INTEGRATION OF CORRESPONDENCES

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1. Introduction

The traditional economic concept of a set of agents, each of which cannot influence the outcome of their collective activity but certain coalitions of which can influence that outcome, has recently received its proper mathematical formulation by means of measure theory. After J. W. Milnor and L. S. Shapley [32] had considered in 1961 a game with a measure space of players (see also [39], [41], [40], [13], [33]), in an article [2] published in 1964, R. J. Aumann showed how two basic concepts for an economy, namely the set of competitive allocations and the core, coincide when the set of consumers is an atomless positive finite measure space. Another solution of this equivalence problem based on Lyapunov's theorem [28] was then given by K. Vind [42]. In the light of this result, measure theory indeed appears as the natural context in which to study economic competition. (The concept of a continuum of agents has also been used in economic theory for different purposes by R. G. D. Allen and A. L. Bowley ([1], pp. 140–141) and H. S. Houthakker [23].)

Now, given a finite set A of agents and a real Banach space S (the commodity space), a standard operation in the analysis of economic equilibrium consists of associating with every element a of A a nonempty subset $\varphi(a)$ of S, that is, of defining a correspondence φ from A to S in Bourbaki's [9] terminology, and of taking the sum $\sum_{a \in A} \varphi(a) = \{z \in S | z = \sum_{a \in A} f(a)$ for some function f from A to S such that, for every $a \in A, f(a) \in \varphi(a)\}$.

In the new measure-theoretic context, the set A of agents is an arbitrary set; the set $\mathcal{A}$ of coalitions if a σ-field of subsets of A; a countably additive nonnegative real-valued function ν is defined on $\mathcal{A}$ with the interpretation that, for a coalition $E \in \mathcal{A}$, $\nu(E)$ is the fraction of the totality of agents contained in E. In this context the sum $\sum_{a \in A} \varphi(a)$ must be replaced by the integral $\int_A \varphi \, d\nu$ of the correspondence φ. Thus it becomes necessary to define this integral and to study its properties, in order to be able to reformulate the theory of economic equilibrium. In [3], Aumann has made to this problem a fundamental contribution which this article proposes to extend in several directions. The first extension aims at replacing his assumption that the set of agents is an analytic set by the assumption that it is a measurable space. From the viewpoint of economic interpretation, this generalization is important, for the identification of

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economic agents with points of an analytic set seems artificial, unlike the assumptions that every countable union of coalitions is a coalition and that the complement of every coalition is a coalition. The second extension consists of introducing three criteria for the measurability of a correspondence in addition to the criterion used by Aumann, the four criteria being essentially equivalent. Since in the various situations encountered in the theory of integration of correspondences one of these criteria is often far easier to apply than the others, this four-fold diversity is of great convenience. The third extension attempts to relax the assumption of finite dimensionality of the space S.

Aside from these extensions, this work differs from Aumann's in its approach which treats the theory of integration of correspondences as a particular case of the theory of integration of functions. The reasons that make such a treatment possible can be outlined here.

Assume that for every $a \in A$, $\varphi(a)$ is compact and convex. The correspondence φ from A to S can be considered as a function from A to the set $\mathcal{L}$ of nonempty, compact, convex subsets of S. As Price [34] (a reference for which I thank L. Le Cam) has remarked, the properties of the set $\mathcal{L}$ endowed with the Hausdorff metric are such that the theory of integration of functions from A to a real Banach space can easily be transposed into a theory of integration of functions from A to $\mathcal{L}$. Actually, one can go further and, following Rådström [35], embed $\mathcal{L}$ in a real Banach space $\hat{\mathcal{L}}$. The transposition then becomes unnecessary.

The preceding program requires the set $\varphi(a)$ to be compact and convex for every $a \in A$. However, if the space S is finite-dimensional, the convexity assumption is inessential, and a theory of integration of compact-valued correspondences is actually obtained. Given the needs of the analysis of economic equilibrium for which the present theory of integration of correspondences is developed, the two restrictions of finite-dimensionality of S and of compactness of $\varphi(a)$ for every $a \in A$ do not seem to be severe.

This article is organized in the following manner. In section 2, S is assumed to be an arbitrary metric space; the Hausdorff distance on the family of nonempty subsets of S is studied. Section 3 reviews certain generalities about measure theory ranging from the universally known (in which case our purpose is to dispel ambiguities in terminology and notation) to the almost unknown. Section 4 is devoted to the question of measurability of compact-valued correspondences from A to S; among its main results are propositions establishing connections between three of the measurability criteria for correspondences mentioned earlier, and a measurability theorem of central importance for equilibrium analysis. In section 5, S is restricted to be a normed real vector space; the family $\mathcal{L}$ of its nonempty, compact, convex subsets endowed with the Hausdorff metric and with the operations of addition of two elements and of multiplication of an element by a nonnegative real number is then embedded in a real Banach space $\hat{\mathcal{L}}$; the fourth measurability criterion for correspondences, specially designed for convex-compact-valued correspondences, is discussed. Section 6 further restricts S to be a real Banach space; it is concerned with the problem of integrating convex-compact-valued correspondences from A to S. Section 7 puts the final restriction of finite-dimensionality on S and studies the integration of compact-valued correspondences from A to S.

Extensive work has been done in statistics by Blackwell [7], [8], Chernoff [12], Dvoretzky, Wald, Wolfowitz [17], [18], [19], Karlin [24], Kudō [25], and Richter [36], [37] (I owe these two references to K. Vind) on mathematical problems closely related to those with which this article deals. With the exception of the references made below to certain points of these contributions, no attempt will be made to compare in detail the mathematical results that were obtained in the two lines of development, the statistical one and the economic one. It may be noted that among the propositions required by the economic theory appear generalizations of several results of the statistical theory.

This paper represents a laying up of many strands, as the bibliography alone indicates, and it could not have achieved its present form without the conversations I had with economists, mathematicians, and statisticians over the last year, and particularly, with R. J. Aumann, S. Kakutani, R. Radner and K. Vind during the summer of 1964. I also wish to acknowledge certain specific debts of this article to Aumann [3]. Several of the arguments below that appeal to his criterion for the measurability of a correspondence, namely the measurability of its graph, have been suggested by [3] or by himself. On the other hand, the proofs of propositions (6.5) and (7.2) use ideas of the proofs of the corresponding propositions in [3].

2. Hausdorff distance

Throughout this article, S denotes a fixed set with a metric s and $\mathcal{K}$ denotes the family of the nonempty, compact subsets of S, and $\mathcal{S}$ denotes the Borel σ-field of S, that is, the σ-field generated by the open subsets of S.

The terminology and the notation of N. Dunford and J. T. Schwartz [16] will be followed as closely as possible.

