FIRST EXIT TIMES FROM A SQUARE ROOT BOUNDARY

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1. Introduction

The thing that motivated the present paper was a curious observation by Blackwell and Freedman [1]. Let $X_1, X_2, \cdots$, be independent ± 1 with probability $\frac{1}{2}$, $S_n = X_1 + \cdots + X_n$, and $T_c = \min \{n; S_n > c\sqrt{n}, c\sqrt{n} \geq 1\}$. Then for all $c < 1$, $ET_c < \infty$, but $c \geq 1$ implies $ET_c = \infty$. In order to understand this better, I wanted to calculate the asymptotic form of $P(T_c > n)$ for large n. It was reasonable to conjecture that $P(T_c > n) \sim \alpha n^{-\beta}$, not only for coin-tossing random variables but for a large class. The first step in the proof of this was to verify the result for Brownian motion. This is done in the second paragraph and follows easily from known results.

To go anywhere from there, one would like to invoke an invariance principle. But the difficulty is clear—for general identically distributed, independent r.v. $X_1, X_2, \cdots$ with $EX_1 = 0$, $EX_1^2 = 1$, the most one could hope for is that $P(T_c > n) \sim \alpha n^{-\beta}$ where β is the same for all distributions, but α depends intimately on the structure of the process. Hence, this is not a situation in which the usual invariance principle is applicable. But the result does hold for all distributions such that $E|X_1|^3 < \infty$, and it is proved by using results of Prohorov [2] which give estimates of the rate of convergence of the relevant invariance theorem. This proof is carried out in the third paragraph.

A dividend of the preceding proof is collected. The conclusion is that

$$P(S_n < \xi\sqrt{n} \,|\, |S_k| < c\sqrt{k}, k = 1, \cdots, n) \to G(\xi) \tag{1.1}$$

where $G(\xi)$ is the corresponding distribution for Brownian motion.

The related works that I know of are the interesting results given by Strassen at this symposium [3], and by Darling and Erdös [4]. There is also a very recent article by Chow, Robbins, and Teicher [5] which generalizes the result of Blackwell and Freedman.

2. First exit distribution for Brownian motion

Take $\xi(t)$ to be Brownian motion with $E\xi(t) = 0$, $E\xi^2(t) = t$. Define $T_c^* = \inf \{t;\ \xi(t) \geq c\sqrt{t}, t \geq 1\}$, that is, T_c^* is the first exit time past $t = 1$.

Preparation of this paper was sponsored in part by National Science Foundation Grant GP-1606.

THEOREM 1. *With the above definition,* $P(T_c^* > t \mid \xi(1) = 0) \sim \alpha t^{-\beta(c)}$ *where*

(i) $\lim_{c \to \infty} \beta(c) = 0$,

(ii) $\lim_{c \to \infty} \beta(c) = \infty$,

(iii) *if* c^2 *is the smallest positive root of*

$$\sum_{n=0}^{m} \frac{(-2c^2)^n}{(2n)!} \frac{m!}{(m-n)!}, \tag{2.1}$$

then $\beta(c) = m$. *In particular* $\beta(1) = 1$, $\beta(\sqrt{3 - \sqrt{6}}) = 2$.

PROOF. Consider the process $Y(u) = \xi(e^{2u})/e^u$. This is the Uhlenbeck process. The problem now is to find the distribution of the first exit time T_Y for $Y(u)$ from the boundaries $\pm c$, given $Y(0) = 0$. This distribution is well known, and its Laplace transform has been given by Bellman and Harris [6] and Darling and Siegert [7]. To wit;

$$\begin{aligned} \Phi(\lambda) = \int_0^\infty e^{-\lambda V} \, dP\,(T_Y > V) &= e^{-c^2/4} \frac{D_{-\lambda}(0) + D_{-\lambda}(0)}{D_{-\lambda}(c) + D_{-\lambda}(-c)} \\ &= \frac{1}{2} \frac{2^{\lambda/2}\Gamma(\lambda/2)}{\int_0^\infty e^{-(1/2)t^2} t^{\lambda-1} \cosh ct \, dt}, \qquad R\ell\lambda > 0, \end{aligned} \tag{2.2}$$

