SOME THEOREMS CONCERNING RESOLVENTS OVER LOCALLY COMPACT SPACES

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1. Introduction and preliminaries

This article is concerned with two problems on resolvents defined over locally compact separable Hausdorff spaces; the generation of a strong Markov process from the given substochastic resolvent (sections 3 and 4) and the representation of excessive measures by means of minimal (or extreme) excessive measures (sections 5 to 11). These problems are closely connected, and our approach to them is based on the results of D. Ray [19] concerning resolvents over compact metric spaces (see section 2) and the metric completion of the original locally compact space with respect to the uniformity generated by a certain family of bounded continuous functions. Two types of the metric completion will be introduced; the completion of F. Knight [14] (in a specific way) in section 3 and the completion of R. S. Martin in section 6.

In the rest of this section we will give some basic definitions as well as a brief description of the first problem. Let $(E, \mathcal{B})$ be a measurable space and $\mathcal{A}$, the σ-field formed by all universally measurable sets, tnat is, sets which, for each finite measure μ over $\mathcal{B}$, differ by at most a set of (μ) measure zero from a set of $\mathcal{B}$. A (nonnegative) real-valued function $R_\alpha(x, A)$ defined for $\alpha > 0$, $x \in E$ and $A \in \mathcal{B}$ is said to be a *resolvent* if the following conditions are satisfied. (R_1) For each $\alpha > 0$ and $x \in E$, $R_\alpha(x, \cdot)$ is a measure over $(E, \mathcal{B})$. (R_2) For each $\alpha > 0$ and $A \in \mathcal{B}$, $R_\alpha(\cdot, A)$ is measurable $(\mathcal{A})$. (R_3) The resolvent equation

$$(1.1)\qquad R_\alpha(x, A) - R_\beta(x, A) + (\alpha - \beta) \int R_\alpha(x, dy) R_\beta(y, A) = 0, \qquad \alpha, \beta > 0$$

is satisfied. The unspecified integral always means the integral over the whole space E. (R_4) The substochastic condition

$$(1.2)\qquad \alpha R_\alpha(x, E) \leq 1 \qquad \text{for every} \quad \alpha > 0 \quad \text{and} \quad x \in E$$

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is satisfied. In particular, if the equality holds in (1.2) for every $\alpha > 0$ and $x \in E$, the resolvent is said to be *stochastic*. Later (section 5), when E is locally compact and $\mathcal{B}$ is the σ-field of all Borel sets, the notion of a generalized resolvent is introduced by replacing the substochastic condition (R_4) by a more general condition. For this reason the above defined "proper" resolvent is sometimes called a *substochastic* resolvent. The measure $R_\alpha(x, \cdot)$ is extended naturally to the sets of $\mathcal{A}$. It is easy to show that, for each A of $\mathfrak{A}$, $R_\alpha(\cdot, A)$ is measurable $(\mathfrak{A})$.

A function $P_t(x, A)$ defined for $t > 0$, $x \in E$ and $A \in \mathcal{B}$ is said to be a *(measurable) transition function* if the following conditions are satisfied. (P_1) For each fixed t and x, $P_t(x, \cdot)$ is a measure over $(E, \mathcal{B})$ such that $P_t(x, E) \leq 1$. (P_2) $P.(\cdot, A)$ is jointly measurable for each A. (P_3) The Chapman-Kolmogorov equation

$$P_{t+s}(x, A) = \int P_t(x, dy)P_s(y, A), \qquad t, s > 0 \tag{1.3}$$

is satisfied. The Laplace transform of a transition function

$$R_\alpha(x, A) = \int_0^\infty e^{-\alpha t}P_t(x, A)\, dt, \qquad \alpha > 0 \tag{1.4}$$

is always a resolvent. In this case $R_\alpha(x, A)$ is stochastic if and only if $P_t(x, A)$ is so, that is, $P_t(x, E) = 1$ for every t and x.

Let $(E, \mathcal{B})$ be a measurable space for which each one-point set is measurable. By a stochastic process (x_t, ζ, P) over E we mean the following. The variable $\zeta(\omega)$ is called the terminal time of the process. It is a nonnegative random variable (allowing the value infinity) defined over a probability space $(\Omega, \mathfrak{F}, P)$. The variable $x_t(\omega)$ is defined for all $0 < t < \zeta(\omega)$ and takes values in E for such t. (If $\zeta(\omega) = 0$, $x_t(\omega)$ is just an empty path.) Every set of the form $\{\omega; x_t(\omega) \in A\}$, $A \in \mathcal{B}$, is in $\mathfrak{F}$.

Consider a family of σ-fields $\{\mathfrak{F}_t, t > 0\}$ such that $\mathfrak{F}_t(\subset \mathfrak{F})$ is a σ-field over $\Omega_t = \{\zeta(\omega) > t\}$ including all the sets of the form $\{x_t \in A\}$, $A \in \mathcal{B}$, and such that $\mathfrak{F}_t$ is relatively increasing with respect to t, that is, $\Omega_t \cap \mathfrak{F}_s \subset \mathfrak{F}_t$ whenever $s \leq t$. The process (x_t, ζ, P) is said to be a *Markov process with* $P_t(x, A)$ *as its transition function*, if there is a family $\{\mathfrak{F}_t, t > 0\}$ (as above) such that, for each $s, t > 0$ and $A \in \mathcal{B}$,

$$P\{x_{s+t} \in A | \mathfrak{F}_s\} = P_t(x_s, A) \tag{1.5}$$

a.e. (= almost everywhere) on the set $\Omega_s = \{x_s \in E\}$. Usually the Markov process is determined by its transition function and its initial distribution. In the present definition, however, the value of x_t at $t = 0$ is not defined, so that one can say nothing about its initial distribution.

If there is a substochastic measure μ over E such that

$$P\{x_t \in A\} = \int \mu(dx)P_t(x, A), \qquad t > 0, \qquad A \in \mathcal{B}, \tag{1.6}$$

such μ may be considered as the initial distribution of the process. But such μ does not exist in general. As a simple example, consider a uniform motion defined over $(0, \infty)$, starting at the origin and proceeding to the right. Moreover, even if the measure μ as above does exist, it may fail to be an "appropriate" initial distribution. Such circumstances will be made clear in the following two sections in a certain general setting.

We will say that a system of substochastic measures $\{Q_t(A), t > 0\}$ is a *system of absolute laws of the transition function* $P_t(x, A)$, if the equation

$$Q_{t+s}(A) = \int Q_t(dx)P_s(x, A), \qquad s, t > 0, \qquad A \in \mathcal{B} \tag{1.7}$$

is satisfied. If the process (x_t, ζ, P) is a Markov process with $P_t(x, A)$ as its transition function, the system of absolute distributions

$$Q_t(A) = P\{x_t \in A\} \tag{1.8}$$

satisfies (1.7).

The strong Markov property is, roughly speaking, that the equation (1.5) is still valid when s is a particular kind of random time called a "stopping time." A natural question arises: when does there exist a strong Markov standard modification of the given Markov process? This problem has been studied by many authors in various settings [1], [2], [8], [19], [23]. We will mention some of them. Ray [19] gave the most general solution when E is compact. Chung ([2] and earlier papers) and Yushkevich [23] solved the problem for Markov chains (namely, when E is a denumerable space). Yushkevich was the first who explicitly indicated that for a reasonable wide definition of strong Markov processes one must allow those processes whose sample paths may take values outside of the original state space (or equivalently, may disappear) for t of a random subset of $(0, \zeta)$. This idea was developed by Ray [19] who stated that the theorem of Chung and Yushkevich for Markov chains can be reduced to the theorem of Ray for the compact case by using a completion of the original denumerable space.

Actually Ray's proof was insufficient. The authors [unpublished] were able to revise the proof by generalizing Ray's results for compact spaces. That proof is applicable to the case of arbitrary locally compact spaces and theorem 1 of the present paper was obtained. Later the paper of F. Knight [14] came out and we found that, as will be given in section 3, theorem 1 can be proved with a slight modification of Knight's completion within the framework of Ray's original results for compact spaces. Recently, Ray informed us that he had found out the slip of his proof and succeeded in revising it by another method.

Knight [12], [13], [14] has studied in the most general setting the same kind of problem, that is, what he calls "the regularization of Markov processes." His final result [14] tells us that if x_t is a Markov process over a measurable space $(E, \mathcal{B})$ with $\mathcal{B}$ a σ-field generated by countably many sets, and if the transition function of x_t separates points in E, there is a strong Markov process $\bar{x}_t$ over certain metric completion $\bar{E}$ of E such that $\bar{x}_t = x_t$, a.e. (P) at each t, except for at most countably many t. Moreover, $\bar{x}_t$ has some other nice properties, like the right continuity of sample paths. Unfortunately, E may not be measurable in the extended space $\bar{E}$. Knight (private communication) informed us of such an example. Therefore, what is really done by constructing $\bar{x}_t$ is not necessarily evident. But if E is a locally compact separable Hausdorff space and if the transition function (or rather its associated resolvent) satisfies certain conditions, E is measurable in $\bar{E}$ and $\bar{x}_t = x_t$ a.e. (P) for each t (with no exceptional set). From this it follows that the process $\bar{x}_t$, when it is only observed over E, is strongly Markov over E with the original transition function (theorem 1).

Finally let us state the strict definition of a strong Markov process. Let $(\hat{E}, \hat{\mathfrak{B}})$ be a measurable space with respect to which each single point is measurable, E a measurable subset of $\hat{E}$, and $\mathfrak{B}$ the induced σ-field $E \cap \hat{\mathfrak{B}}$. Let (x_t, ζ, P) be a stochastic process over $\hat{E}$ and $\{\mathfrak{F}_t; t > 0\}$, a family of σ-fields related to x_t in the previous sense. We assume that the process is *measurable*, namely, that every (t, ω) set of the form $\{(t, \omega); t < s, x_t(\omega) \in \hat{A}\}$, $\hat{A} \in \hat{\mathfrak{B}}$, is in the product σ-field of $\mathfrak{F}_s$ with the σ-field of linear Borel sets over $(0, s]$ for every $s > 0$. (This kind of measurability, which is stronger than the usual one [4], is called *progressively Borel measurable* in [3].) A nonnegative random variable τ is said to be a *stopping time* (relative to $\{\mathfrak{F}_t\}$) if $\{\tau < t < \zeta\} \in \mathfrak{F}_t$ up to a set of (P) measure zero for each $t > 0$. The σ-field $\mathfrak{F}_{\tau+}$ is defined by the collection of subsets Λ of $\Omega_\tau = \{0 < \tau < \zeta\}$ such that the set $\Lambda \cap \{\tau < t < \zeta\}$ is in $\mathfrak{F}_t$ up to a set of (P) measure zero for each $t > 0$. Let $P_t(x, A)$ be a transition function defined over E. The process x_t is said to be *strongly Markov over E with $P_t(x, A)$ as its transition function* if, for each stopping time τ, $t > 0$ and $A \in \mathfrak{B}$,

$$P\{x_{\tau+t} \in A | \mathfrak{F}_{\tau+}\} = P_t(x_\tau, A) \tag{1.9}$$

almost everywhere over the set $\{x_\tau \in E\}$. This definition is equivalent to Yushkevich's of the almost strong Markov property [23].

The notions of measurability, stopping time, and strong Markov property depend on the family $\{\mathfrak{F}_t\}$. Indeed $\{\mathfrak{F}_t\}$ must be properly chosen in each case. However, this is a routine matter, at least in those cases we will encounter in later discussions. From now on we will use terminologies such as "strongly Markov," and so on, without indicating the family $\{\mathfrak{F}_t\}$.

2. Ray's results concerning resolvents over compact spaces

In this section we will summarize, and give some comments on, Ray's results [19] concerning resolvents over compact spaces, their associated transition functions, and strong Markov processes (confining ourselves to the separable case).

Let E be a compact separable Hausdorff space and $\mathfrak{B}$ the σ-field of all Borel subsets of E. Let $R_\alpha(x, A)$ be a resolvent defined over E. For a function f over E we will write $R_\alpha f(x)$ for $\int f(y) R_\alpha(x, dy)$ if the integral is well defined. Denote by $\mathbf{C}$ (resp. $\mathbf{C}^+$) the collection of all real (resp. nonnegative) bounded continuous functions over E. Ray introduced the following hypothesis.

HYPOTHESIS (A). (1) *For every $\alpha > 0$, $R_\alpha(x, A)$ maps $\mathbf{C}$ into itself as an integral operator. In other words, $R_\alpha f \in \mathbf{C}$ whenever $f \in \mathbf{C}$.* (2) *There is a countable subcollection $\mathbf{C}_1$ of $\mathbf{C}^+$, separating points in E and satisfying*

$$\alpha R_{\alpha+1} f \leq f, \qquad \alpha > 0, \quad f \in \mathbf{C}_1. \tag{2.1}$$

This hypothesis is assumed throughout this section. We also assume that $\mathbf{C}_1$ includes a positive constant function, say $\mathbf{1}$ (= the function equal to 1 everywhere), with no loss of generality. In principle it is enough to consider the stochastic case, for the substochastic case can be reduced to a stochastic case without destroying hypothesis (A) by adjoining an absorption point to E as an isolated point. But to avoid irrelevant complication in later discussions, it will be useful to formulate the results for substochastic resolvents.

The first conclusion is this.

(a) *For each x of E, the measures $\alpha R_\alpha(x, \cdot)$ converge weakly to a substochastic measure $\mu(x, \cdot)$ when $\alpha \to \infty$. If $f \in \mathbf{C}$ and if $g = \int f(y)\mu(\cdot, dy)$, then*

$$\int |f(y) - g(y)|\mu(x, dy) = 0 \qquad \text{for every} \quad x. \tag{2.2}$$

For every bounded measurable function f,

$$R_\alpha f(x) = \int R_\alpha f(y)\mu(x, dy), \qquad \alpha > 0, \quad x \in E. \tag{2.3}$$

If $\mu(x, \cdot)$ is not the unit distribution at x, the point x is called a *branching point.*

(b) *The following three conditions are equivalent to each other.* (i) *The point x is a branching point.* (ii) *There is a substochastic measure μ, either $\mu(E) = 0$ or $\mu(\{x\}) < \mu(E)$, such that*

$$f(x) \geq \int f(y)\mu(dy), \qquad f \in \mathbf{C}_1, \tag{2.4}$$

and such that

$$R_1 f(x) = \int R_1 f(y)\mu(dy), \qquad f \in \mathbf{C}. \tag{2.5}$$

(iii) *There is a function g of the form*

$$g = f \wedge c = \min(f(\cdot), c) \tag{2.6}$$

with $f \in \mathbf{C}_1$ and a rational number c such that

$$g(x) > \int g(y)\mu(x, dy). \tag{2.7}$$

Since the proof in [19] that (ii) implies (iii) looks somewhat insufficient (even in the stochastic case), we will here supplement it. Suppose μ satisfies the condition (ii). If $\mu(E) = 0$, (iii) is evident because of $\mathbf{1} \in \mathbf{C}_1$. Assume $\mu(E) > 0$. It is enough to show that there is a function g of the form $g(\cdot) = \min\{f(\cdot), f(x)\}$, $f \in \mathbf{C}_1$, satisfying (2.7). From the assumption there is a point $y_0(\neq x)$ whose arbitrary neighborhood has positive (μ) measure. Let f be a function of $\mathbf{C}_1$ separating the points x and y_0. The case of $f(x) > f(y_0)$ is proved in [19]. Now assume $f(x) < f(y_0)$. Then $\int [f(y) - g(y)]\mu(dy) = \delta > 0$. Since $\alpha R_{\alpha+1} g \leq g$, and since (2.5) implies that

$$R_\alpha f(x) = \int R_\alpha f(y)\mu(dy), \qquad \alpha > 0, \quad f \in \mathbf{C}, \tag{2.8}$$

one has

$$\alpha R_{\alpha+1} g(x) = \int \alpha R_{\alpha+1} g(y)\mu(dy) \leq \int g(y)\mu(dy) \leq f(x) - \delta, \tag{2.9}$$

so that

$$\int g(y)\mu(dy) = \lim_{\alpha\to\infty} \alpha R_{\alpha+1}g(x) < f(x) = g(x). \tag{2.10}$$

If $R_\alpha(x, A)$ is stochastic and if μ is assumed to be a probability measure, (2.5) follows from the following weaker condition,

$$R_1 f(x) \geq \int R_1 f(y)\mu(dy), \qquad f \in \mathbf{C}^+. \tag{2.11}$$

Then the statement (ii) is nothing but the definition of a branching point in [19]. The counterpart of the above condition for substochastic resolvents is this: for any f of $\mathbf{C}$ and any constant $c > 0$,

$$R_1 f(x) - cR_1(x, E) \geq \int R_1 f(y)\mu(dy) - c\int R_1(y, E)\mu(dy). \tag{2.12}$$

Let E_b denote the set of all branching points.