For two nonempty subsets X and Y of S and a point x of S, we define

$$\rho(x, Y) = \inf_{y \in Y} s(x, y), \qquad \rho(X, Y) = \sup_{x \in X} \rho(x, Y). \tag{1}$$

The number $\rho(X, Y)$ is called the *Hausdorff semidistance* [22] of X and Y. It enjoys the following two properties ($\overline{Y}$ denotes the closure of Y):

$$\rho(X, Y) = 0 \Leftrightarrow X \subset \overline{Y}; \tag{2.1}$$

$$\rho(X, Z) \leq \rho(X, Y) + \rho(Y, Z). \tag{2.2}$$

The *Hausdorff distance* of X and Y is defined by

$$\delta(X, Y) = \max \{\rho(X, Y), \rho(Y, X)\}. \tag{2}$$

It satisfies

$$\delta(X, Y) = 0 \Leftrightarrow \overline{X} = \overline{Y}; \tag{2.1$'$}$$

$$\delta(X, Z) \leq \delta(X, Y) + \delta(Y, Z). \tag{2.2$'$}$$

Therefore, δ is a metric on $\mathcal{K}$ which is henceforth endowed with the corresponding metric space structure.

The continuity of ρ is established by the inequality

$$|\rho(X, Y) - \rho(X', Y')| \leq \delta(X, X') + \delta(Y, Y'). \tag{2.3}$$

PROOF. According to (2.2),

$$\rho(X, Y) \leq \rho(X, X') + \rho(X', Y') + \rho(Y', Y). \tag{3}$$

Therefore,

$$\rho(X, Y) - \rho(X', Y') \leq \rho(X, X') + \rho(Y', Y) \leq \delta(X, X') + \delta(Y, Y'). \tag{4}$$

Similarly for $\rho(X', Y') - \rho(X, Y)$, Q.E.D.

The next three results assert that the properties of completeness, compactness, and separability carry over from S to $\mathcal{K}$ ((2.6) is a particular case of proposition 4.5.1 of E. Michael [31]).

(2.4) *If S is complete, then $\mathcal{K}$ is complete.*

PROOF. Let $\{X_p\}$ be a Cauchy sequence of elements of $\mathcal{K}$. There is a nonempty, closed subset X of S such that $\delta(X, X_p) \to 0$ (for example, [27], pp. 314–315). We have to check that X is compact.

To this end it suffices to prove that X is totally bounded. Therefore let ϵ be a strictly positive real number. For some p, one has $\delta(X, X_p) < \epsilon/2$. Since X_p is compact, it can be covered with a finite family of open balls with radius $\epsilon/2$. The finite family of open balls with the same centers and radius ϵ covers X, Q.E.D.

(2.5) *If S is compact, then $\mathcal{K}$ is compact.*

This is a result of Hausdorff ([22], p. 172, proposition VI).

(2.6) *If S is separable, then there is a countable family $\mathfrak{F}$ of finite subsets of S dense in $\mathcal{K}$.*

PROOF. Let $\{x_n\}$ be a countable dense subset of S and let $\mathfrak{F}$ be the family of the nonempty finite subsets of $\{x_n\}$. Clearly, $\mathfrak{F}$ is countable, and we now prove that, given X in $\mathcal{K}$ and $\epsilon > 0$, there is Y in $\mathfrak{F}$ such that $\delta(X, Y) < \epsilon$.

Consider the open balls in S with centers in $\{x_n\}$, radius ϵ and whose intersection with X is not empty. They form a covering of the compact set X. Take a finite subcovering and let Y be the set of the centers of the open balls in that subcovering. The set Y belongs to $\mathfrak{F}$.

Given $y \in Y$, the open ball with center y, radius ϵ has a nonempty intersection with X. Therefore $\rho(y, X) < \epsilon$, and hence $\rho(Y, X) < \epsilon$.

Given $x \in X$, there is $y \in Y$ such that x belongs to the open ball with center y, radius ϵ. Therefore $\rho(x, Y) < \epsilon$; hence $\rho(X, Y) < \epsilon$, Q.E.D.

3. Concepts and results of measure theory

Let M be a set, $\mathfrak{M}$ be a σ-field of subsets of M, T be a metric space, and $\mathfrak{J}$ be the σ-field generated by the open sets of T. A function f from M to T is said to be *measurable with respect to* $\mathfrak{M}$ or to be $\mathfrak{M}$*-measurable,* if, for every $E \in \mathfrak{I}$, the inverse image $f^{-1}(E)$ of E by f belongs to $\mathfrak{M}$.

Let M be a set, $\mathfrak{M}$ be a σ-field of subsets of M, and F be a set. A function f from M to F is said to be $\mathfrak{M}$*-simple* if there is a finite partition of M into sets belonging to $\mathfrak{M}$ such that f is constant in each set of the partition.

Since [16] will be used as a standard reference, it must be pointed out that these two definitions differ slightly from those of Dunford and Schwartz.

For the statement and the proof of the next result we need the following notations:

given a subset X of S, $X^s = \{Y \in \mathcal{K} | Y \subset X\}$ and $X^w = \{Y \in \mathcal{K} | Y \cap X \neq \emptyset\}$;
$\mathcal{E}$ is the σ-field generated by the open sets of $\mathcal{K}$;
$\mathcal{E}^s$ is the σ-field generated by the sets X^s where X is open in S;
$\mathcal{E}^w$ is the σ-field generated by the sets X^w where X is open in S;
$E \backslash F$ denotes the set of elements belonging to E but not to F.

According to an unpublished theorem of Dubins and Ornstein,

(3.1) $\mathcal{E}^s \subset \mathcal{E}$ *and* $\mathcal{E}^w \subset \mathcal{E}$. *If S is separable, then* $\mathcal{E}^s = \mathcal{E}^w = \mathcal{E}$.

Proof (Dubins and Ornstein). We first establish that if X is open in S, then X^s and X^w are open in $\mathcal{K}$.

To see that X^s is open in $\mathcal{K}$, consider an arbitrary element Y of X^s, namely, $Y \in \mathcal{K}$ and $Y \subset X$. Thus $Y \cap (S \backslash X) = \emptyset$. Exclude the trivial case $X = S$; since $S \backslash X$ is closed, every point of Y is at a strictly positive distance from $S \backslash X$ and the number $\epsilon = \min_{y \in Y} \rho(y, S \backslash X)$ is strictly positive. Every $Z \in \mathcal{K}$ for which $\delta(Z, Y) < \epsilon$ satisfies $Z \cap (S \backslash X) = \emptyset$, hence $Z \in X^s$.

