THE MARTIN BOUNDARY OF RECURRENT RANDOM WALKS ON COUNTABLE GROUPS

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1. Introduction

In [16] and [17] Spitzer investigated recurrent random walks on the integers or integral points in the plane from a potential theoretical point of view. Spitzer proved the existence of a potential kernel $A(x, y)$ and found its asymptotic behavior as well as limits of hitting probabilities of finite sets. These results were extended by Spitzer and the author to random walks on countable abelian groups in [13], and the present paper considers recurrent random walks on arbitrary countable groups.

The author is obliged to Professor Spitzer for several discussions on the subject matter of this paper.

An outline of the results follows. The existence of the potential kernel and its simplest properties still hold in the general case, but the asymptotic behavior of the potential kernel was successfully studied only for special groups. More specifically, let $\mathfrak{G}$ be a countable, infinite group with identity element e. A random walk, abbreviated as r.w. in the sequel, on $\mathfrak{G}$ is a (homogeneous) Markov chain $X_0, X_1, \cdots$ with state space $\mathfrak{G}$ and transition probabilities

$$(1.1) \qquad P(x, y) = P\{X_{n+1} = y | X_n = x\} = p(x^{-1}y), \qquad x, y \in \mathfrak{G}$$

where

$$(1.2) \qquad p(z) \geq 0, \qquad \sum_{z \in \mathfrak{G}} p(z) = 1.$$

(The letters x, y, z always denote elements of $\mathfrak{G}$ and x^{-1} is the inverse of x in $\mathfrak{G}$, and xy the product of x and y in $\mathfrak{G}$.) In other words, X_{n+1} is obtained from X_n by right multiplication with the random group element $X_n^{-1}X_{n+1}$ which has the distribution

$$(1.3) \qquad P\{X_n^{-1}X_{n+1} = z\} = p(z).$$

As usual,

$$(1.4) \qquad P_k(x, y) = P\{X_{n+k} = y | X_n = x\} = P_k(e, x^{-1}y)$$

is the x, y entry of the k-th power of P, if $k \geq 1$ and

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$$P_0(x, y) = \delta(x, y) = \begin{cases} 0 & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases} \tag{1.5}$$

Throughout this paper we assume that the r.w. is recurrent (also called persistent) and aperiodic in the space variable, that is,

$$\sum_{k=0}^{\infty} P_k(x, y) = \infty \qquad \text{for all } x, y \in \mathfrak{G} \tag{1.6}$$

(in particular, $P_k(x, y) > 0$ for some k).

We discuss this assumption in section 4. The probability measure induced by the joint probabilities for finite paths starting at $X_0 = x$ will be denoted by P_x, and the expectation with respect to this measure will be denoted by E_x.

The first theorem assures the existence of the potential kernel. In the terminology of Kemeny and Snell, [11] and [12], this states that the Markov chain is normal.

THEOREM 1. *The series $\sum_{k=0}^{\infty} [P_k(e, e) - P_k(x, y)]$ converges for all $x, y \in \mathfrak{G}$, and if this sum is denoted by $A(x, y)$, then*

$$A(x, y) \geq 0, \qquad A(x, x) = 0, \tag{1.7}$$

and (using matrix notation)

$$PA - A = AP - A = \delta. \tag{1.8}$$

Let B be a subset of $\mathfrak{G}$. Following ([17], definition 10.1), we define the hitting probabilities for the r.w. $\{X_n\}$ as follows: if $x \notin B$, $y \in B$, then

$$H_B(x, y) = P_x\,[\text{first visit of } \{X_n\} \text{ to } B \text{ occurs at } y], \tag{1.9a}$$

and

$$H_B(x, y) = \begin{cases} \delta(x, y) & \text{if } x, y \in B, \\ 0 & \text{if } y \notin B. \end{cases} \tag{1.9b}$$

Also, for $x, y \in B$,

$$\begin{aligned} \Pi_B(x, y) &= P_x\,[\text{first return of } \{X_n\} \text{ to } B \text{ occurs at } y] \\ &= P_x\,[\text{for some } n \geq 1, X_n = y \text{ and } X_i \notin B \text{ for } 1 \leq i < n]. \end{aligned} \tag{1.10}$$

Note that $\Pi_B(x, y)$ is not $\delta(x, y)$, because it only measures the return probability after at least one step (compare (1.1) and (1.2) of [16]). Since the stationary measure for the r.w. is the constant measure, that is,

$$\sum_{x \in \mathfrak{G}} P(x, y) = 1, \qquad y \in \mathfrak{G}, \tag{1.11}$$

one has for $B \subseteq \mathfrak{G}$,

$$\sum_{x \in B} \Pi_B(x, y) = 1, \qquad y \in B \tag{1.12}$$

(see theorem 2.3 of [4] together with its proof, or section 0.4 of [15]).

Finally, define

$$\begin{aligned} \tilde{g}(x, y) = E_x\,[&\text{number of visits by } \{X_n\} \text{ to } y \\ &\text{up to and including the time of the first visit to } e]. \end{aligned} \tag{1.13}$$

In this definition we count the visit made at time zero to the starting point of the r.w. Hence,

$$\tilde{g}(e, y) = \delta(e, y). \tag{1.14}$$

Moreover,

$$\tilde{g}(x, e) = P_x[\{X_n\} \text{ visits } e \text{ at some time}] \tag{1.15}$$

$$= 1, \ x \in \mathfrak{G}, \text{ and for } x, y \neq e,$$

$$\tilde{g}(x, y) = g_e(x, y), \tag{1.16}$$

where $g_e(x, y)$ is the notation of Spitzer (formula (3.3) of [16] or definition 10.1 of [17]). The formula $\lim_{x\to\infty} f(x) = \alpha$ will mean that for every $\epsilon > 0$, there exists a finite set $B \subseteq \mathfrak{G}$ such that $|f(x) - \alpha| \leq \epsilon$ whenever $x \in \mathfrak{G} - B$. Similarly, for a sequence $\{x_n\} \subseteq \mathfrak{G}$, $x_n \to \infty$ will mean that x_n is outside every finite set, eventually.

Once theorem 1 has been proved, one can copy from [16] or ([17], chapter 7) the proofs of the following relations:

If $C = \{e, c\}$, $c \in \mathfrak{G} - e$, then there exists a constant ϕ_C, such that $0 \leq \phi_C \leq 1$ and

$$\phi_C = H_C(x, e) - [A(x, c) - A(x, e)]\, \Pi_C(c, e) \qquad \text{for all} \quad x \in \mathfrak{G}. \tag{1.17}$$

For $y \neq e$,

$$\tilde{g}(x, y) = A(x, e) + A(e, y) - A(x, y). \tag{1.18}$$

By (1.6), $\Pi_C(c, e) > 0$, and hence

$$|A(x, c) - A(x, e)| \leq \Pi_C^{-1}(c, e), \tag{1.19}$$

since $0 \leq \phi_C, H_C \leq 1$.

Consequently, $\tilde{g}(x, c)$ is bounded in x for fixed c and one can introduce the metric

$$\rho(x, y) = \sum_{c \in \mathfrak{G}} [\epsilon(c)|\tilde{g}(x, c) - \tilde{g}(y, c)|] + |\eta(x) - \eta(y)|] \tag{1.20}$$

on $\mathfrak{G}$ where $\epsilon(c) > 0$, $\eta(x) > 0$, and

$$\sum_{c \in \mathfrak{G}} \epsilon(c) \sup_{x \in \mathfrak{G}} |\tilde{g}(x, c)| < \infty, \tag{1.21}$$

$$\lim_{x \to \infty} \eta(x) = 0.$$

If $\mathfrak{G}^*$ is the completion of $\mathfrak{G}$ under the metric ρ, the Martin boundary of the r.w. is defined as $\mathfrak{G}^* - \mathfrak{G}$. This is clearly independent of the choice of ϵ and η under the above restrictions. For more details concerning the definition and significance of the Martin boundary, we refer the reader to [12] and [15]. Here we shall only need a few remarks.

A sequence $\{x_n\} \subseteq \mathfrak{G}$ converges to a boundary point if and only if $x_n \to \infty$ and

$$\lim_{n \to \infty} \tilde{g}(x_n, c) \text{ exists for all } c \in \mathfrak{G}. \tag{1.22}$$

By (1.15) it suffices to require (1.22) for $c \neq e$ and, by (1.18), (1.22) is then equivalent to

$$(1.23) \qquad \lim_{n\to\infty} A(x_n, c) - A(x_n, e) \text{ exists for all } c \in \mathfrak{G} - e,$$

which in turn, by (1.17), is equivalent to

$$(1.24) \qquad \lim_{n\to\infty} H_{\{e,c\}}(x_n, e) \text{ exists for all } c \in \mathfrak{G} - e.$$

Boundary points are therefore in a one-to-one correspondence with the different limiting functions (of the variable c) which can be obtained in (1.23) or (1.24), and the only aspect of the boundary investigated in this paper is the number of boundary points. In view of the above remarks, this boils down to a partial investigation of the asymptotic behavior of the potential kernel $A(x, y)$. As pointed out by Spitzer [16], one may regard statements about the limits in (1.23) as analogues of the renewal theorem for transient random walk.

The boundary defined here is usually called the entrance boundary. There is no need to consider the exit boundary separately since this coincides with the entrance boundary of the reversed random walk. This is the r.w. with transition probabilities

$$(1.25) \qquad P^*(x, y) = P(y, x) = p(y^{-1}x).$$

If A^* is the corresponding potential kernel, then

$$(1.26) \qquad A^*(x, y) = A(y, x),$$

and, in obvious notation, by (1.18),

$$(1.27) \qquad \tilde{g}^*(x, y) = \tilde{g}(y, x), \qquad x, y \in \mathfrak{G} - e.$$

The main results of sections 3 and 4 state that if $\mathfrak{G}$ belongs to certain classes of groups, every recurrent r.w. on $\mathfrak{G}$ has one boundary point only. This is by far the neatest situation, because it means that for any recurrent r.w. on $\mathfrak{G}$ and all $c \neq e$,

$$(1.28) \qquad \lim_{x\to\infty} A(x, c) - A(x, e) \quad \text{and} \quad \lim_{x\to\infty} H_{\{e,c\}}(x, e)$$

both exist. It will turn out that in this case necessarily

$$(1.29) \qquad \lim_{x\to\infty} A(x, c) - A(x, e) = 0, \qquad c \neq e,$$

and hence, by (1.18) and (1.15),

$$(1.30) \qquad \lim_{x\to\infty} \tilde{g}(x, y) = A(e, y) + \delta(e, y).$$

It then also follows from the Doob-Hunt representation theorem, which we use in the form of section 5 of [12], especially formula (5.3), that A is the unique solution of

$$(1.31) \qquad AP - A = \delta,$$

which satisfies (1.7). (The uniqueness of the solutions to (1.31) is also considered by Spitzer in section 5 of [16] and section 31 of [17], but those proofs require a knowledge of the possible sets of zeros of $A(x, e)$.) If one prefers, one can also say that A^* is the unique solution of (1.7) and $P^*A^* - A^* = \delta$, which is equivalent to (1.31). As a matter of fact, in [16] and [17] this is the form in which the results occur, because Spitzer considers the exit boundary rather than the entrance boundary.

