SUPPORTS OF CONVOLUTIONS OF IDENTICAL DISTRIBUTIONS

HERMAN RUBIN
MICHIGAN STATE UNIVERSITY, STANFORD UNIVERSITY

1. Introduction

It has long been known that the convolution of two singular univariate distributions can be absolutely continuous, and that this can hold even if the distributions are concentrated on sets of Hausdorff dimension zero. However, the problem is more complicated if the convolvants are required to be identical.

What is done in this paper is to give simple examples of convolutions of singular distributions, even concentrated on sets of Hausdorff dimension zero, which are absolutely continuous. Furthermore, if we define the "dimension" of a distribution to be the smallest Hausdorff dimension of a set with probability one and the distribution is singular, and ∞ if the distribution is absolutely continuous, then if monotonically nondecreasing functions f, g, h are given from $(0, \infty)$ to $[0, 1] \cup \{\infty\}$, such that for each t, $f(t) \leq h(t)$ and $g(t) \leq h(t)$, there exist infinitely divisible distributions F and G such that the dimension of F^t is $f(t)$, the dimension of G^t is $g(t)$, and the dimension of $F^t * G^t$ is $h(t)$. Furthermore, the measure in the Lévy-Khintchine representation of F is purely discrete, and that of G is purely singular. This generalizes results of Tucker [4] and Rubin [3]. We can even insist that the distributions F^t, G^t, $F^t * G^t$ are pure, that is, there is no nonzero component of smaller dimension.

2. An example

For our first example, let

$$X_i = \sum_{j=1}^{\infty} 2^{-j} A_{ij}, \qquad i = 1, 2, \tag{1}$$

where the A_{ij} are independent zero or one random variables and $P(A_{ij} = 1) = p_j$. Then it is known that the distribution of X_i is absolutely continuous if and only if $\sum (p_j - \frac{1}{2})^2 < \infty$. We now show that if $\sum (p_j - \frac{1}{2})^4 < \infty$, the distribution F of $X_1 + X_2$ is absolutely continuous.

We first observe that, since $X_1 + X_2$ is a sum of independent discrete random variables, its distribution is pure [5]. Hence, for F to be absolutely continuous,

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it is necessary and sufficient for F not to be orthogonal to Lebesgue measure. If we now let G_q be the distribution of $X_1 + X_2 \pmod{2^{-q}}$, this condition is equivalent to G_q and Lebesgue measure on $(0, 2^{-q})$ being mutually absolutely continuous for some q, or dG_q/dH_q is nonzero almost everywhere $[H_q]$, where H_q is 2^q times Haar measure on $(0, 2^{-q})$. Let q be so large that if $r > q$, $\frac{1}{3} < p_r < \frac{2}{3}$. Let $E_r(a)$ be the event $A_{1r} + A_{2r} \equiv a \pmod 2$. Then $P(E_r(0)) - \frac{1}{2} = 2(p_r - \frac{1}{2})^2$. Also, let

$$X_1 + X_2 = \sum_{j=0}^{\infty} 2^{-j} B_j. \tag{2}$$

If there is no carry at the r-th place, $B_r = 1$ in the event $B_r(1)$, and if there is a carry, $B_r = 1$ in the event $B_r(0)$. Consequently,

$$P(B_r = 1 | A_{r+i,j}; i = 1, \cdots ; j = 1, 2) = \tfrac{1}{2} \pm 2(p_r - \tfrac{1}{2})^2, \tag{3}$$

and hence

$$|P(B_r = 1 | B_{r+1}, \cdots, B_s) - \tfrac{1}{2}| \leq 2(p_r - \tfrac{1}{2})^2. \tag{4}$$

Consider now the random variables

$$U_{r,s} = \log \frac{d_{rs} G_q}{d_{rs} H_q}, \tag{5}$$

where d_{rs} denotes that the derivatives are taken on the field generated by $B_r, \cdots, B_s$. From (4) it follows that

$$0 \geq E(U_{r,s} - U_{r+1,s} | B_{r+1}, \cdots, B_s) \geq -9(p_r - \tfrac{1}{2})^4, \tag{6}$$

$$V(U_{r,s} - U_{r+1,s} | B_{r+1}, \cdots, B_s) \leq -20(p_r - \tfrac{1}{2})^4. \tag{7}$$