(c) *The set E_b is a K_σ-set ($=$ countable union of compact sets) and*

$$\mu(x, E_b) = 0 \qquad \textit{for every} \quad x. \tag{2.13}$$

The first conclusion of the above was not stated explicitly in [19]. The proof follows from (iii) of the proposition (b). Take a function g of the form (2.6). Then it is easily shown from $\alpha R_{\alpha+1}g \leq g$ that $\alpha R_{\alpha+1}g$ increases with α. Since $\alpha R_{\alpha+1}g$ is continuous, the function

$$\int g(y)\mu(\cdot, dy) = \lim_{\alpha\to\infty} \alpha R_{\alpha+1}g \tag{2.14}$$

is lower semicontinuous. Therefore, the set

$$E_g = \{x; g(x) > \int g(y)\mu(x, dy)\} \tag{2.15}$$

is a K_σ-set. Since E_b is the union of E_g over all such g, it is also a K_σ-set.

The next group of conclusions are concerned with the transition function induced by $R_\alpha(x, A)$.

(d) *There is a unique transition function $P_t(x, A)$ such that the function of t defined by*

$$P_t f(x) = \int f(y)P_t(x, dy), \qquad x \in E, \qquad f \in \mathbf{C} \tag{2.16}$$

is right continuous and such that

$$R_\alpha(x, A) = \int_0^\infty e^{-\alpha t}P_t(x, A)\, dt, \qquad \alpha > 0. \tag{2.17}$$

(e) *If $t \to 0$, then $P_t(x, \cdot)$ converges weakly to $\mu(x, \cdot)$.*

(f) *For every $t > 0$ and for $x \in E$, one has $P_t(x, E_b) = 0$.*

Let $Q_t(A)$ be a system of absolute laws of $P_t(x, A)$, and let $(\tilde{x}_t, \zeta, P)$ be their associated Markov process. It follows that there is a well-defined-limit

$$x_t(\omega) = \lim_{r\downarrow t} \tilde{x}_r(\omega) \qquad (r \text{ rationals}) \tag{2.18}$$

for every $0 < t < \zeta(\omega)$ almost everywhere (P).

Almost all sample paths of the new process x_t have the following properties: (i) the process $x_t(\omega)$, $0 < t < \zeta$, is right continuous; (ii) right and left limits exist at every $0 \leq t < \zeta$; and (iii) if $\zeta(\omega) < \infty$, there is a left-hand limit at $t = \zeta(\omega)$. We will write x_{0+} for $\lim_{t \downarrow 0} x_t$ and $x_{\zeta-}$ for $\lim_{t \uparrow \zeta} x_t$ if they are well defined.

(g) *The process x_t is a standard modification of $\tilde{x}_t$ and strongly Markov over E with $P_t(x, A)$ as its transition function.*

When f is a function over E, we will set $f(x_t) = 0$ if $t = 0$ or $t \geq \zeta$ conventionally.

Take a function f of $\mathbf{C}$. From the strong Markov property we have for $t > 0$ and $\Lambda \in \mathfrak{F}_{\tau+}$,

$$E\{f(x_{\tau+t}); \Lambda\} = E\{P_t f(x_\tau); \Lambda\}. \tag{2.19}$$

Letting $t \to 0$, it follows that

$$E\{f(x_\tau); \Lambda\} = E\left\{\int \mu(x_\tau, dy) f(y); \Lambda\right\}. \tag{2.20}$$

Since (2.20) is true for every f of $\mathbf{C}$, it is also true for every measurable function.

From (g) it follows that the distribution of x_{0+} has no mass on the set E_b and it is the unique solution μ of the equation

$$Q_t(A) = \int_{E - E_b} \mu(dx) P_t(x, A). \tag{2.21}$$

Moreover if ν is an initial measure of $Q_t(A)$, that is, if ν is a solution of the equation

$$Q_t(A) = \int_E \nu(dx) P_t(x, A), \tag{2.22}$$

the above μ is given by

$$\mu(A) = \int_E \nu(dx) \mu(x, A). \tag{2.23}$$

These facts may be expressed as follows.

(h) *For any system of absolute laws of $P_t(x, A)$, there is the unique "appropriate" initial measure.*

(i) *Let τ_n be a sequence of stopping times increasing to the limit τ, and let $x_{\tau-}$ be the left-hand limits of x_{τ_n} for almost all sample paths of the set $\{0 < \tau < \zeta\}$. Then $x_\tau = x_{\tau-}$ almost everywhere over the set $\{0 < \tau < \zeta, x_{\tau-} \in E - E_b\}$.*

This property, which is nothing but the property of Blumenthal [1] when there is no branching point, was not proved in [19].

It is enough to prove it when τ is finite a.e.(P). In the same way as $f(x_t)$, we will set $f(x_{t-}) = 0$ if x_{t-} is undefined, that is, if $t = 0$, $+\infty$ or $t > \zeta$. (Notice that x_{t-} exists when $t = \zeta < +\infty$.) Take a function f of $\mathbf{C}$. From (g), for $\Lambda \in \mathfrak{F}_{\tau_m+}$ and $n \geq m$,

$$\begin{aligned} E\{R_\alpha f(x_{\tau_n}); \Lambda\} &= E\left\{\int_0^\infty e^{-\alpha t} f(x_{\tau_n+t})\, dt; \Lambda\right\} \\ &= E\left\{\int_{\tau_n}^\infty e^{-\alpha(t-\tau_n)} f(x_t)\, dt; \Lambda\right\}. \end{aligned} \tag{2.24}$$

Since τ is finite a.e., when $n \to \infty$ one gets

$$(2.25)\qquad E\{R_\alpha f(x_{\tau-}); \Lambda\} = E\left\{\int_\tau^\infty e^{-\alpha(t-\tau)} f(x_t)\, dt; \Lambda\right\} = E\{R_\alpha f(x_\tau); \Lambda\}.$$

Multiplying α on both sides and letting $\alpha \to \infty$

$$(2.26)\qquad E\left\{\int \mu(x_{\tau-}, dy) f(y); \Lambda\right\} = E\left\{\int \mu(x_\tau, dy) f(y); \Lambda\right\}.$$

From (2.20) the right side equals $E\{f(x_\tau); \Lambda\}$. Therefore, for any set Λ of the σ-field generated by $\bigcup \mathfrak{F}_{\tau_n+}$, we have proved

$$(2.27)\qquad E\left\{\int \mu(x_{\tau-}, dy) f(y); \Lambda\right\} = E\{f(x_\tau); \Lambda\}.$$

Let Λ_0 denote the set $\{x_{\tau-} \in E - E_b\}$. Then

$$(2.28)\qquad E\{f(x_{\tau-}); \Lambda, \Lambda_0\} = E\left\{\int \mu(x_{\tau-}, dy) f(y); \Lambda, \Lambda_0\right\} = E\{f(x_\tau); \Lambda, \Lambda_0\},$$

from which (i) can be obtained by the same argument as in ([18], pp. 5-08 to 5-09).

(j) *Almost all sample paths $x_t(\omega)$, $0 < t < \zeta$, never reach the set E_b.*

Since the proof in [19] is very difficult we will present a more understandable proof which is based on the following general lemma.

LEMMA. *Let E be a metric space, and let $(x_t, \zeta, \mathfrak{F}_t, P)$ be a stochastic process over E, almost all sample paths of which are right continuous. If a subset A of E is open or closed, then the hitting time to the set A defined by*

$$(2.29)\qquad \tau_A = \begin{cases} \inf\{t, x_t \in A\} & \text{if } x_t \in A \text{ for some } 0 < t < \zeta \\ = \zeta & \text{otherwise} \end{cases}$$

is a stopping time relative to $\{\mathfrak{F}_t\}$.

First assuming the lemma we prove (j). Since E_b is a K_σ-set, there is a countable family of compact sets K_n such that $E_b = \bigcup K_n$. Let τ be the hitting time for E_b, and τ_n the hitting time for K_n. Evidently,

$$(2.30)\qquad \{x_t \in E_b \text{ for some } 0 < t < \zeta\} = \{\tau < \zeta\} = \bigcup \{\tau_n < \zeta\}.$$

By the lemma, τ_n is a stopping time, and it is enough to show that the set $\{\tau_n < \zeta\}$ has (P) measure zero. Since $x_{0+} \in E - E_b$ almost everywhere (P), the sets $\{\tau_n < \zeta\}$ and $\{0 < \tau_n < \zeta\}$ have the same probability. By (2.20) and (e),

$$(2.31)\qquad P\{0 < \tau_n < \infty\} = P\{x_{\tau_n} \in E_b, 0 < \tau_n < \zeta\} = E\{\mu(x_{\tau_n}, E_b); 0 < \tau_n < \zeta\} = 0,$$

so that (j) was proved.

To prove the lemma we will introduce a notation. Let $F(\omega)$ be a real-valued function of sample paths, say, $G(x_t(\omega); 0 < t < \zeta)$. (Actually the range of $F(\omega)$ may be an arbitrary space.) Then the value of the function for the path shifted by $s > 0$ is denoted by $F(\theta_s\omega)$; that is, (2.32) $$F(\theta_s\omega) = G(x_{t+s}(\omega); 0 < t < t + s < \zeta).$$

(For a more sophisticated definition, see [6].)

It is well known (and easily proved) that τ_A is a stopping time if A is open. Let A be a closed subset of E, and A_n a sequence of open sets decreasing to A, and τ_n the hitting time for the set A_n. Since τ_n is increasing, the limit τ^1 of τ_n is a stopping time as well as τ_n. For every countable ordinal a, we define a stopping time $\tau^a(\leq \tau_A)$ by transfinite induction as follows: if a is an isolated ordinal, then $\tau^a(\omega) = \tau^{a-1}(\omega) + \tau^1(\theta_{\tau^{a-1}}\omega)$, and if a is a limit ordinal, $\tau^a(\omega) = \sup_{b<a} \tau^b(\omega)$. Since A is closed, one can see that τ^a is strictly increasing until it becomes equal to τ_A. From this observation one can conclude that there is some countable ordinal a (independent of ω) such that $\tau^a = \tau^{a+1}$, a.e. (P), and that for such a, $\tau^a = \tau_A$ a.e. (P). This completes the proof of the lemma.

(k) *For any Borel (or, more generally, analytic) set A, τ_A is a stopping time of the process x_t.*

If A is a subset of $E - E_b$, τ_A is a stopping time by (i): the same argument as in [9] holds. If A is not contained in $E - E_b$, $\tau_A = \tau_{A\cap(E-E_b)}$ almost everywhere (P) because of (j).

REMARK. The most general result on the measurability of hitting times, including the above lemma and proposition (k), is found in P. A. Meyer's book ([24], p. 71).

3. The completion of F. Knight

Let E be a locally compact separable Hausdorff space, $\mathcal{B}$ the σ-field of all Borel subsets of E, and $R_\alpha(x, A)$ a resolvent over E. (Actually all the results of this section hold when E is a σ-compact Hausdorff space.) As in section 2, the collection of all real (positive) bounded continuous functions is denoted by $\mathbf{C}$ $(\mathbf{C}^+)$. Throughout this section we assume that $R_\alpha(x, A)$ satisfies hypothesis (A) of section 2. We will derive a natural extension of $R_\alpha(x, A)$ to certain enlarged compact space and then apply it to the problem of finding a strong Markov standard modification of the given Markov process over E.

First of all, we will state certain general facts on a completion of E based on a family of continuous functions. Let $\mathbf{D}$ be a subfamily of $\mathbf{C}$ which separates points in E and contains a countable dense subfamily, for the uniform norm. Consider the uniformity $\mathcal{U}$ [11] generated by the family of pseudo-metrics $|f(x) - f(y)|$, $f \in \mathbf{D}$ and let $\bar{E}$ be the completion of E with respect to $\mathcal{U}$. It follows from the condition on $\mathbf{D}$ that this is a metric completion. Let $\bar{\mathfrak{I}}$ be the uniform topology of $\bar{E}$ and $\mathfrak{I}$, the topology of the original space E. Since $\mathbf{D}$ is a subfamily of $\mathbf{C}$, the identity mapping $(E, \mathfrak{I}) \to (\bar{E}, \bar{\mathfrak{I}})$ is continuous. From these observations it is easy to prove the following lemma.

LEMMA 3.1. (i) The space $\bar{E}$ *is compact metric.* (ii) *If A is a $\mathfrak{I}$-compact subset of E, it is $\bar{\mathfrak{I}}$-compact.* (iii) *If a function $\bar{f}$ defined over $\bar{E}$ is $\bar{\mathfrak{I}}$-continuous, the restriction f of $\bar{f}$ to E is $\bar{\mathfrak{I}}$-continuous.* (iv) *If $A (\subset E)$ is a $\mathfrak{I}$-Borel set, it is $\bar{\mathfrak{I}}$-Borel.* (v) *If $\bar{A}(\subset \bar{E})$ is $\mathfrak{I}$-Borel, then $A = \bar{A} \cap E$ is $\mathfrak{I}$-Borel.*

Let **B** be the closed (in uniform norm) algebra of functions generated by **D** and the constant function **1**. Clearly, each function f of **B** can be extended to a continuous function $\bar{f}$ over $\bar{E}$. Conversely, if $\bar{f}$ over $\bar{E}$ is $\mathfrak{J}$-continuous, its restriction f to E is a function of **B**. This follows from the Stone-Weierstrass theorem.

Turning to the resolvent we now state the fundamental lemma.

