A SURVEY ON THE MARKOV PROCESS ON THE BOUNDARY OF MULTI-DIMENSIONAL DIFFUSION

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1. Introduction

It has been observed that the Markov process on the boundary of diffusion is related to the solution of a diffusion equation in a domain D, $(\partial/\partial t)u = Au$, with Wentzell's boundary condition, $Lu = 0$. (Precise definitions of A and L are given in section 2, (2.1) and (2.3).) In fact, to obtain the solution, it is sufficient to solve $(\alpha - A)u(x) = 0$, $x \in D$ and $(\lambda - L)u(x) = \varphi(x)$, $x \in \partial D$ for sufficiently many φ on the boundary ∂D, where $\alpha \geq 0$ and $\lambda \geq 0$ are fixed. This provides a class of Markov processes on ∂D ([13], [14]).

A kind of duality between the way of obtaining the diffusion on $\bar{D}$ and the way of obtaining a process of this class naturally leads to a conjecture that the Markov process on the boundary is the trace on the boundary of the trajectory of the diffusion. Moreover, a simple example suggests that this trace is described by a time scale called the local time on the boundary $\mathfrak{t}(t, w)$ in such a way that

$$\mathring{x}(t, w) = x(\mathfrak{t}^{-1}(t, w), w), \tag{1.1}$$

where $x(t)$ denotes the path function of the diffusion, $\mathring{x}(t)$ denotes the Markov process on ∂D, and $\mathfrak{t}^{-1}(t, w)$ is the right continuous inverse of $\mathfrak{t}$ [14]. In fact, K. Sato proved that this is true in the case of reflecting diffusion with sufficient regularities ([10], [11], [13]). Such a process on the boundary had not been explicitly discussed because the boundaries of one-dimensional diffusion are too simple.

However, the concepts of Markov process and local time on the boundary of a diffusion process can also be considered apart from the setup based on elliptic operator A and boundary condition L. In fact, a well-known correspondence between excessive functions and additive functionals insures the existence of a class of additive functionals which increase when and only when $x(t, w)$ is on the boundary, and we obtain a Markov process $\mathring{x}(t, w)$ on ∂D by making use of such a functional as before [8], [12], [13].

From this point of view, a part of the problem Feller solved in one dimension can be formulated in the following way. Given a diffusion process $\mathbf{M}$ on a domain $\bar{D}$, determine the class of all diffusions whose path functions coincide with those of $\mathbf{M}$ before they arrive at ∂D, where jumps from the boundary are permitted. In other words, let $\mathbf{M}^{\min}$ be a diffusion whose path functions vanish as soon as they arrive at ∂D. (Such a process is called a minimal process.) Then, determine the class of diffusions whose path functions coincide with those of $\mathbf{M}^{\min}$ before the arrival at ∂D.

As an approach to the problem, K. Sato [12] proved that a diffusion process on $\overline{D}$ is determined by the minimal process and the Markov process on the boundary, if there are no jumps, from the boundary into D, and if the path stays on ∂D for a set of times having Lebesgue measure zero.

Recently, M. Motoo [7], [8] has made a deep analysis of the behavior of path functions near the boundary, and obtained a result on this problem which is almost complete, as far as his formulation is concerned. In fact, he decomposed a given Markov process on $\overline{D}$ into three factors: the minimal process, the Markov process on ∂D, and some quantities which determine the way of leaving ∂D for D. He called a pair of the last two factors "a boundary system." Moreover, he found, under certain regularities, a necessary and sufficient condition that a given process and a given system are the minimal process and the boundary system of a Markov process on $\overline{D}$, respectively.

We are going to make a simple survey of these results and describe the Markov process on the boundary for the purpose of pursuing the problem of determining all the multidimensional diffusions.

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2. The diffusion satisfying Wentzell's boundary condition

Consider an elliptic operator

$$(2.1)\qquad Au(x) = \frac{1}{\sqrt{a(x)}} \sum_{i,j=1}^{N} \frac{\partial}{\partial x^i}\left(a^{ij}(x)\sqrt{a(x)}\,\frac{\partial u}{\partial x^j}(x)\right) + \sum_{i=1}^{N} b^i(x)\,\frac{\partial u}{\partial x^i}(x) + c(x)u(x), \qquad x \in \overline{D},$$

where D is a bounded domain in an N-dimensional orientable manifold of class C^∞ with boundary ∂D consisting of a finite number of hypersurfaces of class C^3. The $a^{ij}(x)$ and $b^i(x)$ are contravariant tensors of class $C^{2,k}$ and $C^{1,k}$ respectively. The matrix $\{a^{ij}(x)\}$ is symmetric and positive definite and $a(x) = \det(a^{ij}(x))^{-1}$. The function $c(x)$ is nonpositive and is in $C^{0,k}(\overline{D})$.

The set $C(E)$ is the set of all real-valued continuous functions on E and $C^n(E)$ is the set of all functions in $C(E)$ which are n-times continuously differentiable in E. The symbols $C^{n,k}(E)$ denotes the set of all functions in $C^n(E)$, whose n-th derivatives are uniformly Hölder continuous with exponent k in E.

Let $\overline{A}$ be the smallest closed extension of A in $C(\overline{D})$, where A is taken to be defined on $C^2(\overline{D})$. (As we see in Wentzell [16] or [13], $\overline{A}$ actually exists.) A diffusion process satisfying the equation

$$(2.2)\qquad \frac{\partial}{\partial t} u(t, x) = Au(t, x), \qquad x \in D,$$

is expected to correspond in an almost one-to-one way to a semigroup of nonnegative linear operators $\{T_t, t \geq 0\}$ on $C(\bar{D})$ which is strongly continuous at $t = 0$, such that $\|T_t\| \leq 1$, and has a contraction of $\bar{A}$ as its generator, if the process is regular with respect to the topology of $\bar{D}$. (We can drop the strong continuity at $t = 0$, if the regularity at the boundary points is not required.) The path functions of such a process are continuous except for possible jumps from the boundary points. A. D. Wentzell [16] tried to determine all these semigroups and proved that a smooth function u in the domain of the generator of such a semigroup satisfies a boundary condition of type

$$(2.3)\quad \begin{aligned} &Lu(x) = 0, x \in \partial D, \text{ where} \\ &Lu(x) = \sum_{i,j=1}^{N-1} \alpha^{ij}(x) \frac{\partial^2 u}{\partial \xi_x^i \partial \xi_x^j}(x) + \sum_{i=1}^{N-1} \beta^i(x) \frac{\partial u}{\partial \xi_x^i}(x) \\ &\qquad + \gamma(x)u(x) + \delta(x) \lim_{y \to x} Au(y) + \mu(x) \frac{\partial u}{\partial n}(x) \\ &\qquad + \int_{\bar{D}} \left\{ u(y) - u(x) - \sum_{i=1}^{N-1} \frac{\partial u}{\partial \xi_x^i}(x) \xi_x^i(y) \right\} \nu_x(dy). \end{aligned}$$

In this expression $\{\alpha^{ij}(x)\}$ is nonnegative definite, $\gamma(x)$, $\delta(x)$, $-\mu(x)$ are nonpositive, and $\nu_x(\cdot)$ is a measure on $\bar{D}$ satisfying

$$(2.4)\quad \begin{aligned} &\nu_x(\bar{D} - U) < \infty, \qquad \nu_x(\{x\}) = 0, \\ &\int_U \left\{ \sum_{i=1}^{N-1} \xi_x^i(y)^2 + \xi_x^N(y) \right\} \nu_x(dy) < \infty, \end{aligned}$$

U being an arbitrary neighborhood of x. The set $\{\xi_x^i(y), 1 \leq i \leq N\}$ is a class of functions in $C^3(\bar{D})$ and is a local coordinate system in a neighborhood U_x of x such that $\xi_x^N(y) \geq 0$ for all $y \in \bar{D}$, such that ∂D is characterized by $\xi_x^N(y) = 0$ in U_x, and that $\xi_x^i(y) = 0$, $(1 \leq i \leq N)$ if and only if $y = x$ in U_x. Finally, $(\partial/\partial n)$ is the inward-directed normal derivative.

Now, we try to construct the semigroup determined by this boundary condition by extending a method of W. Feller [1] in one dimension. Let $g_\alpha(x, y)$ be the fundamental solution of

$$(2.5)\quad \begin{aligned} (\alpha - A)u(x) &= v(x), \qquad && x \in D; \\ u(x) &= 0, && x \in \partial D. \end{aligned}$$

Define an operator $G_\alpha^{\min}$ on $C(\bar{D})$ by

$$(2.6)\quad G_\alpha^{\min} u(x) = \int_{\bar{D}} g_\alpha(x, y) u(y) m(dy), \qquad u \in C(\bar{D}),$$

where $m(E)$ is given by $\int_E \sqrt{a(x)}\, dx^1 \cdots dx^N$ for a set E in a local coordinate neighborhood U of $(x^1, \cdots, x^N)$. Then, $G_\alpha^{\min}$ maps $C(\bar{D})$ into $C(\bar{D})$ and $C^{0,k}(\bar{D})$ into $C^2(\bar{D})$, and satisfies $(\alpha - \bar{A})G_\alpha^{\min} u = u$, $u \in C(\bar{D})$. (Detailed properties of $G_\alpha^{\min}$, H_α and $p(t, x, y)$ are found in S. Ito [5] and [13].) The operators $\{G_\alpha^{\min}\}$ form a resolvent of a semigroup $\{T_t^{\min}\}$, which satisfies all the conditions for the contraction semigroup of $\overline{A}$, except the strong continuity at $t = 0$, and $\{T_t^{\min}\}$ corresponds to the diffusion satisfying (2.1), whose particle vanishes as soon as it arrives at the boundary. For each $\varphi \in C(\partial D)$ and $\alpha \geq 0$, the equation

$$(2.7)\qquad \begin{aligned} (\alpha - A)u(x) &= 0, & x \in D;\\ u(x) &= \varphi(x), & x \in \partial D, \end{aligned}$$

has a unique solution u, which is represented by

$$(2.8)\qquad u(x) = H_\alpha\varphi(x) = \int_{\partial D} H_\alpha(x, dy)\varphi(y),$$

where $H_\alpha(x, \cdot)$ is a measure on ∂D with total mass at most one.