Now we can use the well-known limiting arguments to establish that appropriate interpretations of (6) and (7) hold as $s \to \infty$. From (6) it then follows that

$$\sum_{q+1}^{\infty} E(U_{r,\infty} - U_{r+1,\infty} | B_{r+1}, \cdots) \tag{8}$$

exists with probability one, and from (7) that

$$\sum_{q+1}^{\infty} U_{r,\infty} - U_{r+1,\infty} - E(U_{r,\infty} - U_{r+1,\infty} | B_{r+1}, \cdots) \tag{9}$$

exists with probability one. But this proves the nonorthogonality of G_q and H_q, q.e.d.

3. Definitions and assumptions

The previous example does not prove anything about Hausdorff dimension because ([1], [2]) $p_n \to \frac{1}{2}$ is sufficient to guarantee that the distribution of X_i is of dimension one. A somewhat more complicated example, but with easier mathematics, gives results in this direction and will form a basis for the more detailed results.

Let $b_0 < b_1 < \cdots < b_n < \cdots$ be an increasing sequence of integers, $b_0 = 0$. Then the integers from $b_{i-1} + 1$ to b_i, inclusive, form the i-th block B_i of integers. Throughout the rest of the paper we will assume that the random variables under consideration are of the form

$$X = \sum_{i=1}^{\infty} \sum_{j=1}^{N_i} \sum_{k \in B_i} Y_{jk} 2^{-k}, \tag{10}$$

where the Y's are independent random variables, each of which is zero or one with probability $\frac{1}{2}$, and the Y's are independent of the N's. It will not always be assumed that the N's are independent of each other.

We will also make some simplifying assumptions. Let r_i be the length of B_i. Then let us assume

$$\lim \frac{b_{2i}}{b_{2i-1}} = \infty, \tag{A}$$

$$\frac{\log N_{i+1}}{r_i} \to 0 \quad \text{a.s.}, \tag{B}$$

$$\sum \exp(-\tfrac{1}{2} r_i) < \infty, \tag{C}$$

$$\underline{\lim}\, N_{2i} > 0 \quad \text{a.s.} \tag{D}$$

Also let $\rho_i = r_{2i}/r_{2i+1}$.

Let us recall that the α-Hausdorff measure of a subset S of a metric space is defined by

$$H_\alpha(S) = \lim_{\epsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} \delta(E_i)^\alpha \colon S \subseteq \bigcup E_i \text{ and } \delta(E_i) < \epsilon \text{ for all } i \right\}. \tag{11}$$

Clearly, if $H_\alpha(S) < \infty$ and $\beta > \alpha$, $H_\beta(S) = 0$, and if $H_\alpha(S) > 0$ and $\beta < \alpha$, $H_\beta(S) = \infty$. If we set $H_0(S) = \infty$ and $H_\infty(S) = 0$, the cut point λ between the 0's and ∞'s of H is called the Hausdorff dimension of S. We call a support of a measure any set whose complement has measure zero and the Hausdorff dimension of a measure the minimum Hausdorff dimension of a support. It can be shown that convolution does not decrease Hausdorff dimension.

4. The auxiliary theorem

We may generalize the results of [3] to obtain the following theorem.

THEOREM. *Under the assumptions* (A)–(D), *the distribution F of the random variable X of* (10) *is absolutely continuous if and only if* $\lim N_i > 0$ a.s., *and its dimension is α if and only if α is the supremum of A, where $0 \in A$ and if $x > 0$, $x \in A$ if and only if for each positive $y < x$,*

$$P(\{i\colon N_i = 0 \text{ and } b_{i-1}/b_i < y\} \text{ is finite}) > 0.$$

From (B) it follows that with probability one for i sufficiently large, the first $(1 - \epsilon)$ of the i-th block of X will be zero unless $N_i > 0$. From (C) it follows that, with probability one for i sufficiently large, the first .8 of the i-th block will not be zero if $N_i > 0$. Thus, if $\epsilon \leq .2$, the first $(1 - \epsilon)$ of the i-th block of X being zero is a sequence of events equivalent to the sequence $N_i = 0$.

