β and for any $n \in Z$. This is the same μ which appears in the formula (12). It is easy to see that $\overline{\text{Orb}}\,(x_0)$ is the carrier of this measure μ, and that μ is nothing but the unique normalized, φ-invariant, countably additive, nonnegative measure for the strictly ergodic dynamical system $(\overline{\text{Orb}}\,(x_0), \varphi)$.

4. Example 1

We consider the shift dynamical system (X, φ) defined on the power space $X = A^Z$, where the base space A is a finite set consisting of two elements: $A = \{-1, +1\}$.

We define a number-theoretic function $\rho(n)$ by

$$\rho(n) = (-1)^{\eta_1+\eta_2+\cdots+\eta_k}, \qquad n = 0, 1, 2, \cdots \tag{15}$$

where

$$n = \eta_1 + \eta_2 \cdot 2 + \cdots + \eta_k \cdot 2^{k-1} \tag{16}$$

is an expansion of a nonnegative integer n with base 2. This means that $\eta_i = 0$ or 1 for $i = 1, \cdots, k$. It is easy to see that $\{\rho(n) | n = 0, 1, 2, \cdots\}$ is completely determined by the relations

$$\rho(0) = 1; \qquad \rho(2^{n-1} + k) = -\rho(k), \qquad k = 0, 1, \cdots, 2^{n-1} - 1;\ n = 1, 2, \cdots. \tag{17}$$

This sequence $\{\rho(n) | n = 0, 1, 2, \cdots\}$ has been discussed by many mathematicians [2], [3], [4], [5], [7], [8], [9], [11], [12] in connection with various problems in different parts of mathematics.

We now define a class of more general sequences as follows: let $\alpha (0 < \alpha \leq 1)$ be a real number, and let

$$\alpha = \frac{\epsilon_1(\alpha)}{2} + \frac{\epsilon_2(\alpha)}{2^2} + \cdots + \frac{\epsilon_n(\alpha)}{2^n} + \cdots \tag{18}$$

be a dyadic expansion of α, where $\epsilon_n(\alpha) = 0$ or 1, $n = 1, 2, \cdots$. This expansion is unique if we require that there are infinitely many n for which $\epsilon_n(\alpha) = 1$. Let us put

$$\rho_\alpha(n) = (-1)^{\eta_1 \cdot \epsilon_1(\alpha) + \eta_2 \cdot \epsilon_2(\alpha) + \cdots + \eta_k \cdot \epsilon_k(\alpha)}, \qquad n = 0, 1, 2, \cdots, \tag{19}$$

where $(\eta_1, \cdots, \eta_k)$ is determined by (16) and $(\epsilon_1(\alpha), \cdots, \epsilon_k(\alpha))$ is determined by (18). It is easy to see that $\{\rho_\alpha(n) | n = 0, 1, 2, \cdots\}$ is completely determined by the relations

(20) $$\rho_\alpha(0) = 1; \qquad \rho_\alpha(2^{n-1} + k) = (-1)^{\epsilon_n(\alpha)}\rho_\alpha(k), \quad k = 0, 1, \cdots, 2^{n-1} - 1; n = 1, 2, \cdots.$$

By comparing (17) with (20), we see that $\rho(n)$ defined by (15) corresponds to the case of $\rho_\alpha(n)$ defined by (19) when $\epsilon_n(\alpha) = 1$ for $n = 1, 2, \cdots$, that is when $\alpha = 1$.

We now define $\rho_\alpha(n)$ for $n = -1, -2, \cdots$ by

(21) $$\rho_\alpha(n) = \rho(-n-1), \qquad n = -1, -2, \cdots.$$

Thus the point $x_\alpha = \{\rho_\alpha(n) | n \in Z\} \in X$ is defined for each real number $\alpha (0 < \alpha \leq 1)$. It is possible to show that, for any block $\beta = (b_1, \cdots b_\ell)$, the uniform density $D_\alpha(\beta)$ exists for the point $x_\alpha = \{\rho_\alpha(n) | n \in Z\}$ and that $D_\alpha(\beta) > 0$ for any block for which $N(\beta, x_\alpha)$ is not empty. We see easily that $D_\alpha(\beta) = \frac{1}{2}$ if β is a block of length 1, but it is in general not true that $D_\alpha(\beta) = \frac{1}{2}^\ell$ if β is a block of length ℓ.