Now, let G_α be the resolvent of the semigroup under consideration. If $G_\alpha u$ is smooth, we obtain $(\alpha - A)(G_\alpha u - G_\alpha^{\min} u)(x) = 0$ for $x \in D$ from

$$(2.9)\qquad (\alpha - \overline{A})(G_\alpha u - G_\alpha^{\min} u) = 0.$$

Since $G_\alpha^{\min} u$ vanishes at $x \in \partial D$, the boundary value of $G_\alpha u - G_\alpha^{\min} u$ is $[G_\alpha u]_{\partial D}$, where $[v]_{\partial D}$ denotes the restriction on ∂D of v. Hence, we have

$$(2.10)\qquad G_\alpha u - G_\alpha^{\min} u = H_\alpha[G_\alpha u]_{\partial D}.$$

Applying L on both sides and noting that $LG_\alpha u(x) = 0$ at $x \in \partial D$ by assumption, we have

$$(2.11)\qquad LG_\alpha u - LG_\alpha^{\min} u = -LG_\alpha^{\min} u = LH_\alpha[G_\alpha u]_{\partial D}.$$

If $(-LH_\alpha)^{-1}$ exists, we have $[G_\alpha u]_{\partial D} = (-LH_\alpha)^{-1}(LG_\alpha^{\min} u)$ formally, and hence $G_\alpha u = G_\alpha^{\min} u + H_\alpha(-LH_\alpha)^{-1}(LG_\alpha^{\min} u)$ by (2.10). This indicates the method of construction of G_α.

To justify the formal computation above, we make the following assumptions on L, noting that for any $u \in C^2(\overline{D})$, $Lu(x)$ is well defined at each $x \in \partial D$:

(L.1) *$Lu(x)$ is continuous in $x \in \partial D$, if u is in $C^2(\overline{D})$;*

(L.2) *$\nu_x(D) = \infty$ for $x \in \partial D$, if $\delta(x) = \mu(x) = 0$.*

(L.2) is a regularity condition to insure the strong continuity at $t = 0$. (Compare with a more general condition in theorem 6 of section 6.) Let $\mathfrak{D}(L)$ be a linear subspace of $C(\overline{D})$ such that $C^2(\overline{D}) \subset \mathfrak{D}(L) \subset \bigcup_{k>0} C^{0,k}(\overline{D})$, consisting of such u that $Lu(x)$ is continuous in $x \in \partial D$. Consider L to be defined on $\mathfrak{D}(L)$ by $u \to Lu(x)$, $x \in \partial D$, and define LH_α by $\varphi \to (LH_\alpha)\varphi = L(H_\alpha\varphi)$ on

$$(2.12)\qquad \mathfrak{D}(LH_\alpha) = \{\varphi \in C(\partial D) | H_\alpha\varphi \in \mathfrak{D}(L)\}.$$

Noting that $LH_\alpha\varphi(x) \leq 0$ if $\varphi \in \mathfrak{D}(LH_\alpha)$ takes a positive maximum at $x \in \partial D$ by an easy computation, it can be proved that LH_α has the *smallest closed extension $\overline{LH}_\alpha$ in $C(\partial D)$*. The operator $LG_\alpha^{\min}$, defined by $u \to (LG_\alpha^{\min})u = L(G_\alpha^{\min} u)$ on $\{u \in C(\overline{D}) | G_\alpha^{\min} u \in \mathfrak{D}(L)\}$, is nonnegative, linear, bounded and it has a dense domain in $C(\overline{D})$. Hence it is extended uniquely to a nonnegative, bounded, linear operator $\overline{LG_\alpha^{\min}}$ on $C(\overline{D})$. From equality

$$(2.13)\qquad H_\alpha\varphi - H_\beta\varphi + (\alpha - \beta)G_\alpha^{\min} H_\beta\varphi = 0, \qquad \varphi \in C(\partial D),$$

which is easily proved, follows

$$\overline{LH}_\alpha\varphi - \overline{LH}_\beta\varphi + (\alpha - \beta)\overline{LG}_\alpha^{\min} H_\beta\varphi = 0, \qquad \varphi \in C(\partial D), \tag{2.14}$$

implying that $\mathfrak{D}(\overline{LH}_\alpha)$ does not depend on $\alpha \geq 0$. Denoting this common domain $\mathfrak{D}(\overline{LH}_\alpha)$, $(\alpha \geq 0)$ by $\tilde{\mathfrak{D}}$, we have the following theorem.

THEOREM 1. *For arbitrarily fixed $\alpha \geq 0$ and $\lambda \geq 0$, assume that*

$$\begin{aligned} (\alpha - A)u(x) &= 0, \qquad & x \in D; \\ (\lambda - L)u(x) &= \varphi(x), \qquad & x \in \partial D \end{aligned} \tag{2.15}$$

has a solution $u \in \mathfrak{D}(L) \cap C^2(D)$ for each φ in a dense subset of $C(\partial D)$ (where $u \in C^2(D)$ means that u is twice continuously differentiable in D). Then, for each $\beta \geq 0$, $\overline{LH}_\beta$ is the generator of a semigroup $\{\tilde{T}_t^\beta, t \geq 0\}$ on $C(\partial D)$. The assumption is equivalent to the assumption that

$$(\lambda - \overline{LH}_\alpha)\psi = \varphi \tag{2.15'}$$

has a solution $\psi \in \tilde{\mathfrak{D}}$ for each φ in a dense subset of $C(\partial D)$.

(By "a semigroup on $C(\overline{D})$ (or $C(\partial D)$)," we mean a semigroup of nonnegative linear operators on $C(\overline{D})$ (or $C(\partial D)$) which is strongly continuous at $t = 0$ and in which the norms of the operators are at most 1.)

We call the semigroup $\{\tilde{T}_t^\alpha\}$ on $C(\partial D)$ with generator $\overline{LH}_\alpha$, *the semigroup on $C(\partial D)$ of order α*, and denote the Green operator by

$$K_\lambda^\alpha\varphi = \int_0^\infty e^{-\lambda t}\tilde{T}_t^\alpha\varphi \, dt, \qquad \lambda > 0. \tag{2.16}$$

Moreover, noting that $\overline{LH}_\alpha 1$ is strictly negative for $\alpha > 0$ by virtue of (L.2), we have the following corollary.

COROLLARY. *Under the assumption of theorem 1, $\overline{LH}_\alpha\psi = \varphi$ has a unique solution for each $\varphi \in C(\partial D)$, if $\alpha > 0$.*

Hence, $(-\overline{LH}_\alpha)^{-1}$ which we write as K_0^α, is defined on $C(\partial D)$. In fact, $K_0^\alpha\varphi = \int_0^\infty \tilde{T}_t^\alpha\varphi \, dt$ converges and it coincides with $(-\overline{LH}_\alpha)^{-1}\varphi$ in this case.

Now, we consider a kind of the *closure* $\hat{L}$ of given L. Let $\mathfrak{D}(\hat{L})$ be the set of all functions in $C(\overline{D})$ written in the form

$$u = \sum_{i=1}^m G_{\alpha_i}^{\min} u_i + \sum_{j=1}^n H_{\beta_j}\varphi_j, \qquad u_i \in C(\overline{D}), \ \varphi_j \in \tilde{\mathfrak{D}} \tag{2.17}$$

and put

$$\hat{L}u = \sum_{i=1}^m \overline{LG}_{\alpha_i}^{\min} u_i + \sum_{j=1}^n \overline{LH}_{\beta_j}\varphi_j. \tag{2.18}$$

Then, $\hat{L}u$ does not depend on the representation (2.17), but only on u by virtue of the resolvent equation for $G_\alpha^{\min}$ and equation (2.13).

PROPOSITION. *Suppose that $C^{2,k}(\overline{D})$ is contained in $\mathfrak{D}(L)$. Let*

$$\hat{L}u = Lu \qquad \text{for} \quad u \in C^{2,k}(\overline{D}),$$
$$(2.19) \qquad \hat{L}G_\alpha^{\min}u = \overline{LG_\alpha^{\min}}u \qquad \text{for} \quad u \in C(\overline{D}),$$
$$\hat{L}H_\alpha\varphi = \overline{LH}_\alpha\varphi \qquad \text{for} \quad \varphi \in \tilde{\mathfrak{D}}.$$

Such an operator $\hat{L}$ is unique.