LEMMA 3.2 (Knight [14]). *There is a closed subalgebra* **A** *of* **C** *which is invariant under the operator* R_α *and includes* $\mathbf{C}_1$, *and whose range* $R(\mathbf{A})$ *by* R_α *(independent of* α*) includes a countable subcollection dense in uniform norm.*

One such family **A** can be constructed by the same procedure as in ([14], lemma 1), starting with $\mathbf{C}_1$ for S_0 there. In general there are many possibilities of **A**. From now on we will fix one of them arbitrarily.

Take the family $R(\mathbf{A}) \cup \mathbf{C}_1$ for **D** at the paragraph preceding lemma 3.1 and consider the corresponding completion $\bar{E}$ and closed algebra **B**. Since $\mathbf{A} \supset \mathbf{B}$, and since $R_\alpha; \mathbf{A} \to \mathbf{B}$, $R_\alpha; \mathbf{B} \to \mathbf{B}$. Therefore, if $\bar{f}$ is a continuous function over $\bar{E}$, and if f is its restriction to E, $R_\alpha f$ is extended to a continuous function $\overline{R_\alpha f}$ over $\bar{E}$. Let $\bar{R}_\alpha(x, \bar{A})$, $x \in \bar{E}$, $\bar{A} \in \bar{\mathfrak{B}}$, be the measure over $\bar{E}$ determined by

$$\bar{R}_\alpha \bar{f}(x) = \overline{R_\alpha f}(x). \tag{3.1}$$

This $\bar{R}_\alpha(x, \bar{A})$, $x \in \bar{E}$, $\bar{A} \in \bar{\mathfrak{B}}$, defines a resolvent over $\bar{E}$ which maps $\bar{\mathbf{C}}$ (= the space of bounded continuous functions over $\bar{E}$) into itself. Also $\bar{R}_\alpha(x, \bar{A})$ is an extension of the original resolvent $R_\alpha(x, A)$ in the following sense: if $x \in E$, then $\bar{R}_\alpha(x, \bar{E} - E) = 0$ and $\bar{R}_\alpha(x, A) = R_\alpha(x, A)$ for every Borel set A of E. To see this, first note that, by definition

$$\bar{R}_\alpha \bar{f}(x) = R_\alpha f(x), \qquad x \in E, \qquad \bar{f} \in \bar{\mathbf{C}}. \tag{3.2}$$

Since the equation is closed under monotone convergence, it remains true when $\bar{f}$ is bounded and $\bar{\mathfrak{J}}$-Borel. But then, since f is also $\bar{\mathfrak{J}}$-Borel by lemma 3.1 (with $f = 0$ over $\bar{E} - E$ conventionally),

$$\bar{R}_\alpha \bar{f}(x) = R_\alpha f(x) = \bar{R}_\alpha f(x), \tag{3.3}$$

which implies our assertion.

Next consider a countable dense subfamily $\mathbf{A}_0^+$ of positive functions of **A** and define $\bar{\mathbf{C}}_1$ by the collection of extensions $\bar{f}$ of $f \in \mathbf{C}_1 \cup R_1(\mathbf{A}_0^+)$, where $R_1(\mathbf{A}_0^+)$ means the range of $\mathbf{A}_0^+$ by the operator R_1. Then, by definition of the completion, $\bar{\mathbf{C}}_1$ separates points in $\bar{E}$ and $\alpha \bar{R}_{\alpha+1} \bar{f} \leq \bar{f}$ over $\bar{E}$ for every function $\bar{f} \in \bar{\mathbf{C}}_1$. Hence, the extended resolvent $\bar{R}_\alpha(x, \bar{A})$, $x \in \bar{E}$, $\bar{A} \in \bar{\mathfrak{B}}$, satisfies hypothesis (A) over $\bar{E}$, so that it determines its associated transition function $\bar{P}_t(x, \bar{A})$ and strong Markov process $\bar{x}_t$.

When $x \in E$, $\bar{R}_\alpha(x, \bar{E} - E) = 0$ implies that $\bar{P}_t(x, \bar{E} - E) = 0$ almost all t (with respect to the Lebesgue measure over $(0, \infty)$). But in general one cannot assert that $\bar{P}_t(x, \bar{E} - E) = 0$ for every $t > 0$. In other words, the original resolvent $R_\alpha(x, A)$ cannot be necessarily expressed as the Laplace transform of certain transition function over E. Such an example is obtained easily by restricting the uniform motion (mod 1) over $[0, 1)$ to $(0, 1)$.

In particular, we will consider the case when there is a transition function $P_t(x, A)$ over E whose Laplace transform is $R_\alpha(x, A)$. Moreover, we assume that $P_t f(x)$ is right continuous in $t > 0$ for each x of E and for each f of $\mathbf{C}$ (or more generally, the range of $\mathbf{C}$ by R_α). Then if $x \in E$ and $\bar{f} \in \overline{\mathbf{C}}$,

$$(3.4) \qquad \int_0^\infty e^{-\alpha t}\overline{P}_t\bar{f}(x)\,dt = \overline{R}_\alpha\bar{f}(x) = R_\alpha f(x) = \int_0^\infty e^{-\alpha t}P_t f(x)\,dt.$$

Notice that f is in the range of $\mathbf{C}$ by R_α. Therefore, both $\overline{P}_t\bar{f}(x)$ and $P_t f(x)$ are right continuous in $t > 0$, so that $\overline{P}_t\bar{f}(x) = P_t f(x)$ for every $t > 0$. By an argument similar to R_α, $\overline{P}_t(x, \overline{A})$ is shown to be an extension of $P_t(x, A)$ in the sense that if $x \in E$, $\overline{P}_t(x, \overline{E} - E) = 0$ and $\overline{P}_t(x, A) = P_t(x, A)$ for $A \in \mathcal{B}$.

Consider a Markov process $(\tilde{x}_t, \zeta, P)$ over E with $P_t(x, A)$ as its transition function. By lemma 3.1 and the above result, $\tilde{x}_t$ may be considered a Markov process over $\overline{E}$ with $\overline{P}_t(x, \overline{A})$ as its transition function. Hence, by the proposition (g) of section 2, there is a strong Markov standard modification x_t (of $\tilde{x}_t$) over $\overline{E}$ with $\overline{P}_t(x, \overline{A})$ as its transition function. Again, from the fact that $\overline{P}_t$ coincides with P_t over E, x_t is strongly Markov over E with $P_t(x, A)$ as its transition function.

Thus we have proved the following theorem.

THEOREM 1. *Let $P_t(x, A)$ be a transition function over E a locally compact separable Hausdorff space, satisfying the following conditions:* (1) *the Laplace transform of $P_t(x, A)$,*

$$(3.5) \qquad R_\alpha(x, A) = \int_0^\infty e^{-\alpha t}P_t(x, A)\,dt$$

is a resolvent satisfying hypothesis (A) *in section* 2. (2) *For each x of E and each f of* $\mathbf{C}$, *$P_t f(x)$ is right continuous in $t > 0$. Let $Q_t(A)$ be a system of absolute laws of $P_t(x, A)$, and let $(\tilde{x}_t, \zeta, P)$ be the Markov process over E with $P_t(x, A)$ as its transition function and $Q_t(A) = P\{\tilde{x}_t \in A\}$. Then there is a strong Markov standard modification x_t (of $\tilde{x}_t$) with $P_t(x, A)$ as its transition function which takes values in a "natural" enlarged state space $\overline{E}$. For any analytic set A, the hitting time τ_A is a stopping time of the process x_t. Moreover, $P_t(x, A)$ can be extended naturally to a transition function $\overline{P}_t(x, \overline{A})$ over $\overline{E}$, and the system of absolute laws $Q_t(A)$ can be written in the form*

$$(3.6) \qquad Q_t(A) = \int_{\overline{E}} \mu(dx)\overline{P}_t(x, A),$$

using a unique "appropriate" initial measure μ over $\overline{E}$.

The last assertion is evident, for $Q_t(A)$ is also a system of absolute laws of $\overline{P}_t(x, \overline{A})$ (with $Q_t(\overline{E} - E) = 0$ conventionally).

Let Δ be the point at infinity of E. Define $y_t = x_t$ if $x_t \in E$ and $y_t = \Delta$ if $x_t \in \overline{E} - E$. Since this change has no effect on the behavior of x_t inside E, y_t is another strong Markov standard modification of $\tilde{x}_t$. One may ask if y_t is "separable" over $E + \{\Delta\}$ (= the minimal compactified state space). The separability is taken with respect to the family of compact sets of $E + \{\Delta\}$ in the same way as in the real case [4]. Following [2], "well separable" means that any dense subset in $(0, \infty)$ is a separability set, and "a standard transition function" means that $P_t(x, \cdot)$ converges weakly to the unit distribution at x as $t \to 0$ for each x of E. The answer to the above question is negative in general. Indeed, if A is compact in E the separability condition holds for any dense subset in $(0, \infty)$, because A is also compact in $\overline{E}$ and x_t is well-separable over $\overline{E}$. But if A is a compact set including Δ, it creates a problem.

We can give an example of a nonstandard transition function for which y_t is not separable over $E + \{\Delta\}$. However, it seems very difficult to find a standard transition function to which corresponds a nonseparable y_t. It is known [2] that if E is denumerable, and if $P_t(x, A)$ is standard (with respect to the discrete topology of E), there is always a well-separable, strong Markov standard modification of $\tilde{x}_t$. It remains open even in this simple case whether or not y_t is well-separable.

We now consider the problem of how to characterize the measure μ in the representation (3.6). More precisely, we want to find when and only when a measure μ over $\overline{E}$ is an initial distribution of the particular kind of system of absolute laws $\overline{Q}_t(\overline{A})$ of $\overline{P}_t(x, \overline{A})$ which has no mass over $\overline{E} - E$ for every $t > 0$. Following the notation of [19], we will denote by $\overline{E}_R$ the set of points x of $\overline{E}$ such that $\overline{R}_1(x, \overline{E} - E) = 0$. Evidently $\overline{E}_R \supset E$. By the resolvent equation it follows that $\overline{R}_\alpha(x, \overline{E} - E) = 0$ for every $\alpha > 0$. Suppose $\overline{Q}_t(\overline{E} - E) = 0$ for every $t > 0$ and

$$\overline{Q}_t(\overline{A}) = \int_{\overline{E}} \mu(dx)\overline{P}_t(x, \overline{A}). \tag{3.7}$$

Then

$$\int_0^\infty e^{-\alpha t}\overline{Q}_t(\overline{A})\, dt = \int_{\overline{E}} \mu(dx)\overline{R}_\alpha(x, \overline{A}), \tag{3.8}$$

so that

$$\int_{\overline{E}} \mu(dx)\overline{R}_\alpha(x, \overline{E} - E) = 0. \tag{3.9}$$

Hence, the total mass of μ must be concentrated on $\overline{E}_R$. Conversely, assume that $\overline{Q}_t(\overline{A})$ is represented in the form (3.7) by a measure μ over $\overline{E}_R$. From (3.8) it follows that $\overline{Q}_t(\overline{E} - E) = 0$ almost all t. For any $t > 0$, take s such that $\overline{Q}_{t-s}(\overline{E} - E) = 0$. Then

$$\begin{aligned} \overline{Q}_t(\overline{E} - E) &= \int_{\overline{E}} \overline{Q}_{t-s}(dy)\overline{P}_s(y, \overline{E} - E) \\ &= \int_{\overline{E}} \overline{Q}_{t-s}(dy)P_s(y, \overline{E} - E) = 0, \end{aligned} \tag{3.10}$$

so that $\overline{Q}_t(\overline{E} - E) = 0$ for every $t > 0$. Hence, we obtained the characterization of μ in (3.6). Incidentally, if $x \in \overline{E}_R$, $\overline{P}_t(x, \overline{E} - E) = 0$ for every $t > 0$.

Next we prove that almost all sample paths $x_t(\omega)$, $0 < t < \zeta$, of the process of theorem 1 never reach the set $\overline{E} - \overline{E}_R$. To the contrary of the conclusion, assume that $P\{\tau_{\overline{E}-\overline{E}_R} < \zeta\} > 0$. Then, by a theorem of Hunt [9], there is a compact set K such that $K \subset \overline{E} - \overline{E}_R$ and $P\{0 < \tau^\epsilon < \zeta\} > 0$ for some $\epsilon > 0$, where τ^ϵ is the first hitting time of the set K after ϵ. This leads to a contradiction as follows:

$$(3.11)\qquad 0 = E\left\{\int_0^\infty e^{-t} I_{\overline{E}-\overline{E}_R}(x_t)\,dt\right\} \geq E\left\{\int_{\tau^\epsilon}^\infty e^{-t} I_{\overline{E}-\overline{E}_R}(x_t)\,dt\right\}$$
$$= E\{e^{-\tau^\epsilon} R_1(x_{\tau^\epsilon}, \overline{E} - \overline{E}_R)\} > 0,$$

where $I_{\overline{E}-\overline{E}_R}$ is the indicator function of the set $\overline{E} - \overline{E}_R$.

Let us denote by $\overline{E}_b$ the set of all branching points in $\overline{E}$. The results of the preceding two paragraphs can be summarized in the following: *the set $\overline{E}_R \cap (\overline{E} - \overline{E}_b)$ is the "real" entrance boundary as well as the "essential" range of sample paths of the process x_t in theorem* 1.

4. An extension of theorem 1

Let $\mathbf{C}_0$ be the collection of real continuous functions with compact support, where a "function with compact support" means a function which vanishes outside of a compact subset of E. We will introduce a hypothesis on resolvents which is slightly weaker than hypothesis (A):

HYPOTHESIS (A_0). (1) *If $f \in \mathbf{C}_0$, then $R_\alpha f \in \mathbf{C}$.* (2) *The same condition as* (2) *of hypothesis* (A).

THEOREM 2. *The conclusions of theorem* 1 *are still valid when $P_t(x, A)$ is a transition function over E a locally compact separable Hausdorff space, satisfying the following conditions:* (1) *the Laplace transform $R_\alpha(x, A)$ of $P_t(x, A)$ is a resolvent satisfying hypothesis* (A_0); (2) *the same condition as* (2) *of theorem* 1; *and* (3) *for each x, $P_t(x, E) \to 1$ $(t \to 0)$, or equivalently, $\alpha R_\alpha(x, E) \to 1$ $(\alpha \to \infty)$.*

From the argument preceding theorem 1, it is enough to prove the following theorem.

THEOREM 2'. *Under the conditions of theorem* 2 *there is a space $\overline{E}$ with its subspace $\overline{E}_b$ satisfying the following conditions:* (i) *the space $\overline{E}$ is the completion of E based on a certain family* $\mathbf{D}$ *of section* 3; (ii) *there is an extension $\overline{P}_t(x, \overline{A})$ over $\overline{E}$ of $P_t(x, A)$*; (iii) *for each x of $\overline{E}$ and any continuous function $\bar{f}$ over $\overline{E}$, $\overline{P}_t\bar{f}(x)$ is right continuous in $t > 0$*; (iv) *$\overline{E}_b$ is a measurable subset of $\overline{E}$, and if $x \in \overline{E} - \overline{E}_b$, $P_t(x, \cdot)$ converges weakly to the unit distribution at x as $t \to 0$*; *and* (v) *for any Markov process $(\bar{x}_t, \zeta, P)$ over $\overline{E}$ with $\overline{P}_t(x, \overline{A})$ its transition function there is a standard modification x_t satisfying* (g) *to* (k) *in section* 2 *with $\overline{E}_b$ in place of E_b there.*

In the present case lemma 3.2, which theorem 1 is based on, does not hold in general, because it depends on the fact that $\mathbf{C}$ is invariant under the iteration of R_α. For the proof we will reduce our case to the case of hypothesis (A), using certain transformations of Markov processes [7], [15].