Let us define $K_i = \min(1, N_i)$,

(12) $$Y = \sum_{i \in R} \sum_{j=1}^{K_i} Y_{jk} 2^{-k},$$

and

(13) $$Z = \sum_{i \in R}^{\infty} \sum_{j=K_i+1}^{N_i} \sum_{k \in B_i} Y_{jk} 2^{-k}.$$

Then Y and Z are independent given $\{i: N_i = 0\} \cap R$. Consequently, if we can find a countable family of subsets of R which form a set of probability one on each of which F is absolutely continuous, or on a set which has positive probability F and dimension of at least y, then the same is true for F. However, it is sufficient to establish the result for the distribution of Y.

First let us look at the case of absolute continuity. Let R be the set of positive integers. If F is absolutely continuous, any event of Lebesgue measure zero must have probability zero. Consider the event that infinitely many blocks have their first .8 identically zero. From (C) it follows, just as before, that this event has probability zero. But this latter condition has already been noted to be probabilistically equivalent to the finiteness of $\{i: N_i = 0\}$.

Conversely, let $P(\{i: N_i = 0\} \text{ finite}) = 1$. Let $\mathcal{S}$ be the set of finite sets of positive integers. If $S \in \mathcal{S}$, then for some m, $N_i > 0$ for all $i > m$. Consequently, Y has an absolutely continuous distribution given $\{i: N_i = 0\} = S$. Therefore, by our previous remark, F is absolutely continuous.

Let us now look at the problem of Hausdorff dimension. First let us show that the α in the statement of the theorem is an upper bound. Let $\beta > \alpha$ and let ϵ be a positive number. Let $S = \{i: N_i = 0 \text{ and } b_{i-1}/b_i < \beta\}$. Then S is infinite with probability one. Consequently, if

(14) $$T = \{i: \text{the first } (1 - \epsilon) \text{ of the } i\text{-th block is } 0 \text{ and } b_{i-1}/b_i < \beta\},$$

T is infinite with probability one. Let

(15) $$c_i = \{b_{i-1} + (1 - \epsilon)r_i\},$$

where $\{x\}$ denotes here the smallest integer not less than x, and let

(16) $$E_{im} = [2^{-b_{i-1}}m, 2^{-b_{i-1}}m + 2^{-c_i}).$$

Since T is infinite with probability one, it follows that if

(17) $$Q_i = 2^{-b_{i-1}}M_i \to \infty,$$

that

(18) $$F_J = \bigcup \{E_{im}: i > J, b_{i-1}/b_i < \beta, \text{ and } 1 \leq m \leq M_i\}$$

has probability one for all J. By (11),

(19) $$H_\gamma(\bigcap_J F_J) \leq \sum_{\substack{b_{i-1}/b_i < \beta \\ i > J}} 2^{-c_i\gamma}M_i \leq \sum_{i=J}^{\infty} 2Q_i \cdot 2^{\eta b_i},$$

where $\eta = 1 - (\gamma/\beta) + \epsilon\gamma((1/\beta) - 1)$. Since b_i increases more than exponentially fast from (A), if Q_i goes to infinity sufficiently slowly (say exponentially), $H_\gamma(\bigcap_J F_J) = 0$ if $\eta < 0$. Thus, the dimension d of F satisfies

$$d \leq \frac{\beta}{1 + \epsilon(\beta - 1)}.$$

Since ϵ is an arbitrary positive number and β an arbitrary number exceeding α, $d \leq \alpha$.

To prove the converse, let $R = \{i: b_{i-1}/b_i < y\}$. Consequently, if $\mathcal{S}$, the set of all finite subsets of $\{i: N_i = 0\} \cap R$ has positive probability, the dimension of F is at least the dimension of the distribution of U. But by the results of Billingsley [1] and Chatterji [2], this dimension is at least y. The conclusion then follows easily.

If we examine the proof carefully, we find we can get the following additional results:

COROLLARY 1. *Under assumptions* (A)–(D), *F has an absolutely continuous component if and only if* $P(\underline{\lim}\, N_i > 0) > 0$.