Moreover, if $u \in \mathfrak{D}(\hat{L})$ satisfies, for some $\alpha \geq 0$ and $\lambda > 0$,

$$(2.20) \qquad \begin{aligned} (\alpha - \overline{A})u(x) &= 0 \qquad \text{for} \quad x \in D, \\ (\lambda - \hat{L})u(x) &= 0 \qquad \text{for} \quad x \in \partial D, \end{aligned}$$

then $u \equiv 0$. Under the condition of theorem 1, this is also true for $\alpha > 0$ and $\lambda = 0$.

THEOREM 2. *Under the assumption of theorem 1, $\overline{A}$ restricted to the subspace $\{u \in \mathfrak{D}(\hat{L}) | \hat{L}u = 0\}$ is the generator of a contraction semigroup on $C(\overline{D})$. (We note that $\mathfrak{D}(L)$ is contained in $\mathfrak{D}(\overline{A})$ by definition.) The Green operator G_α of the semigroup is given by*

$$(2.21) \qquad G_\alpha u = G_\alpha^{\min}u + H_\alpha K_0^\alpha \overline{LG_\alpha^{\min}}u, \qquad u \in C(\overline{D}).$$

Hence, the formal computation has been justified, reducing the problem to the equation of type $(\lambda - \overline{LH}_\alpha)\psi = \varphi$. The main part of the proof is to derive that $\alpha G_\alpha u \to u$ as $\alpha \to \infty$, which is implied essentially by the condition (L.2).

Finally, we note that the equation of type (2.15) above is reduced to an integro-differential equation given on the boundary ∂D by its nature, and that the solution really exists in some important special cases.

The proof of the reduction and examples are given in [13].

3. The Markov process on the boundary—I

To the semigroup on $C(\partial D)$ with generator $\overline{LH}_\alpha$, $(\alpha \geq 0)$ of section 2, there corresponds a Markov process on ∂D with right continuous path functions which have left limits. We now consider the probabilistic meaning of the process, which we call the *Markov process on the boundary of order* α (of the diffusion on $\overline{D}$). As a typical one we take $\overline{LH}_0$, whose resolvent K_λ^0, $(\lambda > 0)$ is obtained in the following way. Solve

$$(3.1) \qquad \begin{aligned} Au(x) &= 0, \qquad & x \in D, \\ (\lambda - L)u(x) &= \varphi(x), \qquad & x \in \partial D, \end{aligned}$$

for $\varphi \in C(\partial D)$, and define $K_\lambda^0: \varphi \to K_\lambda^0\varphi = [u]_{\partial D}$. On the other hand, the resolvent G_α of the diffusion on $\overline{D}$ is obtained by solving

$$(3.2) \qquad \begin{aligned} (\alpha - A)u(x) &= v(x), \qquad & x \in D \\ Lu(x) &= 0, \qquad & x \in \partial D \end{aligned}$$

for $v \in C(\overline{D})$ and defining $G_\alpha: v \to G_\alpha v = u$. Hence, there is an apparent duality between the operations of obtaining these resolvents. This duality can be naturally extended to K_λ^α and G_α^λ, where $K_\lambda^\alpha\varphi$ is obtained from $(\alpha - A)u = 0$

and $(\lambda - L)u = \varphi$, and $G_\alpha^\lambda u$ from $(\alpha - A)v = u$ and $(\lambda - L)v = 0$. (Compare with formula (5.3) of theorem 5.) Since the semigroup corresponding to G_α describes the motion on D (and hence on $\overline{D}$ by continuity), it seems to be natural that the semigroup corresponding to K_λ^0 describes the motion on ∂D, that is, *the trace on the boundary of the trajectory of the diffusion.*

This interpretation is easily justified in a very special case by using one-dimensional local time. Moreover, K. Sato [11] proved that this is also true for reflecting diffusions in the following way.

Assume that the $c(x)$ occurring in the definition of A always vanishes and $L = (\partial/\partial n)$; that is,

$$(3.3)\quad \begin{aligned} Au(x) &= \frac{1}{\sqrt{a(x)}} \sum_{i,j=1}^{N} \frac{\partial}{\partial x^i}\left(a^{ij}(x)\sqrt{a(x)}\,\frac{\partial u}{\partial x^j}(x)\right) + \sum_{i=1}^{N} b^i(x)\,\frac{\partial u}{\partial x^i}(x), \\ Lu(x) &= \frac{\partial}{\partial n}u(x), \qquad x \in \partial D. \end{aligned}$$

Then, there is a Markov process $\mathbf{M} = \{x_t, W, \mathbf{B}_t, P_x, x \in \overline{D}\}$ on $\overline{D}$ with transition probability

$$(3.4)\qquad P(t, x, E) = \int_E p(t, x, y)m(dy),$$

where $p(t, x, y)$ is the fundamental solution of the Cauchy problem for $(\partial/\partial t)u(x) = Au(x), x \in D$ and $(\partial/\partial n)u(x) = 0, x \in \partial D$. The set W is the space of all continuous functions $w(t)$ defined on $[0, \infty)$ with values in $\overline{D}$. The classes $\mathbf{B}$ and $\mathbf{B}_t$ are respectively the smallest Borel fields of subsets of W which make $\{x_s, 0 \leq s < \infty\}$ and $\{x_s, 0 \leq s \leq t\}$ measurable. Here $x_s(w) = w_s$. The function $P_x(\cdot)$ is a probability measure on $\mathbf{B}$ such that $P_x(x_0(w) = x) = 1$ and $P_x(x_t(w) \in E) = P(t, x, E)$. The process $\mathbf{M}$ has the strong Markov property. We call $\mathbf{M}$ the *reflecting diffusion determined by* A. Define

$$(3.5)\qquad \mathfrak{t}_\rho(t, w) = \frac{1}{\rho}\int_0^t \chi_{D_\rho}(x_s(w))\,ds, \qquad \rho > 0$$

where χ_{D_ρ} is the characteristic function of the set $D_\rho = \{x | d(x, \partial D) < \rho\}$, where $d(x, \partial D) = \inf_{y \in \partial D} d(x, y)$, and $d(x, y)$ is the distance between x and y induced by $\{a^{ij}(x)\}$. The number $d(x, y)$ is the infimum of the length of all curves C in $\overline{D}$, which connect x and y and of class C' piecewise. The length of C is given by

$$(3.6)\qquad \int_0^1 \left(\sum_{i,j=1}^{N} a_{ij}(x(\lambda))\,\frac{dx^i(\lambda)}{d\lambda}\,\frac{dx^j(\lambda)}{d\lambda}\right)^{1/2} d\lambda,$$

where C is given by $C: \lambda \in [0, 1] \rightarrow x(\lambda) \in \overline{D}$. Then, K. Sato proved the following theorem.

THEOREM 3. *There is a sequence $\{\rho_n \searrow 0\}$ such that with probability one $\mathfrak{t}_{\rho_n}(t, w)$ converges to a nonnegative, continuous additive functional $\mathfrak{t}(t, w)$ of $\mathbf{M}$, uniformly on any compact time interval. Further $\mathfrak{t}(t, w)$ increases when and only when $x_t(w) \in \partial D$ and, with probability one, it increases to ∞ as $t \rightarrow \infty$. Finally $\mathfrak{t}(t, w)$ satisfies*

(3.7) $$E_x(\mathfrak{t}(t, w) = \int_0^t ds \int_{\partial D} p(s, x, y)\tilde{m}(dy).$$

Such an additive functional is unique up to probability 1.

(The symbol $\tilde{m}$ denotes the surface element on ∂D induced by $\{a^{ij}(x)\}$. The precise definition is given by S. Ito [5] or [13].

We call $\mathfrak{t}(t, w)$ the *local time on the boundary* of $\mathbf{M}$. The proof is based on the technique developed in McKean-Tanaka [6] and on a precise estimation of $p(t, x, y)$.

Since $\mathfrak{t}(t, w)$ increases when and only when $x_t(w)$ is on ∂D, we obtain a process on the boundary by putting $\tilde{x}(t, w) = x(\mathfrak{t}^{-1}(t, w), w)$, where $\mathfrak{t}^{-1}(t, w)$ is the right continuous inverse sup $\{s | \mathfrak{t}_s(w) \leq t\}$ of $\mathfrak{t}(t, w)$. More precisely, let $\tilde{W}$ be the set of all functions $\tilde{w}$ defined on $[0, \infty)$ taking values in ∂D, which are right continuous and have left limits at each $t \in [0, \infty)$. Let $\tilde{x}_t(\tilde{w}) = \tilde{w}_t$ and denote by $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{B}}_t$ the smallest Borel fields which make $\{\tilde{x}_t, 0 \leq t < \infty\}$ and $\{\tilde{x}_s, 0 \leq s \leq t\}$ measurable respectively. Let $\tilde{P}_x$ be the probability measure on $(\tilde{W}, \tilde{\mathbf{B}})$ defined by

(3.8) $$\tilde{P}_x(B) = P_x(x_{\mathfrak{t}^{-1}(t,w)}, (w)) \text{ belongs to } B \text{ as a function of } t), \qquad B \in \mathbf{B}, x \in \partial D.$$

Then, we have theorem 4.