Take a number $\alpha_0 > 0$.

LEMMA 4.1. *There is a positive function ϕ such that $R_\alpha\phi$ is strictly positive everywhere and bounded continuous for every $\alpha \geq \alpha_0$.*

Consider a function of the form

$$(4.1)\qquad \phi = \sum_{n=1}^\infty f_n, \qquad f_n \in \mathbf{C}_0^+,$$

and assume ϕ is strictly positive everywhere. Then it follows from condition (3) that $R_\alpha\phi$ is strictly positive everywhere for every $\alpha > 0$. Choose f_n so that $R_{\alpha_0}f_n(x) < 2^{-n}$ for every x. Then $R_\alpha f_n \in \mathbf{C}$, and when $\alpha \geq \alpha_0$, $\sum_{n=1}^{\infty} R_\alpha f_n$ is convergent uniformly and belongs to $\mathbf{C}^+$.

Define

$$V_\alpha(x, A) = R_{\alpha_0+\alpha}(x, A), \qquad \alpha \geq 0, \tag{4.2}$$

$$V_\alpha^\phi(x, A) = [V_0\phi(x)]^{-1} \int_A V_\alpha(x, dy) V_0\phi(y), \qquad \alpha > 0, \tag{4.3}$$

where ϕ is the function obtained in lemma 4.1. These $V_\alpha(x, A)$ and $V_\alpha^\phi(x, A)$ are resolvents as is easily shown.

LEMMA 4.2. *The resolvent $V_\alpha^\phi(x, A)$ satisfies hypothesis* (A).

It is evident that $V_\alpha^\phi(x, A)$ maps $\mathbf{C}_0$ into $\mathbf{C}$, so that when f is in $\mathbf{C}^+$, $V_\alpha^\phi f$ is lower semicontinuous. We now calculate $V_\alpha^\phi\mathbf{1}$;

$$\begin{aligned} V_\alpha^\phi\mathbf{1}(x) &= [V_0\phi(x)]^{-1}V_\alpha V_0\phi(x) \\ &= [\alpha V_0\phi(x)]^{-1}\{V_0\phi(x) - V_\alpha\phi(x)\} \\ &= [\alpha R_{\alpha_0}\phi(x)]^{-1}\{R_{\alpha_0}\phi(x) - R_{\alpha_0+\alpha}\phi(x)\}, \end{aligned} \tag{4.4}$$

which proves $V_\alpha^\phi\mathbf{1} \in \mathbf{C}^+$. Take a function f of $\mathbf{C}^+$ which is dominated by $\mathbf{1}$. Then both $V_\alpha^\phi f$ and $V_\alpha^\phi(\mathbf{1} - f)$ are lower semicontinuous with their sum $V_\alpha^\phi\mathbf{1}$ being continuous. Therefore $V_\alpha^\phi f$ must be continuous.

Next if $f \in \mathbf{C}_1$, it satisfies

$$\alpha V_{\alpha+1}f \leq \alpha R_{\alpha_0+\alpha+1}f \leq (\alpha_0 + \alpha)R_{\alpha_0+\alpha+1}f \leq f, \tag{4.5}$$

so that the function $(f/V_0\phi)$, as well as the function $(f/V_0\phi) \wedge n$, for each positive integer n satisfies the equation $\alpha V_{\alpha+1}^\phi g \leq g$. The $\mathbf{C}_1^\phi$ denote the subcollection of $\mathbf{C}^+$ of the form $(f/V_0\phi) \wedge n, f \in \mathbf{C}_1, n = 1, 2, \cdots$ with the constant function $\mathbf{1}$ added. This collection separates points in E, because $\mathbf{C}_1$ includes $\mathbf{1}$ and separates points. Now the lemma is proved.

We apply the results of the preceding section to the resolvents $V_\alpha^\phi(x, A)$. Its associated system of the completion space, the extension of $V_\alpha^\phi(x, A)$, and the transition function is denoted by $\{\overline{E}, \overline{V}_\alpha^\phi(x, \overline{A}), \overline{T}_t^\phi(x, \overline{A})\}, x \in \overline{E}, \overline{A} \in \overline{\mathfrak{B}}$. The set of branching points is denoted by $\overline{E}_\delta^\phi$. Let $\overline{U}_t^\phi(\overline{A})$ be a system of absolute laws of $\overline{T}_t^\phi(x, \overline{A})$ and y_t^ϕ the process of (g) of section 2 determined by $\overline{T}_t^\phi(x, \overline{A})$ and $\overline{U}_t^\phi(\overline{A})$. The latter $\overline{U}_t^\phi(\overline{A})$ has the form $\int \nu^\phi(dx)\overline{T}_t^\phi(x, \overline{A})$ by means of a (unique) measure ν^ϕ over $\overline{E} - \overline{E}_\delta^\phi$, which is the distribution of y_{0+}^ϕ.

In the following we will use the results on excessive functions [9], [16] and their associated transformation [7], [15], [17]. For the moment, let $R_\alpha(x, A)$ be a resolvent over a measurable space $(E, \mathfrak{B})$. A nonnegative and $(\mathfrak{B})$ measurable function f, allowing the value $+\infty$, is said to be *excessive relative to* $R_\alpha(x, A)$ if $\alpha R_\alpha f \leq f$, $\alpha > 0$, and if $\lim_{\alpha\to\infty} \alpha R_\alpha f = f$. If the resolvent $R_\alpha(x, A)$ is the Laplace transform of a transition function $P_t(x, A)$, the above definition is equivalent to the condition that $P_t f \leq f$ and $\lim_{t\to 0} P_t f = f$. If only the condition $\alpha R_\alpha f \leq f$, $\alpha > 0$, is assumed, the function f is said to be *quasi-excessive.*

We now evaluate $V_\alpha^\phi(1/V_0\phi)(x)$;

$$(4.6)\qquad \begin{aligned}(\alpha_0+\alpha)\int V_\alpha^\phi(x,dy)[V_0\phi(y)]^{-1} &= [V_0\phi(x)]^{-1}(\alpha_0+\alpha)V_\alpha(x,E)\\ &= [V_0\phi(x)]^{-1}(\alpha_0+\alpha)R_{\alpha_0+\alpha}(x,E)\\ &\le [V_0\phi(x)]^{-1}.\end{aligned}$$

Since $(\alpha_0+\alpha)R_{\alpha_0+\alpha}(x,E)\to 1$ as $\alpha\to\infty$ by the condition (3) of theorem 2, $u=(1/V_0\phi)$ is excessive relative to $V_\alpha^\phi(x,A)$ and satisfies $(\alpha_0+\alpha)V_\alpha^\phi u\le u$. This relation is stricter than that in the definition of an excessive function. Such u may be considered an excessive function of negative exponent $-\alpha_0$.

LEMMA 4.3. *There is an excessive function $\overline{u}$ relative to $\overline{V}_\alpha^\phi(x,\overline{A})$ such that $\overline{u}=(1/V_0\phi)$ over E, and such that*

$$(4.7)\qquad (\alpha_0+\alpha)\overline{V}_\alpha^\phi\overline{u}\le\overline{u}\qquad \text{over}\quad \overline{E}.$$

Set $u=(1/V_0\phi)$ and $u_n=u\wedge n$. Both u_n and $V_\alpha^\phi u_n$ can be extended continuously to $\overline{E}$ by the definition of the completion. We will write $\overline{u}_\infty$ for the increasing limit of the extension $\overline{u}_n$ of u_n. As is easily verified, if α is such that $\alpha_0V_\alpha^\phi\mathbf{1}\le(m-n)/n$, then

$$(4.8)\qquad (\alpha_0+\alpha)V_\alpha^\phi u_n\le u_m\qquad \text{over}\quad E.$$

Hence, by continuity,

$$(4.9)\qquad (\alpha_0+\alpha)\overline{V}_\alpha^\phi\overline{u}_n=(\alpha_0+\alpha)\overline{V_\alpha^\phi u_n}\le\overline{u}_m\qquad \text{over}\quad \overline{E}.$$

Letting $m\to\infty$, one has for every $\alpha>0$,

$$(4.10)\qquad (\alpha_0+\alpha)\overline{V}_\alpha^\phi\overline{u}_n\le\overline{u}_\infty\qquad \text{over}\quad \overline{E}.$$

Letting $n\to\infty$, it follows that $\overline{u}_\infty$ satisfies (4.7). From this, by an argument similar to that in ([16], proposition 4.1), it follows that $(\alpha_0+\alpha)\overline{V}_\alpha^\phi\overline{u}_\infty$ increases to a limit $\overline{u}$ as $\alpha\to\infty$ and $\overline{V}_\alpha^\phi\overline{u}_\infty=\overline{V}_\alpha^\phi\overline{u}$ for every $\alpha\ge 0$. This proves lemma 4.3.

Next define

$$(4.11)\qquad \begin{aligned}\overline{V}_\alpha(x,\overline{A}) &= [\overline{u}(x)]^{-1}\int_{\overline{A}}\overline{V}_\alpha^\phi(x,dy)\overline{u}(y) &&\text{if}\quad 0<\overline{u}(x)<\infty,\\ &=0 &&\text{if}\quad \overline{u}(x)=0\ \text{ or }\ \infty,\end{aligned}$$

$$(4.12)\qquad \begin{aligned}\overline{T}_t(x,\overline{A}) &= [\overline{u}(x)]^{-1}\int_{\overline{A}}\overline{T}_t^\phi(x,dy)\overline{u}(y) &&\text{if}\quad 0<\overline{u}(x)<\infty,\\ &=0 &&\text{if}\quad \overline{u}(x)=0\ \text{ or }\ \infty,\end{aligned}$$

$$(4.13)\qquad \begin{aligned}\overline{U}_t(\overline{A}) &= \left[\int\overline{u}(x)\nu^\phi(dx)\right]^{-1}\int_{\overline{A}}\overline{U}_t^\phi(dy)\overline{u}(y)\\ &\qquad\qquad \text{if}\quad 0<\int\overline{u}(x)\nu^\phi(dx)<\infty,\\ &=0 \qquad\qquad \text{if}\quad \int\overline{u}(x)\nu^\phi(dx)=0\ \text{ or }\ \infty.\end{aligned}$$

It is easily verified that $\overline{T}_t(x,\overline{A})$ is a transition function over $\overline{E}$ and $\overline{V}_\alpha^\phi(x,\overline{A})$ and $\overline{U}_t(\overline{A})$ are its associated resolvent and system of absolute laws. Also, any system of absolute laws of $\overline{T}_t(x,\overline{A})$ can be obtained in the above way up to some constant multiple. (This is not evident. It follows, however, that $[\overline{u}(x)]^{-1}$ is bounded on $\overline{E} - \overline{E}_\delta^\phi$, which implies our assertion.)

Let $\tilde{y}_t$ be a Markov process corresponding to $\overline{T}_t(x, \overline{A})$ and $\overline{U}_t(\overline{A})$. The terminal time of $\tilde{y}_t$ is denoted by η. (In general, the basic probability space of $\tilde{y}_t$ is different from that of y_t^ϕ. But $\tilde{y}_t$ and y_t^ϕ can be always redefined to have the same probability space by taking the product probability space.) Then there is well-defined $y_t(\omega) = \lim_{r \downarrow t} \tilde{y}_r(\omega)$ (with r rationals) simultaneously for every $0 < t < \eta$ for almost all ω (P). This process y_t has the properties (i) and (ii) of the paragraph preceding the proposition (g) of section 2. (The property (iii) might be lost. That is, one cannot assert the existence $y_{\eta-}$ on the set $\{\eta < \infty\}$. This, however, is not important in the present discussion.) The distribution of y_{0+} is given by

$$(4.14) \qquad \left[\int \overline{u}(x)\nu^\phi(dx)\right]^{-1} \int_{\overline{A}} \overline{u}(x)\nu^\phi\,(dx).$$

In particular, it follows that $\alpha\overline{V}_\alpha(x, \cdot)$ converges weakly to a limit measure as $\alpha \to \infty$ for every x. If one defines branching points of $\overline{V}_\alpha(x, \overline{A})$ in terms of those limit measures in the same manner as in section 2, it follows that the set $\overline{E}_b$ of all branching points of $\overline{V}_\alpha(x, \overline{A})$ is $\overline{E}_\delta^\phi \cup \{x; \overline{u}(x) = 0 \text{ or } \infty\}$. With this definition the process y_t satisfies (g) to (k) in section 2. (For the proof of these results, see [15] and [17], proposition 2.3.) Actually, in [15], the above results are stated in a little weaker version. But they can be restated in the present form, using the proposition of "Supplement" of [15].

LEMMA 4.4. *The inequality $\overline{T}_t(x, \overline{E}) \leq e^{-\alpha_0 t}$ holds for all x of $\overline{E}$. Also $\overline{U}_t(\overline{E}) \leq e^{-\alpha_0 t}$ for any system of absolute laws of $\overline{T}_t(x, \overline{A})$.*

For the first statement, noting that $\overline{T}_t(x, \overline{E})$ is right continuous in $t > 0$, it is enough to show that

$$(4.15) \qquad (-1)^n \frac{d^n}{d\alpha^n} \overline{V}_\alpha(x, \overline{E}) \leq (-1)^n \frac{d^n}{d\alpha^n} \left(\frac{1}{\alpha_0 + \alpha}\right), \qquad n = 0, 1, \cdots$$

by the Bernstein theorem in Laplace transform. By a well-known formula on resolvents, this is equivalent to

$$(4.16) \qquad \overline{V}_\alpha^{n+1}(x, \overline{E}) \leq (\alpha_0 + \alpha)^{-n-1},$$

which follows from

$$(4.17) \qquad \overline{V}_\alpha(x, \overline{E}) = [\overline{u}(x)]^{-1}\overline{V}_\alpha^\phi\overline{u}(x) \leq (\alpha_0 + \alpha)^{-1}.$$

Therefore,

$$(4.18) \qquad \overline{U}_{t+s}(\overline{E}) = \int \overline{U}_s(dx)\overline{T}_t(x, \overline{E}) \leq e^{-\alpha_0 t}.$$

When $s \to 0$, one gets $\overline{U}_t(\overline{E}) \leq e^{-\alpha_0 t}$.