where the $D_\lambda(z)$ are parabolic cylinder functions. Now $D_{-\lambda}(c)$ is an entire function of λ (see Erdélyi *et al.* [8], pp. 117 f.f.), so $\Phi(\lambda)$ is entire except for poles on the nonpositive axis. Since $D_{-\lambda}(z) + D_{-\lambda}(-z)$ satisfies the self-adjoint Sturm-Liouville equation

$$\phi'' + (\tfrac{1}{2} - \tfrac{1}{4} x^2)\, \phi = \lambda\phi, \tag{2.3}$$

it follows, under the supplementary condition $\phi(a) = \phi(-a)$, that the characteristic values of this system coincide with the zeroes of $D_{-\lambda}(a) + D_{-\lambda}(-a)$. Therefore, the poles of $\Phi(\lambda)$ are simple and real. Let $-2\beta(c)$ be the position of the largest pole. Then

$$P(T_Y > V) = \alpha e^{-2\beta V} + 0(e^{-(2\beta+\delta)V}), \qquad \delta > 0. \tag{2.4}$$

For the Brownian motion, this translates as

$$P(T_c^* > t) = \alpha t^{-\beta(c)} + 0(t^{-\beta-(\delta/2)}). \tag{2.5}$$

Now to verify (i), (ii), and (iii) by locating the largest zero of

$$\theta(\lambda) = \frac{2^{1-\lambda/2}}{\Gamma(\lambda/2)} \int_0^\infty e^{-t^2/2} t^{\lambda-1} \cosh ct \, dt. \tag{2.6}$$

To continue the integral into $R\ell\lambda < 0$, write

$$\int_0^\infty e^{-t^2/2}t^{\lambda-1}\cosh ct\,dt = \int_0^\infty e^{-t^2/2}t^{\lambda-1}\left[\cosh ct - \sum_0^N \left(\frac{(ct)^{2n}}{(2n)!}\right)\right]dt$$

$$+\,\Gamma\left(\frac{\lambda}{2}\right)2^{\lambda/2-1}\sum_{n=0}^{N}\frac{c^{2n}}{(2n)!}\frac{\Gamma(n+(\lambda/2))}{\Gamma(\lambda/2)} \tag{2.7}$$

$$\theta(\lambda) = \frac{2^{1-\lambda/2}}{\Gamma(\lambda/2)}I_N(\lambda, c) + \sum_{n=0}^{N}\frac{c^{2n}}{(2n)!}\frac{\Gamma(n+(\lambda/2))}{\Gamma(\lambda/2)}.$$

The function $I_N(\lambda, c)$ is analytic for $R\ell\lambda > -2N-2$, real and positive for real λ in this range. For $\lambda = -2m$, the finite sum becomes

$$P(c) = \sum_{n=0}^{m}\frac{(-2c^2)^n}{(2n)!}\frac{m!}{(m-n)!}, \tag{2.8}$$

and the first term in (2.7) vanishes. This gives (iii). As $c \to 0$, $\theta(\lambda) \to 1$ for all λ, so for any number M, and for c sufficiently small, $\theta(\lambda)$ can have no zeroes in $|\lambda| \le M$.

For (i), use $N = 0$ in (2.3) to get

$$\Gamma\left(\frac{\lambda}{2}\right)\theta(\lambda) = 2^{1-\lambda/2}I_0(\lambda) + \Gamma\left(\frac{\lambda}{2}\right), \qquad R\ell\lambda > -2. \tag{2.9}$$

Since $I_0(\lambda) \to 0$ as $c \to \infty$, the root must move toward a pole of $\Gamma(\lambda/2)$. But this can only be at $\lambda = 0$.

3. First exit distribution for sums of independent r.v.

Let $X_1, X_2, \cdots$ be independent, identically distributed r.v. with $EX_1 = 0$, $EX_1^2 = 1$, $E|X_1|^3 < \infty$. Define $T_c = \min\{n, S_n \ge c\sqrt{n}\}$, $S_n = X_1 + \cdots + X_n$.