The case of two boundary points is settled by the following theorem.

THEOREM 4. *The r.w.* $\{X_n\}$ *on* $\mathfrak{G}$ *has two boundary points if and only if* $\mathfrak{G}$ *has an infinite cyclic subgroup* $\mathfrak{H}$ *of finite index such that the imbedded r.w. on* $\mathfrak{H}$ *has finite variance* (*see section* 2 *for definitions*).

The quoted results show that the boundary consists of one or two points, among others, for all solvable groups, or all direct sums of two infinite groups, or groups of finite permutations of the integers (see section 4). We conjecture that this is true in general, but so far the only result we proved for all groups is theorem 5, which states that the boundary always has 1, 2, or infinitely many points.

We end the paper with some open questions.

2. The existence of the potential kernel

If $\mathfrak{H}$ is any subgroup of $\mathfrak{G}$, we define the *imbedded r.w.* on $\mathfrak{H}$ as in [13]. The times of the successive visits by X_n to $\mathfrak{H}$ are denoted by $(0 <)T_1 < T_2 < \cdots$. Thus $X_{T_i} \in \mathfrak{H}$, but $X_n \notin \mathfrak{H}$ when $n \geq 1$ and $n \neq T_i$ for any i. The notation

$$Y_n = X_{T_n}, \qquad n = 1, 2, \cdots, \tag{2.1}$$

will be used throughout for the imbedded r.w., on $\mathfrak{H}$, even though $\mathfrak{H}$ may vary from occasion to occasion.

The distribution of Y_1 depends on X_0, but for $n \geq 1$, the $Y_n^{-1}Y_{n+1}$ are independent, identically distributed random variables, taking values in $\mathfrak{H}$. If $X_0 = e$, then also Y_1 has the same distribution. In particular, if $\mathfrak{H}$ is an infinite cyclic group (definitions of group theoretical terms can be found in [14]), generated by c, say, then $Y_n = c^{k_n}$ for some integer k_n and $k_{n+1} - k_n$, $n = 1, 2, \cdots$ is a sequence of independent, identically distributed integer valued random variables. If X_n is a recurrent r.w., so is Y_n, and in this case $k_{n+1} - k_n$ must either have zero expectation or its first absolute moment is infinite. We shall only talk about the variance of the imbedded r.w. if $\mathfrak{H}$ is infinite cyclic, and in this case it is defined as

$$\sigma^2(Y) = \sigma^2(k_{n+1} - k_n). \tag{2.2}$$

The hitting probabilities for the imbedded r.w. are defined as in (1.9) with $\{X_n\}$ replaced by $\{Y_n\}$ and denoted by $\overline{H}_B(\cdot\,,\cdot)$. Since $\{Y_n\}$ is just $\{X_n\}$ observed only when in $\mathfrak{H}$, one has

$$H_B(x, y) = \overline{H}_B(x, y) \qquad \text{whenever} \quad x, y \in \mathfrak{H}, B \subseteq \mathfrak{H}. \tag{2.3}$$

Another very useful relation between H and $\overline{H}$ is given in

$$H_B(x, y) = \sum_{z \in \mathfrak{H}} H_{\mathfrak{H}}(x, z)\overline{H}_B(z, y), \tag{2.4}$$

valid for $y \in B \subseteq \mathfrak{H}$. Equation (2.4) follows immediately from the fact that the random walk cannot enter B before it enters $\mathfrak{H}$.

If $\mathfrak{H}$ is infinite cyclic with generator c, it is isomorphic to the additive group of integers, so that theorem B of [16] and sections 29 and 30 of [17] apply to $\{Y_n\}$. In our notation the results state for any finite set $B \subseteq \mathfrak{H}$ that

$$\lim_{|k| \to \infty} H_B(c^k, y) = \lim_{|k| \to \infty} \overline{H}_B(c^k, y) \tag{2.5}$$

exists for $y \in B$ if $\sigma^2(Y) = \infty$. If $\sigma^2(Y) < \infty$, then

$$\lim_{k \to \infty} H_B(c^k, y) = \lim_{k \to \infty} \overline{H}_B(c^k, y) \tag{2.6}$$

and

$$\lim_{k \to -\infty} H_B(c^k, y) = \lim_{k \to -\infty} \overline{H}_B(c^k, y) \tag{2.7}$$

both exist and are different for some finite B and y.

As in [17], we introduce

$$A_n(x, y) = \sum_{k=0}^{n} [P_k(e, e) - P_k(x, y)]. \tag{2.8}$$

No change is needed in the proof of proposition 11.3 in [17] to derive for any subset $B \subseteq \mathfrak{G}$ and all $x, y \in \mathfrak{G}$,

$$\sum_{t \in \mathfrak{G}} P_{n+1}(x, t) H_B(t, y) = H_B(x, y) - \sum_{t \in B} A_n(x, t)[\Pi_B(t, y) - \delta(t, y)]. \tag{2.9}$$

Note that (1.12) is needed in the proof. In particular, for $C = \{e, c\}$, $c \neq e$ and $y = e$,

$$\sum_{t \in \mathfrak{G}} P_{n+1}(x, t) H_C(t, e) = H_C(x, e) - [A_n(x, c) - A_n(x, e)]\, \Pi_C(c, e) \tag{2.10}$$

(again (1.12) is used).

For the time being, fix $c \neq e$ and let

$$\mathfrak{H} = \{c^k \colon k = 0, \pm 1, \cdots\} \tag{2.11}$$

be the subgroup generated by c. In the future we shall abbreviate (2.11) to

$$\mathfrak{H} = \langle c \rangle. \tag{2.12}$$

The number of elements in a set B shall be denoted by $|B|$.

The proof of theorem 1 breaks down in three cases:

I. $|\mathfrak{H}| = h < \infty$,

II. $|\mathfrak{H}| = \infty$ and $\sigma^2(Y) = \infty$,

III. $|\mathfrak{H}| = \infty$ and $\sigma^2(Y) < \infty$.

(σ^2 in cases II and III is defined in (2.2)). In all cases we use (2.4) to reduce the problem to a problem about $\overline{H}$ whose behavior is known from (2.5)–(2.7).

LEMMA 1. *Let $y \in B \subseteq \mathfrak{G}$. Then*

$$H_{zB}(zx, zy) = H_B(x, y) \qquad \text{for all} \quad x, z \in \mathfrak{G}. \tag{2.13}$$

If $n_1 < n_2 < \cdots$ is a sequence of integers such that

$$\lim_{i\to\infty} \sum_{t\in\mathfrak{G}} P_{n_i+1}(x_0, t)H_B(t, y) \tag{2.14}$$

exists for some $x_0 \in \mathfrak{G}$, then

$$\lim_{i\to\infty} \sum_{t\in\mathfrak{G}} P_{n_i+j}(x, t)H_B(t, y) \tag{2.15}$$

exists for all integers j and all $x \in \mathfrak{G}$ and is independent of x.

PROOF. Equation (2.13) follows from the one-to-one correspondence between the paths $x_0, x_1, \cdots, x_n$ and $zx_0, zx_1, \cdots, zx_n$ and the relation

$$\begin{aligned} P_{x_0}\{X_i = x_i, 1 \le i \le n\} &= P_{zx_0}\{X_i = zx_i, 1 \le i \le n\} \\ &= \prod_{i=0}^{n-1} p(x_i^{-1}x_{i+1}). \end{aligned} \tag{2.16}$$

The proof of (2.15) is essentially in ([17], pp. 347–348). From (2.9) one has

$$\begin{aligned} &\left|\sum_{t\in\mathfrak{G}} P_{n+j}(x, t)H_B(t, y) - \sum_{t\in\mathfrak{G}} P_{n+1}(x, t)H_B(t, y)\right| \\ &\qquad \le \sum_{t\in B} |A_{n+j}(x, t) - A_{n+1}(x, t)|\, |\Pi_B(t, y) - \delta(t, y)| \end{aligned} \tag{2.17}$$

But, from (2.8) one has

$$|A_{n+j}(x, t) - A_{n+1}(x, t)| \le \sum_{k=n-|j|}^{n+|j|} P_k(e, e) + P_k(x, t) \le 2|j| + 1. \tag{2.18}$$

Since P is double stochastic (see (1.11)),

$$\lim_{k\to\infty} \mathrm{P}_k(x, t) = 0 \tag{2.19}$$

(see [7], p. 358), and from (2.18), (2.19), and (1.12) it follows that the right-hand side of (2.17) tends to zero as $n \to \infty$. Thus the limit in (2.14) exists when $n_i + 1$ is replaced by $n_i + j$. From any subsequence of the n_i we can select a further subsequence $\{n_i'\}$ such that

$$f(x) = \lim_{i\to\infty} \sum_{t\in\mathfrak{G}} P^{n_i'+1}(x, t)H_B(t, y) \tag{2.20}$$

exists for all x. As in ([17], pp. 347–348) f must be a nonnegative solution of $Pf = f$, and hence a constant. Since this can be done for any subsequence of the n_i, (2.15) follows.