PROOF OF THEOREM 2′. We set

$$(4.19) \qquad \overline{P}_t(x, \overline{A}) = e^{\alpha_0 t}\overline{T}_t(x, \overline{A}),$$

$$(4.20) \qquad \overline{Q}_t(\overline{A}) = e^{\alpha_0 t}\overline{U}_t(\overline{A}).$$

Then it is evident that $\overline{P}_t(x, \overline{A})$ is a transition function over $\overline{E}$, and $\overline{Q}_t(\overline{A})$ a system of absolute laws of $\overline{P}_t(x, \overline{A})$. Any system of absolute laws of $\overline{P}_t(x, \overline{A})$ can be obtained in the above way up to a constant multiple. Let $(\tilde{x}_t, \zeta, P)$ be a Markov process over $\overline{E}$ with $\overline{P}_t(x, \overline{A})$ its transition function and $\overline{Q}_t(\overline{A}) = P\{\tilde{x}_t \in \overline{A}\}$. Apparently the functional $e^{\alpha_0 t}$, defined for $0 < t < \eta(\omega)$, is a right continuous multiplicative functional of the process y_t and satisfies

$$(4.21) \qquad E\{e^{\alpha_0 t}; \eta > t\} = \overline{Q}_t(\overline{A}) \leq 1.$$

Hence the results of [15] apply also to the transformation by $e^{\alpha_0 t}$, $0 < t < \eta$, and an argument similar to that for the transformation by $\overline{u}$ leads us to the conclusion that there is a standard modification x_t of $\tilde{x}_t$ satisfying (g) to (k) in section 2 with $\overline{E}_b$ the set of branching points of $\overline{V}_\alpha(x, \overline{A})$. The assertions (iii) and (iv) on $\overline{P}_t(x, \overline{A})$ of theorem 2′ are implied in the construction of the process x_t.

It remains to prove that $\overline{P}_t(x, \overline{A})$ is an extension of the original transition function $P_t(x, A)$ over E. Define $T_t(x, A) = e^{-\alpha_0 t} P_t(x, A)$. It is enough to show that $\overline{T}_t(x, \overline{A})$ is an extension of $T_t(x, A)$. To see this, first note that $\overline{V}_\alpha^\phi(x, \overline{A})$ is an extension of $V_\alpha^\phi(x, A)$. Then since $\overline{u} = (1/V_0\phi)$ over E, it is also easy to see that $\overline{V}_\alpha(x, \overline{A})$ is an extension of $V_\alpha(x, A)$. Let x be a point of E, and $\bar{f}$ a continuous function over $\overline{E}$ and f, the restriction of $\bar{f}$ to E. Since both $\overline{T}_t\bar{f}(x)$ and $T_tf(x)$ are right continuous in $t > 0$, and since their Laplace transforms $\overline{V}_\alpha\bar{f}(x)$ and $V_\alpha f(x)$ coincide, we have $\overline{T}_t\bar{f}(x) = T_tf(x)$ for every $t > 0$, which proves that $\overline{T}_t(x, \overline{A})$ is an extension of $T_t(x, A)$ by the same argument as in section 3. Hence theorem 2′ is proved.

Let $\overline{R}_\alpha(x, \overline{A})$ be the Laplace transform of $\overline{P}_t(x, \overline{A})$. With no help of $P_t(x, A)$, we can easily prove that $\overline{R}_\alpha(x, \overline{A})$ is an extension of $R_\alpha(x, A)$. Therefore the construction given in this section applies to any resolvent $R_\alpha(x, A)$ satisfying hypothesis (A_0) and $\alpha R_\alpha(x, E) \to 1$ as $\alpha \to \infty$ (without the assumption that $R_\alpha(x, A)$ is the Laplace transform of a transition function over E). This fact will be used in later sections.

5. Generalized resolvents and excessive measures

Throughout the rest of the paper, we will be interested in the second problem, the representation of excessive measures. The main results are stated in the next section and their proofs are given in the later sections. In the present section we will give some basic definitions as well as a preliminary theorem due to Hunt [9].

Let E be a locally compact separable Hausdorff space, and $\mathcal{B}$ the σ-field of all Borel sets. The σ-field $\mathcal{A}$ of all universally Borel sets is defined as before. A nonnegative function $R_\alpha(x, A)$ defined for $\alpha > 0$, $x \in E$, and $A \in \mathcal{B}$ is said to be a *generalized resolvent* if it satisfies (R_1) to (R_3) in section 1 and, in place of (R_4), the following condition: (GR_4) [*if A is compact, $R_\alpha(\cdot, A)$ is bounded on every compact set for each $\alpha > 0$ and $R_\alpha(x, A) \to 0$ as $\alpha \to \infty$ for each x*].

In particular, this condition implies that the measure $R_\alpha(x, \cdot)$ is finite on every compact set for each $\alpha > 0$ and $x \in E$. From now on, to avoid confusion, a resolvent in the proper sense is called a *substochastic resolvent*. From the resolvent equation there is always well defined the increasing limit $R(x, A)$ of $R_\alpha(x, A)$ as $\alpha \to 0$. Evidently $R(x, \cdot)$ is a measure over E, but in general, does not satisfy any finiteness condition. If, for each compact set A, $R(\cdot, A)$ is bounded on every compact set, the generalized resolvent $R_\alpha(x, A)$ is said to be *integrable*. Let f be a function over E and λ, a measure over E. We will use the following notations;

$$R_\alpha f = \int R_\alpha(\cdot, dy) f(y), \tag{5.1}$$

$$\lambda R_\alpha = \int \lambda(dx) R_\alpha(x, \cdot). \tag{5.2}$$

A σ-finite measure ν is said to be *excessive relative to* $R_\alpha(x, A)$ if $\nu(A) \geq \alpha(\nu R_\alpha)(A)$ for every $\alpha > 0$ and if $\nu(A) = \lim_{\alpha\to\infty} \alpha(\nu R_\alpha)(A)$ for every Borel set A. The first assumption implies that $\alpha(\nu R_\alpha)(\cdot)$ increases with α. Therefore, when ν is finite over every compact set, the second condition is equivalent to the weak convergence of measures $\alpha(\nu R_\alpha)$ to ν. If only the first condition $\nu \geq \alpha(\nu R_\alpha)$, $\alpha > 0$, is assumed, then ν is said to be *quasi-excessive relative to* $R_\alpha(x, A)$. When no confusion is expected, the phrase "relative to $R_\alpha(x, A)$" may be dropped. For the moment, suppose that $\alpha R_\alpha f \to f$ as $\alpha \to \infty$ for each f of $\mathbf{C}_0$. Then every quasi-excessive measure which is finite on every compact set is excessive. In other words, the second condition in the definition follows from the first condition.

Next consider the case in which $R_\alpha(x, A)$ is the Laplace transform of a transition function $P_t(x, A)$. In such a case, in order that a σ-finite measure ν is excessive, it is necessary and sufficient that $\nu(A) \geq \nu P_t(A) = \int \nu(dx) P_t(x, A)$ for every $t > 0$ and $\nu(A) = \lim_{t\to 0} \nu P_t(A)$ for every Borel set A. Therefore, if $P_t f \to f$ as $t \to 0$ for each f of $\mathbf{C}_0$, and if ν is finite on every compact set, then ν is excessive if and only if it satisfies $\nu \geq \nu P_t$. This turns out to be the original definition of Hunt [9].

THEOREM 5.1. *Suppose the generalized resolvent $R_\alpha(x, A)$ is integrable. Then, for any excessive measure ν, there is a sequence of measures λ_n such that $\lambda_n \leq n\nu$ and such that $\lambda_n R$ increases to ν as $n \to \infty$.*

The proof is similar to that of ([9], part I, p. 86). However, some additional consideration is required, for $R(x, A)$ is not necessarily a bounded operator. Hence we will give a complete proof.

LEMMA. *There is a sequence of quasi-excessive measures ν_n such that ν_n increases to ν and such that, for each n, $\alpha(\nu_n R_\alpha)(A) \to 0$ as $\alpha \to 0$ for every compact set A.*

Take $\epsilon > 0$ and set $\mu_n = n(\nu - n\nu R_{n+\epsilon})$. Then for a set A such that $\nu(A) < \infty$, it is easy to see that

$$\mu_n R_\epsilon(A) = n(\nu R_{n+\epsilon})(A) \leq \nu(A). \tag{5.3}$$

Since ν is σ-finite, (5.3) remains valid for every Borel set A. Consequently, $\mu_n R_\epsilon$ increases to ν as $n \to \infty$. Let $\{A_n\}$ be a sequence of Borel sets with compact closure increasing to E and satisfying $\nu(A_n) < \infty$. Let ξ_n be a finite measure over $(E, \mathcal{B})$ majorized by μ_n, having no mass outside a compact set and satisfying

$$\xi_n R_\epsilon(A_n) > \mu_n R_\epsilon(A_n) - n^{-1}. \tag{5.4}$$

Then the sequence of measures $\nu_n = \min(\nu, \sum_{k=1}^{n} \xi_k R)$ is what is wanted, because

$$\alpha(\xi_n R R_\alpha)(A) = \xi_n R(A) - \xi_n R_\alpha(A) \to 0, \qquad \alpha \to 0, \tag{5.5}$$

for each compact set A.

Proceeding to the proof of theorem 5.1 we will show that the measure $\lambda_n = n(\nu_n - n\nu_n R_n)$ satisfies the conditions of the theorem. It is enough to prove that $\lambda_n R(A)$ increases to $\nu(A)$ as $n \to \infty$ for any Borel set A with compact closure such that $\nu(A) < \infty$. From the resolvent equation it is apparent "formally" that

$$\lambda_n R(A) = n(\nu_n R_n)(A). \tag{5.6}$$

But since $\nu_n R(A)$ may be infinite, a more careful evaluation is needed to verify (5.6). By taking $\alpha > 0$, then

$$\nu_n R_\alpha(A) \le \alpha^{-1}\nu(A) < \infty, \tag{5.7}$$

$$\begin{aligned} \nu_n R_\alpha R_n(A) &= (n - \alpha)^{-1}\{\nu_n R_\alpha(A) - \nu_n R_n(A)\} \\ &\le (n - \alpha)^{-1}\nu_n R_\alpha(A) \qquad \text{for} \quad n > \alpha > 0, \end{aligned} \tag{5.8}$$

$$\begin{aligned} (\nu_n - n\nu_n R_n)R_\alpha(A) &= \{\nu_n - (n - \alpha)\nu_n R_n\}R_\alpha(A) - \alpha(\nu_n R_n R_\alpha)(A) \\ &= \nu_n R_n(A) - \alpha(\nu_n R_n R_\alpha)(A). \end{aligned} \tag{5.9}$$

However, by the preceding lemma,

$$\alpha(\nu_n R_n R_\alpha)(A) \le (n - \alpha)^{-1}\alpha(\nu_n R_\alpha)(A) \to 0, \qquad \alpha \to 0. \tag{5.10}$$

Also $(\nu_n - n\nu_n R_n)$ is a positive measure. Hence, letting $\alpha \to 0$ in (5.9), one gets (5.6) up to the constant multiple n. By (5.6),

$$\begin{aligned} \lambda_{n+1} R(A) &= (n + 1)(\nu_{n+1} R_{n+1})(A) \ge n(\nu_{n+1} R_n)(A) \\ &\ge n(\nu_n R_n)(A) = \lambda_n R(A). \end{aligned} \tag{5.11}$$

Obviously,

$$\lim_{n\to\infty} n(\nu_n R_n)(A) \le \lim_{n\to\infty} n(\nu R_n)(A) = \nu(A). \tag{5.12}$$

On the other hand, for any fixed integer k and m

$$\lim_{n\to\infty} n(\nu_n R_n)(A) \ge m(\nu_k R_m)(A). \tag{5.13}$$

Letting $k \to \infty$ first, and $m \to \infty$ next, one gets

$$\lim_{n\to\infty} n(\nu_n R_n)(A) \ge \nu(A), \tag{5.14}$$

which proves $\lim_{n\to\infty} \lambda_n R(A) = \nu(A)$.

6. The completion of R. S. Martin

Throughout this section and the next two sections, we will assume that the generalized resolvent $R_\alpha(x, A)$ satisfies the following hypothesis.

HYPOTHESIS (B). (1) *The generalized resolvent $R_\alpha(x, A)$ is integrable.* (2) *For every x, $R(x, \cdot)$ is a nontrivial measure, that is,* $0 < R(x, E) \le \infty$. (3) *If $x \ne x'$*

the measures $R(x, \cdot)$ *and* $R(x', \cdot)$ *are not proportional.* (4) *If* $f \in \mathbf{C}_0$, Rf *is continuous.* (5) *If* $f \in \mathbf{C}_0$, $R_\alpha f$ *is continuous for each* $\alpha > 0$.

Similarly to condition (2) of hypothesis (A) or (A_0), condition (3) in the above can be replaced by a little weaker condition: (3)′ *there is a countable collection of nonnegative continuous functions such that each function* f *of the collection satisfies* $\alpha R_\alpha f \leq f$, $\alpha > 0$, *and such that, whenever* $x \neq x'$, *there are certain functions* f *and* g *of the collection, and its ratio* (f/g) *separates* x *and* x'. For the simplicity we will only discuss the case of condition (3).

A nonnegative function ϕ is said to be a *reference function* if $R\phi$ is continuous and if $0 < R\phi(x) < \infty$ for every x. That there exist many continuous reference functions is proved in the same way as in lemma 4.1 (see the proposition (a) following theorem 3). We fix a reference function ϕ and set

$$M(x, A) = R(x, A)/R\phi(x). \tag{6.1}$$

LEMMA 6.1. *When* A *is compact,* $M(\cdot, A)$ *is bounded.*

For the proof we will introduce another kernel;

$$\begin{aligned} R^\phi(x, A) &= [R\phi(x)]^{-1} \int_A R(x, dy) R\phi(y) \\ &= \int_A M(x, dy) R\phi(y). \end{aligned} \tag{6.2}$$

Since $R\phi$ is strictly positive and continuous, it is enough to show that $R^\phi(\cdot, A)$ is bounded for each compact set A. It is evident that $R^\phi(\cdot, A)$ is bounded over A.

Next we define

$$R^\phi_\alpha(x, A) = [R\phi(x)]^{-1} \int_A R_\alpha(x, dy) R\phi(y). \tag{6.3}$$

An *excessive function relative to a generalized resolvent* is defined in the same way as in the case of a substochastic resolvent (section 4). It is easily verified that $R\phi$ is an excessive function relative to $R_\alpha(x, A)$ and that $R^\phi_\alpha(x, A)$ is a substochastic resolvent satisfying $\alpha R^\phi_\alpha(x, E) \to 1$ as $\alpha \to \infty$ for each x. Evidently $R^\phi_\alpha(x, A)$, $\alpha > 0$, maps $\mathbf{C}_0$ into $\mathbf{C}$. Finally we will show that $R^\phi_\alpha(x, A)$ satisfies condition (2) of hypothesis (A_0). Let $\{f_n\}$ be a countable subcollection of $\mathbf{C}_0^+$ dense in uniform norm. Then the collection of functions $\{R^\phi_1 f_n\}$ separates points in E, for $R^\phi_1(x, \cdot)$ and $R^\phi_1(x', \cdot)$ defines different measures for $x \neq x'$ by condition (3) of hypothesis (B).

