SOME PECULIAR SEMI-MARKOV PROCESSES

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SUMMARY. Experience with semi-Markov processes with finite expected *waits* suggests that the behavior of Markov processes is a good guide to understanding the behavior of the more general process. However, examples are given to show that when expected waits are infinite quite surprising behavior is possible. For a two-state aperiodic semi-Markov process the instantaneous state probabilities $P_i(t)$ can have $(C, 1)$-limits but not strict limits; for a three-state (and *irreducible*) process one can have $P_0(t)$ tend to a strict limit as $t \to \infty$ but $P_1(t)$ and $P_2(t)$ not even have $(C, 1)$-limits. For an aperiodic irreducible infinite chain one can have $P_i(t) \to \pi_i > 0$ as $t \to \infty$, for every i, yet $\Sigma\, \pi_i < 1$.

1. Introduction

Semi-Markov processes were introduced simultaneously by Lévy [3] and by Smith [7], [8]. The constructive definition of Smith, which is valuable so long as only a few states are instantaneous, has been given an elaborate and formal treatment by Pyke [5], [6].

For the present note we shall suppose that we are given

(i) the transition matrix $\|p_{ij}\|$ of an irreducible and recurrent Markov chain of, possibly, infinitely many states;

(ii) a sequence $\{\Omega_i(x)\}$ of proper distribution functions of nonnegative random variables and such that there is at least one i such that $\Omega_i(0+) < 1$.

We imagine the process develops as follows. An initial state, say i_0, is selected, and the process stays in this state for a period of time governed by the distribution function $\Omega_{i_0}(x)$. At the end of this *wait* in the state i_0 the process then selects a fresh state in accordance with the transition matrix $\|p_{ij}\|$; thus, with probability $p_{i_0 i_1}$ the system now moves to state i_1. Having reached state i_1 the system waits there a period of time governed by $\Omega_{i_1}(x)$, and so on. It is assumed that successive waits are independent. Under the assumptions we are presently making (especially the recurrence of $\|p_{ij}\|$) there will be, with probability one, finitely many transitions in any finite time period. To avoid ambiguity we may suppose that at the instant of a transition the system is in the state in which it will next reside for a strictly positive amount of time.

For purposes of discussion, let us suppose (with no loss of generality) that the

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initial state is 0, and let us write $P_i(t)$ for the probability that the system is in state i at time t. We shall also suppose that the semi-Markov process is *aperiodic;* that is, the time intervals between successive returns to state 0 are not, with probability one, multiples of some fixed span $\tilde{\omega} > 0$. Subject to these understandings, and when the distribution functions $\{\Omega_i(x)\}$ all have finite mean values, Smith [7] showed that the probabilities $P_i(t)$ all tend to limits: $P_i(t) \to \pi_i \geq 0$ as $t \to \infty$. In this respect, and in many others, semi-Markov processes behave so very much like Markov processes that one might be excused for supposing them unworthy of study. For instance, much of the interest in Markov processes arises from the existence of instantaneous states, and it seems unlikely that, were one to adopt a suitably changed definition for the semi-Markov process so as to allow an abundance of instantaneous states, the resulting more "general" processes would exhibit any behavior not an obvious copy of that already found in familiar Markov processes. However, there is one possibility for semi-Markov processes which is precluded from Markov processes; this is the possibility that the waits in the states may have infinite expectation.

In a very short, but interesting, paper Derman [1] considered a semi-Markov process of only two states, 0 and 1 say, in which $p_{0,0} = p_{1,1} = 0$. He showed that, if we denote Laplace-Stieltjes transforms, for real $s > 0$,

$$\Omega_i^*(s) = \int_{0-}^{\infty} e^{-sx}\, d\Omega_i(x), \tag{1.1}$$

then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T P_i(t)\, dt = \lim_{s\to 0+} \frac{1 - \Omega_i^*(s)}{1 - \Omega_1^*(s)\Omega_2^*(s)} \tag{1.2}$$

if either of the limits exist. In other words, $(C, 1)$-limits, at least, can exist for the probabilities $P_i(t)$ even when wait-distributions have infinite means.

Derman's result raises a number of questions about semi-Markov processes. Do strict limits, as opposed to $(C, 1)$-limits, exist for the probabilities $P_i(t)$? Do semi-Markov processes of infinitely many states exhibit similarly pleasant asymptotic behavior? In another paper we shall discuss sufficient conditions for the existence of strict limits to the $P_i(t)$ when the $\Omega_i(x)$ have infinite means. In this paper we shall exhibit examples of semi-Markov processes for which the asymptotic behavior of the $P_i(t)$ is disturbing, if not surprising; the examples serve to show that, unlike what happens with semi-Markov processes with finite mean waits, when the mean waits are infinite analogy with ordinary Markov processes is quite useless.