THEOREM 4. *The process* $\tilde{\mathbf{M}} = \{\tilde{x}_t, \tilde{W}, \tilde{\mathbf{B}}_t, \tilde{P}_x, x \in \partial D\}$ *is a strong Markov process on* ∂D. *The transition probability of* $\mathbf{M}$ *induces a semigroup* $\tilde{T}_t$ *on* $C(\partial D)$ *with generator* $\overline{LH}_0$. *The semigroup* $\{\tilde{T}_t\}$ *and its resolvent operator* K_λ^0 *are given as follows*:

(3.9)
$$\tilde{T}_t\varphi(x) = \tilde{E}_x(\varphi(\tilde{x}_t)) = E_x(\varphi(x(\mathfrak{t}^{-1}(t, w), w))),$$
$$K_\lambda^0\varphi(x) = \int_0^\infty e^{-\lambda t}\tilde{T}_t\varphi(x)\, dt = E_x\left(\int_0^\infty e^{-\lambda \mathfrak{t}(t,w)}\varphi(x_t)\mathfrak{t}(dt, w)\right).$$

Hence, the Markov process on the boundary of order zero is the trace on ∂D of the diffusion described by the time scale $\mathfrak{t}^{-1}(t, w)$, completing the justification of the probabilistic interpretation.

To prove this, we put

(3.10) $$v(t, x) = E_x\left(\int_0^t e^{-\lambda \mathfrak{t}(s)}\varphi(x_s)\mathfrak{t}(ds)\right), \qquad \varphi \in C^{0,k}(\partial D)$$

and prove that $v(t, x)$ satisfies

(3.11)
$$\left(\frac{\partial}{\partial t} - A + \alpha\right) v(t, x) = 0,$$
$$\left(\lambda - \frac{\partial}{\partial n}\right) v(t, x) = \varphi(x),$$
$$\lim_{t \downarrow 0} v(t, w) = 0.$$

Then, by a result on differential equations (see sections 2 and 8 of [13]),

(3.12) $$u(x) = \lim E_x\left(\int_0^\infty e^{-\lambda \mathfrak{t}(s)}\varphi(x_s)\mathfrak{t}(ds)\right)$$

exists and satisfies

$$(3.13)\qquad \begin{aligned} &(\alpha - A)u(x) = 0, \\ &\left(\lambda - \frac{\partial}{\partial n}\right) u(x) = \varphi(x), \qquad x \in \partial D, \end{aligned}$$

implying by theorem 1 that there exists a semigroup on $C(\partial D)$ with generator $\overline{LH_\alpha}$ for any $\alpha \geq 0$, and that u must coincide with $K_\lambda^0\varphi$.

Moreover, we can construct the diffusion determined by A, and

$$(3.14)\qquad L^*u(x) = \gamma(x)u(x) + \delta(x) \lim_{y \to x} Au(y) + \mu(x) \frac{\partial}{\partial n} u(x) = 0, \qquad \gamma \text{ and } \delta \in C^{2,k}(\partial D),$$

using $t(t, w)$. In fact, by a time change, and killing induced by

$$(3.15)\qquad \begin{aligned} \mathfrak{A}(t, w) &= t + \int_0^t |\delta(x_s(w))| \mathfrak{t}(ds, w), \\ \mathfrak{b}(t, w) &= \int_0^t |\gamma(x_s(w))| \mathfrak{t}(ds, w), \end{aligned}$$

respectively, we obtain the corresponding Green operator G_α^* by

$$(3.16)\qquad G_\alpha^* u(x) = E_x\left(\int_0^\infty u(x_s) e^{-\alpha \mathfrak{A}(s) - \mathfrak{b}(s)} \mathfrak{A}(ds)\right).$$

The proof relies on a little more general setup for time change and killing and a computation similar to that in the proof of theorem 4.

4. Markov process on the boundary—II

We have introduced the Markov process and the local time on the boundary of a diffusion, which is determined by an elliptic operator A and L. But, apart from this setup, these concepts should be considered also for general diffusion processes. In fact, for a diffusion process $\mathbf{M}$, for which jumps from boundary points are permitted, we define $G_\alpha^{\min}$ by

$$(4.1)\qquad G_\alpha^{\min} u(x) = E_x\left(\int_0^{\sigma_{\partial D}(w)} e^{-\alpha t} u(x_t(w))\, dt\right),$$

where $\sigma_{\partial D}(w)$ is the time when path w first arrives at ∂D; that is,

$$(4.2)\qquad \sigma_{\partial D}(w) = \inf \{t > 0 | x_t(w) \in \partial D\} \wedge \zeta(w),$$

$\zeta(w)$ being the life time for w. By $\mathbf{M}$ we mean a usual Markov process $\mathbf{M} = \{W, \mathbf{B}_t, P_x, x \in D \cup \{\Delta\}\}$, with sufficient regularities. The path functions are assumed to be continuous except for jumps from ∂D into D and jumps from $\overline{D}$ to the death point Δ. Also, $\zeta(w)$ is the life time in the usual sense: $w_t = \Delta$, if $t \geq \zeta(w)$, and $w_t \in \overline{D}$ if $t < \zeta(w)$.

Then, $\{G_\alpha^{\min}\}$ satisfies a resolvent equation and determines a diffusion process $\mathbf{M}^{\min}$ such that the path functions coincide with those of $\mathbf{M}$ before they arrive at ∂D, and vanish as soon as they arrive at the boundary. We call $\mathbf{M}^{\min}$ the *minimal process* of $\mathbf{M}$. Assume that the transition probability of $\mathbf{M}$ induces a semigroup $\{T_t, t \geq 0\}$ on $C(\overline{D})$, and take a strictly positive $u \in C(\overline{D})$. Then, $u_\alpha(x) = G_\alpha u(x) - G_\alpha^{\min} u(x)$ is regularly α-excessive with respect to M. Hence, there is a unique nonnegative, continuous additive functional $\mathfrak{t}_u^\alpha(t, w)$ such that

$$u_\alpha(x) = E_x\left(\int_0^\infty e^{-\alpha t} \alpha \mathfrak{t}_u^\alpha(t, w)\right). \tag{4.3}$$

The functional $\mathfrak{t}_u^\alpha(t, w)$ increases only when $x_t(w)$ is on ∂D. Although $\mathfrak{t}_u^\alpha$ clearly depends on the choice of α and u, it can be proved that, for another such pair α' and u', there is a bounded positive function φ on ∂D such that

$$\mathfrak{t}_u^\alpha(t, w) = \int_0^t \varphi(x_s(w)) \mathfrak{t}_{u'}^{\alpha'}(ds, w), \tag{4.4}$$

$\varphi(x)$ being bounded away from zero. Thus, $\mathfrak{t}_{u'}^{\alpha'}$ is also represented similarly by $\mathfrak{t}_u^\alpha$. So, denote by T the set of all additive functionals which are equivalent to one $\mathfrak{t}_{u_0}^{\alpha_0}$ in the sense of (4.4). Then, all $\mathfrak{t}_u^\alpha$ are contained in T. Moreover, in the special case of the reflecting diffusion, the local time on the boundary introduced by theorem 3 is in the class, because we have by an easy computation,

$$\mathfrak{t}(t, w) = \int_0^t \frac{1}{\psi(x_s(w))} \mathfrak{t}_u^\alpha(ds, w), \tag{4.5}$$

where $\psi = (\partial/\partial n) G_\alpha^{\min} > 0$. Hence, it seems natural to call a member $\mathfrak{t}$ of T a *local time on the boundary of the diffusion* M, and

$$\mathfrak{x}_t(w) = x_{\mathfrak{t}^{-1}(t,w)}(w) \tag{4.6}$$

a Markov process on the boundary of the diffusion M. The function $\mathfrak{t}^{-1}(t, w)$ is the right continuous inverse of $\mathfrak{t}(t, w)$ as in section 3.

Of course, the local time and the Markov process on the boundary depends on the choice of $\mathfrak{t}$ in T. But, this dependence corresponds exactly to the situation that in the case of section 2, L is not uniquely determined by the diffusion. In fact, the boundary condition $Lu = 0$ and $L'u = 0$ coincides if $L' = \varphi(x)L$, where $\varphi(x)$ is a bounded positive function on ∂D and is bounded away from zero.

Now, from this point of view, the problem we considered in section 2 can be formulated as follows. Given a diffusion process M on $\overline{D}$, determine the class of all diffusion processes whose path functions coincide with those of M before they arrive at the boundary; in other words, determine the class of all diffusion processes which have the minimal process $\mathsf{M}^{\min}$ in common with M, where jumps from the boundary are permitted. A formal consideration, combined with an observation of analytical representation of G_α in (2.19) suggests that the behavior of a path function of such a process is determined by three factors: the behavior before it arrives at ∂D, the trajectory on ∂D, and the way of leaving the boundary into D. If we take this for granted, then the above problem reduces to find all the Markov processes on the boundary and the ways of leaving ∂D which are consistent with the given minimal process $\mathsf{M}^{\min}$.