LEMMA 2. *For every integer k and any $x \in \mathfrak{G}$*

$$\lim_{n\to\infty} \sum_{t\in\mathfrak{G}} P_{n+1}(x, t)H_{\mathfrak{H}}(t, c^k) = \frac{1}{h} \qquad \text{in case I,} \tag{2.21}$$

and

$$\lim_{n\to\infty} \sum_{t\in\mathfrak{G}} P_{n+1}(x, t)H_{\mathfrak{H}}(t, c^k) = 0 \qquad \text{in cases II and III.} \tag{2.22}$$

PROOF. By (2.13) and (1.1),

$$(2.23)\qquad \sum_{t\in\mathfrak{G}} P_{n+1}(x, t)H_{\mathfrak{H}}(t, c^k) = \frac{1}{h}\sum_{m=0}^{h-1}\sum_{t\in\mathfrak{G}} P_{n+1}(c^m x, c^m t)H_{\mathfrak{H}}(c^m t, c^{m+k})$$
$$= \frac{1}{h}\sum_{m=0}^{h-1}\sum_{t\in\mathfrak{G}} P_{n+1}(c^m x, t)H_{\mathfrak{H}}(t, c^{m+k}).$$

Any sequence of integers contains a subsequence n_i for which the limits (2.15), with B replaced by $\mathfrak{H}$ and y by c^{m+k}, exist. Then, by lemma 1 and (2.23),

$$(2.24)\qquad \lim_{i\to\infty}\sum_{t\in\mathfrak{G}} P_{n_i+1}(x, t)H_{\mathfrak{H}}(t, c^k) = \lim_{i\to\infty}\sum_{t\in\mathfrak{G}} P_{n_i+1}(x, t)\,\frac{1}{h}\sum_{m=0}^{h-1} H_{\mathfrak{H}}(t, c^{m+k}).$$

However, in case I,

$$(2.25)\qquad \sum_{m=0}^{h-1} H_{\mathfrak{H}}(t, c^{m+k}) = \sum_{y\in\mathfrak{H}} H_{\mathfrak{H}}(t, y) = 1,$$

since $\mathfrak{H} = \{c^{m+k}, 0 \le m \le h-1\}$ for every k. Thus (2.21) follows. Similarly, one proves (2.22) from (2.23), (2.24), and the fact that in cases II and III,

$$(2.26)\qquad \sum_{m=0}^{h-1} H_{\mathfrak{H}}(t, c^{m+k}) \le 1$$

for arbitrarily large h.

Cases I and II can now be finished as in [13]. In fact, by (2.4) with B replaced by $C = \{e, c\}$, one has in case I

$$(2.27)\qquad \lim_{n\to\infty}\sum_{t\in\mathfrak{G}} P_{n+1}(x, t)H_C(t, e) = \lim_{n\to\infty}\sum_{t\in\mathfrak{G}} P_{n+1}(x, t)\sum_{c^k\in\mathfrak{H}} H_{\mathfrak{H}}(t, c^k)\overline{H}_C(c^k, e)$$
$$= \frac{1}{h}\sum_{k=0}^{h-1}\overline{H}_C(c^k, e).$$

Similarly, in case II (compare the argument immediately following (3.9) of [13]),

$$(2.28)\qquad \lim_{n\to\infty}\sum_{t\in\mathfrak{G}} P_{n+1}(x, t)H_C(t, e) = \lim_{|k|\to+\infty}\overline{H}_C(c^k, e)$$

(the right-hand side exists by (2.5)). In both cases, therefore,

$$(2.29)\qquad \phi_C = \lim_{n\to\infty}\sum_{t\in\mathfrak{G}} P_{n+1}(x, t)H_C(t, e)$$

exists, is independent of x by lemma 1, and, since $0 \le H_C \le 1$, satisfies $0 \le \phi_C \le 1$.

It follows from (2.10) with $x = c$ that

$$(2.30)\qquad \phi_C = \Pi_C\,(c, e)\lim_{n\to\infty} A_n(c, e),$$

since $A_n(c, c) = 0$ and $H_C\,(c, e) = 0$. This proves the existence of

$$(2.31)\qquad A(c, e) = \sum_{k=0}^{\infty}[P_k(e, e) - P_k(c, e)].$$

Case III was treated in [13] by first establishing that the index of $\mathfrak{H}$ in $\mathfrak{G}$ is finite (lemma 3.4 in [13]). We believe this to be true even if $\mathfrak{G}$ is not abelian, but we only have a partial result in this direction (lemma 5 below). This difficulty is circumvented by the use of the following theorem.

THEOREM 2. (*Central limit theorem with a random number of summands.*) *Let* (U_k, V_k), $k \geq 1$, *be a sequence of independent, identically distributed 2-vectors for which*

$$P\{V_k \geq 0\} = 1, \qquad P\{V_k = 0\} < 1, \tag{2.32}$$
$$EU_k = 0, \qquad \sigma^2 = \sigma^2(U_k) < \infty.$$

Put

$$R(n) = \min\left\{m: \sum_{k=1}^{m} V_k > n\right\}. \tag{2.33}$$

Then

$$\lim_{n\to\infty} P\left\{\frac{\sum_{k=1}^{R(n)} U_k}{\sigma\sqrt{R(n)}} \leq \alpha\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha} e^{-t/2^2}\,dt. \tag{2.34}$$

Even though we believe this theorem to be new, we shall not give its proof here since it is unrelated to the boundary theory. We apply this theorem in case III with $U_n = k_n - k_{n-1}$, $V_n = T_n - T_{n-1}$, $n \geq 1$ where we take $X_0 = e$, $k_0 = 0$, $T_0 = 0$ (see beginning of this section for definitions). The conditions of the theorem are fulfilled for these variables and σ^2 is just as in (2.2). Moreover, in the notation of theorem 2,

$$\sum_{k\leq K}\sum_{t\in\mathfrak{G}} P_{n+1}(e, t)H_{\mathfrak{H}}(t, c^k) \tag{2.35}$$
$$= P_e\{\text{first entrance to } \mathfrak{H} \text{ after the } n\text{-th step occurs at some } c^k \text{ with } k \leq K\}$$
$$= P\left\{\sum_{k=1}^{R(n)} U_k \leq K\right\}.$$

Since $R(n) \to \infty$ in probability as $n \to \infty$, it follows that the limit as $n \to \infty$ of the left-hand side of (2.35) equals $\frac{1}{2}$ for each fixed K. This takes the place of (3.25) and (3.26) in [13], and as in [13], one obtains from the first equality in (2.27), (2.22), (2.6), and (2.7),

$$\lim_{n\to\infty}\sum_{t\in\mathfrak{G}} P_{n+1}(e, t)H_C(t, e) = \tfrac{1}{2}\lim_{k\to\infty}\,[\overline{H}_C(c^k, e) + \overline{H}_C(c^{-k}, e)] = \phi_C, \text{ say}. \tag{2.36}$$

As in cases I and II, this establishes the existence of ϕ_C in (2.29), independent of x by lemma 1, and the existence of $A(c, e)$.

The existence of $A(x, y)$ for general x, y now follows from

$$A_n(x, y) = A_n(e, x^{-1}y) = A_n(y^{-1}x, e) \tag{2.37}$$

(see (2.8)) by taking limits as $n \to \infty$. Similarly, (1.17) follows from (2.10). The proof of (1.7) and $PA - A = \delta$ as given in [16] or ([17], propositions 1.3 and 13.3) need only trivial changes. Finally, when this is applied to the reversed r.w., one obtains $P^*A^* - A^* = \delta$ which is the same as $AP - A = \delta$. The proof of theorem 1 is therefore complete.

We end this section with the following remark.

REMARK 1. If $\{x_n\} \subseteq \mathfrak{G}$ converges to a boundary point, then

$$\lim_{n\to\infty} H_B(x_n, y) \text{ exists} \tag{2.38}$$

for all finite sets $B \subseteq \mathfrak{G}$ and $y \in B$. The proof is the same as in theorem 30.1 of [17]. One takes limits as $n \to \infty$ in (2.9). The right-hand side has a limit, and thus the left-hand side has a limit, which by lemma 1 is independent of x. Using (1.12) one has therefore, for a suitable $\phi = \phi_B(y)$,

$$\phi_B(y) = H_B(x, y) - \sum_{t\in B} [A(x, t) - A(x, e)][\Pi_B(t, y) - \delta(t, y)]. \tag{2.39}$$

If one substitutes x_n for x and uses (1.23), one obtains (2.38).

3. The number of boundary points

The general form of the results of this section is that the boundary of $\{X_n\}$ has one or two points whenever there exists an infinite normal subgroup $\mathfrak{H}$ of $\mathfrak{G}$ such that the boundary of the imbedded r.w. $\{Y_n\}$ has one or two points.

Throughout the remainder of this paper we shall use the following notation. If $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$, $[\mathfrak{G} : \mathfrak{H}]$ = index of $\mathfrak{H}$ in $\mathfrak{G}$, and $\mathfrak{H} \lhd \mathfrak{G}$ shall mean that $\mathfrak{H}$ is a normal subgroup of $\mathfrak{G}$. For a subgroup $\mathfrak{H}$, we shall always use $z_0 = e$, $z_1, \cdots$ for a set of representatives of its cosets, such that each $x \in \mathfrak{G}$ has a unique representation

$$x = hz_i \tag{3.1}$$

with

$$h = h(x) \in \mathfrak{H} \quad \text{and} \quad i = i(x). \tag{3.2}$$

From now on z_i and $i(x)$ will only be used in this sense.