2. A two-state process with no $(C, 1)$-limits to the $P_i(t)$ as $t \to \infty$

Consider the Derman result (1.2). We have

$$\frac{1 - \Omega_1^*(s)}{1 - \Omega_1^*(s)\Omega_2^*(s)} = \frac{1}{1 + \Omega_1^*(s)\left(\dfrac{1 - \Omega_2^*(s)}{1 - \Omega_1^*(s)}\right)}, \tag{2.1}$$

and since $\Omega_1^*(s) \to 1$ as $s \to 0+$, a necessary and sufficient condition for the existence of

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T P_1(t)\, dt \tag{2.2}$$

is that

$$\lim_{s\to 0+} \frac{1 - \Omega_2^*(s)}{1 - \Omega_1^*(s)} \tag{2.3}$$

should exist. At this point the following theorem is relevant.

THEOREM 1. *If $F(x)$ and $G(x)$ are distribution functions of nonnegative random variables and $\lambda > 0$ is some finite constant, then a necessary and sufficient condition to ensure that*

$$\int_0^x \{1 - F(y)\}\, dy \sim \lambda \int_0^x \{1 - G(y)\}\, dy \qquad \text{as} \quad x \to \infty, \tag{A}$$

is that

$$\{1 - F^*(s)\} \sim \lambda\{1 - G^*(s)\} \qquad \text{as} \quad s \to 0+. \tag{B}$$

PROOF. This theorem is in fact an easy consequence of some general Abelian-Tauberian results of Feller [2]. However, it is easy if $F(x)$ has a finite mean, and it is also easy to show that if $F(x)$ has an infinite mean (so that the left-hand member of (A) diverges as $x \to \infty$), then (A) implies (B). The only really difficult part is to prove that (B) implies (A) when F has an infinite mean and it is of interest to see that a slight modification of the classical Karamata argument (see, for example, Widder [9]) can accomplish what we need.

Define $h(x) = 0$ for $x < \frac{1}{2}$, $h(x) = 1$ for $x \geq 1$. Choose a small $\epsilon > 0$ and let $P(x)$ be any polynomial such that $P(0) = 0$, $P(1) = 1$, and

$$\int_0^1 \frac{|P(x) - h(x)|}{x} \left(\frac{1}{\log 2} + \frac{1}{|\log x|}\right) dx < \epsilon. \tag{2.4}$$

It is not difficult to see that this can be done. Then, if we assume (B), and note particularly that $P(0) = 0$, as $s \to 0+$

$$\lambda \int_0^\infty P(e^{-st})\{1 - G(t)\}\, dt \sim \int_0^\infty P(e^{-st})\{1 - F(t)\}\, dt. \tag{2.5}$$

From this, by obvious maneuvers, we have

(2.6)

$$\lambda \int_0^T \{1 - G(t)\}\, dt + \lambda T \int_0^1 \frac{\{P(x) - h(x)\}}{x} \left\{1 - G\left(T \log \frac{1}{x}\right)\right\} dx$$
$$\sim \int_0^T \{1 - F(t)\}\, dt + T \int_0^1 \frac{\{P(x) - h(x)\}}{x} \left\{1 - F\left(T \log \frac{1}{x}\right)\right\} dx$$

as $T \to \infty$. But, for $0 \leq x \leq \frac{1}{2}$,

$$\frac{T\left\{1 - G\left(T \log \frac{1}{x}\right)\right\}}{\int_0^{T \log 2} \{1 - G(t)\}\, dt} \leq \frac{1}{\log 2}, \tag{2.7}$$

while, for $\frac{1}{2} \leq x \leq 1$, a similar inequality holds if we replace the right member by $1/(\log 1/x)$. Thus, in view of (2.4),

$$\left| T \int_0^1 \frac{\{P(x) - h(x)\}}{x} \{1 - G(T \log 1/x)\} \right| < \epsilon \left| \int_0^{T \log 2} \{1 - G(t)\} \, dt \right| \cdot \tag{2.8}$$

Clearly a similar inequality holds with F in place of G, and the theorem follows from (2.6).

With the aid of theorem 1 and Derman's result we can easily construct an example of the sort required in this section. We have only to construct distribution functions $\Omega_1(x)$ and $\Omega_2(x)$ such that

$$\frac{\int_0^T \{1 - \Omega_1(x)\} \, dx}{\int_0^T \{1 - \Omega_2(x)\} \, dx} \tag{2.9}$$

does not tend to a limit as $T \to \infty$. Such a task is too easy for words.