To see that X^w is open in $\mathcal{K}$, consider an arbitrary element Y of X^w, that is, $Y \in \mathcal{K}$ and $Y \cap X \neq \emptyset$. Select a point $x \in Y \cap X$. There is in S an open ball with center x, radius $\epsilon > 0$ that is contained in X. Every $Z \in \mathcal{K}$ for which $\delta(Y, Z) < \epsilon$ intersects that ball, hence $Z \cap X \neq \emptyset$ and $Z \in X^w$.

Thus for every X open in S, one has $X^s \in \mathcal{E}$ and $X^w \in \mathcal{E}$. Consequently,

(i) $$\mathcal{E}^s \subset \mathcal{E} \quad \text{and} \quad \mathcal{E}^w \subset \mathcal{E}.$$

Having established the first assertion, we now assume that S is separable. Given a nonempty subset X of S and $\epsilon > 0$, we denote by $(X; \epsilon)$ the open set $\{x \in S | \rho(x, X) < \epsilon\}$ and by $[X, \epsilon]$ the closed set $\{x \in S | \rho(x, X) \leq \epsilon\}$.

If Y belongs to $\mathcal{K}$, then

(5) $$Y \subset (X; \epsilon) \Leftrightarrow \rho(Y, X) < \epsilon \quad \text{and} \quad Y \subset [X; \epsilon] \Leftrightarrow \rho(Y, X) \leq \epsilon.$$

Given $X \in \mathcal{K}$ and $\epsilon > 0$, we introduce the further notation

(6) $$\begin{aligned} (X; \epsilon)_w &= \{Y \in \mathcal{K} | X \subset (Y; \epsilon)\} = \{Y \in \mathcal{K} | \rho(X, Y) < \epsilon\}, \\ [X; \epsilon]_w &= \{Y \in \mathcal{K} | X \subset [Y; \epsilon]\} = \{Y \in \mathcal{K} | \rho(X, Y) \leq \epsilon\}. \end{aligned}$$

We will also need the remark that

(ii) *for every subset X of S, $\mathcal{K} \backslash X^s = (S \backslash X)^w$ and $\mathcal{K} \backslash X^w = (S \backslash X)^s$.*

Proof of (ii). The following relations hold:

$$\begin{aligned}\mathcal{K}\backslash X^{s} &= \{Y \in \mathcal{K} \mid Y \not\subset X\} = \{Y \in \mathcal{K} \mid Y \cap (S\backslash X) \neq \emptyset\} = (S\backslash X)^{w},\\ \mathcal{K}\backslash X^{w} &= \{Y \in \mathcal{K} \mid Y \cap X = \emptyset\} = \{Y \in \mathcal{K} \mid Y \subset (S\backslash X)\} = (S\backslash X)^{s},\end{aligned} \tag{7}$$

Q.E.D.

The space S is separable; therefore, according to (2.6), $\mathcal{K}$ is separable. Consequently, the open balls of $\mathcal{K}$ generate the σ-field $\mathcal{E}$. However, in $\mathcal{K}$, the open ball with center X, radius ϵ is

$$\{Y \in \mathcal{K} \mid \rho(X, Y) < \epsilon \quad\text{and}\quad \rho(Y, X) < \epsilon\} = (X; \epsilon)_w \cap (X; \epsilon)^s. \tag{8}$$

Thus, in order to complete the proof of the theorem, it suffices to establish

(iii) $\qquad (X; \epsilon)_w \in \mathcal{E}^s, \qquad (X; \epsilon)^s \in \mathcal{E}^s, \qquad (X; \epsilon)_w \in \mathcal{E}^w, \qquad (X; \epsilon)^s \in \mathcal{E}^w,$

for this implies that every open ball in $\mathcal{K}$ belongs to $\mathcal{E}^s$ and to $\mathcal{E}^w$, which implies in turn

(iv) $\qquad \mathcal{E} \subset \mathcal{E}^s \quad\text{and}\quad \mathcal{E} \subset \mathcal{E}^w.$

The proof of (iii) consists of the series of assertions (v)–(x):

(v) $\qquad (X; \epsilon)^s \in \mathcal{E}^s$

because $(X; \epsilon)$ is open;

(vi) $\qquad [X; \epsilon]^s \in \mathcal{E}^w$

because, by (ii), $\mathcal{K}\backslash[X; \epsilon]^s = (S\backslash[X; \epsilon])^w$ and $[X; \epsilon]$ is closed;

(vii) $\qquad [X; \epsilon]^w \in \mathcal{E}^s$

because, by (ii), $\mathcal{K}\backslash[X; \epsilon]^w = (S\backslash[X; \epsilon])^s$ and $[X; \epsilon]$ is closed;

(viii) $\qquad [X; \epsilon]^w \in \mathcal{E}^w$

because if $\{\epsilon_p\}$ is a strictly decreasing sequence converging to ϵ, one has $[X; \epsilon]^w = \{Y \in \mathcal{K} \mid Y \cap [X; \epsilon] \neq \emptyset\} = \{Y \in \mathcal{K} \mid \text{there is } y \in Y \text{ such that } \rho(y, X) \leq \epsilon\}$. By an immediate compactness argument on Y, the last set is seen to equal $\{Y \in \mathcal{K} \mid \text{for every } p, \text{ there is } y \in Y \text{ such that } \rho(y, X) < \epsilon_p\}$ which, in turn, equals $\bigcap_p \{Y \in \mathcal{K} \mid \text{there is } y \in Y \text{ such that } \rho(y, X) < \epsilon_p\} = \bigcap_p (X; \epsilon_p)^w$. However, every $(X; \epsilon_p)$ is open. Therefore, every $(X; \epsilon_p)^w$ belongs to $\mathcal{E}^w$ and so does their countable intersection. The following assertion also holds:

(ix) $\qquad [X; \epsilon]_w \in \mathcal{E}^s \quad\text{and}\quad [X; \epsilon]_w \in \mathcal{E}^w.$

To prove (ix), consider a countable dense subset $\{x_p\}$ of X. Given a nonempty subset Y of S, one has $[\rho(x, Y) \leq \epsilon$ for every $x \in X] \Leftrightarrow [\rho(x_p, Y) \leq \epsilon$ for every $p]$.

Therefore, $[X; \epsilon]_w = \{Y \in \mathcal{K} \mid \rho(X, Y) \leq \epsilon\} = \{Y \in \mathcal{K} \mid \textit{for every } p, \rho(x_p, Y) \leq \epsilon\} = \bigcap_p \{Y \in \mathcal{K} \mid \rho(x_p, Y) \leq \epsilon\} = \bigcap_p [x_p; \epsilon]_w$. However, $[x_p; \epsilon]_w = [x_p; \epsilon]^w$ which belongs to $\mathcal{E}^s$ by (vii) and to $\mathcal{E}^w$ by (viii).