THEOREM 1. *For each real number $\alpha (0 < \alpha \leq 1)$, $(\overline{\text{Orb}}\,(x_\alpha), \varphi)$ is a strictly ergodic dynamical system.*

Let $\mathcal{B}_\alpha$ be the σ-field of all Borel subsets of $\overline{\text{Orb}}\,(x_\alpha)$ and let μ_α be the unique, normalized, φ-invariant, countably additive, nonnegative measure defined on $\mathcal{B}_\alpha$. It is clear that φ is an ergodic measure preserving transformation on the measure space $(\overline{\text{Orb}}\,(x_\alpha), \mathcal{B}_\alpha, \mu_\alpha)$.

Let τ be a mapping of X onto itself defined by

(22) $$\pi_n(\tau(x)) = -\pi_n(x) \qquad \text{for all} \quad n \in Z.$$

It is easy to see that τ is a homeomorphism of X onto itself with period 2 (that is, $\tau^2(x) = x$ for any $x \in X$), and that τ commutes with φ (that is, $\tau\varphi(x) = \varphi\tau(x)$ for any $x \in X$). It is also easy to show that τ is a homeomorphism of $\overline{\text{Orb}}\,(x_\alpha)$ onto itself and that τ is a measure preserving transformation on the measure space $(\overline{\text{Orb}}\,(x_\alpha), \mathcal{B}_\alpha, \mu_\alpha)$.

Let $\mathcal{H}_\alpha = L^2(\overline{\text{Orb}}\,(x_\alpha), \mathcal{B}_\alpha, \mu_\alpha)$ be the complex L^2-space over the measure space $(\overline{\text{Orb}}\,(x_\alpha), \mathcal{B}_\alpha, \mu_\alpha)$. Let V_α^φ, V_α^τ be the unitary operators defined on $\mathcal{H}_\alpha$ by $V_\alpha^\varphi f(x) = f(\varphi(x))$, $V_\alpha^\tau f(x) = f(\tau(x))$, respectively. Further, let $\mathfrak{M}_\alpha^1$, $\mathfrak{M}_\alpha^{-1}$ be the closed linear subspaces of $\mathcal{H}_\alpha$ consisting of all $f \in \mathcal{H}_\alpha$ such that $V_\alpha^\tau f = f$, $V_\alpha^\tau f = -f$, respectively. It is easy to see that $\mathfrak{M}_\alpha^1$ and $\mathfrak{M}_\alpha^{-1}$ are orthogonal to each other and together span the space $\mathcal{H}_\alpha$. It is also easy to see that both $\mathfrak{M}_\alpha^1$ and $\mathfrak{M}_\alpha^{-1}$ are invariant under V_α^φ.

THEOREM 2. *For each real number $\alpha (0 < \alpha \leq 1)$, V_α^φ has a pure point spectrum on $\mathfrak{M}_\alpha^1$, and a continuous singular spectrum on $\mathfrak{M}_\alpha^{-1}$. Further, for any two real numbers α and α' $(0 < \alpha < \alpha' \leq 1)$, V_α^φ on $\mathfrak{M}_\alpha^1$ is spectrally isomorphic with $V_{\alpha'}^\varphi$ on $\mathfrak{M}_{\alpha'}^1$, while V_α^φ on $\mathfrak{M}_\alpha^{-1}$ and $V_{\alpha'}^\varphi$ on $\mathfrak{M}_{\alpha'}^{-1}$ are not spectrally isomorphic if $\alpha' - \alpha$ is a dyadically irrational number.*

Thus we obtained a concrete example of a continuum number of strictly ergodic dynamical systems, no two of which are spectrally isomorphic.

5. Example 2

We now consider the shift dynamical system (X, φ) defined on the power space $X = A^Z$, where the base space A is a finite set consisting of four elements: $A = \{2, 4, 6, 8\}$.

We define a number-theoretic function $\lambda(n)$ by

(23) $\lambda(n) =$ the last nonzero digit in the decimal expansion of $n!$, $n = 2, 3, \cdots$.

For example, $\lambda(2) = 2$, $\lambda(3) = 6$, $\lambda(4) = 4$, $\lambda(5) = 2$, $\lambda(6) = 2, \cdots$.