As an approach to the problem along this line, K. Sato [12] proved that *the diffusion process* M *on* $\overline{D}$ *is determined only by the minimal process* $\mathsf{M}^{\min}$ *and the*

Markov process on the boundary, if (i) *there is no jump into D and* (ii) *there is no stay on ∂D for a set of times with positive Lebesgue measure,* that is

$$E_x\left(\int_0^\infty \chi_{\partial D}(x_s(w))\, ds\right) = 0, \tag{4.7}$$

under certain regularity conditions (which are mainly concerned with the kernel $g_\alpha(x, y)$ of G_α with respect to a certain measure). This is natural in view of the analytical treatment in section 2. In fact,

$$G_\alpha u = G_\alpha^{\min} u + H_\alpha K_0^\alpha \overline{L G_\alpha^{\min}} u, \tag{4.8}$$

where $G_\alpha^{\min}$ and H_α are determined by $\mathbf{M}^{\min}$ and K_0^α by the Markov process on the boundary. Here $\overline{LG_\alpha^{\min}}$ is given by

$$\begin{aligned}\overline{LG_\alpha^{\min}} u &= \delta(x) \lim_{y \to x} A G_\alpha^{\min} u(y) + \mu(x) \frac{\partial}{\partial n} G_\alpha^{\min} u(x) \\ &\quad + \int_{\overline{D}} G_\alpha^{\min} u(y) \nu_x(dy), \\ &= -\delta(x) u(x) + \mu(x) \frac{\partial}{\partial n} G_\alpha^{\min} u(x) + \int_{\overline{D}} G_\alpha^{\min} u(y) \nu_x(dy)\end{aligned} \tag{4.9}$$

noting that $G_\alpha^{\min} u$ vanishes on ∂D. Conditions (i) and (ii) correspond to $\nu_x(\cdot) = 0$ and $\delta(x) = 0$, respectively. Moreover, we can replace μ by 1 by replacing L by some other operator without changing the diffusion. Then, we can consider $\overline{LG_\alpha^{\min}} = (\partial/\partial n) G_\alpha^{\min} u$, and hence G_α is determined by $\mathbf{M}^{\min}$ and K_0^α. Here, we considered $(\partial/\partial n)$ to be determined by $\mathbf{M}^{\min}$, since $(\partial/\partial n)$ is given by $\{a^{ij}(x)\}$. Sato proved the result by making use of a time reversion of diffusion processes, apart from the analytical setup above.

As for the general case, M. Motoo [8] recently obtained a result on this problem, which is almost complete as far as his formulation is concerned. We make a survey of his result in the next section.

5. A probabilistic approach by M. Motoo—I

The content of sections 5 and 6 is based on M. Motoo [7], [8]. However, notations and formulations are slightly modified for our present use. Here, the domain $\overline{D}$ is assumed only to be a compact metric space. Let $\mathbf{M} = (W, \mathbf{B}_t, P_x, x \in \overline{D} \cup \{\Delta\})$ be a strong Markov process on $\overline{D}$ with right continuous path functions having left limits, and let $\mathbf{M}$ be *quasi-left continuous:* if $\{\sigma_n\}$ is an increasing sequence of Markov times and $\sigma = \lim_{n\to\infty} \sigma_n$, then $P_x(x_\sigma = \lim_{n\to\infty} x_{\sigma_n}, \sigma < \infty) = Px(\sigma < \infty)$. Here, the Markov times and strong Markov property are defined on the basis of Borel fields $\mathbf{F} = \bigcap_\mu \mathbf{B}^\mu$, $\mathbf{F}_\sigma = \{A | A \in \mathbf{F}, A \cap \{\sigma < t\} \in \mathbf{F}_t^*, \text{ for any } t\}$, $\mathbf{F}_t^* = \bigcap_\mu \mathbf{B}_t^\mu$, where $\mathbf{B}^\mu$ and $\mathbf{B}_t^\mu$ are the completion of $\mathbf{B}$ and $\mathbf{B}_t$ with respect to $\mu P(\cdot) = \int_{\overline{D}} \mu(dx) P_x(\cdot)$, μ being a bounded positive measure on $\overline{D}$. We take P_x to be extended on $\mathbf{B}^\mu$. We assume that $\mathbf{M}$ has a *reference measure* ν on $\overline{D}$; that is, there is a measure ν on $\overline{D}$ such that

$u(x) = 0$ almost every x with respect to ν implies $u \equiv 0$ for any α-excessive function u of $\mathbf{M}$. Moreover, we assume that *every point of ∂D is regular to ∂D* with respect to $\mathbf{M}$; that is, $P_x(\sigma_{\partial D} = 0) = 1$ for $x \in \partial D$. By a Markov process on $\overline{D}$, we understand that the conditions for $\mathbf{M}$ above are always satisfied, unless it is otherwise specifically stated.

Let $\sigma = \sigma_{\partial D}$ be the hitting time to ∂D given by (4.2). We assume that path function $x_t(w)$ is continuous before $\sigma_{\partial D}(w)$ with P_x probability 1. Let $\mathbf{M}^{\min} = (W^{\min}, \mathbf{B}_t, P_x^{\min}, x \in D \cup \{\Delta\})$ be the minimal process of $\mathbf{M}$, that is, the process on D obtained from M by the killing at $\sigma = \sigma_{\partial D}$. The Green operator of $\mathbf{M}^{\min}$ is given by (4.1). The kernel H_α is the hitting measure of $\mathbf{M}$ to ∂D of order α:

$$H_\alpha(x, E) = E_x(e^{-\alpha\sigma}\chi_E(x_\sigma)), \tag{5.1}$$

which is determined by $\mathbf{M}^{\min}$ by the quasi-left continuity of $\mathbf{M}$.

We define $\mathfrak{t} = \mathfrak{t}_1^{\alpha_0}$ the *local time on the boundary* of $\mathbf{M}$ for a fixed $\alpha_0 > 0$, and define *Markov process on the boundary* $\tilde{\mathbf{M}}$ by $\tilde{x}_t(w) = x_{\mathfrak{t}^{-1}}(t, w)$ as in section 4. The semigroup and the Green operator K_λ^0 are given by (3.9) also in this case. Then, Motoo [7], [8] has proved the following.

THEOREM 5. (i) *Both $\tilde{\mathbf{M}}$ and $\mathbf{M}^{\min}$ are strong Markov processes on ∂D and D respectively, and have quasi-left continuity and reference measures.*

(ii) *Assume that $\mathbf{M}^{\min}$ satisfies the following conditions:*

($\mathbf{M}^{\min}$.1) *$G_\alpha^{\min}u \in C(\overline{D})$, if $u \in C(\overline{D})$, and $H_\alpha\varphi \in C(\overline{D})$, if $\varphi \in C(\partial D)$;*

($\mathbf{M}^{\min}$.2) *$\dfrac{G_\alpha^{\min}u}{G_{\alpha_0}^{\min}1}$ can be extended to $\hat{H}_\alpha u \in C(D)$ for any $u \in C(\overline{D})$ and $\alpha > 0$.*

Then, resolvent G_α of $\mathbf{M}$ is given for $u \in B(\overline{D})$ by

$$G_\alpha u(x) = G_\alpha^{\min}u(x) + H_\alpha K_0^\alpha\Big(-\delta\cdot u + \mu\hat{H}_\alpha u + \int_D \nu(dy)G_\alpha^{\min}u(y)\Big)(x), \tag{5.2}$$

where $B(E)$ is the set of all real-valued bounded measurable functions on E, and where $-\delta(x)$ and $\mu(x)$ are nonnegative functions in $B(\partial D)$, $\nu_x(\cdot)$ is a measure on D such that $\pi(x) = \int_D G_{\alpha_0}^{\min}1(y)\nu_x(dy)$ is finite, and $-\delta(x) + \mu(x) + \pi(x) = 1$, except the set of points $E \subset \partial D$ such that $E_x(\int_0^\infty \chi_E(x_t)\, d\mathfrak{t}(t, w)) = 0$. The operator K_0^α is a special case of K_λ^α defined by

$$K_\lambda^\alpha\varphi(x) = E_x\Big(\int_0^\infty e^{-\alpha t - \lambda\mathfrak{t}(t)}\varphi(x_t)\, d\mathfrak{t}\Big). \tag{5.3}$$

(Similarly, G_α^λ is given by $G_\alpha^\lambda u(x) = E_x\,(\int_0^\infty e^{-\alpha t - \lambda\mathfrak{t}(t)}u(x_t)\, dt)$, which is considered to be dual to K_λ^α in a certain sense.)

(iii) *A system $(\tilde{\mathbf{M}}, \delta, \mu, \nu)$ called the boundary system is unique in the following sense. If $\mathbf{M}$ and $\mathbf{M}'$ are processes satisfying the conditions for $\mathbf{M}$ above, and have boundary systems $(\tilde{\mathbf{M}}, \delta, \mu, \nu)$ and $(\tilde{\mathbf{M}}', \delta', \mu', \nu')$ respectively (more precisely, the space of path functions and the assigned Borel fields of $\mathbf{M}$ and $\mathbf{M}'$ are common, though this is not an essential restriction). Then $\mathbf{M} = \mathbf{M}'$, if and only if $\tilde{\mathbf{M}} = \tilde{\mathbf{M}}'$, and $\delta - \delta'$, $\mu - \mu'$ and $\int_D f(y)(\nu_x(dy) - \nu_x'(dy))$ belongs to the common null space of the Green kernel of $\tilde{\mathbf{M}} = \tilde{\mathbf{M}}'$.*

The use of $\hat{H}_\alpha$ was introduced by K. Sato [12] in a different setup. He considered this as a dual notion of H_α on the basis of time reversion. Motoo gives good probabilistic reasons for the conditions $\mathbf{M}^{\min}.2$ in [8].