3. A two-state aperiodic process with $(C, 1)$-limits but not strict limits to the $P_i(t)$

We consider the basic two-state process of Derman once more ($p_{0,0} = p_{1,1} = 0$) and shall arrange that

$$\frac{1}{T} \int_0^T P_1(t) \, dt \tag{3.1}$$

tends to a limit as $T \to \infty$, while $P_1(t)$ does nothing of the sort. It is vital, of course, to note that we are not relying on any periodicity to achieve our ends.

Suppose that $\Omega_1(0+) = 0$ and that, for some large $\xi > 0$, both $\Omega_0(x)$ and $\Omega_1(x)$ have been defined for all $x < \xi$. Suppose further that

$$\Omega_0(\xi) < 1, \qquad \Omega_1(\xi) < 1. \tag{3.2}$$

Let us write

$$\Omega_i^c(x) = 1 - \Omega_i(x), \qquad i = 0, 1, \tag{3.3}$$

and

$$\eta = \min \{\Omega_0^c(\xi - 0), \Omega_1^c(\xi - 0)\}. \tag{3.4}$$

Choose $t_0 \gg \xi$, but unspecified for the moment. Define

$$\Omega_0^c(x) = \Omega_1^c(x) = \eta \qquad \text{for all} \quad \xi \leq x < t_0, \tag{3.5}$$

and set

$$F(x) = \int_0^x \Omega_0(x - z) \, d\Omega_1(z). \tag{3.6}$$

Let $H(x)$ be the renewal function based on the distribution function $F(x)$. If we assume there was a transition into state 0 at $t = 0$, then

$$P_0(t) = \Omega_0^c(t) + \int_0^t \Omega_0^c(t-\tau)\, dH(\tau). \tag{3.7}$$

Suppose we introduce improper distribution functions as follows:

$$\bar{\Omega}_0^c(x) = \begin{cases} \Omega_0^c(x), & x < \xi, \\ \eta, & x \geq \xi, \end{cases} \tag{3.8}$$

$$\bar{\Omega}_1^c(x) = \begin{cases} \Omega_1^c(x), & x < \xi, \\ \eta, & x \geq \xi, \end{cases} \tag{3.9}$$

$$\bar{F}(x) = \int_0^x \bar{\Omega}_0(x-z)\, d\bar{\Omega}_1(z). \tag{3.10}$$

Then $\bar{F}(\infty) = (1-\eta)^2$ and if $\bar{H}(x)$ is the renewal function based on the improper distribution function $\bar{F}(x)$, then $\bar{H}(x) \uparrow ((1-\eta)^2/1 - (1-\eta)^2)$ as $x \uparrow \infty$.

Choose $T > \xi$ such that $\bar{H}(\infty) - \bar{H}(T) < \eta$ and take $t_0 > 2T$. Then $H(x) = \bar{H}(x)$ for all $x < t_0$. Furthermore, we can choose t_0 sufficiently large such that

$$\frac{\int_0^\xi \{1 - \Omega_0(z)\}\, dz + t_0\eta + T\eta}{\int_0^\xi \{1 - \Omega_1(z)\}\, dz + t_0\eta + 2T\eta} \tag{3.11}$$

differs from unity by less than η.

Suppose we set

$$R(x) = \frac{\int_0^x \{1 - \Omega_0(z)\}\, dz}{\int_0^x \{1 - \Omega_1(z)\}\, dz}, \tag{3.12}$$

and suppose $|1 - R(\xi)| < \delta$, say. Then, if we define

$$\begin{aligned} \Omega_0^c(x) &= \tfrac{1}{2}\eta, && t_0 \leq x \leq t_0 + 2T, \\ \Omega_1^c(x) &= \eta, && \text{in the same range,} \end{aligned} \tag{3.13}$$

it will follow that

$$|1 - R(x)| \leq \max(\delta, \eta) \tag{3.14}$$

for all x in the range $\xi \leq x \leq t_0 + 2T$.