(x) *If $\{\epsilon_p\}$ is a strictly increasing sequence converging to ϵ, then $(X; \epsilon)^s = \bigcup_p [X; \epsilon_p]^s$ and $(X; \epsilon)_w = \bigcup_p [X; \epsilon_p]_w$*

because $(X; \epsilon)^s = \{Y \in \mathcal{K} \mid \rho(Y, X) < \epsilon\} = \{Y \in \mathcal{K} \mid \text{for some } p, \rho(Y, X) \leq \epsilon_p\} =$

$\bigcup_p \{Y \in \mathcal{K} | \rho(Y, X) \leq \epsilon_p\} = \bigcup_p [X; \epsilon_p]^s$ and $(X; \epsilon)_w = \{Y \in \mathcal{K} | \rho(X, Y) < \epsilon\} = \{Y \in \mathcal{K} |$ for some p, $\rho(X, Y) \leq \epsilon_p\} = \bigcup_p \{Y \in \mathcal{K} | \rho(X, Y) \leq \epsilon_p\} = \bigcup_p [X; \epsilon_p]_w$.

The assertions (x) and (vi) establish that $(X; \epsilon)^s \in \mathcal{E}^w$; (x) and (ix) establish that $(X; \epsilon)_w \in \mathcal{E}^s$ and $(X; \epsilon)_w \in \mathcal{E}^w$; (v) asserts that $(X; \epsilon)^s \in \mathcal{E}^s$, Q.E.D.

Let M be a set, $\mathfrak{M}$ be a σ-field of subsets of M, and L be a finite-dimensional topological real vector space. A *measure* μ defined on $\mathfrak{M}$ with values in L is a function from $\mathfrak{M}$ to L such that, for every sequence $\{E_n\}$ of pairwise disjoint elements of $\mathfrak{M}$, one has $\sum_{n=1}^{\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n)$. Again this differs slightly from the terminology of [16].

A set E in $\mathfrak{M}$ is an *atom* for the measure μ if $\mu(E) \neq 0$ and $[F \in \mathfrak{M}, F \subset E] \Rightarrow [\mu(F) = 0$ or $\mu(E \backslash F) = 0]$. The following theorem is due to Lyapunov [28], [29] (see also [21] and [15]).

(3.2) *The range of μ is compact. If μ has no atom, then its range is also convex.*

Note: *in the remainder of this section the values of μ are assumed to be real and nonnegative.*

We have the result (see, for instance [6]) that

(3.3) *M can be partitioned into a countable family of atoms and an atomless part.*

(The set of atoms and/or the atomless part may be empty).

A subset E of M is said to be *null* if there is $F \in \mathfrak{M}$ such that $\mu(F) = 0$ and $E \subset F$. An assertion about the elements of M is said to be true *almost everywhere*, or *for almost every element of M*, if it is true except for the elements of a null set. The *Lebesgue extension* of $\mathfrak{M}$ is the family $\mathfrak{M}^*$ of the sets of the form $E \cup F$ where E belongs to $\mathfrak{M}$ and F is a null subset of M.

Given n sets $M_1, \cdots, M_n$ and for every $j = 1, \cdots, n$ a family $\mathfrak{M}_j$ of subsets of M_j, we denote by $\mathfrak{M}_1 \times \cdots \times \mathfrak{M}_n$ the family of the subsets of $M_1 \times \cdots \times M_n$ of the form $E_1 \times \cdots \times E_n$ where $E_j \in \mathfrak{M}_j$ for every $j = 1, \cdots, n$. This is still another slight departure from the notation of [16]. Given a set M and a family $\mathfrak{M}$ of subsets of M, we denote by $\mathcal{B}(\mathfrak{M})$ the σ-field generated by $\mathfrak{M}$. Finally, we denote the projection on M of a subset E of $M \times S$ by $\operatorname{proj}_M E$. The last result of this section is a generalization by D. A. Freedman of a lemma of D. Bierlein [5]. The proof that we give is due to D. A. Freedman and J. Neveu.

(3.4) *If S is complete and separable, then $E \in \mathcal{B}(\mathfrak{M}^* \times \mathcal{S})$ implies $\operatorname{proj}_M E \in \mathfrak{M}^*$.*

Proof. (1) Assume first that S is compact. Then $\mathcal{S}$ is the σ-field generated by the compact subsets of S. According to the theorem of E. Marczewski and C. Ryll-Nardzewski [30], if E belongs to the Suslin class generated by $\mathfrak{M}^* \times \mathcal{S}$, then $\operatorname{proj}_M E$ belongs to the Suslin class generated by $\mathfrak{M}^*$. However, this class coincides with $\mathfrak{M}^*$ by a classic result (for example, [38], chapter 2, section 5). Therefore there only remains to prove that $\mathcal{B}(\mathfrak{M}^* \times \mathcal{S})$ is contained in the Suslin class generated by $\mathfrak{M}^* \times \mathcal{S}$. This fact is established in lemma 2.a of [5] for the case in which S is the real line. It can be established for the present case of a compact S by the trivial substitution of the family of compact subsets of S for the family $\mathcal{I}$ of compact real intervals in Bierlein's proof.

(2) Consider now the case of a complete, separable S. Let H be the Hilbert cube and let $\mathcal{H}$ be its Borel σ-field. By ([27], p. 119), the space S is homeomorphic to a subspace S' of H. Let $\mathcal{S}'$ be the Borel σ-field of S'. By ([27], p. 337), S' is a countable intersection of open sets of H, and hence belongs to $\mathcal{H}$. Thus $\mathcal{S}' \subset \mathcal{H}$, and consequently, $E' \in \mathcal{B}(\mathfrak{M}^* \times \mathcal{S}')$ implies $E' \in \mathcal{B}(\mathfrak{M}^* \times \mathcal{H})$. Finally, since H is compact (for instance, [27], p. 91), $E' \in \mathcal{B}(\mathfrak{M}^* \times \mathcal{H})$ imples $\text{proj}_M E' \in \mathfrak{M}^*$ by (1), Q.E.D.

4. Measurable compact-valued correspondences

Throughout this article, A denotes a given set endowed with a σ-field $\mathcal{A}$ of subsets and a nonnegative real measure ν on $\mathcal{A}$.

Given two sets E and F, a *correspondence* φ from E to F associates with every element x of E a nonempty subset $\varphi(x)$ of F. Its *graph* is

$$G(\varphi) = \{(x, y) \in E \times F | y \in \varphi(x)\}. \tag{9}$$

The correspondence φ can alternatively be considered as a function from E to the set of nonempty subsets of F. The ability to study φ from either point of view is very valuable and will often be called upon. But one must guard against the confusion that would arise if at any time the point of view from which φ is considered were not explicit. For instance, the graph of the *correspondence* φ from E to F is the subset of $E \times F$ defined above, whereas the graph of the *function* φ from E to $\mathcal{P}(F)$, the set of subsets of F, is a subset of $E \times \mathcal{P}(F)$, namely, $\{(x, Y) \in E \times \mathcal{P}(F) | Y = \varphi(x)\}$.