LEMMA 3. *If $\{X_n\}$ has one boundary point only, then*

$$\lim_{x\to\infty} A(x, c) - A(x, e) = 0, \qquad c \in \mathfrak{G}. \tag{3.3}$$

Moreover, if $B_1, B_2, \cdots$ is an increasing sequence of finite subsets of $\mathfrak{G}$ such that $\mathfrak{G} = \bigcup B_j$, then for each fixed $y \in \mathfrak{G}$ and $\epsilon > 0$ there is a j_0 such that $y \in B_j$ and

$$\lim_{x\to\infty} H_{B_j}(x, y) \leq \epsilon \tag{3.4}$$

for all $j \geq j_0$. If $\mathfrak{G}$ is an infinite cyclic group, then (3.4) *holds without the condition that $\{X_n\}$ has one boundary point only.*

PROOF. If $\{X_n\}$ has one boundary point only, then with $C = \{e, c\}$,

$$\lim_{x\to\infty} H_C(x, e) \tag{3.5}$$

exists (see (1.24)), and it follows from (2.29) and (2.19) that ϕ_C must have the same value as (3.5) (see [13], beginning of proof of theorem 3.1). Equation (3.3) follows, therefore, from (1.17) by taking limits ($|x| \to \infty$). As remarked in the introduction (1.18) was proved in lemma 3.1 of [16] for $x, y \neq e$ (see (1.16)), and for $x = e$, $y \neq e$ both sides of (1.18) are zero (see (1.14) and (1.7)). We therefore conclude from (1.18), (3.3), (1.7), and (1.15), that

$$\lim_{x \to \infty} \tilde{g}(x, y) = A(e, y) + \delta(e, y). \tag{3.6}$$

On the other hand, for $x \neq e$, $e, y \in B \subseteq \mathfrak{G}$,

$$\tilde{g}(x, y) \geq H_B(x, y) + \sum_{t \in B - e} \tilde{g}(x, t) P(t, y). \tag{3.7}$$

In fact, the left-hand side represents the expected number of visits to y, up to and including the first visit to e when the r.w. starts at x. In the right-hand side of (3.7), $H_B(x, y)$ is the expected number of visits to y at the first entrance to B, and $\sum \tilde{g}(x, t)P(t, y)$ represents the expected number of visits to y, coming in one step from a point of B, but without having visited e. Because $e \in B$, both terms in the right-hand side count only visits up to and including the first visit to e. If one takes limits as $|x| \to \infty$ in (3.7) and uses (3.6), (1.7), and (1.8), one obtains

$$\begin{aligned} \limsup_{x \to \infty} H_B(x, y) \leq \delta(e, y) &+ A(e, y) \\ &- \sum_{t \in B} A(e, t) P(t, y) = \sum_{t \notin B} A(e, t) P(t, y). \end{aligned} \tag{3.8}$$

Since $A \geq 0$ and, again by (1.8), $\sum_{t \in \mathfrak{G}} A(e, t)P(t, y) < \infty$, one can make the last member of (3.8) small by taking B large. This proves (3.4).

If $\mathfrak{G} = \langle c \rangle$ is infinite cyclic, $\{X_n\}$ has either one boundary point or, if its variance σ^2 (in the sense of (2.2)) is finite, $\{X_n\}$ has two boundary points and (3.6) has to be replaced by

$$\lim_{k \to \pm\infty} \tilde{g}(c^k, c^m) = A(e, c^m) \pm \frac{m}{\sigma^2} + \delta(e, c^m) \tag{3.9}$$

(see theorem B of [16] or section 29 of [17]). Only the last case has to be considered. But because a recurrent r.w. on the integers with finite variance has zero mean, one has then

$$\sum_{m=-\infty}^{+\infty} mP(c^m, c^\ell) = \sum_{m=-\infty}^{+\infty} mP(c^{-\ell}, c^{-m}) = \ell. \tag{3.10}$$

This allows us to repeat the proof of (3.4) with a few trivial changes, even when $\sigma^2 < \infty$.

The following simple remark will be useful in some of the proofs.

REMARK 2 (Compare lemma 3.1 of [13]). If $B \subseteq \mathfrak{G}$ and $u \in \mathfrak{G}$, then

$$\begin{aligned} H_B(x, y) &\geq H_{B \cup xu}(x, xu) H_B(xu, y) \\ &= H_{x^{-1}B \cup u}(e, u) H_B(xu, y), \end{aligned} \tag{3.11}$$

since the right-hand side represents the probability of going from x to xu before entering B, and then to enter B at y. In the last step, (2.13) was used. If one replaces x by xu and u by u^{-1} in (3.11), a similar inequality is obtained which, together with (3.11), leads to

$$|H_B(x, y) - H_B(xu, y)| \leq 1 - H_{x^{-1}B \cup u}(e, u) + 1 - H_{x^{-1}B \cup e}(u, e). \tag{3.12}$$

In applications of (3.12), we will have a sequence $\{x_n\} \subseteq \mathfrak{G}$ such that for every finite set $D \subseteq \mathfrak{G}$,

$$x_n^{-1}B \cap D = \varnothing \qquad \text{for sufficiently large } n. \tag{3.13}$$

The right-hand side of (3.12) with x_n instead of x will then tend to zero so that

$$\lim_{n\to\infty} |H_B(x_n, y) - H_B(x_n u, y)| = 0. \tag{3.14}$$

LEMMA 4. *If $\mathfrak{H} \lhd \mathfrak{G}$ and $|\mathfrak{H}| = h < \infty$, then*

$$\lim_{x\to\infty} H_{\mathfrak{H}}(x, y) = \frac{1}{h}, \qquad y \in \mathfrak{H}. \tag{3.15}$$

Assume now that $\mathfrak{H}$ is an arbitrary infinite subgroup of $\mathfrak{G}$ and let $g_0 = e, g_1, \cdots$ be a numbering of the elements of $\mathfrak{G}$. If there exist (possibly finite) sequences $M^i = \{m_n^i\}$ of indices such that

$$m_1^i < m_2^i < \cdots, \tag{3.16}$$

$$z_i g_m z_i^{-1} \in \mathfrak{H} \qquad \text{whenever} \quad m \in M^i \tag{3.17}$$

(see (3.1) and (3.2) for notation), and such that

$$\lim_{i\to\infty} |M^i| = \infty, \qquad \text{whereas} \quad \limsup_{i\to\infty} m_n^i < \infty \tag{3.18}$$

for each fixed n, then

$$\lim_{x\to\infty} H_{\mathfrak{H}}(x, y) = 0, \qquad y \in \mathfrak{H}. \tag{3.19}$$

In particular, (3.19) holds if $|\mathfrak{H}| = \infty$ and $\mathfrak{H} \lhd \mathfrak{G}$, or $[\mathfrak{G}:\mathfrak{H}] < \infty$.

PROOF. This is an imitation of the proofs of lemmas 3.2 and 3.3 in [13]. By remark 2,

$$\begin{aligned}\lim_{i\to\infty} \sup_{h\in\mathfrak{H}} &|H_{\mathfrak{H}}(hz_i, y) - H_{\mathfrak{H}}(hz_i u, y)| \\ &\leq \limsup_{i\to\infty} [1 - H_{z_i^{-1}\mathfrak{H}\cup u}(e, u) + 1 - H_{z_i^{-1}\mathfrak{H}\cup e}(u, e)] = 0\end{aligned} \tag{3.20}$$

for $y \in \mathfrak{H}$ and $u \in \mathfrak{G}$. Thus, for fixed $u_1, \cdots, u_k$,

$$\lim_{i\to\infty} H_{\mathfrak{H}}(hz_i, y) = \lim_{i\to\infty} \frac{1}{k} \sum_{r=1}^{k} H(hz_i u_r, y). \tag{3.21}$$

If $\mathfrak{H} \lhd \mathfrak{G}$ and $|\mathfrak{H}| < \infty$, we take $k = |\mathfrak{H}|$ and $\{u_1, \cdots, u_k\} = \mathfrak{H}$. Then, using (2.13),

$$\sum_{r=1}^{k} H_{\mathfrak{H}}(hz_i u_r, y) = \sum_{u\in\mathfrak{H}} H_{\mathfrak{H}}(uz_i, y) = \sum_{u\in\mathfrak{H}} H(z_i, u) = 1, \tag{3.22}$$

since $h, y \in \mathfrak{H}$ and $\mathfrak{H}$ is a normal subgroup.

For the second part of the lemma, we take $u_j = g_{m_j^i}$. Equation (3.20) remains valid, even though u_j now depends on i, because of (3.18). Instead of (3.22), we now obtain from (3.16) and (3.17),

$$\sum_{r=1}^{k} H_{\mathfrak{H}}(hz_i g_{m_r^i}, y) \leq \sum_{u\in\mathfrak{H}} H_{\mathfrak{H}}(z_i, u) \leq 1. \tag{3.23}$$

If $|\mathfrak{H}| < \infty$, then $x \to \infty$ if and only if $i(x) \to \infty$ so that (3.15) follows from (3.21) and (3.22). To prove (3.19) under the assumptions (3.16)–(3.18), one only needs to observe that for any k, by (3.21)–(3.23),

$$\lim_{i\to\infty} \sup_{h\in\mathfrak{H}} H_{\mathfrak{H}}(hz_i, y) \le \frac{1}{k}, \tag{3.24}$$

whereas for each fixed i,

$$\lim_{h\to\infty, h\in\mathfrak{H}} H_{\mathfrak{H}}(hz_i, y) = \lim_{h\to\infty, h\in\mathfrak{H}} H(z_i, h^{-1}y) = 0. \tag{3.25}$$

Finally, if $|\mathfrak{H}| = \infty$ but $[\mathfrak{G}:\mathfrak{H}] < \infty$, then there are only finitely many possible values for z_i and (3.18) is vacuous, but (3.19) follows from (3.25). If $|\mathfrak{H}| = \infty$ and $\mathfrak{H} \lhd \mathfrak{G}$, we can take M^i independent of i. In fact, if

$$\mathfrak{H} = \{g_{m_1}, g_{m_2}, \cdots\}, \qquad m_1 < m_2 < \cdots, \tag{3.26}$$

we take $m_n^i = m_n$. With this choice, (3.17) and (3.18) hold automatically.

THEOREM 3. *If there exists an infinite subgroup* $\mathfrak{H}$ *of* $\mathfrak{G}$ *such that*

$$\lim_{x\to\infty} H_{\mathfrak{H}}(x, y) = 0 \qquad \text{for all} \quad y \in \mathfrak{H} \tag{3.27}$$

and such that $\{Y_n\}$, *the imbedded r.w. on* $\mathfrak{H}$, *has only one boundary point, then the r.w. on* $\mathfrak{G}$ *has only one boundary point.*

(Note that lemma 4 is useful in checking condition (3.27). Some applications appear in section 4.)