Therefore, by the remark at the final paragraph of section 4, one can apply the results of section 4 to this substochastic resolvent $R^\phi_\alpha(x, A)$. Let $\overline{E}^\phi$ be it associated completion of E. In particular, for each x of E, there is a right continuous strong Markov process $(x_t^{(x)}, \zeta^{(x)}, P)$ defined over $\overline{E}^\phi$ such that

$$R^\phi_\alpha(x, A) = E\left\{\int_0^\infty e^{-\alpha t} I_A(x_t^{(x)})\, dt\right\}, \qquad \alpha > 0, \tag{6.4}$$

where I_A is the indicator function of the set A. Letting $\alpha \to 0$, one has

$$R^\phi(x, A) = E\left\{\int_0^\infty I_A(x_t^{(x)})\, dt\right\}. \tag{6.5}$$

Let A be a compact set of E, and τ the hitting time of A. By the strong Markov property,

$$R^{\phi}(x, A) = E\{R^{\phi}(x_{\tau}^{(x)}, A)\}. \tag{6.6}$$

Since A is also compact in $\overline{E}^{\phi}$, $x_{\tau}^{(x)}$ is in A almost everywhere (P). Therefore, the right side of the above equation is dominated (independently of x) by $\sup R^{\phi}(y, A)$, $y \in A$, which is known to be finite.

By condition (3), the measures $M(x, \cdot)$ and $M(x', \cdot)$ are different whenever $x \neq x'$. Hence the family of functions, $M(\mathbf{C}_0) = \{Mf;\ f \in \mathbf{C}_0\}$, separates points in E. By the preceding lemma, every function of $M(\mathbf{C}_0)$ is bounded and continuous. We take $M(\mathbf{C}_0)$ for $\mathbf{D}$ in section 3. The associated completion of E is denoted by $\overline{E}$ and called the *Martin completion*. For each x of $\overline{E}$, $\overline{Mf}(x)$ defines a linear functional over $\mathbf{C}_0$, so that there is a unique measure $\overline{M}(x, A)$ over $\mathcal{B}$ satisfying

$$\overline{Mf}(x) = \int \overline{M}(x, dy) f(y), \qquad f \in \mathbf{C}_0. \tag{6.7}$$

The measure $\overline{M}(x, A)$ is an extension of $M(x, A)$ from $E \times \mathcal{B}$ to $\overline{E} \times \mathcal{B}$, that is, $\overline{M}(x, \cdot) = M(x, \cdot)$ when $x \in E$.

An excessive measure ν is said to be *minimal* (or *extreme*) if ν cannot be expressed as the sum of two excessive measures which are not proportional to ν.

Our main representation theorem is this.

THEOREM 3. *Under hypothesis* (B) *the following conclusions are true.* (i) *For each x of $\overline{E}$, $\overline{M}(x, \cdot)$ is excessive relative to $R_\alpha(x, A)$.* (ii) *Let $\overline{E}_1$ be the set of points x of $\overline{E}$ such that $\overline{M}(x, \cdot)$ is minimal and satisfies $\int \overline{M}(x, dy)\phi(y) < \infty$ and such that, if there is some other point x' for which $\overline{M}(x', \cdot)$ is proportional to $\overline{M}(x, \cdot)$, then $\overline{M}(x, \cdot) \geq \overline{M}(x', \cdot)$. Then, such $\overline{E}_1$ exists uniquely.* (iii) *The set $\overline{E}_1$ is a measurable subset of $\overline{E}$ and*

$$\int \overline{M}(x, dy)\phi(y) \geq 1 \qquad \text{for each} \quad x \text{ of } \overline{E}_1. \tag{6.8}$$

(iv) *Any excessive measure ν such that $\int \phi(x)\nu(dx) < \infty$ is represented uniquely in the form*

$$\nu(A) = \int_{\overline{E}_1} \mu(dx)\overline{M}(x, A), \qquad A \in \mathcal{B}, \tag{6.9}$$

using a finite measure μ over $\overline{E}_1$.

The proof will be given in the next two sections.

In the rest of this section, assuming theorem 3, we will discuss several results obtained from, or related to, the theorem.

(a) *Let ν be an excessive measure. Then the following three conditions are equivalent.* (i) *The measure ν is finite over every compact set.* (ii) *There is a continuous reference function ϕ such that $\int \phi(x)\nu(dx) < \infty$.* (iii) *There is a reference function ϕ such that $\int \phi(x)\nu(dx) < \infty$.*

Suppose (i) holds. Let A_n be a sequence of open sets with compact closure increasing to E. Choose f_n of $\mathbf{C}_0^+$ so that $0 < f_n$ over A_n, $f_n < 2^{-n}$ over E,

$Rf_n < 2^{-n}$ over A_n, and $\int f_n(x)\nu(dx) < 2^{-n}$. Then the function $\phi = \sum f_n$ satisfies (ii). Evidently (ii) implies (iii). It follows from (6.9) that (iii) implies (i).

(b) *If the reference function ϕ is continuous (or more generally, lower semicontinuous), then the point x is in $\overline{E}_1$ if and only if $\overline{M}(x, \cdot)$ is minimal and $\int \overline{M}(x, dy)\phi(y) = 1$. In this case, for any finite measure μ over $\overline{E}$, the measure defined by*

$$\nu(A) = \int_{\overline{E}} \mu(dx)\overline{M}(x, A), \tag{6.10}$$

is an excessive measure satisfying $\int \phi(x)\nu(dx) < \infty$.

It is enough to show that $\overline{M}\phi(x) \leq 1$ for every x of $\overline{E}$; then (b) follows immediately from theorem 3. If ϕ is lower semicontinuous, $\overline{M}\phi$ is lower semicontinuous over $\overline{E}$. Since $\overline{M}\phi = 1$ over E, $\overline{M}\phi \leq 1$ over $\overline{E}$.

Theorem 3 can be restated in terms of the weak (or weak*) topology of measures. We will just give the version of the conclusion (iv) for a continuous reference function. Let $\mathcal{E}^\phi$ be the space of minimal excessive measures e such that $\int \phi(x)e(dx) = 1$ provided the weak topology. By (b), $\mathcal{E}^\phi$ is homeomorphic to $\overline{E}_1$ (corresponding to the same ϕ). Hence we have that

(c) *if ϕ is a continuous reference function and if ν is excessive and satisfies $\int \phi(x)\nu(dx) < \infty$, then ν can be written uniquely in the form*

$$\nu = \int_{\mathcal{E}^\phi} \mu(de)e \tag{6.11}$$

using a finite measure μ over $\mathcal{E}^\phi$. In particular, if there is a reference function ϕ_0 belonging to $\mathbf{C}_0^+$, the above conclusion holds for any excessive measure finite over every compact set.

By (a), if $\int \nu(dx)\phi(x) < \infty$ for some reference function ϕ, then $\int \nu(dx)\phi_0(x) < \infty$. In this sense ϕ_0 may be considered as a universal one of all reference functions.

7. Proof of theorem 3: a special case

In this section we will prove theorem 3 for a very special case, where the generalized resolvent $R_\alpha(x, A)$ and its associated reference function ϕ satisfy the following conditions (S_1). *Conditions* (1) *to* (4) *of hypothesis* (B) *are satisfied.* (S_2) *The resolvent $R_\alpha(x, A)$ is substochastic.* (S_3) *If A is compact, $R(\cdot, A)$ is bounded.* (S_4) *For each $\epsilon > 0$, there is a compact set A such that $R(x, E - A) < \epsilon$ for every x.* (S_5) *For every x of E, one has $R\phi(x) = 1$.*

For the convenience of later discussion (sections 8 and 9), condition (5) of hypothesis (B) is not assumed explicitly. But as will be seen below, it is implied by (S_1) to (S_4).

By (S_4), any function of $R(\mathbf{C})$ (= the range of $\mathbf{C}$ by $R(x, A)$) can be uniformly approximated by some function of $R(\mathbf{C}_0)$. In other words, the uniform enclosures of $R(\mathbf{C})$ and $R(\mathbf{C}_0)$ are the same. This implies that $R(x, A)$ maps $\mathbf{C}$ into itself. Hence, from a routine work (for instance, [9], part II, p. 352), it follows that $R_\alpha(x, A)$ maps $\mathbf{C}$ into itself for every $\alpha > 0$ and $R_\alpha(\mathbf{C}) = R(\mathbf{C})$. Therefore, we can take $\mathbf{C}$ for $\mathbf{A}$ in lemma 3.2. We will take a countable dense subcollection in

$R(\mathbf{C}_0^+)$ with the constant function $\mathbf{1}$ added for the corresponding $\mathbf{C}_1$. By (S_5),

$$M(x, A) = R(x, A), \tag{7.1}$$

so that the Knight completion based on $R_\alpha(x, A)$ is the same as the Martin completion $\bar{E}$ based on $M(x, A)$. Notice that $R(x, A)$ can be extended to $\bar{R}(x, \bar{A})$ over $\bar{E}$ in the same way as $R_\alpha(x, A)$ to $\bar{R}_\alpha(x, \bar{A})$ for $\alpha > 0$.

LEMMA 7.1. *For each x of $\bar{E}$, $\bar{R}(x, \bar{A})$ is a trivial extension of $\bar{M}(x, A)$, that is, $\bar{R}(x, \bar{E} - E) = 0$ and $\bar{R}(x, A) = \bar{M}(x, A)$ for $A \in \mathcal{B}$.*

For each f of $\mathbf{C}^+$

$$\overline{Mf}(x) = \overline{Rf}(x) \qquad \text{for every } x \text{ of } \bar{E}. \tag{7.2}$$

But when $f_n \in \mathbf{C}_0^+$ increases to f, Mf_n increases uniformly over E to Mf, so that for each x of $\bar{E}$,

$$\begin{aligned} \overline{Mf}(x) &= \lim_{n\to\infty} \overline{Mf_n}(x) = \lim_{n\to\infty} \int \bar{M}(x, dy) f_n(y) \\ &= \int \bar{M}(x, dy) f(y). \end{aligned} \tag{7.3}$$

Let $\bar{f}$ be a function of $\bar{\mathbf{C}}^+$, and f the restriction of $\bar{f}$ to E. Since $f \in \mathbf{C}^+$, by (7.2) and (7.3)

$$\overline{Rf}(x) = \bar{M}f(x), \qquad x \in \bar{E}, \tag{7.4}$$

which proves the lemma. (Use the same argument as in section 3.)

It follows from the lemma that any excessive measure relative to $\bar{R}_\alpha(x, \bar{A})$ has no mass over $\bar{E} - E$, and its restriction to E is excessive relative to $R_\alpha(x, A)$. The converse statement that the trivial extension of any excessive measure relative to $R_\alpha(x, A)$ is excessive relative to $\bar{R}_\alpha(x, \bar{A})$ is always true for any Knight completion. In the rest of this section we will write $\bar{R}(x, A)$ for $\bar{M}(x, A)$.

We now proceed to prove theorem 3 in this special case.

PROOF OF (i). $\bar{R}(x, \bar{A})$ is excessive relative to $\bar{R}_\alpha(x, \bar{A})$, so that $\bar{R}(x, A)$ is excessive relative to $R_\alpha(x, A)$.

Let $\bar{E}_b$ be the set of branching points of $\bar{R}_\alpha(x, \bar{A})$ and $\bar{E}_\infty$, the set of points x such that $\int \bar{R}(x, dy)\phi(y) = \infty$ and set $\bar{E}_1 = \bar{E} - \bar{E}_b \cup \bar{E}_\infty$.

PROOF OF (iv). Let ν be excessive relative to $R_\alpha(x, A)$ and satisfy $\int \phi(x)\nu(dx) < \infty$. For the proof of the existence of the representation (6.9), it is enough to show that ν can be expressed as

$$\nu(A) = \int_{\bar{E}-\bar{E}_b} \mu(dx)\bar{R}(x, A), \qquad A \in \mathcal{B}. \tag{7.5}$$

for then it is evident that μ has no mass over the set $\bar{E}_\infty$.

Take $\{\lambda_n\}$, a sequence of measures over E such that $\lambda_n R$ increases to ν (theorem 5.1). Then

$$\begin{aligned} \lambda_n(E) &= \int \lambda_n(dx) \int R(x, dy)\phi(y) \\ &= \int (\lambda_n R)(dy)\phi(y) \le \int \nu(dy)\phi(y) < \infty, \end{aligned} \tag{7.6}$$

so that a subsequence $\{\lambda_{n'}\}$ of $\{\lambda_n\}$ converges weakly over $\overline{E}$ to a measure λ over $\overline{E}$, namely, for each $\bar{f} \in \overline{\mathbf{C}}$ (with f as its restriction to E),

$$\int \lambda_{n'}(dy)f(y) \to \int_{\overline{E}} \lambda(dy)\bar{f}(y). \tag{7.7}$$

Since $\overline{R}f \in \overline{\mathbf{C}}$,

$$\int \lambda_{n'}(dy)Rf(y) \to \int_{\overline{E}} \lambda(dy)\overline{R}f(y). \tag{7.8}$$

On the other hand,

$$\int \lambda_{n'}(dy)Rf(y) = \int (\lambda_{n'}R)(dz)f(z) \to \int \nu(dz)f(z), \tag{7.9}$$

so that

$$\int \nu(dz)f(z) = \int_{\overline{E}} \lambda(dy)\overline{R}f(y). \tag{7.10}$$

Let $\bar{\mu}(x, \cdot)$ be the limit measure of $\alpha\overline{R}_\alpha(x, \cdot)$ as $\alpha \to \infty$. Letting $\alpha \to 0$ in (2.3), and using the proposition (c) of section 2,

$$\overline{R}f(y) = \int_{\overline{E}-\overline{E}_b} \bar{\mu}(y, dx)\overline{R}f(x). \tag{7.11}$$

Setting

$$\mu = \int_{\overline{E}} \lambda(dy)\bar{\mu}(y, \cdot), \tag{7.12}$$

one gets

$$\int \nu(dz)f(z) = \int \left\{\int_{\overline{E}-\overline{E}_b} \mu(dx)\overline{R}(x, dz)\right\} f(z), \tag{7.13}$$

which proves (7.5).

For the uniqueness, suppose ν is written in the form (7.5), using a finite measure μ over $\overline{E} - \overline{E}_b$. (One need not assume $\int \nu(dx)\phi(x) < \infty$.) Recalling that for each $x \in \overline{E} - \overline{E}_b$, $\alpha\overline{R}_\alpha f(x) \to \bar{f}(x)$ (boundedly) as $\alpha \to \infty$ for every $\bar{f} \in \overline{\mathbf{C}}$, and using the resolvent equation of $\overline{R}_\alpha(x, \overline{A})$, one has for $\bar{f} \in \overline{\mathbf{C}}$,

$$\begin{aligned} \int_{\overline{E}-\overline{E}_b} \mu(dx)\bar{f}(x) &= \lim_{\alpha\to\infty} \int_{\overline{E}-\overline{E}_b} \mu(dx)\{\alpha\overline{R}_\alpha f(x)\} \\ &= \lim_{\alpha\to\infty} \alpha \int_{\overline{E}-\overline{E}_b} \mu(dx)\{\overline{R}f(x) - \alpha\overline{R}\overline{R}_\alpha f(x)\} \\ &= \lim_{\alpha\to\infty} \alpha \left\{\int \nu(dy)f(y) - \alpha \int \nu(dy)R_\alpha f(y)\right\}, \end{aligned} \tag{7.14}$$

so that μ is uniquely determined by ν.