The *inverse* φ^{-1} of the function φ is defined as usual: let $\mathfrak{F}$ be a family of subsets of F, then

$$\varphi^{-1}(\mathfrak{F}) = \{x \in E | \varphi(x) \in \mathfrak{F}\}. \tag{10}$$

The *strong-inverse* φ^s of the correspondence φ is defined as follows: let Y be a subset of F; then

$$\varphi^s(Y) = \{x \in E | \varphi(x) \subset Y\}. \tag{11}$$

The *weak-inverse* φ^w of the correspondence φ is defined as follows: let Y be a subset of F; then

$$\varphi^w(Y) = \{x \in E | \varphi(x) \cap Y \neq \varnothing\}. \tag{12}$$

Let P be an attribute defined for the subsets of F. The correspondence φ from E to F is said to be *P-valued* if, for every $x \in E$, $\varphi(x)$ is P.

Let T be a metric space and φ be a compact-valued correspondence from T to S. We follow Kuratowski [26] in saying that the correspondence φ is *upper semicontinuous* at a point x_0 of T if, for every sequence $\{x_n\}$ of points of T converging to x_0, $\rho[\varphi(x_n), \varphi(x_0)]$ converges to zero; φ is *lower semicontinuous* at x_0 if $x_n \to x_0$ implies $\rho[\varphi(x_0), \varphi(x_n)] \to 0$; and φ is *continuous* at x_0 if $x_n \to x_0$ implies $\delta[\varphi(x_0), \varphi(x_n)] \to 0$.

A compact-valued correspondence φ from A to S is said to be *$\mathcal{A}$-measurable (resp. $\mathcal{A}^*$-measurable)* if the function φ from A to $\mathcal{K}$ is $\mathcal{A}$-measurable (resp. $\mathcal{A}^*$-measurable). Propositions (4.2)–(4.4) will study the relationship between three criteria for the measurability of the correspondence φ, the first one being the definition of measurability adopted here, and the third one being the generalization of Aumann's [3] definition of measurability to the present situation:

(a) for every $E \in \mathcal{E}$, $\varphi^{-1}(E)$ belongs to $\mathcal{A}$ [or to $\mathcal{A}^*$];

(b) for every $X \in \mathcal{S}$, $\varphi^s(X)$ and $\varphi^w(X)$ belong to $\mathcal{A}$ [or to $\mathcal{A}^*$];

(c) $G(\varphi)$ belongs to $\mathcal{B}(\mathcal{A} \times \mathcal{S})$ [or to $\mathcal{B}(\mathcal{A}^* \times \mathcal{S})$].

By trivial transpositions of the proofs of classic theorems about functions from A, endowed with its measure space structure, to a Banach space (see [16], lemma 9, p. 147, and corollary 13, p. 150) to the present case of functions from A to a metric space, one can establish that, *for a separable S,* (a) [*φ is $\mathcal{A}^*$-measurable*] *is equivalent to* (a′) [*there is a sequence $\{\varphi_p\}$ of $\mathcal{A}^*$-simple functions from A to $\mathcal{K}$ converging almost everywhere to φ*] *which is equivalent to* (a″) [*there is a sequence $\{\varphi'_p\}$ of $\mathcal{A}^*$-simple functions from A to $\mathcal{K}$ converging in measure to φ*]. Thus two standard additional measurability criteria are immediately obtained.

This remark permits an easy proof of a result on which we will call in section 7.

(4.1) *If S is separable, φ is an $\mathcal{A}$-measurable function from A to $\mathcal{K}$ and E is an atom for ν, then E contains an atom E' for ν on which φ is constant.*

Proof. There are a null set $A_0 \in \mathcal{A}$ and a sequence $\{\varphi_1, \varphi_2, \cdots\}$ of $\mathcal{A}$-simple functions from A to $\mathcal{K}$ converging to φ outside of A_0. Choose for φ_0 some constant function from A to $\mathcal{K}$.

Let $E_0 = E \backslash A_0$. Obviously E_0 is an atom, φ_0 is constant on E_0, and $\{\varphi_n\}$ converges to φ on E_0. Construct the sequence $\{E_q\}$ inductively as follows. Make the induction hypothesis (satisfied for $q = 0$) that E_q is an atom contained in E_p for every $p \leq q$, and that φ_q is constant on E_q. Consider a finite partition of E_q into $\mathcal{A}$-sets associated with the $\mathcal{A}$-simple function φ_{q+1}. One of these sets, denote it by E_{q+1}, is an atom; all the others are null. Therefore, the inductive construction can be carried out. Let now $E' = \bigcap_{q=0}^{\infty} E_q$. The sets E_q are non-increasing. For every q, one has $\nu(E_q) = \nu(E)$. Hence $\nu(E') = \nu(E)$. Thus E' is an atom. On E' every φ_n is constant and $\{\varphi_n\}$ converges to φ. Therefore, φ is constant on E', Q.E.D.

The next result concerns the connection between criteria (a) and (b).

(4.2) *Consider a compact-valued correspondence φ from A to S and the following three assertions:*

(α) *φ is $\mathcal{A}$-measurable;*

(β) *for every X open in S, $\varphi^s(X)$ belongs to $\mathcal{A}$;*

(β') *for every X open in S, $\varphi^w(X)$ belongs to $\mathcal{A}$;*

(α) *implies* (β) *and* (β').

If, in addition, S is separable, then the three assertions (α), (β), *and* (β') *are equivalent.*

PROOF. For any subset X of S, one has $\varphi^s(X) = \{a \in A | \varphi(a) \subset X\} = \varphi^{-1}(X^s)$. Moreover, according to (3.1), the σ-field $\mathcal{E}^s$ generated by the sets X^s where X is open in S is contained in $\mathcal{E}$. Therefore (α) implies (β) since [X open in S] implies [$X^s \in \mathcal{E}^s(\subset \mathcal{E})$], whereas $\varphi^s(X) = \varphi^{-1}(X^s)$.

Assume now that S is separable. According to (3.1), $\mathcal{E}^s = \mathcal{E}$. Therefore ($\beta$) implies ($\alpha$) since [for every X open in S, $\varphi^s(X) \in \mathcal{A}$] is equivalent to [for every X open in S, $\varphi^{-1}(X^s) \in \mathcal{A}$] which implies [for every $\mathfrak{F} \in \mathcal{E}^s(= \mathcal{E})$, $\varphi^{-1}(\mathfrak{F}) \in \mathcal{A}$].

The implications relating (α) and (β') are established in a similar manner, Q.E.D.