PROOF. As remarked in the introduction, we have to show that for every $c \in \mathfrak{G} - e$,

$$\lim_{x\to\infty} H_C(x, e) \text{ exists,} \tag{3.28}$$

whereas before $C = \{e, c\}$. Let $\mathfrak{H}$ satisfy the assumptions of the theorem and let $c \in \mathfrak{G} - e$ be fixed. Pick a $y_0 \in \mathfrak{H}$ such that $H_{\mathfrak{H}}(c, y_0) > 0$. Just as in (3.11), one has

$$H_{\mathfrak{H}}(x, y_0) \ge H_{\mathfrak{H}\cup c}(x, c) H_{\mathfrak{H}}(c, y_0), \tag{3.29}$$

and thus, by (3.27),

$$\lim_{x\to\infty} H_{\mathfrak{H}\cup c}(x, c) = 0. \tag{3.30}$$

Choose an increasing sequence of finite sets $B_j \subseteq \mathfrak{H}$ such that $\mathfrak{H} = \bigcup_{j=1}^{\infty} B_j$. Again as in (3.11),

$$\begin{aligned} H_{B_j}(x, y_0) &\ge H_{B_j\cup c}(x, c) H_{B_j}(c, y_0) \\ &\ge H_{B_j\cup c}(x, c) H_{\mathfrak{H}}(c, y_0), \qquad \text{if} \quad y_0 \in B_j. \end{aligned} \tag{3.31}$$

By lemma 3, applied to $\{Y_n\}$, we can for any given $\epsilon > 0$ find a j_0 such that $e, y_0 \in B_{j_0}$ and

$$\lim_{t\to\infty, t\in\mathfrak{H}} \overline{H}_{B_{j_0}}(t, y_0) \le \epsilon H_{\mathfrak{H}}(c, y_0), \tag{3.32}$$

and thus by (3.31) and (2.3),

$$\limsup_{t\to\infty, t\in\mathfrak{H}} H_{B_{i_0}\cup c}(t, c) \le \epsilon. \tag{3.33}$$

Consequently, there is a j_1 such that for $j \ge j_1$,

$$H_{B_{i_0}\cup c}(t, c) \le 2\epsilon \qquad \text{for} \quad t \in \mathfrak{H} - B_j. \tag{3.34}$$

Now (3.30) states that when the starting point x is far out, X_n will hit $\mathfrak{H}$ before c with a probability close to one. By (3.27), the point where $\mathfrak{H}$ is entered first will be outside B_j with a probability close to one, and by (3.26), X_n will then enter B_{j_0} before c except for a set of probability at most 2ϵ.

Formally, this argument runs as follows. For some $|\theta_i| \le 1$,

$$\begin{aligned} H_C(x, e) &= \sum_{t\in\mathfrak{H}\cup c} H_{\mathfrak{H}\cup c}(x, t) H_C(t, e) \\ &= \sum_{t\in\mathfrak{H}} H_{\mathfrak{H}}(x, t) H_C(t, e) + \theta_1 H_{\mathfrak{H}\cup c}(x, c) \\ &= \sum_{t\in\mathfrak{H}-B_j} H_{\mathfrak{H}}(x, t) H_C(t, e) + \theta_1 H_{\mathfrak{H}\cup c}(x, c) \\ &\quad + \theta_2 \sum_{t\in B_j} H_{\mathfrak{H}}(x, t) \\ &= \sum_{t\in\mathfrak{H}-B_j} H_{\mathfrak{H}}(x, t) \sum_{z\in B_{i_0}} H_{B_{i_0}}(t, z) H_C(z, e) \\ &\quad + \theta_1 H_{\mathfrak{H}\cup c}(x, c) + \theta_2 \sum_{t\in B_j} H_{\mathfrak{H}}(x, t) \\ &\quad + \theta_3 \sup_{t\in\mathfrak{H}-B_j} H_{B_{i_0}\cup c}(t, c). \end{aligned} \tag{3.35}$$

For $j \ge j_1$ and x sufficiently far out, the last three terms of (3.35) are together at most 3ϵ (by (3.30), (3.27), and (3.34)). Since $\{Y_n\}$ has one boundary point only,

$$\lim_{u\to\infty, u\in\mathfrak{H}} H_{B_{i_0}}(u, z) = \lim_{u\to\infty, u\in\mathfrak{H}} \overline{H}_{B_{i_0}}(u, z) \text{ exists,} \tag{3.36}$$

(see remark 1 and (2.3)). One can therefore choose $j \ge j_1$ so large that

$$\begin{aligned} &\Big| \sum_{z\in B_{i_0}} H_{B_{i_0}}(t, z) H_C(z, e) \\ &\qquad - \lim_{u\to\infty, u\in\mathfrak{H}} \sum_{z\in B_{i_0}} H_{B_{i_0}}(u, z) H_C(z, e) \Big| \le \epsilon \qquad \text{for} \quad t \in \mathfrak{H} - B_j. \end{aligned} \tag{3.37}$$

With such a choice of j one concludes from (3.35) and (3.27) that for x sufficiently far out,

$$\begin{aligned} &\Big| H_C(x, e) - \lim_{u\to\infty, u\in\mathfrak{H}} \sum_{z\in B_{i_0}} H_{B_{i_0}}(u, z) H_C(z, e) \Big| \\ &\qquad \le 4\epsilon + \sum_{t\in B_j} H_{\mathfrak{H}}(x, t) \le 5\epsilon. \end{aligned} \tag{3.38}$$

Since $\epsilon > 0$ was arbitrary, (3.28) follows.

THEOREM 4. *The r.w. $\{X_n\}$ on $\mathfrak{G}$ has two boundary points if and only if there exists an infinite cyclic subgroup $\mathfrak{H} = \langle c \rangle$, such that $[\mathfrak{G}:\mathfrak{H}] < \infty$ and $\sigma^2(Y) < \infty$, where $\sigma^2(Y)$ is the variance of the imbedded r.w. $\{Y_n\}$ on $\mathfrak{H}$ (see (2.1) and 2.2)).*

Note. If a subgroup $\mathfrak{H}$ as specified exists, but with $\sigma^2(Y) = \infty$, then $\{Y_n\}$ has only one boundary point by (2.5), and then $\{X_n\}$ has only one boundary point by theorem 3 and lemma 4. Thus the condition $\sigma^2(Y) < \infty$ turns out to be independent of the choice of $\mathfrak{H}$.

PROOF. First assume that there exists an $\mathfrak{H} = \langle c \rangle$ as specified in the theorem. Let $z_0 = e, z_1, \cdots, z_{\lambda-1}$ be representatives of its cosets where $\lambda = [\mathfrak{G}:\mathfrak{H}] < \infty$ (see (3.1) and (3.2)). Every $x \in \mathfrak{G}$ can now be written uniquely as

$$x = c^k z_i, \qquad k = k(x) \text{ integer}, \qquad 0 \le i = i(x) \le \lambda - 1. \tag{3.39}$$

Since $\lambda < \infty$, $x \to \infty$ if and only if $|k(x)| \to \infty$, and by remark 2,

$$\lim_{|k| \to \infty} |H_B(c^k z_i, e) - H_B(c^k, e)| = 0 \tag{3.40}$$

for $B = \{e, b\}$, $b \in \mathfrak{G} - e$ fixed. Thus, if we show that for each $b \in \mathfrak{G} - e$,

$$\lim_{k \to \infty} H_B(c^k, e) \quad \text{and} \quad \lim_{k \to -\infty} H_B(c^{-k}, e) \tag{3.41}$$

exist, then $\{X_n\}$ has at most two boundary points. Since for $\sigma^2(Y) < \infty$ there also are at least two boundary points by (2.6), (2.7), (2.3), and remark 1, this will prove one implication of the theorem. However, if one takes $B_j = \{c^k\colon |k| \le j\}$, then, by (2.6), (2.7), and (2.3),

$$\lim_{k \to \infty} H_{B_j}(c^k, z) \quad \text{and} \quad \lim_{k \to -\infty} H_{B_j}(c^k, z) \tag{3.42}$$

both exist for $z \in B_j$. The proof of the existence of the limits in (3.41) is now exactly the same as that of (3.28) (starting immediately after 3.30).

To prove the converse we assume that $\mathfrak{G}$ has an element c of infinite order. The case where no such element exists will be postponed to the proof of theorem 5 below. Let then

$$\mathfrak{H} = \langle c \rangle \subseteq \mathfrak{G}, \qquad |\mathfrak{H}| = \infty. \tag{3.43}$$

As before, $\{Y_n\}$ will be the imbedded r.w. on $\mathfrak{H}$. We distinguish two cases. First,

$$\lim_{x \to \infty} H_{\mathfrak{H}}(x, y) = 0 \qquad \text{for all} \quad y \in \mathfrak{H}. \tag{3.44}$$

We may then assume that $\sigma^2(Y) < \infty$, for otherwise $\{Y_n\}$ and $\{X_n\}$ have one boundary point only by (2.5) and theorem 3. By the part of the theorem which has been proved already, we may also assume

$$[\mathfrak{G}:\mathfrak{H}] = \infty. \tag{3.45}$$

Second, there exists a sequence $\{x_n\} \in \mathfrak{G}$ such that $x_n \to \infty$ $(n \to \infty)$, and for some $y_0 \in \mathfrak{H}$,

$$\lim_{n \to \infty} H_{\mathfrak{H}}(x_n, y_0) \ge \eta > 0. \tag{3.46}$$

These cases clearly exhaust the situations in which the boundary may have more than two points. It therefore suffices to show that in these cases $\{X_n\}$ has infinitely many boundary points.

Case (1). Choose k_0 such that with $B = \{e, c^{k_0}\}$,

$$\lim_{k\to\infty} \overline{H}_B(c^k, e) \neq \lim_{k\to-\infty} \overline{H}_B(c^k, e). \tag{3.47}$$

Such a k_0 exists since $\{Y_n\}$ has two boundary points (see (2.6) and (2.7)). Define

(i) $\alpha_N(x) = P_x$ [first entrance of $\{X_n\}$ to $\mathfrak{H}$ is at c^k with $k \geq N$],
(ii) $\beta_N(x) = P_x$ [first entrance of $\{X_n\}$ to $\mathfrak{H}$ is at c^k with $k \leq -N$].