THEOREM 2. *Either there exists an integer n such that $P(T_c > n) = 0$, or $P(T_c > n) \sim \alpha n^{-\beta(c)}$, where $\beta(c)$ is the same function appearing in theorem 1, but $\alpha > 0$ will not, in general, be the same.*

PROOF. The proof is constructed around the use of the identity

$$P_n(\gamma) = \int Q_{n,m}(\gamma, \eta)P_m(d\eta), \qquad m < n, \tag{3.1}$$

where

$$P_n(\gamma) = P\left(\frac{S_n}{\sqrt{n}} < \gamma, T_c > n\right), \tag{3.2}$$

$$Q_{n,m}(\gamma, \eta) = P\left(\frac{S_n}{\sqrt{n}} < \gamma, \left|\frac{S_k}{\sqrt{k}}\right| < c, k = m, \cdots, n \,\Big|\, \frac{S_m}{\sqrt{m}} = \eta\right). \tag{3.3}$$

Take $m = [e^{2u}]$ (the largest integer $\le e^{2u}$) and $n = [e^{2(u+u_0)}]$, then the invariance principle results in

$$\lim_{n\to\infty} Q_{n,m}(\gamma,\eta) = P(Y(u_0) < \gamma, |Y(V)| < c, 0 \le V \le u_0 | Y(0) = \eta) \tag{3.4}$$
$$= Q(\gamma, \eta)$$

(see [7]). However, what is needed is a uniform error bound.

PROPOSITION 1. *There is a constant D such that*

$$\sup_{\gamma,\eta} |Q_{m,n}(\gamma,\eta) - Q(\gamma,\eta)| \le De^{-u/16}, \qquad n > 0. \tag{3.5}$$

PROOF. Write

(3.6)
$$P\left(\frac{S_n}{\sqrt{n}} < \gamma, \left|\frac{S_k}{\sqrt{k}}\right| < c, k = m, \cdots, n \left| \frac{S_m}{\sqrt{m}} = \eta\right.\right)$$
$$= P\left(\frac{S_N + \eta\sqrt{m}}{\sqrt{n}} < \gamma, \left|\frac{S_k + \eta\sqrt{m}}{\sqrt{k+m}}\right| < c, k = 0, \cdots, N = n - m | S_0 = 0\right)$$
$$= P\left(\frac{S_N}{\sqrt{N}} < \sigma_N, \left|\frac{S_k}{\sqrt{N}} + \eta\sqrt{\lambda_N}\right| < c\sqrt{\frac{k}{N} + \lambda_N}, k = 0, \cdots, N | S_0 = 0\right),$$

where $\lambda_N^{-1} = N/m$, $\sigma_N = \gamma\sqrt{1+\lambda_N} - \eta\sqrt{\lambda_N}$. Let $\xi_N(t)$ be the process obtained by interpolating linearly with $\xi_N(k/N) = S_k/\sqrt{N}$, and $\xi_N(0) = 0$. Therefore,

$$Q_{n,m}(\gamma,\eta) = P(\xi_N(1) < \sigma_N, |\xi_N(t) + \eta\sqrt{\lambda_N}| < c\sqrt{t+\lambda_N}, 0 \le t \le 1). \tag{3.7}$$

Now, we use an estimate due to Prohorov [2], that is, consider the set of all continuous functions on $[0, 1]$ as a metric space with the sup norm topology. For any closed set S, let S_ϵ be the open set consisting of all points whose distance to S is less than ϵ. If

$$\delta_1 = \inf_\epsilon \{\epsilon; P(\xi_N \in S) \le P(\xi \in S_\epsilon) + \epsilon, \text{ all closed } S\}, \tag{3.8}$$

$$\delta_2 = \inf_\epsilon \{\epsilon; P(\xi \in S) \le P(\xi_N \in S_\epsilon) + \epsilon, \text{ all closed } S\}, \tag{3.9}$$

then $\max(\delta_1, \delta_2) < kN^{-1/8}(\log N)^2 = \rho_N$, where k is a constant not depending on N. For $|\eta| < c - \rho_N$,

$$\begin{aligned} &P(\xi(1) \le \sigma_N - \rho_N, |\xi(t) + \eta\sqrt{\lambda_N}| \le c\sqrt{t+\lambda_N} - \rho_N, 0 \le t \le 1) - \rho_N \\ &\le Q_{n,m}(\gamma,\eta) \\ &\le P(\xi(1) \le \sigma_N + \rho_N, |\xi(t) + \eta\sqrt{\lambda_N}| \le c\sqrt{t+\lambda_N} + \rho_N, 0 \le t \le 1) + \rho_N. \end{aligned} \tag{3.10}$$