COROLLARY 2. *Under assumptions* (A)–(D), *if* $P(\{i: N_i = 0 \text{ and } b_{i-1}/b_i < y\}$ is finite) *is zero or one for each y with at most one exception, every set of positive probability has the same dimension as a support.*

5. Further examples and the principal theorem

From the theorem and corollaries, it is clear that we merely have to select the distribution of the N_i so that there is a suitable collection $\mathcal{K}$ of subsets of the integers so that for $K \in \mathcal{K}$ the behavior of $\{i: N_i = 0\} \cap K$ is probabilistically different from that of $\{i: N_{1i} + N_{2i} = 0\} \cap K$.

EXAMPLE 2. Let the N_i be independent, $P(N_i = 0) = p_i$, $P(N_i = 1) = 1 - p_i$. Then F is singular if and only if $\sum p_i = \infty$, while the convolution $F * F$ is singular if and only if $\sum p_i^2 = \infty$.

We may even obtain more. For let $b_n = n!$ Then when F is singular, it is even zero-dimensional. Also, if we let $b_{2n} = (2n)!$, $b_{4n+1} = 2\cdot(4n)!$, $b_{4n+3} = 3\cdot(4n+2)!$, and

$$\sum p_{2n} < \infty, \tag{20}$$

$$\begin{cases} \sum p_{4n+3} = \infty, \\ \sum p_{4n+3}^2 < \infty, \end{cases} \tag{21}$$

$$\begin{cases} \sum p_{4n+1}^2 = \infty, \\ \sum p_{4n+1}^3 < \infty, \end{cases} \tag{22}$$

we have F is $\frac{1}{4}$-dimensional, $F * F$ is $\frac{1}{3}$-dimensional, and $F * F * F$ is absolutely continuous.

To obtain our more detailed results, let us specify the b's.

(E) If $n \geq 2$, let $b_{2n} = (2n)!$; if $2^k < n \leq 2^{k+1}$, $k \geq 0$, let $b_{2n+1} = [(2n)!\cdot 2^k/(n - 2^k)] + n$.

Then for $n \geq 5$, $r_n > (n-1)/2$; therefore, (C) is satisfied. Also, if $2n \geq 6$,

$$b_{2n-1} = (2n-2)!2^k/(n - 1 - 2^k) + n - 1 \leq (2n-2)!(n-2) + (n-1) < (2n-1)!, \tag{23}$$

so that $b_{2n}/b_{2n-1} > 2n$, and (A) is satisfied. For each odd integer $n \geq 2$ let $\varphi(n)$ be the largest integer x such that $2^x < n$, and let $\psi(n) = (n - 2^{\varphi(n)})/2^{\varphi(n)}$. We observe that $\psi(n) > \rho_n > \psi(n) - 1/(2n - 1)!$. Consequently, in theorem 1 we may replace b_{2n}/b_{2n+1} by $\psi(n)$.

What we shall do now is to let each N_i be a sum of finitely many Poisson random variables, all the Poisson variables being independent. Let, for $n \geq 1$, $z_{1n} = 2^n$, $z_{2n} = 2^{z_{1n}} + z_{1n}$, $z_{3n} = 2^{z_{2n}} + z_{2n}$, and so on. Then every z_{kn} is even, and $z_{kn} \neq z_{jm}$ unless $k = j$, $m = n$. Also $z_{kn} > n$.

Now we shall let $X_1(t)$ and $X_2(t)$ have the form (10) where, for X_1, the N_{1i} are independent Poisson with mean $t\lambda_i$. For X_2 the situation is more complicated. Let M_i be independent Poisson with mean $t\mu_i$. Let q be an idempotent function defined on the integers n such that $\varphi(n)$ is divisible by 3. If $i = 2n + 1$ and $\varphi(n)$ is not divisible by 3, let $N_{2i} = M_i$. If $\varphi(n)$ is divisible by 3 and $i = 2n + 1$, and if $q(n) = j$, let $N_{2i} = M_{2j+i}$. If i is even but not equal to any z_{kn}, let $N_{2i} = M_i$. If i is even and $i = z_{kn}$, let $N_{2i} = M_i + N_{2n}$. In other words, each N is a sum of at most two independent Poisson variables, but each M appears in an infinite number of N's.