The connection between criteria (a) and (c) is dealt with in two separate propositions.

(4.3) *If S is separable and ϕ is an $\mathcal{A}$-measurable compact-valued correspondence from A to S, then $G(\phi)$ belongs to $\mathcal{B}(\mathcal{A} \times \mathcal{S})$.*

PROOF. Let Y be a countable dense subset of S and denote by $B(x, r)$ the open ball in S with center $x \in S$ and positive radius r. Define the sequence $\{\phi_n\}$ of correspondences from A to S by $\phi_n(a) = \{x \in S |$ there is $y \in Y$ such that $\rho(y, \phi(a)) < (1/n)$ and $s(x, y) < (1/n)\}$ for every $a \in A$. Clearly, for every a, $\phi(a) = \bigcap_{n=1}^{\infty} \phi_n(a)$. Hence $G(\phi) = \bigcap_{n=1}^{\infty} G(\phi_n)$. Moreover, $G(\phi_n) = \bigcup_{y \in Y} \{a \in A | \rho(y, \phi(a)) < (1/n)\} \times \{x \in S | s(x, y) < (1/n)\} = \bigcup_{y \in Y} \phi^w(B(y, (1/n))) \times B(y, (1/n))$. In this last product the first set belongs to $\mathcal{A}$, by (4.2), and the second set belongs to $\mathcal{S}$. Since $G(\phi_n)$ is the union of a countable family of sets belonging to $\mathcal{A} \times \mathcal{S}$, it belongs to $\mathcal{B}(\mathcal{A} \times \mathcal{S})$. So does $G(\phi)$, Q.E.D.

(4.4) *If S is complete and separable and φ is a correspondence from A to S such that $G(\varphi)$ belongs to $\mathcal{B}(\mathcal{A}^* \times \mathcal{S})$, then for every $X \in \mathcal{S}$, $\varphi^s(X)$ and $\varphi^w(X)$ belong to $\mathcal{A}^*$.*

PROOF. We have $\varphi^w(X) = \{a \in A | \varphi(a) \cap X \neq \varnothing\} = \mathrm{proj}_A [G(\varphi) \cap (A \times X)]$. Since $A \times X$ belongs to $\mathcal{A}^* \times \mathcal{S}$, the set $G(\varphi) \cap (A \times X)$ belongs to $\mathcal{B}(\mathcal{A}^* \times \mathcal{S})$. By (3.4), $\varphi^w(X)$ belongs to $\mathcal{A}^*$. We also have $\varphi^s(X) = \{a \in A | \varphi(a) \subset X\} = \{a \in A | \varphi(a) \cap (S \backslash X) = \varnothing\} = A \backslash \varphi^w(S \backslash X)$. Since $S \backslash X$ belongs to $\mathcal{S}$, $\varphi^w(S \backslash X)$ belongs to $\mathcal{A}^*$. So does $A \backslash \varphi^w(S \backslash X)$, Q.E.D.

We now prove a theorem that is basic to the theory of economic equilibrium. This proposition generalizes some of the results of ([14], section 2.16) and of ([4], lemma 5.6 and end of proof of lemma 5.10).

(4.5) *Given an $\mathcal{A}^*$-measurable compact-valued correspondence ϕ from A to S and a function u from $G(\phi)$ to the real line R, measurable with respect to $\mathcal{B}(\mathcal{A}^* \times \mathcal{S})$ and upper semicontinuous on $\phi(a)$ for every $a \in A$, let $v(a) = \max_{x \in \phi(a)} u(a, x)$ and $\psi(a) = \{x \in \phi(a) | u(a, x) = v(a)\}$. If S is complete and separable then the function v from A to R and the function ψ from A to $\mathcal{K}$ are $\mathcal{A}^*$-measurable.*

PROOF. According to (4.3), $G(\phi)$ belongs to $\mathcal{B}(\mathcal{A}^* \times \mathcal{S})$. Let c be a real number. The set $\{a \in A | v(a) \geqq c\}$ is the projection on A of the set $\{(a, x) \in G(\phi) | u(a, x) \geqq$

$c\}$ which belongs to $\mathcal{B}(\mathcal{A}^* \times \mathcal{S})$. By (3.4), $\{a \in A | v(a) \geqq c\}$ belongs to $\mathcal{A}^*$. Therefore v is $\mathcal{A}^*$-measurable.

The graph $G(\psi)$ of the correspondence ψ from A to S is the set $\{(a, x) \in G(\phi) | u(a, x) = v(a)\}$, which belongs to $\mathcal{B}(\mathcal{A}^* \times \mathcal{S})$. By (4.4) and (4.2), ψ is $\mathcal{A}^*$-measurable, Q.E.D.

5. Embedding operations

Henceforth, S is a normed real vector space. The norm of an element x of S is denoted by $|x|$ and O denotes the origin of S. Also θ denotes the one-element set $\{O\}$ and $\mathcal{L}$ denotes the family of the nonempty, compact, convex subsets of S.

We have already appealed to a transposition of results established for functions from a measure space to a real Banach space to the context of functions from a measure space to the metric space $\mathcal{K}$. In the sequel a similar transposition from the case of a real Banach space to the case of the metric space $\mathcal{L}$ endowed with a certain algebraic structure would be necessary on many more occasions. While there is little doubt that these transpositions can be carried out in a trivial fashion, a complete solution requires that this long and tedious work be actually performed. An alternative approach will be followed here. The metric space $\mathcal{L}$ with its algebraic structure will be embedded in a real Banach space. The theorems of the standard theory of integration (for example, [16]) will then be directly applicable. In the same manner the operation of transposition could be dispensed with in section 4 if one consented to work with the set $\mathcal{L}$ (instead of with the set $\mathcal{K}$), which presupposes that one had introduced a vector space structure on S. However, it has seemed worth emphasizing that the measurability results of section 4 depend only on the metric space structure of S.

The *sum* $X + Y$ of two subsets X, Y of S is defined by

$$(\mathfrak{A}) \qquad X + Y = \{z \in S | z = x + y \text{ for some } (x, y) \text{ in } X \times Y\}.$$

This addition has the properties (X, Y, Z are arbitrary subsets of S)

$$(\mathfrak{A}') \qquad X + (Y + Z) = (X + Y) + Z; \qquad X + Y = Y + X; \qquad X + \theta = X.$$

The *product* αX of a *nonnegative* real number α and a subset X of S is defined by

$$(\mathfrak{M}) \qquad \alpha X = \{z \in S | z = \alpha x \text{ for some } x \text{ in } X\}.$$

This multiplication has the properties (α, β are arbitrary nonnegative real numbers; X, Y are arbitrary subsets of S)

$(\mathfrak{M}')$ $\alpha(\beta X) = (\alpha\beta)X$; $1X = X$; $\alpha(X + Y) = \alpha X + \alpha Y$; if X is convex, then $(\alpha + \beta)X = \alpha X + \beta X$.