The difference between the right- and left-hand sides above is dominated by

$$\begin{aligned} &P(|\xi(1) - \sigma_N| \le 2\rho_N) + P(|\xi(t) + \eta\sqrt{\lambda_N}| < c\sqrt{t+\lambda_N} + \rho_N) \\ &\qquad - P(|\xi(t) + \eta\sqrt{\lambda_N}| \le c\sqrt{t+\lambda_N} - \rho_N) + 2\rho_N. \end{aligned} \tag{3.11}$$

At this point, we need another result from Prohorov [2]; that is, if functions $a_1(t)$, $a_2(t)$ satisfy $a_1(0) < 0 < a_2(0)$, $a_1(t) < a_2(t)$, $0 < t \le 1$, and $|a(t'') - a(t')| \le K|t'' - t'|$, then there is a constant J depending only on K, and $\sup|a_1(t)|$, $\sup|a_2(t)|$, $0 \le t \le 1$, such that for all $\epsilon > 0$ small enough

$$(3.12)\qquad \begin{aligned}&P(a_1(t) \leq \xi(t) \leq a_2(t), 0 \leq t \leq 1)\\ &\qquad \geq P(a_1(t) - \epsilon \leq \xi(t) \leq a_2(t) + \epsilon, 0 \leq t \leq 1) - J\epsilon.\end{aligned}$$

Applying this to (3.11), the difference is dominated by $(3 + J)\rho_N$. Since

$$(3.13)\qquad \lambda_N^{-1} = \frac{[e^{2(u+u_0)}] - [e^{2u}]}{[e^{2u}]} = e^{2u_0} + 0(e^{-2u}),$$

by a similar argument we can replace λ_N by $\lambda = e^{2u_0}$, $\sigma_N = \sigma = \gamma\sqrt{1 + \lambda} - \eta\sqrt{\lambda}$ and conclude that for $|\eta| < c - \rho_N$,

$$(3.14)\qquad |Q_{n,m}(\gamma, \eta) - P(\xi(t) \leq \sigma, |\xi(t) + \eta\sqrt{\lambda}| \leq c\sqrt{t + \lambda}, 0 \leq t \leq 1)| \leq H\rho_N,$$

where H does not depend on γ or η. If $\eta = c - s\rho_N$, $0 \leq s \leq 1$, then

$$(3.15)\qquad \begin{aligned} Q_{n,m}(\gamma, \eta) &\leq P(\xi_N(t) \leq c(\sqrt{t + \lambda_N} - \sqrt{\lambda_N}) + \rho_N\sqrt{\lambda_N}, 0 \leq t \leq 1)\\ &\leq P\left(\xi_N(t) \leq \frac{c}{2}\frac{t}{\sqrt{\lambda_N}} + \rho_N\sqrt{\lambda_N}, 0 \leq t \leq 1\right)\\ &\leq P\left(\xi(t) \leq \frac{c}{2}\frac{t}{\sqrt{\lambda_N}} + \rho_N\sqrt{\lambda_N} + \rho_N, 0 \leq t \leq 1\right) + \rho_N\\ &\leq P(\xi(t) \leq c_N\rho_N, 0 \leq t < \rho_N) + \rho_N,\end{aligned}$$

where $c_N \to c/2\sqrt{\lambda} + \sqrt{\lambda} + 1$. As is well known,

$$(3.16)\qquad \begin{aligned} P(\xi(t) \leq c_N\rho_N, 0 \leq t \leq \rho_N) &= 1 - 2P(\xi(\rho_N) \geq c_N\rho_N)\\ &= 2P(0 \leq X \leq c_N\sqrt{\rho_N}) \leq H'\sqrt{\rho_N}\end{aligned}$$

where X is normal $N(0, 1)$. This completes the proof.

To complete the proof of theorem 2, write

$$(3.17)\qquad P(\gamma, u) = P_{[e^{2u}]}(\gamma), \qquad Q(\gamma, \eta, u, u_0) = Q_{[e^{2u+2u_0}],[e^{2u}]}(\gamma, \eta)$$

and (2.8) as

$$(3.18)\qquad P(\gamma, u + u_0) = \int Q(\gamma, \eta, u, u_0)P(d\eta, u)$$

or

$$(3.19)\qquad P(\gamma, u + u_0) = \int Q(\gamma, \eta)P(d\eta, u) + \int \Delta(\gamma, \eta, u, u_0)P(d\eta, u).$$