The representation (5.2) exhibits a complete correspondence between the analytical expression (2.19) in section 2, if we consider

$$\hat{H}_\alpha u(x) = \lim_{y \to x} \frac{G_\alpha^{\min} u(y)}{G_{\alpha_0}^{\min} 1(y)} \tag{5.4}$$

as a normal derivative of $G_\alpha^{\min} u(y)$ at x. Here, $\nu_x(\cdot)$ is the restriction to D of $\nu_x(\cdot)$ in section 2. In fact, the mass on ∂D of $\nu_x(\cdot)$ has nothing to do with integrating a function of type $G_\alpha^{\min} u$; all the influence of $\nu_x(\cdot)$ on ∂D is contained in K_0^α. In this case the probabilistic meanings of δ, μ, ν are clear by the following construction. Here, in order to describe the situation, we sketch an outline of the proof of (5.2), though it is just an incomplete repetition of a part of Motoo's work [7], [8].

By an *additive functional* $\mathfrak{A}(t, w)$ of $\mathbf{M}$, we understand a $[0, \infty]$ valued function defined on $[0, \infty] \times W$, nonnegative, right continuous in t, continuous at $t = \zeta$, finite for finite t almost surely, $\mathbf{F}_t$-measurable in w for fixed t, and satisfying

$$\mathfrak{A}(t + s, w) = \mathfrak{A}(t, w) + \mathfrak{A}(s, w_t^+). \tag{5.5}$$

Here w_t^+ is the shifted path in the usual sense $w_t(s) = w(t + s)$, for all s. The functional $f \cdot \mathfrak{A}$ given by $(f \cdot \mathfrak{A})(t, w) = \int_0^t f(x_s)\, d\mathfrak{A}(s, w)$ is also an additive functional for nonnegative $f \in B(\overline{D})$. For a continuous additive functional $\mathfrak{A}$ of $\mathbf{M}$, for which $E_x\left(\int_0^\infty e^{-\alpha t}\, d\mathfrak{A}(t)\right)$ is finite, there exists a unique additive functional $\hat{\mathfrak{A}}_\alpha$ such that

$$E_x\left(\int_0^\infty e^{-\alpha t}\, d\hat{\mathfrak{A}}_\alpha\right) = E_x\left(\int_\sigma^\infty e^{-\alpha t}\, d\mathfrak{A}(t)\right). \tag{5.6}$$

Motoo called this $\hat{\mathfrak{A}}_\alpha$ the *α-th order sweeping out of* $\mathfrak{A}$. Then, $\mathfrak{t}$ is the α_0-th order sweeping out of the additive functional

$$\hat{t}(t, w) = t \wedge \xi(w) = \int_0^{t \wedge \xi(w)} 1\, ds. \tag{5.7}$$

Similarly, we define $\mathfrak{t}_0$ and $\mathfrak{t}_1$, as the α_0-th order sweeping out of $\chi_D \cdot \hat{t}$ and $\chi_{\partial D} \cdot \hat{t}$. Then, we have clearly $\mathfrak{t} = \mathfrak{t}_0 + \mathfrak{t}_1$ and $\chi_{\partial D} \cdot \mathfrak{t}_0 = 0$.

To see the structure of $\mathfrak{t}_0$, consider a Markov time $\rho = \rho(k)$ such that $P_x(\rho > 0) = 1$ for $x \in \partial D$ and

$$\rho \le \frac{1}{k} \wedge \sigma_{D_k} \wedge \inf\left\{t \middle| d(x_0, x_t) \ge \frac{1}{k}\right\}, \tag{5.8}$$

where σ_{D_k} is the hitting time to $D_k = \{x \in D | d(x, \partial D) > 1/k\}$. Define

$$\begin{aligned} \sigma_1 = \sigma_1(k) = \sigma_{\partial D}, \qquad \rho_n = \rho_n(k) = \sigma_n + \rho(w_{\sigma_n}^+), \\ \sigma_{n+1} = \sigma_{n+1}(k) = \rho_n + \sigma(w_{\rho_n}^+). \end{aligned} \tag{5.9}$$

Then, we have the following lemma.

LEMMA 1. *For a continuous additive functional such that $\chi_{\partial D} \cdot \mathfrak{A} = 0$, and an f in $B(\overline{D})$ such that $f(x_{t-})$ exists and $|f(x_t) - f(x_0)| \le 1/k$ if $0 \le t < \rho(k)$, we have*

$$(5.10)\qquad \lim_{k\to\infty} E_x\left(\sum_{n=1}^{\infty} e^{-\alpha\rho_n(k)}f(x_{\rho_n(k)}-)u_{\beta,\mathfrak{A}}(x_{\rho_n(k)})\right) = E_x\left(\int_0^\infty e^{-\alpha t}f(x_t)\,d\tilde{\mathfrak{A}}_\beta\right)\qquad \text{for any}\quad \alpha,\beta>0$$

where $u_{\beta,\mathfrak{A}}(x) = E_x\left(\int_0^\zeta e^{-\beta t}\,d\mathfrak{A}(t)\right)$.

Now, consider the set of all discontinuities of $\mathfrak{t}^{-1}(t, w) = \tau(t, w)$,

$$(5.11)\qquad T = \{s > 0 | \tau(s-) < \tau(s),\ \tau(s-) < \zeta\}.$$

Such discontinuities occur because the path function $x_t(w) = w_t$ leaves ∂D and spends time in D for the time interval $(\tau(s-), \tau(s))$, and such leaves are divided into two classes: one induced by jumps from ∂D, and one induced by continuous leave from ∂D. In fact, if we put

$$(5.12)\qquad \begin{aligned} T_d &= \{s | x_{\tau(s-)-} \neq x_{\tau(s-)},\ s \in T\}, \\ T_c &= \{s | x_{\tau(s-)-} = x_{\tau(s-)},\ s \in T\}, \end{aligned}$$

then, by making use of $\{\rho_n\}$ we have,

$$(5.13)\qquad \{\tau(s-) | s \in T_d\} = \{t | x_{t-} \in \partial D,\ x_t \in D\};$$

$$(5.14)\qquad \lim_{k\to\infty} E_x\left(\sum_1^\infty e^{-\alpha\rho_n}\chi_{\partial D}(x_{\rho_n}-)f(x_{\rho_n}-)\chi_D(x_{\rho_n})h(x_{\rho_n})\right) = E_x\left(\sum_{s\in T_d} e^{-\alpha\tau(s-)}\chi_{\partial D}(x_{\tau(s-)-})f(x_{\tau(s-)-})\chi_D(x_{\tau(s-)})h(x_{\tau(s-)})\right);$$

$$(5.15)\qquad \lim_{k\to\infty} E_x\left(\sum_1^\infty e^{-\alpha\rho_n}\chi_D(x_{\rho_n}-)f(x_{\rho_n}-)h(x_{\rho_n})G_{\alpha_0}^{\min}1(x_{\rho_n})\right) = E_x\left(\sum_{S\in T_c} e^{-\alpha\tau(s-)}f(x_{\tau(s-)})h(x_{\tau(s-)})\int_{\tau(s-)}^{\tau(s)} e^{-\alpha_0(t-\tau(s-))}\,dt\right),$$

where $f, h \in B(\overline{D})$ for (5.14) and $f, h \in C(\overline{D})$ for (5.15).

Now, we use the Lévy system of $\mathbf{M}$ introduced by S. Watanabe [15].

LEMMA 2. *There exists a pair* (P, L) *of a continuous additive functional* L *and a measure* $P(x, \cdot)$ *(not necessarily bounded) such that* $P(x, \{x\}) = 0$ *and*

$$(5.16)\qquad E_x\left(\sum_{s\le t} f(x_{s-}, x_s)\right) = E_x\left(\int_0^t \int_{\overline{D}} P(x_s, dy)f(x_s, y)\,dL(s)\right),$$

where $f(x, y) \in B(\overline{D}\times\overline{D})$ *such that* $f(x, x) = 0$. *Moreover,*

$$(5.17)\qquad E_x\left(\sum_{s\le\eta} e^{-\mathfrak{A}(s)}f(x_{s-}, x_s)\right) = E_x\left(\int_0^\eta e^{-\mathfrak{A}(s)}\int_{\overline{D}} P(x_s, dy)f(x_s, y)\,dL\right),$$

where $\mathfrak{A}$ *is a continuous additive functional and* η *is a Markov time.*

Such (P, L), *called the Lévy system of* M, *is unique in the sense that* $Pf\cdot L$ *is unique up to the equivalence of additive functionals for any such* f *cited above.*

Then, applying lemma 2 to $f(x, y) = \chi_{\partial D}(x)f(x)\,\chi_D(y)G_{\alpha_0}^{\min}1(y)$ in (5.14) and noting (5.13), we have

(5.18) $$\lim_{k\to\infty} E_x\left(\sum_1^\infty e^{-\alpha\rho_n}\chi_{\partial D} f(x_{\rho_n}-)\chi_D h(x_{\rho_n})\right) = E_x\left(\int_0^\infty e^{-\alpha t}\chi_{\partial D}(x_t) f(x_t) P(\chi_D h)(x_t)\, dL\right),$$

where $P(f)(x) = \int_{\overline{D}} p(x, dy) f(y)$. Thus, we have, by lemma 1, (5.14), and (5.15),