Proof of (iii). The first half is evident. For the second half, to the contrary of the conclusion, assume that $\int \overline{R}(x, dy)\phi(y) < 1$ for some point x of $\overline{E} - \overline{E}_b$. Then, apply the same procedure as in the proof of (iv) to the excessive measure $\nu = \overline{R}(x, \cdot)$. The total mass $\lambda(\overline{E})$ of the corresponding limit measure λ of $\lambda_{n'}$ must equal $\overline{R}\,\phi(x) < 1$. Hence, by (7.12), $\mu(\overline{E}) < 1$, which is impossible, because μ must be the unit distribution at x according to the uniqueness of the representation (7.5).

PROOF OF (ii). For the moment let us denote by $\overline{E}_2$ the set of points x of $\overline{E}$ satisfying the conditions in (ii). We will prove $\overline{E}_1 = \overline{E}_2$. Assume $x \in \overline{E}_1$. Evidently, $\int \overline{R}(x, dy)\phi(y) < \infty$. The minimality of the excessive measure $\overline{R}(x, \cdot)$ follows easily from the uniqueness of the representation (7.5). Suppose there is some $x' \in \overline{E}$ such that $\overline{R}(x, \cdot) = c\overline{R}(x', \cdot)$ with a positive constant c. Then, by the resolvent equation, $\overline{R}_\alpha(x, \cdot) = c\overline{R}_\alpha(x', \cdot)$ for every $\alpha > 0$. Consequently,

$$(7.15) \qquad 1 = \lim_{\alpha \to \infty} \alpha \overline{R}_\alpha(x, \overline{E}) = c \lim_{\alpha \to \infty} \alpha \overline{R}_\alpha(x', \overline{E}) \geq c,$$

so that $x \in \overline{E}_2$.

Conversely, assume $x \in \overline{E}_2$. By the first two conditions and the conclusion (iv), there is a (unique) point x' of $\overline{E}_1$ such that $\overline{R}(x, \cdot) = c\overline{R}(x', \cdot)$ with a positive constant c. From the maximal property of both $\overline{R}(x, \cdot)$ and $\overline{R}(x', \cdot)$, it follows that $\overline{R}(x, \cdot) = \overline{R}(x', \cdot)$, which means $x = x'$.

8. Proof of theorem 3: reduction of the general case to the special case of section 7

For a nonnegative function $a(x)$ over E, define

$$(8.1) \qquad M^a(x, A) = \int_A M(x, dy)a(y).$$

LEMMA 8.1. *There is a strictly positive everywhere, bounded and continuous function $a(x)$ such that $M^a(x, A)$ satisfies (S_4) of section 7.*

The proof is similar to that of lemma 4.1. Let $\{A_n\}$ be a sequence of open sets with compact closures increasing to E such that the closure of A_n is included in A_{n+1}. Choose f_n so that $f_n > 0$ over A_n, $f_n = 0$ outside A_{n+1}, $f_n < 2^{-n}$ everywhere and $Mf_n < 2^{-n}$ everywhere. Then the function $a(x) = \sum f_n(x)$ is the desired one.

In the following we will assume that $a(x)$ satisfies the condition $a(x) \leq R\phi(x)$ for every x as well as the conditions in lemma 8.1.

Our reduction is based on the following

LEMMA 8.2. *There is a (unique) substochastic resolvent $M^a_\alpha(x, A)$ which converges to $M^a(x, A)$ as $\alpha \to 0$.*

(Since $M^a(x, A)$ is a bounded operator, the uniqueness is evident.) Consider $R^\phi(x, A)$ and $R^\phi_\alpha(x, A)$ defined by (6.2) and (6.3). For a nonnegative function $b(x)$, define

$$(8.2) \qquad R^{\phi, b}(x, A) = \int_A R^\phi(x, dy)b(y).$$

When $b(x) = a(x)/R\phi(x)$ (≤ 1),

$$(8.3) \qquad R^{\phi, b}(x, A) = M^a(x, A).$$

We will show that, for any bounded nonnegative function $b(x)$, there is a substochastic resolvent such that

$$R^{\phi,b}(x, A) = \lim_{\alpha\to 0} R_\alpha^{\phi,b}(x, A). \tag{8.4}$$

In the proof of lemma 6.1, it was shown that the results of section 4 are applicable to $R_\alpha^\phi(x, A)$. Its associated extended system is denoted by $\{\overline{E}^\phi, \overline{R}_\alpha^\phi(x, \overline{A}), \overline{P}_t^\phi(x, \overline{A})\}$, $x \in \overline{E}^\phi$, $\overline{A} \in \overline{\mathfrak{B}}^\phi$. For each x of $\overline{E}^\phi$, let us denote by $(x_t^{(x)}, \zeta^{(x)}, P)$ the right continuous strong Markov process of theorem 2′ such that $P\{x_t^{(x)} \in \overline{A}\} = \overline{P}_t^\phi(x, \overline{A})$, $\overline{A} \in \overline{\mathfrak{B}}^\phi$. Let $\overline{b}(x)$ be an extension of $b(x)$ to $\overline{E}$, bounded and nonnegative. Define

$$A^{(x)}(t, \omega) = \int_0^t \overline{b}(x_s^{(x)}(\omega))\, ds, \tag{8.5}$$

$$\tau^{(x)}(t, \omega) = \sup\{t';\ 0 < t' < \zeta^{(x)},\ A^{(x)}(t', \omega) \le t\}, \tag{8.6}$$

$$y_t^{(x)}(\omega) = x^{(x)}(\tau^{(x)}(t, \omega), \omega), \qquad 0 < t < A^{(x)}(\zeta^{(x)}, \omega), \tag{8.7}$$

$$\overline{P}_t^{\phi,b}(x, \overline{A}) = P\{y_t^{(x)} \in \overline{A}\}. \tag{8.8}$$

(In the definition of $y_t^{(x)}$, we used the notation $x^{(x)}(t, \omega)$ for $x_t^{(x)}(\omega)$.)

Then, by the Volkonsky theorem on random time change ([20], theorem 1.4), $\overline{P}_t^{\phi,b}(x, \overline{A})$ defines a transition function over $\overline{E}^\phi$. Set

$$\overline{R}_\alpha^{\phi,b}(x, \overline{A}) = \int_0^\infty e^{-\alpha t}\overline{P}_t^{\phi,b}(x, \overline{A})\, dt = E\left\{\int_0^\infty e^{-\alpha t} I_{\overline{A}}(y_t^{(x)}(\omega))\, dt\right\}. \tag{8.9}$$

Therefore,

$$\begin{aligned}\lim_{\alpha\to 0} \overline{R}_\alpha^{\phi,b}(x, \overline{A}) &= E\left\{\int_0^\infty I_{\overline{A}}(y_t^{(x)}(\omega))\, dt\right\} \\ &= E\left\{\int_0^\infty I_{\overline{A}}(x_t^{(x)}(\omega))\overline{b}(x_t^{(x)}(\omega))\, dt\right\} \\ &= \lim_{\alpha\to 0} \overline{R}_\alpha^\phi(I_{\overline{A}}\overline{b})(x).\end{aligned} \tag{8.10}$$

In particular, if $x \in E$, setting $A = \overline{A} \cap E$

$$\overline{R}_\alpha^\phi(I_{\overline{A}}\overline{b})(x) = \overline{R}_\alpha^\phi(I_A b)(x), \tag{8.11}$$

so that $\overline{R}_\alpha^{\phi,b}(x, \overline{A})$ has no mass over $\overline{E}^\phi - E$, which implies that $R_\alpha^{\phi,b}(x, A)$, the restriction of $\overline{R}_\alpha^{\phi,b}(x, \overline{A})$ to E, is a substochastic resolvent over E. By (8.10), this resolvent $R_\alpha^{\phi,b}(x, A)$ satisfies (8.4).

LEMMA 8.3. *The measure ν is excessive relative to $R_\alpha(x, A)$ if and only if the measure ν^a defined by*

$$\nu^a(A) = \int_A \nu(dx)a(x) \tag{8.12}$$

is excessive relative to $M_\alpha^a(x, A)$.

Let ν be excessive and $\{\lambda_n R\}$ a sequence of excessive measures increasing to ν. Set $\lambda_n^\phi(dx) = \lambda_n(dx)R\phi(x)$. Then

$$\begin{aligned}\lambda_n^\phi M^a(A) &= \int \lambda_n^\phi(dx) \int_A M(x, dy)a(y) \\ &= \int \lambda_n(dx) \int_A R(x, dy)a(y) \uparrow \int_A \nu(dy)a(y) = \nu^a(A),\end{aligned} \tag{8.13}$$

so that ν^a is excessive relative to $M_\alpha^a(x, A)$. The other half is proved in the same manner.

PROOF OF THEOREM 3. Set

$$\psi(x) = \phi(x)/a(x). \tag{8.14}$$

It is easily verified that $M_\alpha^a(x, A)$ and its reference function $\psi(x)$ of (8.14) satisfy (S_1)–(S_5) of section 7. Hence, theorem 3 is valid for $\{M_\alpha^a(x, A), \psi(x)\}$. We will denote by $\{\overline{E}^a, \overline{E}_1^a, \overline{M}^a(x, \overline{A})\}$ the system in the theorem corresponding to $\{M_\alpha^a(x, A), \psi(x)\}$. But, since $M(\mathbf{C}_0) = M^a(\mathbf{C}_0)$, $\overline{E}^a$ is identical with $\overline{E}$ (as a topological space). It is also evident that, for each x of $\overline{E}$ ($= \overline{E}^a$),

$$\overline{M}^a(x, A) = \int_A \overline{M}(x, dy)a(y). \tag{8.15}$$

Set $\overline{E}_1 = \overline{E}_1^a$. Then all the assertions of theorem 3 are true for $\{\overline{M}(x, \overline{A}), \phi(x)\}$, because they are invariant under transformation from $\{\overline{M}^a(x, A), \psi(x)\}$ to $\{\overline{M}(x, \overline{A}), \phi(x)\}$. Such invariance follows easily from lemma 8.3 and the formula (8.15).

9. Other sufficient conditions

We will now give two alternatives for condition (5) of hypothesis (B).

HYPOTHESIS (B′). *Conditions* (1) *to* (4) *of hypothesis* (B) *and the following condition are satisfied.* (5′) *$R_\alpha(x, A)$ is the Laplace transform of a transition function $P_t(x, A)$ over E. Moreover, for each x of E, there is a right continuous and strong Markov* (*with $P_t(\cdot, \cdot)$ as its transition function*) *process* $(x_t^{(x)}, \zeta^{(x)}, P)$ *defined over E such that $P\{x_t^{(x)} \in A\} = P_t(x, A)$. Here a "right continuous" process over E means that almost all sample paths $x_t^{(x)}(\omega)$, $0 < t < \zeta(\omega)$, are right continuous in E.*

HYPOTHESIS (B″). *Conditions* (1) *to* (4) *of hypothesis* (B) *and the following conditions are satisfied.* (5″) *$R(x, A)$ satisfies the "complete principle of the maximum," that is, if $f \in \mathbf{C}_0$ and if $Rf \leq 1$ over the set $\{x; f(x) > 0\}$, then $Rf \leq 1$ everywhere* (*over E*).

As is easily seen, the above principle of the maximum is equivalent to either of the following condition. (i) *Let f be a function of $\mathbf{C}_0$ and c, a nonnegative constant. If $Rf \leq c$ over the set $\{x; f(x) > 0\}$, then $Rf \leq c$ everywhere.* (ii) *Let f, g be functions of $\mathbf{C}_0^+$ and c, a nonnegative constant. If $Rf \leq Rg + c$ over $\{x; f(x) > 0\}$, then $Rf \leq Rg + c$ everywhere.*

The effect of the complete principle of the maximum in the theory of Markov processes was discovered by Hunt ([9], part II, section 15). In his paper an additional (but critical) assumption which is referred to as (γ) is required. However, we need only a partial result of his, so that we can do without his assumption (γ).

THEOREM 3′. *Under hypothesis* (B′) (*resp.* (B″)), *lemma* 6.1 *and theorem* 3 *are valid for any reference* (*resp. continuous reference*) *function.*

It is enough to prove lemma 6.1 and lemma 8.2 in both cases; in other parts we did not use condition (5) of hypothesis (B).

CASE OF HYPOTHESIS (B′). Set

$$P_t^\phi(x, A) = [R\phi(x)]^{-1} \int_A P_t(x, dy) R\phi(y). \tag{9.1}$$

Since this is a transformation by the excessive function $R\phi$, $P_t^\phi(x, A)$ is a transition function and its Laplace transform is $R_\alpha^\phi(x, A)$. By the theorem of [15], for each x of E, there is a right continuous strong Markov (with $P_t^\phi(\cdot, \cdot)$ as its transition function) process $(y_t^{(x)}, \eta^{(x)}, P)$, defined over E, such that $P\{y_t^{(x)} \in A\} = P_t^\phi(x, A)$. Then the proof of lemma 6.1 is carried out in the same way as in hypothesis (B) by using the strong Markov property and the right continuity of sample paths of the process $(y_t^{(x)}, \eta^{(x)}, P)$. The proof of lemma 8.2 is also similar to the case of hypothesis (B); to obtain the substochastic resolvent $R_\alpha^{\phi,b}(x, A)$ in (8.4), it is enough to consider the random time change of $(y_t^{(x)}, \eta^{(x)}, P)$ by means of the function $b(x)$.

CASE OF HYPOTHESIS (B″). The proofs in this case stand on a different basis from the previous cases.

To prove lemma 6.1 it is enough to show that the kernel $M(x, A)$ satisfies the complete principle of the maximum. Take a function f of $\mathbf{C}_0$ and assume

$$Mf \leq 1 \qquad \text{over the set} \quad \{x; f(x) > 0\}, \tag{9.2}$$

which is equivalent to $Rf \leq R\phi$ on the same set. Since ϕ is assumed to be continuous, there is a sequence of functions ϕ_n of $\mathbf{C}_0^+$ increasing to ϕ. By the Dini theorem, $R\phi_n$ converges uniformly to $R\phi$ on the set $\{x; f(x) > 0\}$. Therefore, for any $\epsilon > 0$,

$$Rf \leq R\phi_n + \epsilon \qquad \text{on} \quad \{x; f(x) > 0\}, \tag{9.3}$$

for sufficiently large n. By the complete principle of the maximum relative to $R(x, A)$, (9.3) holds everywhere. This proves

$$Rf \leq R\phi + \epsilon \qquad \text{everywhere.} \tag{9.4}$$

Since ϵ is arbitrary, $Rf \leq R\phi$ everywhere.