If we denote by $\gamma(n)$ the number of consecutive zeros at the right end of the decimal expansion of $n!$, then we may write

(24) $$n! \equiv 0 \pmod{10^{\gamma(n)}}; \qquad n!/10^{\gamma(n)} \equiv \lambda(n) \pmod{10}.$$

We observe that $\lambda(n)$ is even, and hence $\lambda(n) \in A$ for $n = 2, 3, \cdots$. This follows from the fact that if $n! = 2^a 3^b 5^c 7^d \cdots p^e$ is a representation of n as the product of powers of a finite number of different prime numbers, then $a \geq c$, and hence $\gamma(n) = c$, and consequently, $\gamma(n) \equiv 2^{a-c} 3^b 7^d \cdots p^e \pmod{10}$.

We now want to find a general rule to compute the values of $\lambda(n)$. For this purpose, we introduce a cyclic permutation T of the base space $A = \{2, 4, 6, 8\}$ of order 4 defined by

(25) $$T = \begin{pmatrix} 2 & 4 & 6 & 8 \\ 4 & 8 & 2 & 6 \end{pmatrix}.$$

We also consider the set $B = \{1, 2, 3, 4\}$.

First, let $n \equiv b \in B \pmod 5$, $n \geq 3$. In this case, $\lambda(n)$ is obtained from $\lambda(n - 1)$ by the relation

(26) $$\lambda(n) \equiv b \cdot \lambda(n - 1) \pmod{10},$$

or equivalently, by

(27) $$\lambda(n) \equiv T^{\eta(b)} \lambda(n - 1),$$

where T is a permutation defined by (25), and η is a function defined on B by

(28) $$\eta(1) = 0, \qquad \eta(2) = 1, \qquad \eta(3) = 3, \qquad \eta(4) = 2.$$

If we put $\lambda(0) = \lambda(1) = 6$, then the relations (26) and (27) hold for $n = 1$ and $n = 2$. For this reason, we use these values of $\lambda(0)$ and $\lambda(1)$ even though they do not satisfy (23). Thus the relations (26) and (27) are valid for $n = 1, 2, \cdots$ if $n \not\equiv 0 \pmod 5$. If $n \equiv 0 \pmod 5$, then the situation is a little more complicated.

Next, let $n \equiv 0 \pmod 5$ and $n/5 \equiv b \in B \pmod 5$. In this case, we have $\lambda(n) = \lambda(n - 1) + 1$, and hence $2\lambda(n) \equiv b \cdot \lambda(n - 1) \pmod{10}$. From this follows that $T \cdot \lambda(n) = T^{\eta(b)} \lambda(n - 1)$, or equivalently,

(29) $$\lambda(n) = T^{\eta(b)+3} \lambda(n - 1).$$

For example, we can compute the value of $\lambda(15)$ if we know that $\lambda(14) = 2$.

Since $15 \equiv 0 \pmod 5$ and $15/5 \equiv 3 \pmod 5$, we have $\lambda(15) = T^{\eta(3)+3}\lambda(14) = T^{3+3}2 = T^2 2 = 8$.

Finally, we discuss the general case:

$$n \equiv 0 \pmod{5^k}, \qquad n/5^k \equiv b \in B \pmod 5; \qquad k = 0, 1, 2, \cdots. \tag{30}$$

In this case, we have $\gamma(n) = \gamma(n-1) + k$, and hence $2^k\lambda(n) \equiv b \cdot \lambda(n-1) \pmod{10}$. From this follows that $T^k\lambda(n) = T^{\eta(b)}\lambda(n-1)$, or equivalently,

$$\lambda(n) = T^{\xi(n)}\lambda(n-1), \tag{31}$$

where $\xi(n)$ takes one of the four values 0, 1, 2, 3, such that

$$\xi(n) \equiv \eta(b) + 3k \pmod 4 \tag{32}$$

and $\eta(b)$ is defined by (28).

Equation (31) is a general formula by which we can compute the value of $\lambda(n)$ from that of $\lambda(n-1)$ for $n = 1, 2, \cdots$. We now want a formula by which we can compute the value of $\lambda(n)$ from that of $\lambda(0)$.