(5.19) $$E_x\left(\int_0^\infty e^{-\alpha t} f(x_t)\, d\mathfrak{t}_0(t)\right) = E_x\left(\int_0^\infty e^{-\alpha t} f(x_t)\, d(\chi_D \hat{\mathfrak{t}})_{\alpha_0}(t)\right)$$
$$= \lim_{k\to\infty} E_x\left(\sum_1^\infty e^{-\alpha\rho_n} f(x_{\rho_n}-) G_{\alpha_0}^{\min} 1(x_{\rho_n})\right)$$
$$= \lim_{k\to\infty} E_x\left(\sum_1^\infty e^{-\alpha\rho_n} f(x_{\rho_n}-)(\chi_{\partial D}(x_{\rho_n}-) + \chi_D(x_{\rho_n}-)) G_{\alpha_0}^{\min} 1(x_{\rho_n})\right)$$
$$= E_x\left(\int_0^\infty e^{-\alpha t}\chi_{\partial D}(x_t) f(x_t) P(\chi_D G_{\alpha_0}^{\min} 1)(x_t)\, dL\right)$$
$$+ E_x\left(\sum_{s\in T_c} e^{-\alpha\tau(s-)} f(x_{\tau(s-)}) \int_{\tau(s-)}^{\tau(s)} e^{-\alpha_0(t-\tau(s-))}\, dt\right), f \in C(\overline{D}).$$

Since

(5.20) $$\mathfrak{t}_3(t, w) = \chi_{\partial D} P(\chi_D G_{\alpha_0}^{\min} 1) L(t, w)$$

is clearly a continuous additive functional majorized by $\mathfrak{t}_0(t, w)$, the difference $\mathfrak{t}_2(t, w) = \mathfrak{t}_0(t, w) - \mathfrak{t}_3(t, w)$ is also a continuous additive functional such that

(5.21) $$E_x\left(\int_0^\infty e^{-\alpha t} f(x_t)\, d\mathfrak{t}_2(t)\right) = E_x\left(\sum_{s\in T_c} e^{-\alpha\tau(s-)} f(x_{\tau(s-)}) \int_{\tau(s-)}^{\tau(s)} e^{-\alpha_0(t-\tau(s-))}\, dt\right).$$

Hence, there are nonpositive functions $-\delta$, μ, π in $B(\partial D)$ such that

(5.22) $$\mathfrak{t}_1 = -\delta\mathfrak{t}, \qquad \mathfrak{t}_2 = \mu\mathfrak{t}, \qquad \mathfrak{t}_3 = \pi\mathfrak{t}, \qquad -\delta + \mu + \pi = 1,$$

where δ, μ, π are determined uniquely except on the set of points E on ∂D such that $E_x\left(\int_0^\infty \chi_E(x_t)\, dt\right) = 0$. Define a measure $\nu_x(\cdot)$ by

(5.23) $$\nu_x(E) = P(\chi_D \cdot G_{\alpha_0}^{\min} 1)^{-1}(x) P(x, D \cap E), E \in \mathbf{B}_{\overline{D}}.$$

Then, by a computation similar to (5.19), and noting that (5.20) and (5.22) hold, we have

(5.24) $$\lim_{k\to\infty} E_x\left(\sum_1^\infty e^{-\alpha\rho_n}\chi_{\partial D}\cdot f(x_{\rho_n}-)\chi_D\cdot h(x_{\rho_n})\right) = E_x\left(\int_0^\infty e^{-\alpha t} f(x_t)\left(\int_D \nu_{x_t}(dy) h(y)\right) d\mathfrak{t}_3(t)\right),$$

(5.25) $$\lim_{k\to\infty} E_x\left(\sum_1^\infty e^{-\alpha\rho_n}\chi_D\cdot f(x_{\rho_n}-) h(x_{\rho_n}) G_{\alpha_0}^{\min} 1(x_{\rho_n})\right) = E_x\left(\int_0^\infty e^{-\alpha t} f(x_t) h(x_t)\, d\mathfrak{t}_2(t)\right).$$

Now, the proof of (5.2) is obtained in the following way:

$$(5.26)\qquad G_\alpha u(x) - G_\alpha^{\min} u(x) = E\left(\int_0^\infty e^{-\alpha t} u(x_t)\, d\hat{\ell}(t)\right)$$

$$= E_x\left(\int_0^\infty e^{-\alpha t} u(x_t)\, d(\widetilde{\chi_{\partial D}\hat{\ell}})\right)_{\alpha_0}(t) + E_x\left(\int_0^\infty e^{-\alpha t} (\widetilde{\chi_D u\cdot t})_{\alpha_0}(t)\right)$$

$$= E_x\left(\int_0^\infty e^{-\alpha t} u(x_t)\, dt_1(t)\right) + \lim_{k\to\infty} E_x\left(\sum_1^\infty e^{-\alpha\rho_n} E_{x_{\rho_n}}\left(\int_0^\sigma \chi_D u(x_t)\, d\hat{\ell}(t)\right)\right)$$

$$= E_x\left(\int_0^\infty e^{-\alpha t} u(x_t)(-\delta(x_t))\, dt(t)\right) + \lim_{k\to\infty} E_x\left(\sum_1^\infty e^{-\alpha\rho_n} G_\alpha^{\min} u(x_{\rho_n})\right).$$

Since

$$(5.27)\qquad G_\alpha^{\min} u(x_{\rho_n}) = \chi_{\partial D}(x_{\rho_n -}) G_\alpha^{\min} u(x_{\rho_n}) + \chi_D(x_{\rho_n -}) G_\alpha^{\min} u(x_{\rho_n})$$

$$= \chi_{\partial D}(x_{\rho_n -}) G_\alpha^{\min} u(x_{\rho_n}) + \chi_D(x_{\rho_n -}) \hat{H}_\alpha u(x_{\rho_n}) \cdot G_{\alpha_0}^{\min} 1(x_{\rho_n}),$$

we have, by (5.22), (5.24), (5.25), (5.26),

$$(5.28)\qquad G_\alpha u(x) - G_\alpha^{\min} u(x)$$

$$= E_x\left(\int_0^\infty e^{-\alpha t}\{(-\delta u)(x_t)\, dt + \hat{H}_\alpha u(x_t)\, dt_2 + \int_D \nu_{x_t}(dy) G_\alpha^{\min} u(y)\, dt_3\right),$$

$$= H_\alpha K_\alpha^0\left(-\delta u + \mu \hat{H}_\alpha u + \int_D \nu(dy) G_\alpha^{\min} u(y)\right),$$

completing the proof.

6. A probabilistic approach by M. Motoo—II

The boundary system $(\tilde{\mathbf{M}}, \delta, \mu, \nu)$ of $\mathbf{M}$ in section 5 determines the trajectory of path functions of $\mathbf{M}$ on ∂D and the way of leaving ∂D into D; therefore it determines $\mathbf{M}$ by (5.2), combined with the minimal process $\mathbf{M}^{\min}$ of M. Motoo found more detailed properties of the boundary systems, which, except for the use of the regularity conditions ($\mathbf{M}^{\min}$.3), ($\mathbf{M}^{\min}$.4), and (B.5) in theorem 7, combined with the properties in theorem 5, almost characterize the *consistency condition for boundary systems for a given minimal process* $\mathbf{M}^{\min}$.

It can be proved that right-hand sides of

$$(6.1)\qquad \textcircled{H}_\alpha f(x) = \lim_{\beta\to\infty} \beta\, \hat{H}_{\alpha+\beta} H_\alpha f(x)$$

$$= \lim_{\beta\to\infty} \beta \int_{\bar{D}} \hat{H}_{\alpha+\beta}(x, dy) \int_{\partial D} H_\alpha(y, dz) f(x, z)$$

$$(6.2)\qquad \textcircled{H}_0 f(x) = \lim_{\alpha\to 0} \textcircled{H}_\alpha f(x)$$

exist for nonnegative f in $B(\bar{D}\times\partial D)$ such that $f(x, x) = 0$ for $x\in\partial D$, as monotone limits. We have also

$$(6.3)\qquad \textcircled{H}_\alpha f(x) = \frac{H_\alpha f(x)}{G_{\alpha_0}^{\min} 1(x)} = \frac{\int_{\partial D} H_\alpha(x, dy) f(x, y)}{G_{\alpha_0}^{\min} 1(x)}, \qquad x\in D.$$

Similarly, we have the existence of θ and the following relations:

$$\theta(x) = \lim_{\alpha\to\infty} \alpha\hat{H}_\alpha(I - H_0)1(x), \tag{6.4}$$

$$\theta(x) = \frac{1 - H_\alpha 1(x)}{G_{\alpha_0}^{\min}1(x)}, \qquad x \in D. \tag{6.5}$$

The *killing functional* $\mathfrak{A}_\infty$ of M is defined by

$$\tilde{P}_x(\xi \le t) = P_x(\tilde{\xi} \le t) = \tilde{E}_x(\mathfrak{A}_\infty(t)), \tag{6.6}$$

where $\tilde{\xi} = \inf\{s|x_{\tau(s)} = \Delta\}$. Both $\tilde{P}_x$ and $\tilde{E}_x$ correspond to $\tilde{\mathsf{M}}$, and $(\tilde{P}, \tilde{L})$ is the Lévy system of $\tilde{\mathsf{M}}$. Then, we have the following result.