Take Laplace transforms of both sides with

$$\hat{P}(\gamma, s) = \int_0^\infty e^{-su}P(\gamma, u)\,du,$$

$$(3.20)\qquad \begin{aligned} e^{u_0 s}\hat{P}(\gamma, s) = \int Q(\gamma, \eta)\hat{P}(d\eta, s) &+ e^{u_0 s}\int_0^{u_0} e^{-us}P(\gamma, u)\,du\\ &+ \int_0^\infty \int \Delta(\gamma, \eta, u, u_0)e^{-us}P(d\eta, u)\,du.\end{aligned}$$

The second term on the right in (3.20) is entire in s. Put $I_c = [-c, +c]$; if $R\ell s > s_0$ is the maximal half-plane of analyticity for $\hat{P}(I_c, s)$, then the third term is analytic in the plane $Rl s > s_0 - \frac{1}{16}$. The function $e^{(\gamma^2/2)}(\partial/\partial\gamma)Q(\gamma, \eta)$ satisfies the equation

$$\frac{\partial f}{\partial u_0} = \frac{\partial^2 f}{\partial y^2} - y\frac{\partial f}{\partial y}, \qquad f(c) = f(-c) = 0 \tag{3.21}$$

for y either variable γ or η. For $u_0 \to 0$, $\int h(\gamma)Q(d\gamma, \eta) \to h(\gamma)$ for reasonable $h(\gamma)$, hence

$$Q_\gamma(\gamma, \eta) = e^{-(\gamma^2/2)} \sum_{k=0}^{\infty} e^{\lambda_k u_0}\psi_k(\gamma)\psi_k(\eta) \tag{3.22}$$

where $\psi_k(y)$, λ_k satisfy the eigenvalue equation,

$$\frac{d^2\psi}{dy^2} - y\frac{d\psi}{dy} = \lambda\psi, \quad \psi(c) = \psi(-c) = 0. \tag{3.23}$$

Write $(f, g) = \int_{-c}^{+c} e^{-(y^2/2)}f(y)g(y)\,dy$; then $(\psi_k, \psi_j) = \delta_{k,j}$. In (3.20) write $I(\gamma, s)$ for the second and third terms. Therefore,

$$e^{u_0 s}\int \psi_k(\gamma)\hat{P}(d\gamma, s) = e^{\lambda_k u_0}\int \psi_k(\gamma)\hat{P}(d\gamma, s) + \int \psi_k(\gamma)I(d\gamma, s), \tag{3.24}$$

and for a sufficiently smooth function $h(\gamma) = \sum_k (h, \psi_k)\psi_k(\gamma)$,

$$\int f(\gamma)\hat{P}(d\gamma, s) = \sum_{k=0}^{\infty} \frac{1}{e^{u_0 s} - e^{\lambda_k u_0}}(f, \psi_k)\int \psi_k\, dI. \tag{3.25}$$

Now, integrating by parts, using (3.20),

$$\left|\int \psi_k I(d\gamma, s)\right| \le c\int e^{-(\gamma^2/2)}(\psi_k'(\gamma))^2\, d\gamma \cdot \sup_\gamma I(\gamma, s) = c|\lambda_k| \sup_\gamma I(\gamma, s). \tag{3.26}$$

Take f so that the sum of $|(f, \psi_k)\lambda_k|$ is absolutely convergent. In particular, this will be true if $f \equiv 1$. Thus, (3.25) becomes

$$P(I_c, s) = \sum_k \frac{1}{e^{u_0 s} - e^{\lambda_k u_0}}(1, \psi_k)\int \psi_k\, dI. \tag{3.27}$$

From the foregoing, for real s, $\int \psi_k\, dI \to d|\lambda_k|\hat{P}(I_c, s - (\frac{1}{16}))$. The point is that if $s_0 > -\infty$, then s_0 must be one of the values $\lambda_0, \lambda_1, \cdots$, say λ_j, $\hat{P}(I_c, s)$ has a single pole at $s = s_0$, and is otherwise regular in the half-plane $Rl s > s_0 - \delta$, $\delta > 0$. (The poles at $s_0 \pm 2n\pi i/u_0$, $n \neq 0$, are ruled out because $\hat{P}(I_c, s)$ does not depend on u_0.) The statement is also true for $\int f(\gamma)\hat{P}(d\gamma, s)$ for any smooth f such that $(f, \psi_j) \neq 0$. For such smooth f, one can write

$$\int f(\gamma)P(d\gamma, u) = L(f)e^{\lambda_j u} + 0(e^{(\lambda_j - \delta)u}). \tag{3.28}$$

