ON QUASI-COMPACT PSEUDO-RESOLVENTS

J. G. BASTERFIELD
EMMANUEL COLLEGE

1. Introduction

Let $\{R_\lambda: \lambda > 0\}$ be a family of continuous linear operators on a Banach space such that

$$(1)\qquad \left.\begin{array}{l} \|\lambda R_\lambda\| \leq 1 \\ R_\lambda - R_\mu + (\lambda - \mu)R_\lambda R_\mu = 0 \end{array}\right\} \qquad \text{for} \quad \lambda, \mu > 0.$$

Suppose that, for some $\alpha > 0$, αR_α is quasi-compact; that is, there exist a positive integer m, a linear D with $\|D\| < 1$, and a compact K such that $(\alpha R_\alpha)^m = K + D$. Then we shall show that λR_λ is quasi-compact for *all* $\lambda > 0$. (In the special case where $\{R_\lambda\}$ is the resolvent family of a strongly continuous Markov semigroup on ℓ_1, the result is proved in the accompanying paper by David Williams [1].)

2. A lemma on pseudo-resolvents

LEMMA. *Suppose* $\{S_\lambda: \lambda > 0\}$ *is a family of elements of a Banach algebra with identity* (*of norm* 1) *such that*

$$(2)\qquad \left.\begin{array}{l} \|\lambda S_\lambda\| \leq 1 \\ S_\lambda - S_\mu + (\lambda - \mu)S_\lambda S_\mu = 0 \end{array}\right\} \qquad \text{for} \quad \lambda, \mu > 0.$$

Then for all $\lambda, \mu > 0$,

(*i*) *the points* z_λ *in* $\sigma(S_\lambda)$ (*the spectrum of* S_λ) *are precisely those points of the form*

$$(3)\qquad z_\lambda = \frac{z_\mu}{1 - (\mu - \lambda)z_\mu} \quad \text{where} \quad z_\mu \in \sigma(S_\mu),$$

and (ii) $\sigma(\mu S_\mu) \subseteq \{z: |z - \tfrac{1}{2}| \leq \tfrac{1}{2}\}$.

PROOF. By repeated substitution of $S_\lambda = (1 + (\mu - \lambda)S_\lambda)S_\mu$ into itself, we get

$$(4)\qquad S_\lambda = \sum_{n=0}^{N} (\mu - \lambda)^n S_\mu^{n+1} + (\mu - \lambda)^{N+1} S_\lambda S_\mu^{N+1}.$$

Therefore, provided that $(\mu - \lambda)^N \|S_\mu^N\| \to 0$ as $N \to \infty$,

$$(5)\qquad S_\lambda = \sum_{n=0}^{\infty} (\mu - \lambda)^n S_\mu^{n+1}.$$

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In particular, since $r(S_\mu)$ (the spectral radius of S_μ) does not exceed $1/\mu$, the result holds for $|\lambda - \mu| < \mu/2$.

Now consider $f(z) = \sum_{n=0}^{\infty} (\mu - \lambda)^n z^{n+1}$. For any λ satisfying $|\lambda - \mu| < \mu/2$, f is holomorphic in $\{z\colon |z| < 2/\mu\}$ which is an open set containing the spectrum of S_μ. Hence, provided $|\lambda - \mu| < \mu/2$, the spectral mapping theorem implies that the points z_λ in $\sigma(S_\lambda)$ are precisely those points of the form

$$z_\lambda = \frac{z_\mu}{1 - (\mu - \lambda)z_\mu} \qquad \text{for some} \quad z_\mu \in \sigma(S_\mu). \tag{6}$$

However, substituting $z_\mu = z_\nu/(1 - (\nu - \mu)z_\nu)$ in $z_\lambda = z_\mu/(1 - (\mu - \lambda)z_\mu)$ gives $z_\lambda = z_\nu/(1 - (\nu - \lambda)z_\nu)$. Hence, for any fixed λ, $\mu > 0$, the proof of (i) may be completed in a finite number of steps.

If the point $0 \in \sigma(\mu S_\mu)$, then it satisfies (ii). Suppose μz_μ is a nonzero point in $\sigma(\mu S_\mu)$. Put $\mu z_\mu = ae^{i\theta}$, and let $\lambda = \mu + \delta$ $(-\mu < \delta < \infty)$. Then

$$z_\lambda = e^{i\theta}/(\mu a^{-1} + \delta e^{i\theta}) \tag{7}$$

must be in $\sigma(S_\lambda)$, and so

$$|z_\lambda| \leq 1/\lambda = (\mu + \delta)^{-1}. \tag{8}$$

Hence,

$$(\mu + \delta)^2 \leq |\mu a^{-1} + \delta e^{i\theta}|^2 = \frac{\mu^2}{a^2} + \frac{2\mu\delta}{a} \cos\theta + \delta^2. \tag{9}$$

Since this is true for arbitrarily large δ, we must have $\cos\theta/a \geq 1$, or $a \leq \cos\theta$, and so (ii) again holds.

3. Proof of the main result

The set of compact operators forms a closed two-sided ideal in the algebra of operators. Consider the quotient algebra. Its elements are equivalence classes (under the equivalence relation $A \sim B$ if and only if $A - B$ is compact), and the norm of an equivalence class is the infimum of the norms (in the original algebra) of its members. Provided that the identity of the original algebra is not a compact operator (and if it is, then each element of the algebra has this property, and the λR_λ's are trivially quasi-compact), the quotient algebra has the equivalence class containing the identity of the original algebra as an identity of norm 1 (the norm of an identity cannot be strictly less than 1, and the norm of the identity of the original algebra is equal to 1).

Let S_λ denote the equivalence class containing R_λ. Then $\{S_\lambda\colon \lambda > 0\}$ satisfies the hypotheses of the lemma. Moreover, λR_λ is quasi-compact in the original algebra if and only if there is a positive integer n such that $\|(\lambda S_\lambda)^n\| < 1$, and hence, if and only if $r(\lambda S_\lambda) < 1$. Thus, from part (ii) of the lemma and the fact that $\sigma(\lambda S_\lambda)$ is closed, it follows that *λR_λ is quasi-compact if and only if $1 \notin \sigma(\lambda S_\lambda)$.*

Suppose that λR_λ is not quasi-compact. Then $1/\lambda \in \sigma(S_\lambda)$, and hence,

$$\frac{\lambda^{-1}}{1 - (\lambda - \alpha)\lambda^{-1}} \in \sigma(S_\alpha); \tag{10}$$

that is, $1/\alpha \in \sigma(S_\alpha)$, which contradicts the assumption that αR_α is quasi-compact. This completes the proof of the result claimed in section 1.

I should like to thank Professor D. G. Kendall for bringing this problem to my notice, and Dr. David Williams who proved the result in a special case and conjectured that it was true in general.

REFERENCE

[1] D. WILLIAMS, "Uniform ergodicity in Markov chains," *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1966, Vol. II, Part II, pp. 187–191.