Next we will prove lemma 8.2. It is evident that $M^a(x, A)$ satisfies the complete principle of the maximum. Moreover, since $M^a(x, A)$ satisfies (S_4) of section 7, it is easily verified that $M^a(x, A)$ satisfies the complete principle of the maximum for those functions of $\mathbf{C}$. Noting that $M^a(x, E)$ is bounded in x, we set

$$0 < \alpha_0 = [\sup_{x \in E} M^a(x, E)]^{-1}, \tag{9.5}$$

$$M_\alpha^a(x, A) = \sum_{k=0}^{\infty} (-\alpha)^k [M^a]^{k+1}(x, A), \qquad 0 < \alpha < \alpha_0, \tag{9.6}$$

where $[M^a]^k(x, A)$ is the k-times convolution of $M^a(x, A)$. As is easily seen, $M_\alpha^a(x, A)$ is well defined and maps $\mathbf{C}$ into itself. We will show that $M_\alpha^a(x, A)$, $0 < \alpha < \alpha_0$, has the same properties as a substochastic resolvent. The resolvent equation follows immediately from the definition. Take a function f of $\mathbf{C}^+$. From the resolvent equation

$$M_\alpha^a(-f) = M^a(\alpha M_\alpha^a f - f). \tag{9.7}$$

Assume $\sup_{x\in E} M_\alpha^a(-f)(x) = c > 0$. Then the complete principle of the maximum for those functions of $\mathbf{C}$ implies that there is a sequence of points x_n of E such that $M_\alpha^a(-f)(x_n) \to c$ as $n \to \infty$ and $\alpha M_\alpha^a f(x_n) - f(x_n) \geq 0$ for every n. This is impossible, so that $M_\alpha^a(x, A) \geq 0$ for every $A \in \mathcal{B}$. To prove $\alpha M_\alpha^a(x, E) = \alpha M_\alpha^a \mathbf{1}(x) \leq 1$ for every x of E, assume that $\alpha^{-1} < c = \sup_{x\in E} M_\alpha^a \mathbf{1}(x)$. Then an argument similar to the above, applied to the equation

$$M_\alpha^a \mathbf{1} = M^a(\mathbf{1} - \alpha M_\alpha^a \mathbf{1}), \tag{9.8}$$

leads to the contradiction that there must be a sequence of points x_n of E such that $M_\alpha^a \mathbf{1}(x_n) \to c > \alpha^{-1}$ as $n \to \infty$ and $1 - \alpha M_\alpha^a \mathbf{1}(x_n) \geq 0$ for every n. The extension of the above $M_\alpha^a(x, A)$ to arbitrary $\alpha > 0$ is carried through by setting inductively

$$M_\alpha^a(x, A) = \sum_{k=0}^{\infty} (\beta - \alpha)^k [M_\beta^a]^{k+1}(x, A), \qquad 0 < \alpha < 2\beta. \tag{9.9}$$

REMARK 1. We will note that hypothesis (B′) is stronger than hypothesis (B″). For the proof, take a function f of $\mathbf{C}_0$ and assume $Rf \leq 1$ over the set $\{x; f(x) > 0\}$. Set $A =$ the closure of $\{x; f(x) > 0\}$. By continuity, $Rf \leq 1$ over A. Let x be any point of E and τ, the hitting time for the set A of the process $x_t^{(x)}$. By the strong Markov property,

$$\begin{aligned} Rf(x) &= E\left\{\int_0^\infty f(x_t^{(x)})\, dt\right\} \\ &= E\left\{\int_0^\tau f(x_t^{(x)})\, dt\right\} + E\{Rf(x_\tau^{(x)})\} \\ &\leq E\{Rf(x_\tau^{(x)})\} \leq 1. \end{aligned} \tag{9.10}$$

REMARK 2. The use of the complete principle of the maximum gives us another proof of lemma 8.2 under hypothesis (B). In fact the same argument as above applied to the process associated with the extended system $\{\overline{E}^\phi, \overline{R}_\alpha^\phi(x, \overline{A}), P_t^\phi(x, \overline{A})\}$, leads us to the conclusion that that $R^\phi(x, A)$ satisfies the complete principle of the maximum. Then the same proof as in hypothesis (B″) is valid.

10. Reference functions allowing the value infinity of $R\phi$

In the definition of a reference function ϕ in section 6 we imposed that $R\phi$ is finite everywhere over E. We now remove the restriction, and we will again say that a nonnegative function ϕ is a *reference function* if $R\phi(x)$ is strictly positive everywhere and continuous, allowing the value infinity. We will show that theorem 3 and theorem 3′ are still valid with a slight change of the definition of $\overline{E}$.

Set

$$E_\infty^\phi = \{x; R\phi(x) = \infty\}, \qquad E^\phi = E - E_\infty^\phi \tag{10.1}$$

(E^ϕ has no relation with $\overline{E}^\phi$, which appeared in the proofs of lemma 6.1 and lemma 8.2).

LEMMA 10.1. (i) *For each x of E^ϕ, $R(x, E_\infty^\phi) = 0$.* (ii) *Any excessive measure ν, such that $\int \phi(x)\, \nu(dx) < \infty$, has no mass over E_∞^ϕ.*

Fix $\alpha > 0$ and $x \in E^\phi$. Then

$$\infty > R\phi(x) \geq \alpha \int R_\alpha(x, dy)R\phi(y) \geq \alpha \int_{E^\phi_\infty} R_\alpha(x, dy)R\phi(y), \tag{10.2}$$

so that $R_\alpha(x, E^\phi_\infty) = 0$. Since $R(x, E^\phi_\infty) = \lim_{\alpha\to 0} R_\alpha(x, E^\phi_\infty)$, $R(x, E^\phi_\infty) = 0$. To prove the second assertion, take $\lambda_n R$, a sequence of measures increasing to ν. Then

$$\int \lambda_n(dx)\, R\phi(x) \leq \int \nu(dx)\, \phi(x) < \infty, \tag{10.3}$$

so that λ_n has no mass on the set E^ϕ_∞. Hence, by (i), $\lambda_n R(E^\phi_\infty) = 0$, which proves $\nu(E^\phi_\infty) = 0$.

Since E^ϕ_∞ is a closed subset of E, E^ϕ is again a locally compact separable Hausdorff space. By the above lemma the restriction of $R_\alpha(x, A)$ to E^ϕ is also an integrable generalized resolvent over E^ϕ. Moreover, a measure ν excessive relative to $R_\alpha(x, A)$ such that $\int \nu(dx)\, \phi(x) < \infty$ can be identified with its restriction ν^ϕ to E^ϕ, which is known to be excessive relative to the restriction of $R_\alpha(x, A)$ to E^ϕ. We can now apply the previous results to the restricted generalized resolvent $R_\alpha(x, A)$, $x \in E^\phi$, $A \in \mathcal{B}^\phi$ and obtain the representation of ν^ϕ (and therefore, of ν).

THEOREM 4. *Theorem 3 and 3′ are still valid for the reference function allowing the value infinity of $R\phi$ if one considers the Martin completion of E^ϕ (based on $M(\mathbf{C}^\phi_0)$) for $\overline{E}$.*

11. Examples

Let E be a denumerable space carrying the discrete topology. Then conditions (4) and (5) of hypothesis (B) are trivially satisfied for any integrable generalized resolvent. Therefore, theorem 3 applies to any generalized resolvent $R_\alpha(x, A)$ satisfying (1) to (3) of hypothesis (B). In this special case, theorem 3 gives also the representation of excessive functions relative to $R_\alpha(x, A)$, that is, nonnegative functions f such that $\alpha R_\alpha f \leq f$ and $\lim_{\alpha\to\infty} \alpha R_\alpha f = f$. To suggest the extension to more general cases, we will use a too heavy formulation to dispose of the present special case.

Let $R_\alpha(x, A)$ be an integrable generalized resolvent. Let $m(A)$ be the uniform measure over E and let $r_\alpha(x, y)$ (resp. $r(x, y)$), the density of $R_\alpha(x, A)$ (resp. $R(x, A)$) relative to $m(A)$; that is, $r_\alpha(x, y) = R_\alpha(x, \{y\})$ and $r(x, y) = R(x, \{y\})$. Define $R^*_\alpha(x, A) = \int_A r_\alpha(y, x)m(dy)$. The resolvent $R^*_\alpha(x, A)$ is an integrable generalized resolvent which satisfies

$$\int_A R^*_\alpha(x, B)m(dx) = \int_B R_\alpha(x, A)m(dx). \tag{11.1}$$

Let $\Phi(A)$ be a measure such that $0 < \int r(x, y)\Phi(dx) < \infty$ for every y. Such $\Phi(A)$ is called a *reference measure.* Set $\phi(x) = (d\Phi/dm)(x) = \Phi(\{x\})$. This becomes a reference function for $R^*_\alpha(x, A)$. For an excessive function f such that $\int f(x)\Phi(dx) < \infty$, define $\nu(A) = \int_A f(x)m(dx)$. Then the measure ν is excessive relative to $R^*_\alpha(x, A)$ and satisfies $\int \nu(dx)\phi(x) < \infty$. The inverse correspondence is also valid. Assuming that $R^*_\alpha(x, A)$ satisfies (2) and (3) of hypothesis (B), we apply theorem 3 to $R^*_\alpha(x, A)$. Its M-kernel is denoted by

$$M^*(x, A) = R^*(x, A)/R^*\phi(x) = \int_A [r(y, x)/\int r(z, x)\Phi(dz)]m(dy), \tag{11.2}$$

and its associated system in completion by $(\overline{M}^*(x, A), \overline{E}^*, \overline{E}^*_1)$.

For each x of $\overline{E}^*$, set $\kappa(y, x) = (d\overline{M}^*(x, \cdot)/dm)(y)$, which is an excessive function in y relative to $R_\alpha(y, A)$ and, when $x \in E$, coincides with the integrand of the last side of (11.2). Since ϕ is trivially continuous, x is in $\overline{E}^*_1$ if and only if $\kappa(\cdot, x)$ is a minimal excessive function and $\int \kappa(y, x)\Phi(dy) = 1$. Thus the unique representation of f is obtained;

$$f = \int_{\overline{E}^*_1} \mu(dx)\kappa(\cdot, x). \tag{11.3}$$

This generalizes to a considerable extent the result of [21]. Indeed, it is easily verified that both $R_\alpha(x, A)$ and its generalized co-resolvent $R^*_\alpha(x, A)$ satisfy hypothesis (B) if $R_\alpha(x, A)$ is the Laplace transform of a transient and standard transition function $P_t(x, A)$. In [21] we studied the case when $P_t(x, A)$ is a special type of transient and standard transition function.

The preceding argument to obtain the representation (11.3) of excessive functions applies to more general cases. In those cases we need assume the existence of the measure m and the generalized co-resolvent $R^*_\alpha(x, A)$ satisfying (11.2). Moreover, we assume that $R^*_\alpha(x, A)$ satisfies either hypothesis (B), (B′), or (B″). Then the kernels $r(x, y)$ and $r_\alpha(x, y)$ are uniquely determined in a certain way. If the reference measure Φ is absolutely continuous with respect to m, the class of excessive functions f relative to $R_\alpha(x, A)$ such that $\int f(x)\Phi(dx) < \infty$ is in one-to-one correspondence with the class of excessive measures ν relative to $R^*_\alpha(x, A)$ such that $\int \nu(dx)(d\Phi/dm)(x) < \infty$ through the relation

$$\nu(A) = \int_A f(x)m(dx). \tag{11.4}$$

We will note that (11.4) never involves the ambiguity of (m) measure zero on $f(x)$. This follows from the fact that $f(x)$ is excessive. (For these discussions refer to [16] and [17].) Consequently, we can obtain the Martin representation theorem of excessive functions with respect to a smooth reference measure under a weaker hypothesis than that in [16] and [17].

The Martin representations of excessive measures and excessive functions of discrete parameter Markov chains were obtained by Doob [5] and Hunt [10]. It has been known, with no explicit mention in references, that the discrete parameter case can be reduced, or rather essentially equivalent, to the special continuous parameter case of [21]. So far, such reduction has not been so important, for the previous methods [10], [21] can be applied in a parallel way to both the discrete parameter case and the above-mentioned special continuous parameter case. However, our present method, involving the transformation of an unbounded operator to a bounded operator by time change, has no counterpart in the discrete parameter case. Hence, it will be useful to give a brief description of such reduction.

Let $P(x, A)$ be a one-step transition function and $\delta(x, A)$, the unit distribution at the point x. Define $q(x) = 1 - P(x, \{x\})$ and $\Pi(x, A) = [q(x)]^{-1} P(x, A - \{x\})$. Then the system (q, Π) satisfies the equation

$$P(x, A) - \delta(x, A) = q(x)\{\Pi(x, A) - \delta(x, A)\}. \tag{11.5}$$

We will denote by $\mathcal{G}(x, A)$ the above operator. The system (q, Π) is the unique solution of the equation (11.5), with the additional condition $\Pi(x, \{x\}) = 0$. Then the equation

$$(\alpha - \mathcal{G})R_\alpha(x, A) = \delta(x, A), \qquad \alpha > 0 \tag{11.6}$$

has the unique solution $R_\alpha(x, A)$. This $R_\alpha(x, A)$ satisfies also the adjoint equation

$$R_\alpha(\alpha - \mathcal{G})(x, A) = \delta(x, A), \qquad \alpha > 0. \tag{11.7}$$

Moreover, $R_\alpha(x, A)$ is a substochastic resolvent satisfying conditions (2), (3) and (5′) of hypothesis (B′).

The corresponding continuous parameter transition function is standard. We will denote by τ the hitting time of the set $E - \{x\}$ of the process $(x_t^{(x)}, \zeta^{(x)}, P)$. Then $[q(x)]^{-1} = E\{\tau\}$ and $\Pi(x, A) = P\{x_\tau^{(x)} \in A\}$. A measure ν or a nonnegative function f, is excessive relative to $R_\alpha(x, A)$ if and only if it is excessive relative to $P(x, A)$, namely, if $\nu(A) \geq \int \nu(dx)P(x, A)$, or respectively $f(x) \geq \int f(y)P(x, dy)$. The proofs of all the above results are given in [21]. In particular, if $P(x, A)$ is transient, then conditions (1) and (4) of hypothesis (B) are also satisfied and

$$R(x, A) = \lim_{\alpha \to 0} R_\alpha(x, A) = \sum_{n=0}^{\infty} P^n(x, A). \tag{11.8}$$

Therefore,

$$M(x, A) = \sum_{n=0}^{\infty} P^n(x, A) \Big/ \left[\int \sum_{n=0}^{\infty} P^n(x, dy)\phi(y) \right], \tag{11.9}$$

which is nothing but the Martin dual kernel corresponding to the transition function $P(x, A)$ in [5], [10]. Moreover, the integrand of the last side of (11.2) turns out to be

$$\sum_{n=0}^{\infty} P^n(y, x) \Big/ \sum_{n=0}^{\infty} \int P^n(z, x)\Phi(dz), \tag{11.10}$$

which is the Martin kernel of $P(x, A)$.

The above argument applies to discrete parameter Markov processes (that is, one-step transition functions) over general state spaces. Their corresponding continuous parameter Markov processes are right continuous and strongly Markov and proceed with only simple jumps [22]. Eventually we can conclude that the Martin representation theorem of excessive measures relative to the discrete parameter transition function $P(x, A)$ over a locally compact separable

Hausdorff space is valid if $P(x, A)$ is transient and if the kernel $\sum_{n=0}^{\infty} P^n(x, A)$ maps $\mathbf{C}_0$ into $\mathbf{C}$.

Finally we will consider the case when E is a group. If an integrable generalized resolvent is invariant under translation, namely, $R_\alpha(x, A) = R_\alpha(xy^{-1}, Ay^{-1})$, conditions (4) and (5) of hypothesis (B) are always satisfied. In particular, if $R_\alpha(x, A)$ corresponds to a standard continuous parameter transition function $P_t(x, A)$, or to a discrete parameter one, $P(x, A)$, conditions (2) and (3) are also satisfied. Therefore, the Martin representation theorem of excessive measures is valid for every spatially homogeneous transient Markov process either with continuous parameter or discrete parameter.

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