In the present more special context the Hausdorff semidistance ρ and the Hausdorff distance δ have additional properties. In (5.1)–(5.4), (5.1′), and (5.2′) capital letters are arbitrary nonempty subsets of S, α is an arbitrary nonnegative real number (properties (5.2) and (5.4) are given by Price [34]:

$$(5.1) \qquad \rho(\alpha X, \alpha Y) = \alpha \rho(X, Y);$$

(5.2) $$\rho(X_1 + X_2, Y_1 + Y_2) \leq \rho(X_1, Y_1) + \rho(X_2, Y_2).$$

PROOF OF (5.2). Let x_1, x_2, y_1, y_2 be arbitrary points of X_1, X_2, Y_1, Y_2, respectively. One has

(13) $$|(x_1 + x_2) - (y_1 + y_2)| = |(x_1 - y_1) + (x_2 - y_2)| \leq |x_1 - y_1| + |x_2 - y_2|.$$

Therefore $\rho(x_1 + x_2, Y_1 + Y_2) \leq \rho(x_1, Y_1) + \rho(x_2, Y_2)$. Hence the result.

From (5.1) and (5.2), one obtains the following assertion.

(5.3) *On the set of pairs of nonempty, convex subsets of S, the function ρ is convex.*

Denoting by $\dot{X}$ *the convex hull of* X, one has

(5.4) $$\rho(\dot{X}, \dot{Y}) \leq \rho(X, Y).$$

PROOF. For every $x \in X$, $\rho(x, \dot{Y}) \leq \rho(X, \dot{Y})$. According to (5.3), $\rho(x, \dot{Y})$ is a convex function of x on S. Thus for every $x \in \dot{X}$, one also has $\rho(x, \dot{Y}) \leq \rho(X, \dot{Y})$. Therefore $\rho(\dot{X}, \dot{Y}) \leq \rho(X, \dot{Y})$. Finally $Y \subset \dot{Y}$ implies $\rho(X, \dot{Y}) \leq \rho(X, Y)$, Q.E.D.

Assertions (5.1)–(5.4) obviously remain true if ρ is replaced by δ therein. We make explicit for future reference

(5.1′) $$\delta(\alpha X, \alpha Y) = \alpha\delta(X, Y);$$

(5.2′) $$\delta(X_1 + X_2, Y_1 + Y_2) \leq \delta(X_1, Y_1) + \delta(X_2, Y_2).$$

Since the sum of two elements of $\mathfrak{L}$, as well as the product of a nonnegative real number and an element of $\mathfrak{L}$, belong to $\mathfrak{L}$, the set $\mathfrak{L}$ is endowed with an algebraic structure satisfying ($\mathfrak{A}'$) and ($\mathfrak{M}'$) and with a metric δ satisfying (5.1′) and (5.2′). According to a theorem of Rådström [35],

(5.5) *$\mathfrak{L}$ can be embedded as a convex cone with vertex θ in a normed real vector space $\hat{\mathfrak{L}}$ in such a way that*
 (i) *the embedding is isometric,*
 (ii) *addition in $\hat{\mathfrak{L}}$ induces addition in $\mathfrak{L}$,*
 (iii) *multiplication by nonnegative real numbers in $\hat{\mathfrak{L}}$ induces the corresponding operation in $\mathfrak{L}$,*
 (iv) *$\mathfrak{L}$ spans $\hat{\mathfrak{L}}$.*

The norm of an element X of $\hat{\mathfrak{L}}$ will be denoted by $|X|$. To prevent confusion with a widespread usage, one must emphasize that, given an element X of $\hat{\mathfrak{L}}$, $-X$ denotes its negative, namely, the element of $\hat{\mathfrak{L}}$ which, if added to X, gives θ. For an element X of $\mathfrak{L}$, $-X$ coincides with the set $\{z \in S | z = -x \text{ for some } x \text{ in } X\}$ if and only if X is a one-element subset of S. We can also say that an element X and its negative $-X$ both belong to the cone $\mathfrak{L}$ if and only if X is a one-element subset of S. Therefore, the greatest vector subspace of $\hat{\mathfrak{L}}$ contained in the cone $\mathfrak{L}$ is the set $\mathfrak{L}_0$ of the one-element subsets of S, which can be identified with S.

(5.6) *If S is complete, then $\mathfrak{L}$ is complete.*

Proof. Let $\{X_p\}$ be a Cauchy sequence of elements of $\mathcal{L}$. According to (2.4), there is a nonempty, compact subset X of S such that $\delta(X, X_p) \to 0$. We have to check that X is convex.

Notice that, by (2.2), for every p, $\rho(\dot{X}, X) \leq \rho(\dot{X}, X_p) + \rho(X_p, X)$. One also has by (5.4), since X_p is convex, $\rho(\dot{X}, X_p) \leq \rho(X, X_p)$ for every p. Therefore $\rho(\dot{X}, X) \leq 2\delta(X, X_p)$ for every p. Consequently, $\rho(\dot{X}, X) = 0$ and, by (2.1), $\dot{X} \subset X$. Hence, $\dot{X} = X$, Q.E.D.

(5.7) *If S is separable, then $\hat{\mathcal{L}}$ is separable.*

Proof. The space $\mathcal{L}$ is a subspace of the metric space $\mathcal{K}$. Since the latter is separable by (2.6), so is the former. Let then $\{X_n\}$ be a countable dense subset of $\mathcal{L}$. It will be shown that the countable set of the $Y_{pq} = X_p - X_q$ is dense in $\hat{\mathcal{L}}$. Consider an arbitrary element Y of $\hat{\mathcal{L}}$ and an arbitrary $\epsilon > 0$. Since $\mathcal{L}$ spans $\hat{\mathcal{L}}$, Y can be written in the form $Y = X^1 - X^2$ with X^1 and X^2 belonging to $\mathcal{L}$. There are X_p, X_q belonging to $\{X_n\}$ such that $\delta(X^1, X_p) < \epsilon/2$ and $\delta(X^2, X_q) < \epsilon/2$. Therefore, $|Y - Y_{pq}| = |(X^1 - X^2) - (X_p - X_q)| = |(X^1 - X_p) + X_q - X^2)| \leq |X^1 - X_p| + |X^2 - X_q| < \epsilon$, Q.E.D.