Let $\{x_n\} \subset \mathfrak{G}$ be a sequence for which

$$\alpha = \lim_{n\to\infty} \alpha_N(x_n) \quad \text{and} \quad \beta = \lim_{n\to\infty} \beta_N(x_n) \tag{3.48}$$

exist and such that (see (3.2) for notation)

$$i(x_n) \to \infty, \qquad (n \to \infty). \tag{3.49}$$

By (3.44), these limits are independent of N and $\alpha + \beta = 1$. From (2.4) it follows that

$$\lim_{n\to\infty} H_B(x_n, e) = \alpha \lim_{k\to+\infty} \overline{H}_B(c^k, e) + (1 - \alpha) \lim_{k\to-\infty} \overline{H}_B(c^k, e) \tag{3.50}$$

(cf. the argument following (3.9) in [13]). We now construct other sequences $\{x_n'\}$ for which the corresponding α' takes any value in $[0, 1]$. This is done as follows. By (2.13),

$$\alpha_N(c^{-r}x_n) = \alpha_{N+r}(x_n), \tag{3.51}$$

and for fixed N, n,

$$\lim_{r\to\infty} \alpha_{N+r}(x_n) = 0, \qquad \lim_{r\to-\infty} \alpha_{N+r}(x_n) = 1. \tag{3.52}$$

Moreover, by (2.13), (3.49), and (3.44),

$$\begin{aligned} 0 \leq \alpha_{N+r}(x_n) - \alpha_{N+r+1}(x_n) &= H_{\mathfrak{H}}(x_n, c^{N+r}) \\ &= H_{\mathfrak{H}}(c^{-N-r}x_n, e) = o(1), \qquad (n \to \infty). \end{aligned} \tag{3.53}$$

From (3.51)–(3.53), one sees that one can choose r_n such that

$$\lim_{n\to\infty} \alpha_N(c^{-r_n}x_n) = \alpha' \tag{3.54}$$

for any given $\alpha' \in [0, 1]$. Moreover, by (3.49), $c^{-r_n}x_n \to \infty$ as $n \to \infty$. In view of (3.47) and (3.50), different values of α' correspond to different boundary points (see remark 1) so that $\{X_n\}$ must have infinitely many boundary points.

Case (2). We may assume that $\{x_n\}$ has been chosen (by use of the diagonal procedure) such that in addition to (3.46),

$$\lim_{n\to\infty} H_B(x_n, y) \text{ exists,} \tag{3.55}$$

for all finite sets $B \subseteq \mathfrak{G}$ and all $y \in B$. Let $B_0 \subseteq \mathfrak{H}$ be any set of at least $2m/\eta$ points and such that $y_0 \in B_0$. Inequality (3.46) implies

$$\lim_{n\to\infty} H_{B_0}(x_n, y_0) \geq \eta. \tag{3.56}$$

But

$$\lim_{n\to\infty} H_{B_0}(x_n, y) \geq \frac{\eta}{2} \tag{3.57}$$

can hold for at most $2/\eta$ points $y \in B_0$, since for any x and B,

$$\sum_{y\in B} H_B(x, y) = 1. \tag{3.58}$$

Thus we can find at least $(m - 1)2/\eta$ points $y_1 \in B_0$ for which

$$\lim_{n\to\infty} H_{B_0}(x_n, y_1) \leq \frac{\eta}{2}. \tag{3.59}$$

Fix $y_1 \in B_0$ such that (3.59) holds. On the other hand, by (2.13) and (3.46),

$$\begin{aligned}\liminf_{n\to\infty} H_{B_0}(y_1y_0^{-1}x_n, y_1) &\geq \lim_{n\to\infty} H_{\mathfrak{H}}(y_1y_0^{-1}x_n, y_1) \\ &= \lim_{n\to\infty} H_{\mathfrak{H}}(x_n, y_0) \geq \eta.\end{aligned} \tag{3.60}$$

Again we may assume, after selecting a subsequence if necessary, that also for all finite sets B and $y \in B$,

$$\lim_{n\to\infty} H_B(y_1y_0^{-1}x_n, y) \text{ exists.} \tag{3.61}$$

From remark 1, (3.59), and (3.60), it follows that $\{x_n\}$ and $\{y_1y_0^{-1}x_n\}$ converge to different boundary points. Still, there are at most $2\cdot 2/\eta$ points $y \in B_0$ for which either (3.57) or

$$\lim_{n\to\infty} H_{B_0}(y_1y_0^{-1}x_n, y) \geq \frac{\eta}{2} \tag{3.62}$$

holds. There are, therefore, at least $(m - 2)2/\eta$ possible choices for y_2 satisfying

$$\lim_{n\to\infty} H_{B_0}(x_n, y_2) \leq \frac{\eta}{2} \quad \text{and} \quad \lim_{n\to\infty} H_{B_0}(y_1y_0^{-1}x_n, y_2) \leq \frac{\eta}{2}, \tag{3.63}$$

and as above we find that, perhaps after selecting a subsequence once more, $\{y_2y_0^{-1}x_n\}$ converges to a boundary point which differs from the limits of both $\{x_n\}$ and $\{y_1y_0^{-1}x_n\}$. Proceeding in this manner we find m sequences $\{y_iy_0^{-1}x_n\}$, $i = 0, \cdots, m - 1$, converging to different boundary points. Since m is arbitrary, $\{X_n\}$ must have infinitely many boundary points in case (2) as well.

REMARK 3. It will be apparent from the applications how to combine the results of this section if one has a chain of subgroups

$$\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \cdots \qquad \supset \mathfrak{G}_n = \{e\}, \tag{3.64}$$

such that at each step $\mathfrak{G}_i$ and $\mathfrak{G}_{i+1}$ form a group and a subgroup to which one of these results applies. Particularly useful is the fact that

$$\lim_{x\to\infty, x\in\mathfrak{G}_i} H_{\mathfrak{G}_{i+2}}(x, y) = 0, \qquad y \in \mathfrak{G}_{i+2} \tag{3.65}$$

whenever

$$\lim_{x\to\infty, x\in\mathfrak{G}_i} H_{\mathfrak{G}_{i+1}}(x, z) = 0, \qquad z \in \mathfrak{G}_{i+1}, \tag{3.66}$$

as well as

$$\lim_{z\to\infty, z\in\mathfrak{G}_{i+1}} H_{\mathfrak{G}_{i+2}}(z, y) = 0, \qquad y \in \mathfrak{G}_{i+2}, \tag{3.67}$$

hold. This is an immediate consequence of (2.4) and (2.3) with $\mathfrak{H} = \mathfrak{G}_{i+1}$, $B = \mathfrak{G}_{i+2}$.

THEOREM 5. *Any recurrent random walk $\{X_n\}$ on a group $\mathfrak{G}$ has 1, 2, or infinitely many boundary points.*

PROOF. In theorem 4 and its proof all cases were covered where $\mathfrak{G}$ has an element of infinite order. We may therefore assume that all elements of $\mathfrak{G}$ have finite order. We shall prove this theorem and, at the same time, complete the proof of theorem 4 by showing that $\mathfrak{G}$ must have an infinite subgroup $\mathfrak{H}$ such that $\{Y_n\}$, the imbedded r.w. on $\mathfrak{H}$, has one or infinitely many boundary points. This will indeed be sufficient, for if $\{Y_n\}$ has infinitely many boundary points, then a fortiori this is true for $\{X_n\}$ by (2.3), and if $\{Y_n\}$ has one boundary point and (3.44) holds, then $\{X_n\}$ has one boundary point by theorem 3. Finally, if $\{Y_n\}$ has one boundary point but (3.44) does not hold, then the proof of case (2) in theorem 4 goes through without any change, and $\{X_n\}$ must have infinitely many boundary points.

If $\mathfrak{G}$ is locally finite, that is (see [14]) if every finite subset generates a finite group, it is known, [9] or [10], to have an infinite abelian subgroup $\mathfrak{H}$. Moreover, any r.w. on $\mathfrak{H}$ has one boundary point only by [13]. This leaves only one case to investigate, namely where $\mathfrak{G}$ has an infinite subgroup $\mathfrak{H}$, generated by the finite set $\{c_1, \cdots, c_r, c_1^{-1}, \cdots, c_r^{-1}\}$ and such that $\{Y_n\}$ has at least two boundary points. In this case let $\{x_n\}$, $\{y_n\} \subseteq \mathfrak{H}$ be two sequences for which $x_n \to \infty$, $y_n \to \infty$ as $n \to \infty$ and such that

$$\lim_{n\to\infty} \overline{H}_C(x_n, e) \neq \lim_{n\to\infty} \overline{H}_C(y_n, e) \tag{3.68}$$

for some $C = \{e, c\} \subseteq \mathfrak{H}$. If I is any interval between $\lim \overline{H}_C(x_n, e)$ and $\lim \overline{H}_C(y_n, e)$, we shall find a sequence $\{z_n\} \subseteq \mathfrak{H}$ for which $z_n \to \infty$ as $n \to \infty$, and

$$\overline{H}_C(z_n, e) \in I. \tag{3.69}$$

This will of course show that $\{Y_n\}$ must have infinitely many boundary points, since one can choose infinitely many disjoint intervals I between $\lim \overline{H}_C(x_n, e)$ and $\lim \overline{H}_C(y_n, e)$. In order to prove (3.69), we choose an increasing sequence $\{B_j\}$ of subsets of $\mathfrak{H}$ such that $\mathfrak{H} = \bigcup B_j$ and determine j_0 such that

$$|\overline{H}_C(z, e) - \overline{H}_C(zd, e)| \leq \epsilon = \tfrac{1}{2}\cdot\text{length of } I \tag{3.70}$$

whenever

$$z \notin B_{j_0} \quad \text{and} \quad d \in \{c_1, \cdots, c_r, c_1^{-1}, \cdots, c_r^{-1}\}. \tag{3.71}$$

Such a j_0 can be found by remark 2. Since $\mathfrak{H}$ is finitely generated and has no elements of infinite order, it has only one end point, in the terminology of Freudenthal ([8], especially section 7.6). This means that for each j and sufficiently large n there exists a "path" from x_n to y_n outside B_j. Here we mean by a path from x_n to y_n outside B_j, a finite sequence $u_0 = x_n, u_1, u_2, \cdots, u_m = y_n$ of elements of $\mathfrak{H}$ such that

$$u_{i+1} = u_i d_i \quad \text{for some} \quad d_i \in \{c_1, \cdots, c_r, c_1^{-1}, \cdots, c_r^{-1}\}, \tag{3.72}$$

and

$$u_i \notin B_j. \tag{3.73}$$

For $j \geq j_0$ we have, therefore,

$$|\overline{H}_C(u_i, e) - \overline{H}_C(u_{i+1}, e)| \leq \epsilon, \qquad i = 0, \cdots, m-1, \tag{3.74}$$

and hence, for some i,

$$\overline{H}_C(u_i, e) \in I. \tag{3.75}$$

Choose some u_i satisfying (3.73) and (3.75) and denote it by z_j. Then $z_j \to \infty$ as $j \to \infty$ by (3.73), and (3.69) is satisfied because of (3.75). This completes the proof of theorem 5.