We have introduced a probability $\frac{1}{2}\eta$ at $x = t_0$ in the construction of Ω_0. Thus, it is not true that $H(x) = \bar{H}(x)$ for $t_0 \leq x < t_0 + 2T$. However, for $0 \leq x < 2T$ we do have (recall $t_0 > 2T$)

$$\begin{aligned} H(x + t_0) &= \bar{H}(x + t_0) + \tfrac{1}{2}\eta \int_0^x \bar{H}(x - z)\, d\Omega_1(z) \\ &\leq \bar{H}(x + t_0) + \tfrac{1}{2}\eta\bar{H}(x). \end{aligned} \tag{3.15}$$

Now consider

$$(3.16)\qquad \begin{aligned} P_0(t_0 + 2T) &< \tfrac{1}{2}\eta + \int_0^{t_0+2T} \Omega_0^c(t_0 + 2T - z)\, dH(z) \\ &< \tfrac{1}{2}\eta + \int_0^{2T} + \int_{2T}^{t_0} + \int_{t_0}^{t_0+T} + \int_{t_0+T}^{t_0+2T} \\ &< \tfrac{1}{2}\eta + \tfrac{1}{2}\left[\frac{(1-\eta)^2}{1-(1-\eta)^2}\right] + \eta^2 \\ &\quad + \eta\{H(t_0+T) - H(t_0)\} + \{H(t_0+2T) - H(t_0+T)\}. \end{aligned}$$

But,

$$(3.17)\qquad \begin{aligned} H(t_0+T) - H(t_0) &\le \overline{H}(T+t_0) + \tfrac{1}{2}\eta\overline{H}(T) - \overline{H}(t_0) \\ &\le \eta + \tfrac{1}{2}\eta\left[\frac{(1-\eta)^2}{1-(1-\eta)^2}\right]. \end{aligned}$$

Also,

$$(3.18)\qquad \begin{aligned} &H(t_0+2T) - H(t_0+T) \\ &\quad = \overline{H}(t_0+2T) + \tfrac{1}{2}\eta \int_0^{t_0+2T} \overline{H}(t_0+2T-z)\, d\Omega_1(z) \\ &\quad - \overline{H}(t_0+T) - \tfrac{1}{2}\eta \int_0^{t_0+T} \overline{H}(t_0+T-z)\, d\Omega_1(z) \\ &\quad < \eta + \tfrac{1}{2}\eta \int_0^{\xi} \{\overline{H}(t_0+2T-z) - H(t_0+T-z)\}\, d\Omega_1(z) \\ &\quad < \eta + \tfrac{1}{2}\eta^2(1-\eta). \end{aligned}$$

Thus,

$$(3.19)\qquad P_0(t_0+2T) < \Lambda(\eta),$$

say, where

$$(3.20)\qquad \Lambda(\eta) = \tfrac{3}{2}\eta + 2\eta^2 + \tfrac{1}{2}\eta^2(1-\eta) + \tfrac{1}{2}(\eta+\eta^2)\left[\frac{(1-\eta)^2}{1-(1-\eta)^2}\right].$$

We note that $\Lambda(\eta) \to \frac{1}{4}$ as $\eta \downarrow 0$. Thus we can continue our constructive process in a sequential manner and ensure that

$$(3.21)\qquad \liminf_{t\to\infty} P_0(t) \le \tfrac{1}{4}.$$

However,

$$(3.22)\qquad \frac{\int_0^x \{1 - \Omega_0(z)\}\, dz}{\int_0^x \{1 - \Omega_1(z)\}\, dz} \to 1$$

as $x \to \infty$, so that,

$$(3.23)\qquad \frac{1}{T}\int_0^T P_0(t)\, dt \to \tfrac{1}{2} \qquad \text{as} \quad T \to \infty.$$

By modifications to the above argument (we omit details), one can also demonstrate the existence of a two-state semi-Markov process such that

$$\frac{1}{T}\int_0^T P_1(t)\,dt \to 1 \qquad \text{as} \quad T \to \infty, \tag{3.24}$$

but

$$\liminf_{t\to\infty} \left(\frac{t}{\log t}\right) P_1(t) = 0. \tag{3.25}$$

4. A three-state semi-Markov process in which $P_0(t) \to \pi_0 > 0$ while $P_1(t)$ and $P_2(t)$ do not even have $(C, 1)$-limits

Choose two distribution functions $\Omega_1(x)$ and $\Omega_2(x)$ such that

$$\frac{\int_0^T \Omega_1^c(x)\,dx}{\int_0^T \Omega_2^c(x)\,dx} \tag{4.1}$$

does *not* tend to a limit as $T \to \infty$. Define $\Omega_0(x) = \frac{1}{2}\Omega_1(x) + \frac{1}{2}\Omega_2(x)$ and let the process have transition matrix

$$\begin{array}{lll} p_{0,0} = 0 & p_{0,1} = \frac{1}{2} & p_{0,2} = \frac{1}{2} \\ p_{1,0} = 1 & p_{1,1} = 0 & p_{1,2} = 0 \\ p_{2,0} = 1 & p_{2,1} = 0 & p_{2,2} = 0. \end{array} \tag{4.2}$$