From (31) follows that

$$\lambda(n) = T^{\zeta(n)}\lambda(0), \tag{33}$$

where $\zeta(n)$ takes one of the four values 0, 1, 2, 3, such that

$$\zeta(n) \equiv \xi(1) + \xi(2) + \cdots + \xi(n) \pmod 4. \tag{34}$$

Let now

$$n = c_1 + c_2 5 + \cdots + c_k 5^{k-1} \tag{35}$$

be the expansion of a nonnegative integer n in base 5, where $c_i = 0, 1, 2, 3,$ or 4 for $i = 1, 2, \cdots, k$. Then

$$\zeta(n) \equiv \sum_{m=1}^{n} \xi(m) \equiv \sum_{m=1}^{c_k \cdot 5^{k-1}} \xi(m) + \sum_{m=1}^{c_{k-1} \cdot 5^{k-2}} \xi(m + c_k 5^{k-1}) + \cdots + \sum_{m=1}^{c_1} \xi(m + c_2 5 + \cdots + c_k 5^{k-1}) \pmod 4. \tag{36}$$

If we observe that

$$\xi(m + c_i 5^{i-1} + c_{i+1} 5^i + \cdots + c_k 5^{k-1}) = \xi(m) \tag{37}$$

for $m = 1, 2, \cdots, 5^{i-1} - 1$, then (36) becomes

$$\begin{aligned} \zeta(n) &\equiv \sum_{m=1}^{c_k \cdot 5^{k-1}} \xi(m) + \sum_{m=1}^{c_{k-1} \cdot 5^{k-2}} \xi(m) + \cdots + \sum_{m=1}^{c_1} \xi(m) \\ &\equiv \zeta(c_k 5^{k-1}) + \zeta(c_{k-1} 5^{k-2}) + \cdots + \zeta(c_1) \\ &\equiv \zeta_k(c_k) + \zeta_k(c_{k-1}) + \cdots + \zeta_1(c_1) \pmod 4 \end{aligned} \tag{38}$$

where

$$\zeta_k(c) = \zeta(c\, 5^{k-1}), \qquad c = 0, 1, 2, 3, 4; \ k = 1, 2, \cdots. \tag{39}$$

We calculate the values of $\zeta_k(c)$ for $c = 0, 1, 2, 3, 4$ and $k = 1, 2, 3, 4$, and obtain the following table:

(40)

c	0	1	2	3	4
$\zeta_1(c)$	0	0	1	0	2
$\zeta_2(c)$	0	1	3	3	2
$\zeta_3(c)$	0	2	1	2	2
$\zeta_4(c)$	0	3	3	1	2

We also observe (by computation) that

$$\xi(5^4) = \zeta(5^4) = 0. \tag{41}$$

On the other hand, from (32) it follows that

$$\xi(c\,5^{k+4}) = \xi(c\,5^k), \qquad c = 1, 2, 3, 4;\ k = 0, 1, 2, \cdots. \tag{42}$$

From (41) and (42) it is easy to show, by mathematical induction, that

$$\zeta_{k+4}(c) = \zeta_k(c), \qquad c = 0, 1, 2, 3, 4;\ k = 0, 1, 2, \cdots. \tag{43}$$

Thus, (40) and (43) together give all the values of $\zeta_k(c)$ for $c = 0, 1, 2, 3, 4$ and $k = 0, 1, 2, \cdots$. Combined with (33) and (38), we have now a fairly simple method to calculate the values of $\lambda(n)$ for $n = 0, 1, 2, \cdots$.

We can restate the above result in the following form. If

$$n = c_1^* + c_2^*\,625 + c_3^*\,(625)^2 + \cdots + c_k^*\,(625)^{k-1} \tag{44}$$

is the expansion of a nonnegative integer n with base $625 = 5^4$, where $c_i^* = 0, 1, 2, \cdots, 624$ for $i = 1, 2, \cdots, k$, then there exists a function $\zeta^*(c^*)$ defined for $c^* = 0, 1, 2, \cdots, 624$ and taking the values 0, 1, 2, 3, such that

$$\zeta(n) \equiv \zeta^*(c_1^*) + \zeta^*(c_2^*) + \cdots + \zeta^*(c_k^*) \pmod 4. \tag{45}$$