Letting $P^*(d\gamma, u) = e^{-\lambda_j u}P(d\gamma, u)$, one obtains

$$\int f(\gamma)P^*(d\gamma, u) = L(f) + 0(e^{-\delta u}). \tag{3.29}$$

The measures $P^*(d\gamma, u)$ converge weakly to $P^*(d\gamma)$, where $(\psi_k, dP^*) = 0$,

$k \neq j$. The latter implies $P^*(d\gamma) = e^{-(\gamma^2/2)}\psi_j(\gamma)\, d\gamma$. However, $P^*(d\gamma)$ is a nonnegative measure. The only nonnegative $\psi_j(\gamma)$ is $\psi_0(\gamma)$; thus $j = 0$.

Finally, the transformation $\phi(x) = e^{-(x^2/4)}\psi(x)$ changes (3.23) into the system (2.3) so that $\lambda_0 = -2\beta(c)$. The only thing remaining is to verify that $s_0 > -\infty$, if there is no n such that $P(T_c > n) = 0$.

For any u_0, take I such that for $u \geq u_1$,

$$\inf_{\eta \in I} Q(I, \eta, u, u_0) \geq \delta > 0. \tag{3.30}$$

From (3.18), $P(I, u + u_0) \geq \delta P(I, u)$, $u \geq u_1$. This will imply that either there is an M > 0 such that $P(I, u) \geq e^{-Mu}$, all $u \geq 0$, or that there is a u_2 such that $P(I, u) = 0$, $u \geq u_2$. The first alternative is impossible because of the assumed analyticity of $\hat{P}(I_c, s)$. Put another way, there is an interval J and an m such that $P(S_m \in J, T_c > m) = 0$. Then argue that J can certainly be taken large enough such that $P(X_1 \in J - x) > 0$, all $x \in J$. Therefore,

$$P(S_m \in J, T_c > m) = 0 \Rightarrow P(S_{m-1} \in J, T_c > m - 1) = 0. \tag{3.31}$$

By reduction, an n is arrived at with $P(T_c > n) = 0$. Note that n is not necessarily one by considering X_1 to take on the two values $1/M$ and M^2, for M large. The smallest n satisfying $P(T_c > n) = 0$ is the largest n satisfying $n/M < c\sqrt{n}$.

In the course of this proof we have incidentally proven the following theorem.

THEOREM 3. *If there exists no n such that $P(T_c > n) = 0$, then*

$$P(S_n < \gamma\sqrt{n} | T_c > n) \to \theta \int_{-c}^{\gamma} e^{-(\xi^2/2)}\psi_0(\xi)\, d\xi, \tag{3.32}$$

where θ is a normalizing constant.

4. Remarks

Some unresolved problems that are left in this area are concerned with what happens to the tail of the T_c distribution as the boundaries are moved out. More specifically, let $T_c(\tau) = \min \{n; S_n > c\sqrt{n + \tau}\}$. By invoking the usual invariance principle, it is easy to show that

$$\lim_{\tau \to \infty} P(T_c(\tau) \geq \tau t) = P(T_c^* > t | \xi(1) = 0) \tag{4.1}$$

where T_c^* is the first exit time for Brownian motion past $t = 1$. This is not as interesting as asking the following questions.

(i) By theorem 2, $P(T_c(\tau) > \tau t) \sim \alpha(\tau) t^{-\beta(c)}$. As $\tau \to \infty$ show that $\alpha(\tau)$ converges to the corresponding constant for Brownian motion.

(ii) A bit more strongly, is it true that $P(T_c(\tau) > \tau t)/P(T_c^*(\tau) > \tau t) \to 1$ uniformly as $\tau, t \to \infty$?

The condition $E|X_1|^3 < \infty$ can be easily weakened down to $E|X_1|^{2+\delta} < \infty$ and the same methods will work. I suspect that theorem 2 may even be true under only $E|X_1|^2 < \infty$, but that would require better tools.

What happens with more general boundaries, for instance, $t^{1/3}$, or $\sqrt{t}\,\phi(t)$, where $\phi(t)$ does not increase too fast? The results for the $c\sqrt{t}$ boundaries rely very heavily on the fact that these transform into constant boundaries for the Uhlenbach process. Hence the simple methods used here do not generalize.

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