4. Applications

A. Let $\mathfrak{G}$ have a finite normal series (see [14])

$$\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \cdots \supset \mathfrak{G}_n = \{e\} \tag{4.1}$$

(that is, $\mathfrak{G}_{i+1} \lhd \mathfrak{G}_i$), and let r be the highest index for which

$$[\mathfrak{G}_r : \mathfrak{G}_{r+1}] = \infty. \tag{4.2}$$

If any recurrent r.w. on $\mathfrak{G}_r/\mathfrak{G}_{r+1}$ has one boundary point only, then the recurrent r.w. $\{X_n\}$ on $\mathfrak{G}$ has one boundary point only.

Proof. By repeated application of lemma 4 and remark 3, one has

$$\lim_{x \to \infty} H_{\mathfrak{G}_r}(x, y) = 0 \qquad \text{for all} \quad y \in \mathfrak{G}_r. \tag{4.3}$$

Let $\{Y_n\}$ be the imbedded r.w. on $\mathfrak{G}_r$. If $r + 1 = n$ or $\mathfrak{G}_{r+1} = \{e\}$, then $\{Y_n\}$ has only one boundary point by assumption, and the result is then contained in theorem 3. In general, $\mathfrak{G}_{r+1}$ is finite by the definition of r. Let

$$\mathfrak{G}_{r+1} = \{g_0 = e, g_1, \cdots, g_{h-1}\} \tag{4.4}$$

and choose representatives $z_0 = e, z_1, \cdots$ of the cosets of $\mathfrak{G}_{r+1}$ in $\mathfrak{G}_r$ such that each element x of $\mathfrak{G}_r$ can be written uniquely as

$$x = g_k z_i, \qquad 0 \leq k = k(x) \leq h - 1 \quad \text{and} \quad 0 \leq i = i(x). \tag{4.5}$$

Since $\mathfrak{G}_{r+1} \lhd \mathfrak{G}_r$, the cosets $\{\mathfrak{G}_{r+1} z_i\}$ form the group $\mathfrak{G}_r/\mathfrak{G}_{r+1}$ under the usual multiplication. The induced r.w. on $\mathfrak{G}_r/\mathfrak{G}_{r+1}$ is defined as the sequence of random variables $\{Z_n\}_{n \geq 1}$, where $Z_n \in \mathfrak{G}_r/\mathfrak{G}_{r+1}$ and

$$Z_n = \mathfrak{G}_{r+1} z_i \quad \text{when} \quad Y_n \in \mathfrak{G}_{r+1} z_i. \tag{4.6}$$

One easily checks that Z_n is indeed a random walk on $\mathfrak{G}_r/\mathfrak{G}_{r+1}$, and by assumption it has one boundary point only. In other words, if

$$B = \mathfrak{G}_{r+1} z_0 \cup \mathfrak{G}_{r+1} z_{i_0}, \qquad i_0 \neq 0, \tag{4.7}$$

then for $j = 0$ or i_0,

$$\lim_{i \to \infty} P\,[\text{first visit of } \{Z_n\} \text{ to } B \text{ is in } \mathfrak{G}_{r+1}z_j | Z_0 = \mathfrak{G}_{r+1}z_i\} \tag{4.8}$$

exists. In the previous notation this means

$$\lim_{i(x) \to \infty} \sum_{k=0}^{h-1} \overline{H}_B(x, g_k z_j) \text{ exists.} \tag{4.9}$$

($\overline{H}$ denotes the hitting probabilities for $\{Y_n\}$). We also know from lemma 4, that

$$\lim_{x \to \infty, x \in \mathfrak{G}_r} \overline{H}_{\mathfrak{G}_{r+1}}(x, g_k) = \frac{1}{h}, \qquad 0 \leq k \leq h - 1. \tag{4.10}$$

A proof entirely analogous to that of (4.10), or rather (3.15), shows that $1/h$ is the limit

$$\lim_{x \to \infty, x \in \mathfrak{G}_r} P\,\{\text{first visit of } \{Y_n\} \text{ to } B \text{ is at } g_k z_j | Y_1 = x \text{ and first visit of } \{Y_n\} \text{ to } B \text{ is in } \mathfrak{G}_{r+1}z_j\} \tag{4.11}$$

(4.9)–(4.11) combined show that

$$\lim_{x \to \infty, x \in \mathfrak{G}_r} \overline{H}_B(x, g_k) \quad \text{and} \quad \lim_{x \to \infty, x \in \mathfrak{G}_r} \overline{H}_B(x, g_k z_{i_0}) \tag{4.12}$$

exist for $0 \leq k \leq h - 1$. This is even true if $i_0 = 0$ by (4.10). Thus for $c = g_{k_0} z_{i_0}$, $i_0 \neq 0$ and $C = \{e, c\}$ (compare (2.4)),

$$\lim_{x \to \infty, x \in \mathfrak{G}_r} \overline{H}_C(x, e) = \lim_{x \to \infty, x \in \mathfrak{G}_r} \sum_{k=0}^{h-1} [\overline{H}_B(x, g_k)\overline{H}_C(g_k, e) + \overline{H}_B(x, g_k z_{i_0})\overline{H}_C(g_k z_{i_0}, e)] \tag{4.13}$$

exists, and similarly for $i_0 = 0$. Thus $\{Y_n\}$ has only one boundary point, and the result is again included in theorem 3.

B. Let $\mathfrak{G}$ have a normal series as in A and let r be defined as in A. If now $\mathfrak{G}_r/\mathfrak{G}_{r+1}$ is abelian, then $\{X_n\}$ has one or two boundary points. The last case can occur only in the situation described in theorem 4. *This example covers all groups with a finite solvable normal series.*

Proof. Since $\mathfrak{G}_r/\mathfrak{G}_{r+1}$ is commutative, the results of section 3 of [13] apply. Therefore, a recurrent r.w. $\{Z_n\}$ on $\mathfrak{G}_r/\mathfrak{G}_{r+1}$ has one boundary point only, unless $\mathfrak{G}_r/\mathfrak{G}_{r+1}$ has an infinite cyclic subgroup $\mathfrak{H}$ of finite index, and the imbedded r.w. $\{U_n\}$ of $\{Z_n\}$ on $\mathfrak{H}$ has finite variance. In this last case, $\{Z_n\}$ has two boundary points. As in example A we take for $\{Z_n\}$ the induced r.w. on $\mathfrak{G}_r/\mathfrak{G}_{r+1}$ (see (4.6)). If $\{Z_n\}$ has only one boundary point, then the same is true for $\{X_n\}$ as in example A. We may assume, therefore, that $\{Z_n\}$ has two boundary points. Then there must exist an element $c \in \mathfrak{G}_r$ of infinite order such that

$$[\mathfrak{G}_r/\mathfrak{G}_{r+1} : \mathfrak{H}] < \infty, \tag{4.14}$$

where $\mathfrak{H}$ is the subgroup of $\mathfrak{G}_r/\mathfrak{G}_{r+1}$ generated by $\mathfrak{G}_{r+1}c$. Since $|\mathfrak{G}_{r+1}| < \infty$ one must also have $[\mathfrak{G}_r : \mathfrak{H}_1] < \infty$, where $\mathfrak{H}_1 = \langle c \rangle$, the infinite cyclic group generated by c. It follows from a familiar group theoretical argument, see ([14], Vol. I, pp. 83–84) that $\mathfrak{H}_1$ must contain a subgroup $\mathfrak{H}_2$ such that

$$\mathfrak{H}_2 \lhd \mathfrak{G}_r \quad \text{and} \quad [\mathfrak{G}_r : \mathfrak{H}_2] < \infty. \tag{4.15}$$

Again, $\mathfrak{H}_2$ is necessarily an infinite cyclic group, generated by $d = c^s$ for $s \neq 0$. Thus $\mathfrak{G}$ has the normal series

$$\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \cdots \supset \mathfrak{G}_r \supset \mathfrak{H}_2 \supset \{e\}. \tag{4.16}$$

Let $\{V_n\}$ be the imbedded r.w. on $\mathfrak{H}_2$. If $\sigma^2(V) = \infty$, then $\{V_n\}$ has one boundary point only, and by example A, $\{X_n\}$ again has one boundary point only. Finally, if $\sigma^2(V) < \infty$, one has necessarily $[\mathfrak{G} : \mathfrak{H}_2] < \infty$ (see lemma 5 below), and it follows from theorem 4 that the boundary for $\{X_n\}$ consists of two points.

LEMMA 5. *Let*

$$\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \cdots \supset \mathfrak{G}_{n-1} \supset \mathfrak{G}_n = \{e\} \tag{4.17}$$

be a normal series such that $\mathfrak{G}_{n-1}$ is an infinite cyclic group. Let $\{X_n\}$ be a recurrent r.w. on $\mathfrak{G}$ satisfying (1.6), *and let $\{Y_n\}$ be the imbedded r.w. on $\mathfrak{G}_{n-1}$. Then $\sigma^2(Y) < \infty$ implies $[\mathfrak{G} : \mathfrak{G}_{n-1}] < \infty$.*

This lemma generalizes lemma 3.4 of [13]. Its proof consists of a few simple group theoretical reductions, plus a repetition of the proof given in [13] for lemma 3.4. We skip the details because, among other reasons, we believe a more general result to be true (see section 5).

C. If $\mathfrak{G} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$ with $|\mathfrak{G}_1| = |\mathfrak{G}_2| = \infty$, then $\{X_n\}$ has one boundary point only.