In fact, if

$$c^* = c_1 + c_2\,5 + c_3\,5^2 + c_4\,5^3 \tag{46}$$

is an expansion of c^* with base 5, where $c_i = 0, 1, 2, 3, 4$ for $i = 1, 2, 3, 4$, then

$$\zeta^*(c^*) \equiv \zeta_1(c_1) + \zeta_2(c_2) + \zeta_3(c_3) + \zeta_4(c_4) \pmod 4. \tag{47}$$

We observe that there is some similarity in the properties of two sequences $\{\rho(n)|n = 0, 1, 2, \cdots\}$ and $\{\lambda(n)|n = 0, 1, 2, \cdots\}$. For example, formulas (15) and (45) are similar to each other. Instead of the expansion of n with base 2 in (16), we use the expansion with base 625 in (44), and instead of the involution of the set $A = \{-1, +1\}$ which interchanges -1 and $+1$, we use the cyclic permutation T of order 4 of our set $A = \{2, 4, 6, 8\}$.

We now define $\lambda(n)$ for $n = -1, -2, \cdots$ by

$$\lambda(n) = \lambda(-n - 1), \qquad n = -1, -2, \cdots \tag{48}$$

and consider the point x_0 in $X \in A^Z$ defined by $x_0 = \{\lambda(n)|n \in Z\}$. It is possible to show that, for any block $\beta = (b_1, \cdots, b_l)$, the uniform density $D(\beta)$ exists for the point $x_0 = \{\lambda(n)|n \in Z\}$ and that $D(\beta) > 0$ for any block β for which

$N(\beta, x_0)$ is not empty. We see easily that $D(\beta) = \frac{1}{4}$ for any block of length 1, and that $D(\beta) = \frac{1}{16}$ for any block β of length 2, but it is in general not true that $D(\beta) = \frac{1}{4}^{\ell}$ for any block of length ℓ.

THEOREM 3. *$(\overline{\text{Orb}}\,(x_0), \varphi)$ is a strictly ergodic dynamical system.*

Let $\mathcal{B}_0$ be the σ-field of all Borel subsets of $\overline{\text{Orb}}\,(x_0)$ and let μ_0 be the unique, normalized, φ-invariant, countably additive, nonnegative measure defined on $\mathcal{B}_0$. It is clear that φ is an ergodic measure preserving transformation on the measure space $(\overline{\text{Orb}}\,(x_0), \mathcal{B}_0, \mu_0)$.

Let τ be a mapping of X onto itself defined by

$$\pi_n(\tau(x)) = T\pi_n(x) \qquad \text{for all} \quad n \in Z. \tag{49}$$

It is easy to see that τ is a homeomorphism of X onto itself with period 4 (that is, $\tau^4(x) = x$ for any $x \in X$), and that τ commutes with φ (that is, $\tau\varphi(x) = \varphi\tau(x)$ for any $x \in X$). It is also easy to see that τ is a homeomorphism of $\overline{\text{Orb}}\,(x_0)$ onto itself, and that τ is a measure-preserving transformation on the measure space $(\overline{\text{Orb}}\,(x_0), \mathcal{B}_0, \mu_0)$.

Let $\mathcal{H}_0 = L^2(\overline{\text{Orb}}\,(x_0), \mathcal{B}_0, \mu_0)$ be the complex L^2-space over the measure space $(\overline{\text{Orb}}\,(x_0), \mathcal{B}_0, \mu_0)$. Let V_0^{φ}, V_0^{τ} be the unitary operators defined on $\mathcal{H}_0$ by $V_0^{\varphi}f(x) = f(\varphi(x))$, $V_0^{\tau}f(x) = f(\tau(x))$, respectively. Further, let $\mathfrak{M}_0^k$ be the closed linear subspace of $\mathcal{H}_0$ consisting of all $f \in \mathcal{H}_0$ such that $V_0^{\tau}f = e^{2\pi ik/4}f$, $k = 0, 1, 2, 3$. It is easy to see that $\mathfrak{M}_0^k$, $k = 0, 1, 2, 3$, are mutually orthogonal and together span the space $\mathcal{H}_0$. It is also easy to see that each $\mathfrak{M}_0^k$ is invariant under V_0^{φ}.

THEOREM 4. *The operator V_0^{φ} has a pure point spectrum on $\mathfrak{M}_0^0$, and continuous singular spectra on $\mathfrak{M}_0^k$, $k = 1, 2, 3$.*

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