THEOREM 6. *Under the conditions of theorem 5 for* M *and* $\mathsf{M}^{\min}$,

(B.1) $K_0^\lambda(\mu\hat{H}_\alpha\chi_{\partial D})(x) = \int_{\partial D} K_0^\lambda(x, dy)\mu(y)(\hat{H}_\alpha\chi_{\partial D})(y) = 0;$

(B.2) $\tilde{P}f\cdot\tilde{L}$ *majorizes* $(\mu Ⓗ_0 f + \nu H_0 f)\cdot t$ *as an additive functional for nonnegative* f *in* $B(\bar{D}\times\partial D)$ *such that* $f(x, x) = 0$;

(B.3) $\mathfrak{A}_\infty$ *majorizes* $(\mu\cdot\theta + \nu\cdot\theta)\cdot t$;

(B.4) $P_x\left(\int_0^\infty \left\{\frac{1}{1 - \chi_E(\tilde{x}_t)} + \nu_{\tilde{x}_t}(D)\right\} dt = \infty\right) = 1$ *for* $x \in \partial D$, *and*

$$E = \{x \in \partial D | \delta(x) + \mu(x) > 0\}.$$

Statement (B.1) follows from the direct computation of $G_\alpha\chi_{\partial D}$. This means that there is no reflection from pure exit points. To obtain (B.2), we have

$$\tilde{E}_x\left(\int_0^t \tilde{P}f(\tilde{x}_s)\, d\tilde{L}(s)\right) = \tilde{E}_x\left(\sum_{s\le t} f(\tilde{x}_{s-}, \tilde{x}_s)\right) = E_x\left(\sum_{s\le t} f(x_{\tau(s-)-}, x_{\tau(s)})\right), \qquad x \in \partial D, \tag{6.7}$$

by definition. Then, divide the discontinuities of $\tilde{x}_t$ into two classes; one induced by jumps of path functions of M from ∂D to ∂D and the other induced by jumps from ∂D into D and continuous leave from ∂D. Thus,

$$E_x\left(\sum_s f(x_{\tau(s-)-}, x_{\tau(s)})e^{-\alpha\tau(s)}\right) = E_x\left(\int_0^\infty e^{-\alpha t}\chi_{\partial D}(x_t)P(\chi_{\partial D}\cdot f)(x_t)\, dL(t)\right) + E_x\left(\sum_{s\in T} f(x_{\tau(s-)-}, x_{\tau(s)})e^{-\alpha\tau(s)}\right). \tag{6.8}$$

The essential part of the proof lies in

$$E_x\left(\sum_{s\in T} f(x_{\tau(s-)-}, x_{\tau(s-)})(1 - e^{-\beta(\tau(s)-\tau(s-))})e^{-\alpha\tau(s-)}\right) = E_x\left(\int_0^\infty e^{-\alpha t}\beta(\mu\hat{H}_{\alpha+\beta}H_\alpha f + \nu(G_{\alpha+\beta}^{\min}H_\alpha f))\, dt(t)\right), \tag{6.9}$$

which is derived by a similar computation to (5.26) using (5.14)–(5.15). Then, by letting $\beta\to\infty$, we have

$$E_x\left(\sum_{s\in T} f(x_{\tau(s-)-}, x_{\tau(s)})e^{-\alpha\tau(s)}\right) = E_x\left(\int_0^\infty e^{-\alpha t}(\mu Ⓗ_\alpha f + \nu H_\alpha f)(x_t)\, dt(t)\right). \tag{6.10}$$

Hence, we have

$$(6.11)\qquad \tilde{E}_x\Big(\int_0^\infty e^{-\alpha t}\tilde{P}f(\tilde{x}_s)\,d\tilde{L}(s)\Big) = E_x\Big(\int_0^\infty e^{-\alpha t}\chi_{\partial D}(x_t)P(\chi_{\partial D}\cdot f)(x_t)\,dL(t)\Big) + E_x\Big(\int_0^\infty e^{-\alpha t}(\mu Ⓗ_\alpha f + \nu H_\alpha f)(x_t)\,dt(t)\Big).$$

Then, noting that $\tau(s)$ is a Markov time and satisfies $\tau(s+t) = \tau(s) + \tau(t, w^+_{\tau(s)})$, we have, by (6.8) and (6.11),

$$(6.12)\qquad E_x\Big(\sum_{s\le T} e^{-\alpha\tau(s)}f(x_{\tau(s-)-}, x_{\tau(s)})\Big) = E_x\Big(\int_0^{(t)} e^{-\alpha s}\chi_{\partial D}(x_s)\cdot P(\chi_{\partial D}\cdot f)(x_s)\,dL(s)\Big) + E_x\Big(\int_0^t e^{-\alpha s}(\mu Ⓗ_\alpha f + \nu H_\alpha f)(x_s)\,dt(s)\Big).$$

Now, letting $\alpha \to \infty$, we obtain

$$(6.13)\qquad \tilde{E}_x\Big(\int_0^t \tilde{P}f(\tilde{x}_s)\,d\tilde{L}(s)\Big) = E_x\Big(\int_0^{\tau(t)} \chi_{\partial D}\cdot P(\chi_{\partial D}\cdot f)\,dL\Big) + \tilde{E}_x\Big(\int_0^t (\mu Ⓗ_0 f + \nu H_0 f)(\tilde{x}_s)\,ds\Big),$$

which implies (B.2), since f is nonnegative. Statement (B.3) is implied by

$$(6.14)\qquad \tilde{P}_x(\xi \le s) = P_x(\xi \le \tau(s), x_{\xi-} \in \partial D, \xi < \infty) + \tilde{E}_x\Big(\int_0^s (\mu\cdot\theta + \nu\cdot\theta)\,dt\Big),$$

which is proved in a similar way as the proof of (B.2). The second term concerns killing which is caused by traveling through D, and the first is caused by traveling on ∂D. Statement (B.4) assures a kind of regularity of boundary points. In fact, a little stronger condition "$\delta(x) + \mu(x) = 0$ implies $\nu_x(D) = \infty$" is exactly the condition (L.2) in section 2. The converse of theorems 5 and 6 is formulated as follows.

THEOREM 7. *Let* $\mathbf{M}^{\min}$ *be a Markov process on* D, *and let* $(\tilde{\mathbf{M}}, \delta, \mu, \nu)$ *be a system, which satisfies the conditions in theorems* 5 *and* 6. *Assume that* $\mathbf{M}^{\min}$ *satisfies*

($\mathbf{M}^{\min}$.3) $\{\hat{H}_\alpha u, u \in C(\bar{D})\}$ *is dense in* $C(\bar{D})$;

($\mathbf{M}^{\min}$.4) $\mathbf{M}^{\min}$ *can be extended to a stopped process* $\mathbf{M}^0$ *on* $\bar{D}$ *at* ∂D.

Moreover, we assume that the resolvent K^0_λ *of* $\tilde{\mathbf{M}}$ *satisfies*

(B.5) $K^0_\lambda\varphi \in C(\partial D)$, *if* $\varphi \in C(\partial D)$, *or if* $\varphi = (\delta + \mu\hat{H}_\alpha + \nu G^{\min}_\alpha)f$ *for some* $f \in C(\bar{D})$.

Then, there is a Markov $\mathbf{M}$ *process on* $\bar{D}$, *whose minimal process is* $\mathbf{M}^{\min}$ *and whose boundary system is* $(\tilde{\mathbf{M}}, \delta, \mu, \nu)$.

(By a stopped process $\mathbf{M}^0$ on $\bar{D}$ we mean a Markov process on $\bar{D}$ as described at the beginning of section 5 such that $P^0_x(x_t = x, 0 \le t < \infty) = 1$ for any $x \in \partial D$. This condition is easily verified in important cases.

To prove this, Motoo constructs kernels K_0^α on the basis of a delicate probabilistic observation and then G_α on $C(\overline{D})$. This latter is proved to correspond to the desired Markov process $\mathbf{M}$ by making use of the Hille-Yosida theorem. Condition ($\mathbf{M}^{\min}$.3) and ($\mathbf{M}^{\min}$.4) are assumed for the use of Hille-Yosida theorem, and ($\mathbf{M}^{\min}$.4) is the regularity condition for $\mathbf{M}^{\min}$. The essential part of the consistency condition between $\mathbf{M}^{\min}$ and the boundary system consists of those in theorems 5 and 6.

As we have seen above, Motoo's result seems to be almost final as far as his formulation is concerned (excepting the betterment of some regularity conditions). Hence, the construction of path functions in the sense of Itô-McKean [4], and the correspondence between the quantities of concrete analysis are left to be solved.

On the other hand, the case we have considered above corresponds to the "two regular boundary cases" in one dimension, while nothing remarkable is known corresponding to the classification of boundary points or boundary conditions for nonregular boundaries. Only the result of Feller [3] in the case of Markov processes with countable states is at hand. As for this problem, it seems to be reduced, as Motoo suggests in [8] also, to obtain the exit and entrance boundaries for given minimal process, and to make suitable identification of certain parts of these boundaries, and then to determine the class of all consistent boundary systems on the boundary thus constructed.

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