From the point of view of state 1 the other two states can be pooled together into a single state A_α, say, with

$$\Omega_\alpha^*(s) = \frac{\Omega_0^*(s)}{1 - \Omega_0^*(s)\Omega_1^*(s)}. \tag{4.3}$$

For $P_1(t)$ to have a $(C, 1)$-limit we need $(1 - \Omega_1^*(s)/1 - \Omega_\alpha^*(s)\Omega_1^*(s))$ to approach a limit as $s \downarrow 0$. It is a matter of computation to show that such a limit does not exist, since $(1 - \Omega_2^*(s)/1 - \Omega_1^*(s))$ will not (by construction) approach a limit. Similarly $P_2(t)$ cannot have a $(C, 1)$-limit.

Suppose a transition occurred into state 0 at $t = 0$. Then

$$P_0(t) = \Omega_0^c(t) + \int_0^t \Omega_0^c(t - z)\,dH(z) \tag{4.4}$$

where $H(t)$ is the renewal function based on the distribution function

$$\int_0^t \Omega_0(t - z)\,d\Omega_0(z). \tag{4.5}$$

Thus,

$$P_0^*(s) = [1 - \Omega_0^*(s)] + \frac{\Omega_0^*(s)}{1 + \Omega_0^*(s)}. \tag{4.6}$$

We can easily arrange that $\Omega_0(x)$ shall have a nonnull absolutely continuous component. This will ensure that

$$\inf |1 + \Omega_0^*(s)| > 0, \qquad \text{Re } s \geq 0, \tag{4.7}$$

and we can then deduce from a Mercerian theorem (Pitt [4], theorem 11, p. 118), that $P_0(t)$ is of bounded variation. But $P_0^*(s) \to \frac{1}{2}$ as $s \downarrow 0+$. Thus, $P_0(t) \to \frac{1}{2}$ as $t \to \infty$, and this demonstration is complete.

5. A semi-Markov process with infinitely many states, an irreducible transition matrix, such that $P_i(t) \to \pi_i > 0$ as $t \to \infty$, for all i, but $\sum \pi_i < 1$

Let $R(x)$ be the distribution function of a rectangular distribution over (0, 1). Let $U(x) = P\{0 \leq x\}$; let $\{p_n\}$, $n = 1, 2, 3, \cdots$, be probabilities such that $\sum p_n = 1$ and $\sum np_n = \infty$. Define

$$G(x) = \sum_{n=1}^{\infty} p_n U(x - n). \tag{5.1}$$

Suppose the process is initially in state 0 and remains there for a wait with distribution function

$$\int_0^x G(x - z)\, dR(z). \tag{5.2}$$

At the end of this wait let the system move to state "n" with probability p_n ($n > 0$), and remain there for a wait with distribution function $G(x - n)$, followed by two independent waits with distribution function $R(x)$. After these three waits the system returns to state 0 and the cycle repeats itself, and so on. Between sojourns in state 0 the system spends periods outside state 0 with distribution function $\Lambda_\alpha(x)$, say, where

$$\Lambda_\alpha(x) = \int_0^x \Lambda_\beta(x - z)\, dR_2(z), \tag{5.3}$$

$$R_2(x) = \int_0^x R(x - z)\, dR(z), \tag{5.4}$$

$$\Lambda_\beta(x) = \sum_{n=1}^{\infty} p_n G(x - n) = \int_0^x G(x - z)\, dG(z). \tag{5.5}$$

From all this we can show

$$P_0^*(s) = [1 - R^*(s)G^*(s)] + \frac{[R^*(s)G^*(s)]}{1 + [R^*(s)G^*(s)] + [R^*(s)G^*(s)]^2}. \tag{5.6}$$

If $1 + \delta + \delta^2 = 0$, then $|\delta| = 1$, and so the presence of $R^*(s)$ ensures that $\inf |1 + [R^*(s)G^*(s)] + [R^*(s)G^*(s)]^2| > 0$ in the region $Rs \geq 0$. Thus, by the same Mercerian argument as before we can deduce that $P_0(t)$ is of bounded variation and $P_0(t) \to \frac{1}{3}$ as $t \to \infty$.

On the other hand, one can show by slightly more involved, but essentially similar, calculations that $P_n(t) \to \frac{1}{3}p_n$ as $t \to \infty$. Thus,

$$\sum_{n=0}^{\infty} P_n(\infty) = \tfrac{1}{3} + \tfrac{1}{3} \sum_{n=1}^{\infty} p_n = \tfrac{2}{3}. \tag{5.7}$$

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