This is done in order to guarantee that the Lévy-Khintchine representation of $X_2(t)$ is purely singular. The reason for the complications for $\varphi(n)$ divisible by 3 will appear later.

We now commence the proof of the following principal theorem.

PRINCIPAL THEOREM. *Let f, g, and h be three monotone nondecreasing functions from $(0, \infty)$ to $[0, 1] \cup \{\infty\}$ such that $f \leq h$, $g \leq h$. Then there exist infinitely divisible distributions F and G so that the Lévy-Khintchine representation of F is purely discrete, that of G is purely singular, and the dimensions of F^t, G^t, and $F^t * G^t$ are $f(t)$, $g(t)$ and $h(t)$.*

The idea of the proof is to choose λ, μ, and q in the preceding discussion so that theorem 1 gives the desired results, except that condition (B) is violated, and then to modify them so that condition (B) will be satisfied. To show that this modification can be done, notice that condition (B) follows from $\lambda_n + \mu_n = O(n)$, and if we define $\lambda_n' = \min(\lambda_n, n)$, $\mu_n' = \min(\mu_n, n)$, the convergence or divergence of series of the form $\sum_{n\in S} e^{-t\sigma_n}$ will be equivalent to that of $\sum_{n\in S} e^{-t\sigma_n'}$, where σ is λ, μ, or $\lambda + \mu$.

The structure of the proof is as follows. If the parameters are chosen so condition (D) is satisfied, $\{i\colon N_i = 0 \text{ and } i \text{ even}\}$ is finite. To do this, let us choose $\lambda_{2n} = \mu_{2n} = 2n$. Let us also make $\mu_{2n+1} = 2n + 1$ if $\varphi(n) \equiv 1 \pmod 3$ and $\lambda_{2n+1} = 2n + 1$ if $\varphi(n) \equiv 2 \pmod 3$. Thus the i's of the form $2k + 1$ with $\varphi(k) \equiv 1 \pmod 3$ will contribute nothing to the dimension of G^t, and those with $\varphi(k) \equiv 2 \pmod 3$ will contribute nothing to the dimension of F^t, and only those with $\varphi(k)$ divisible by 3 need to be considered in the dimension of $F^t * G^t$. Notice also that if $Q_{ij} = \{k\colon N_{i,2k+1} = 0 \text{ and } b_{2k}/b_{2k+1} < y \text{ and } \varphi(k) \equiv j \pmod 3\}$, the dimension of F^t is the minimum of that determined by Q_{11} and Q_{13}, and that of G^t by Q_{22} and Q_{23}, and for $F^t * G^t$ by $Q_{13} \cap Q_{23}$. Consequently, if we can construct the λ's, μ's, and q so that consideration of Q_{11} makes the dimension of F^t take the value $f(t)$, Q_{22} makes the dimension of G^t equal $g(t)$, and Q_{13}, Q_{23}, and $Q_{13} \cap Q_{23}$ cause the dimensions of F^t, G^t, and $F^t * G^t$ all to assume the value $h(t)$. The proofs of all but the last part closely follow [3], and will now be given.

It is enough to show how the dimension of F^t can be made equal to $f(t)$ by suitably choosing λ_{2k+1} for $\varphi(k) \equiv 1 \pmod 3$. For each binary rational r, let

(24) $$T_r = \{k: \varphi(k) \equiv 1 \pmod 3 \text{ and } \psi(k) = r\}.$$

If $r \leq 1$, let $\tau(r) = \inf \{t: f(t) > r\}$, so that if $r = 1$, $\tau(r) = \inf \{t: f(t) = \infty\}$.