PROOF. The elements of $\mathfrak{G}$ are the ordered pairs (g_1, g_2), $g_i \in \mathfrak{G}_i$. Let $c \in \mathfrak{G}_1$ and let

$$\mathfrak{H} = \{(c^k, g_2) : k \text{ integer}, g_2 \in \mathfrak{G}_2\}. \tag{4.18}$$

The set $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$, and one can choose the representatives z_i of its cosets to be of the form $\{w_i, e_2\}$ with $w_i \in \mathfrak{G}_1$, and e_2 the identity element of $\mathfrak{G}_2$. Each of the z_i's commutes with every element $g \in \mathfrak{H}$ of the form (e_1, g_2) where e_1 is the identity element of $\mathfrak{G}_1$ and $g_2 \in \mathfrak{G}_2$. Thus, by lemma 4 (see also end of proof of lemma 4)

$$\lim_{x \to \infty} H_{\mathfrak{H}}(x, y) = 0, \qquad y \in \mathfrak{H}. \tag{4.19}$$

Let

$$\mathfrak{H}_1 = \langle (c, e) \rangle = \{(c^k, e) : k \text{ integer}\}. \tag{4.20}$$

Then $\mathfrak{H}_1 \lhd \mathfrak{H}$. Assume first that the cyclic group $\mathfrak{H}_1$ is infinite and let $\{Y_n\}$ and $\{Z_n\}$ be the imbedded random walks on $\mathfrak{H}$ and $\mathfrak{H}_1$ respectively. Also $\{Z_n\}$ is the imbedded r.w. of $\{Y_n\}$ on $\mathfrak{H}_1$. By lemma 5 applied with $\{Y_n\}$ and $\mathfrak{H}$ as the original r.w. and original group, $\sigma^2(Z) = \infty$, since

$$[\mathfrak{H} : \mathfrak{H}_1] = |\mathfrak{G}_2| = \infty. \tag{4.21}$$

Thus $\{Z_n\}$ has one boundary point only, and by lemma 4 and theorem 3 the same is true for $\{Y_n\}$. From (4.19) and theorem 3, it now follows that $\{X_n\}$ has one boundary point only.

If, however, $|\mathfrak{H}_1| = h_1 < \infty$, then by lemma 4

$$\lim_{y\to\infty, y\in\mathfrak{H}} H_{\mathfrak{H}_1}(y, (c^k, e)) = \frac{1}{h_1}. \tag{4.22}$$

From (4.19), (4.22), and (2.4), it follows that

$$\lim_{x\to\infty} H_{\mathfrak{H}_1}(x, (c^k, e)) = \lim_{x\to\infty} \sum_{y\in\mathfrak{H}} H_{\mathfrak{H}}(x, y) H_{\mathfrak{H}_1}(y, (c^k, e)) = \frac{1}{h_1}. \tag{4.23}$$

If $C = \{(e, e), (c, e)\}$, then again by (2.4),

$$H_C(x, (e, e)) = \sum_{k=0}^{h_1-1} H_{\mathfrak{H}_1}(x, (c^k, e)) H_C((c^k, e), (e, e)), \tag{4.24}$$

so that

$$\lim_{x\to\infty} H_C(x, (e, e)) \text{ exists.} \tag{4.25}$$

Property (4.25) holds for any $c \in \mathfrak{G}_1$ of finite order. If $\mathfrak{G}_1$ has any element of infinite order, we already know by the first part of this proof that $\{X_n\}$ has only one boundary point. We may assume, therefore, that (4.25) holds for all $c \in \mathfrak{G}_1$, and, in particular, the imbedded r.w. on

$$\tilde{\mathfrak{G}}_1 = \{(g_1, e_2) : g_1 \in \mathfrak{G}_1\} \tag{4.26}$$

has one boundary point only. Since $\tilde{\mathfrak{G}}_1 \lhd \mathfrak{G}$, it follows from one more application of lemma 4 and theorem 3 that $\{X_n\}$ has only one boundary point.

D. Let $\mathfrak{G}$ be a group of finite permutations of the integers, namely, if $g(n)$ represents the image of the integer n under the permutation $g \in \mathfrak{G}$, then for each fixed g, $g(n) \neq n$ for finitely many n only. If $|\mathfrak{G}| = \infty$, then $\{X_n\}$ has one boundary point only.

Proof. Define

$$\operatorname{spt}(g) = \text{support of } g = \{n : g(n) \neq n\}. \tag{4.27}$$

Let $c \in \mathfrak{G}$ be fixed and let $\mathfrak{H} = \langle c \rangle$ and

$$\mathfrak{H}_1 = \{g \in \mathfrak{G} : \operatorname{spt}(g) \cap \operatorname{spt}(c) = \varnothing\}. \tag{4.28}$$

Since c is a finite permutation, $|\mathfrak{H}| = h < \infty$. On the other hand, it is not hard to show that $|\mathfrak{H}_1| = \infty$ and that $\mathfrak{H} \lhd \mathfrak{H}_2$, where $\mathfrak{H}_2$ is the group generated by $\mathfrak{H} \cup \mathfrak{H}_1$. Thus, by lemma 4,

$$\lim_{x\to\infty, x\in\mathfrak{H}_2} H_{\mathfrak{H}}(x, c^k) = \frac{1}{h}. \tag{4.29}$$

Let $z_0 = e, z_1, \cdots$ be representatives of the cosets of $\mathfrak{H}_2$ in $\mathfrak{G}$. One easily sees that

$$z_i g z_i^{-1} \in \mathfrak{H}_1 \subseteq \mathfrak{H}_2 \quad \text{whenever} \quad \operatorname{spt}(g) \cap z_i^{-1}(\operatorname{spt}(c)) = \varnothing. \tag{4.30}$$

Since $|z_i^{-1}(\operatorname{spt}(c))| = |(\operatorname{spt}(c)| < \infty$ is independent of i, it is not hard to show that sequences $M^i = \{m_n^i\}$ satisfying (3.16)–(3.18) with $\mathfrak{H}$ replaced by $\mathfrak{H}_2$ exist.

Hence, by lemma 4,

$$\lim_{x \to \infty} H_{\mathfrak{H}_2}(x, y) = 0, \qquad y \in \mathfrak{H}_2. \tag{4.31}$$

Equations (4.29) and (4.31) take the place of (4.22) and (4.19), and just as in the proof of (4.25), it follows that

$$\lim_{x \to \infty} H_C(x, e) \text{ exists} \tag{4.32}$$

for $C = \{e, c\}$, $c \in \mathfrak{G}$ arbitrary. This completes the proof.

D′. Let $\mathfrak{G}$ be a group of permutations of the integers and $\mathfrak{G}_1$ the subgroup of finite permutations of the integers. If $|\mathfrak{G}_1| = \infty$ then $\{X_n\}$ has one boundary point only. This is an immediate consequence of lemma 4, example D, and theorem 3 because $\mathfrak{G}_1 \lhd \mathfrak{G}$.

5. Some open problems

Our main object has been to prove the following conjectures.

Conjecture 1. If $\{X_n\}$ is a recurrent r.w. on a countable infinite group $\mathfrak{G}$ such that (1.6) is satisfied, then $\{X_n\}$ has one or two boundary points.

The results of this paper are only approximations to this conjecture, and even the following weaker conjectures have not been proved.

Conjecture 2. The entrance boundary of $\{X_n\}$ is the same as the exit boundary (which equals the entrance boundary of the reversed r.w. (see section 1).

Conjecture 3. Let $\mathfrak{H} \subseteq \mathfrak{G}$ be an infinite cyclic group. If the imbedded r.w. on $\mathfrak{H}$ has finite variance, then $[\mathfrak{G}:\mathfrak{H}] < \infty$.

Conjecture 3 seems the right generalization of lemma 5. The difficulty here is that we only have a partial answer to the following general question.

Question. Which groups $\mathfrak{G}$ permit a recurrent r.w. $\{X_n\}$ satisfying (1.6)?

If $\mathfrak{G}$ is a free group on two or more generators, it does not have such an r.w. (see [6] for the proof of a special case; in general, one can show that a group which permits a recurrent r.w. must be amenable (for example, by means of corollary e″, p. 101 of [3]) and free groups on more than one generator are not amenable ([2], p. 516). For abelian groups $\mathfrak{G}$, the full answer was given by Dudley [5]. A slight variation of his argument (see [5], pp. 448–450) shows that such an r.w. exists for every countable, locally finite group.

It is amusing to point out a possible relation with the theory of group ends, discussed in [8] and already used in the proof of theorem 5. Freudenthal proves in [8] that every finitely generated group has 1, 2, or infinitely many ends. Two ends occur if and only if $\mathfrak{G}$ has an infinite cyclic subgroup of finite index. If

$$\mathfrak{G} = \mathfrak{G}_1 \oplus \mathfrak{G}_2, \qquad |\mathfrak{G}_i| = \infty, \tag{5.1}$$

then $\mathfrak{G}$ has only one end. The similarity with theorems 5, 4, and example C is striking but probably not significant. It is possible to prove that a finitely generated group with infinitely many ends does not permit a recurrent r.w. satisfying (1.6). This is actually a special case of the next conjecture.

Conjecture 4. Let $\mathfrak{G}$ be a finitely generated group. Let $\{c_1, \cdots, c_m\}$ be a set of generators and let $N(L)$ be the number of different group elements in the set

$$\{c_{i_1}^{\eta_1} \cdots c_{i_L}^{\eta_L} : 1 \leq i_j \leq m, \eta_j = \pm 1\}. \tag{5.2}$$

If

$$\limsup N(L)^{1/L} > 1, \tag{5.3}$$

then $\mathfrak{G}$ does not have a recurrent r.w. satisfying (1.6).

If this conjecture is true, then even a group $\mathfrak{G}$ generated by $\{c_1, c_2\}$ such that c_1 and c_2 generate a free semigroup does not allow a recurrent r.w. satisfying (1.6). (In this case, c_1 and c_2 do not have to be free generators of a free group; see [1].)

As a final, and seemingly simple, situation, consider a group $\mathfrak{G}$ generated by $\{c_1, c_2, c_3\}$ such that no nonzero power of c_i belongs to the subgroup generated by c_j and c_k, where (i, j, k) is any permutation of $(1, 2, 3)$. Does $\mathfrak{G}$ permit a recurrent r.w.? (If $\mathfrak{G}$ is commutative, it is $Z \oplus Z \oplus Z$ where Z is the group of integers, and hence does not allow a recurrent r.w.).

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