If $\tau(r) = \infty$, set $\eta(r, m) = 0$; if $\tau(r) = 0$, set $\eta(r, m) = m$; if $f(\tau(r)) > r$, set $\eta(r, m) = (\log m + 2 \log (1 + \log m))/\tau(r)$; and if $f(\tau(r)) \leq r$, set $\eta(r, m) = \log m/\tau(r)$. Then if k is the m-th value for which $\psi(k) = r$ and $\varphi(k) \equiv 1 \pmod 3$, set $\zeta(2k + 1) = \eta(r, m)$. Thus we have for every rational number r and every $t > 0$,

(25) $$\theta(t, r) = \sum_{\substack{\varphi(k) \equiv 1 \pmod 3 \\ \psi(k) = r}} e^{-t\zeta(2k+1)}$$

is finite precisely when $f(t) > r$. If the ζ's were to be used as λ's, it would immediately follow from theorem 1 that the dimension of F cannot exceed $f(t)$, since for any $r \geq f(t)$,

(26) $$P(\{k: N_{1,2k+1} = 0 \text{ and } \psi_k = r\} \text{ is finite}) = 0.$$

Now let $T = \{t: t \text{ rational or } f \text{ is discontinuous at } t \text{ and } t > 0\}$, and let $t_1, \cdots, t_n, \cdots$ be the elements of T. Also let $r_1, \cdots, r_n, \cdots$ be an enumeration of the binary rationals. Define C_n to be the smallest number so that

(27) $$C_n t_k - \log \theta(t_k, r_n) \geq n$$

for all $k \leq n$ for which $\theta(t_k, r_n)$ is finite. Now if $q(k) \equiv 1 \pmod 3$ and $\psi(k) = r_n$, let

(28) $$\lambda_{2k+1} = C_n + \zeta(2k + 1).$$

Now let u be any positive number and let $\epsilon > 0$. Choose $t \in T$ so that $t < u$ and $f(t) > f(u) - \epsilon$. Then by (27) and (28),

(29) $$\sum_{\substack{\varphi(k) \equiv 1 \bmod 3 \\ \psi(k) < f(t)}} P(N_{1,2k+1} = 0) < \infty.$$

Therefore F^u is of dimension at least $f(u) - \epsilon$, q.e.d. Note that condition (B) may not be satisfied, but we have already shown that this can be remedied. A similar calculation holds for μ_{2k+1} for $\varphi(k) \equiv 2 \pmod 3$.

The more difficult case is that of μ_{2k+1} for $\varphi(k)$ divisible by 3. Use the above procedure to define λ_{2k+1} for those k's with f replaced by h and "$\equiv 1 \pmod 3$" replaced by "divisible by 3."

Let k_j be the 2^{j-1} value n with $\psi(n) = r$ and $\varphi(n)$ divisible by 3, and for those k's with $\psi(k) = r$ and $\varphi(k)$ divisible by 3 between k_j and k_{j+1}, set $q(2k + 1) = 2k_j + 1$ and $\mu_{2k+1} = \lambda_{2s+1}$, where s is the j-th number u with $\psi(u) = r$ and $\varphi(u)$ divisible by 3. Now if the dimension exceeds r, the previous proof certainly remains valid since $N_{1i} + N_{2i} \geq N_{1i}$. Also the previous proof that F^t has dimension $\leq f(t)$ now shows that the dimension is $\leq h(t)$ for both F^t and G^t.

Now let r be a binary rational and $h(t) \leq r$. If k is the m-th number with $\varphi(k) \equiv 0 \pmod 3$ and $\psi(k) = r$, for sufficiently large m we have

$$\exp(-t\lambda_{2k+1}) > c/m. \tag{30}$$

Consequently, the probability that some $N_{1,2k+1}$ is zero for such a k with $2^{j-1} \leq m < 2^j$ is

$$R_j > 1 - \prod_{m=2^{j-1}}^{2^{j}-1} (1 - c/m) > v > 0, \tag{31}$$

for m sufficiently large. Now the probability that *all* $N_{2,2k+1}$ are zero for those k's is, since they are all the same, the j-th element of a divergent series. Consequently, it is almost certain that for infinitely many k's with $\varphi(k) \equiv 0 \pmod 3$ and $\psi(k) = r$, $N_{1,2k+1} + N_{2,2k+1} = 0$, and hence dimension $(F^t * G^t) \leq h(t)$.

This concludes the proof of the theorem.

6. Concluding remarks

It is clear that the principal theorem can be generalized to include n convolvants, some of which may be specified to have a discrete representation and some a singular representation. It is hoped that some clever argument can be found to simplify the detailed manipulation required by the author to obtain the principal theorem.

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