Proposition (5.6) asserts that if S is complete, then $\mathcal{L}$ is complete. One naturally wonders whether, in that case, $\hat{\mathcal{L}}$ is also complete. The following example due to Aumann and Kakutani answers the question negatively. Let S be the Euclidean plane R^2. Let $\{\alpha_i\}$ be a decreasing sequence of positive real numbers such that $\alpha_1 < \pi/2$ and $\sum_{i=1}^{\infty} \sin \alpha_i < +\infty$. Given an angle α, denote by E_α the closed straight line segment whose extremities have coordinates (0, 0) and(cos α, sin α). Let $X_p = \sum_{i=1}^{p} E_{\alpha_i}$, $Y_p = pE_0$ and ${}_pZ = X_p - Y_p$. It is easy to see that $\{Z_p\}$ is a Cauchy sequence in $\hat{\mathcal{L}}$. And one can prove that there is no $Z \in \hat{\mathcal{L}}$ to which $\{Z_p\}$ converges.

However, *$\hat{\mathcal{L}}$ can be embedded as a dense subspace of a real Banach space $\bar{\mathcal{L}}$* by a standard operation ([16], p. 89). (I thank K. Vind for this reference.)

When the correspondence φ from A to S is compact-convex-valued, the measurability criterion (d) introduced by Kudō [25] in the case of a finite-dimensional S (see also Richter [36]) is available.

Criterion (d): for every continuous linear form f on S, the function μ_f defined for every $a \in A$ by $\mu_f(a) = \max_{x \in \varphi(a)} f(x)$ is $\mathcal{A}$-measurable (or $\mathcal{A}^*$-measurable).

(5.8) *If φ is an $\mathcal{A}$-measurable function from A to $\mathcal{K}$, then, for every continuous linear form f on S, μ_f is $\mathcal{A}$-measurable.*

Proof. Consider a continuous linear form f on S. Let c be a real number, and let X be the open set $\{x \in S | f(x) < c\}$. One has $\{a \in A | \mu_f(a) < c\} = \{a \in A | \varphi(a) \subset X\} = \varphi^s(X)$ which belongs to $\mathcal{A}$ by (4.2). Therefore, μ_f is $\mathcal{A}$-measurable, Q.E.D.

In order to prove a converse of (5.8), we need the following.

(5.9) *Let S be separable. Then there is a countable set C of continuous linear forms on S such that if X is a nonempty, compact, convex subset of S and B is a closed ball in S disjoint from X, then there is $f \in C$ for which* $\max f(X) < \inf f(B)$.

PROOF. Let $\mathfrak{F} = \{X_p\}$ be the countable family of proposition (2.6). Let $\{y_i\}$ be a countable dense subset of S. Let $\{B_q\}$ be the countable family of closed balls with centers in $\{y_i\}$ and positive rational radii. The set of pairs (X_p, B_q) is countable, and whenever $\dot{X}_p \cap B_q = \emptyset$, there is, by ([16], p. 417), a continuous linear form f_{pq} with norm 1 strictly separating the compact, convex set $\dot{X}_p$ and the closed, convex set B_q. We define C as the set of all the f_{pq} and the $-f_{pq}$ so obtained.

Let X and B be as in the statement of the proposition. Denote by ϵ the positive number $\min_{x \in X} \rho(x, B)$.

According to (2.6), there is $X' \in \mathfrak{F}$ such that $\delta(X, X') < \epsilon/6$. *A fortiori*, by (5.4), $\delta(X, \dot{X}') < \epsilon/6$. Thus

$$\text{(i)} \qquad x \in \dot{X}' \quad \textit{implies} \quad \rho(x, X) < \frac{\epsilon}{6}.$$

Let y and r be the center and the radius of the ball B. Select y' in $\{y_i\}$ such that $|y - y'| < \epsilon/6$. Consider now the closed ball B' with center y' and rational radius r' so chosen that $r + \epsilon/3 < r' < r + 2/3\epsilon$. One has $\delta(B, B') = |y' - y| + |r' - r|$. Therefore, $\delta(B, B') < 5/6\epsilon$. Consequently,

$$\text{(ii)} \qquad x \in B' \quad \textit{implies} \quad \rho(x, B) < \tfrac{5}{6}\epsilon.$$

From (i) and (ii) one obtains $\dot{X}' \cap B' = \emptyset$, for if x were in this intersection, there would be x^1 in X and x^2 in B such that $|x^1 - x^2| < \epsilon$, a contradiction of the definition of ϵ. Therefore, there is $f \in C$ such that $\max f(\dot{X}') < \inf f(B')$. However, since the norm of f is unity, $\delta(X, \dot{X}') < \epsilon/6$ implies $\max f(X) < \max f(\dot{X}') + \epsilon/6$ while $\inf f(B) = f(y) - r$ and $\inf f(B') = f(y') - r'$. Moreover, $|y - y'| < \epsilon/6$ implies $f(y') - f(y) < \epsilon/6 < r' - r - \epsilon/6$; hence $\inf f(B') + \epsilon/6 < \inf f(B)$. Consequently, $\max f(X) < \max f(\dot{X}') + \epsilon/6 < \inf f(B') + \epsilon/6 < \inf f(B)$, Q.E.D.

(5.10) *Let S be separable and let φ be a compact-convex-valued correspondence from A to S. If for every continuous linear form f on S, μ_f is $\mathfrak{A}$-measurable, then φ is $\mathfrak{A}$-measurable.*

PROOF. Given a closed ball B in S, let $E_B = \{(f, r) | f \in C$ and r is a rational number less than $\inf f(B)\}$. According to (5.9), one has $\{a \in A | \varphi(a) \cap B = \emptyset\} = \bigcup_{(f,r) \in E_B} \{a \in A | \mu_f(a) < r\}$. Since E_B is countable and μ_f is $\mathfrak{A}$-measurable, this union belongs to $\mathfrak{A}$.

Let now X be an open set in S. The set X is the union of a sequence $\{B_n\}$ of closed balls. Moreover, $A \backslash \varphi^w(X) = \{a \in A | \varphi(a) \cap (\bigcup_{n \in N} B_n) = \emptyset\} = \bigcap_{n \in N} \{a \in A | \varphi(a) \cap B_n = \emptyset\}$. According to the last paragraph, each set in this intersection belongs to $\mathfrak{A}$. Therefore, so does $A \backslash \varphi^w(X)$, hence also $\varphi^w(X)$. By (4.2), φ is $\mathfrak{A}$-measurable, Q.E.D.

6. Integrable compact-convex-valued correspondences

Henceforth S is a real Banach space.

For two $\mathfrak{A}$-simple functions φ, ψ from A to $\mathfrak{L}$, let

$$\Delta(\varphi, \psi) = \int |\varphi(a) - \psi(a)| \, d\nu(a). \tag{14}$$

A sequence $\{\varphi_p\}$ of $\mathfrak{A}$-simple functions from A to $\hat{\mathfrak{L}}$ is said to be *Δ-Cauchy* if $\Delta(\varphi_p, \varphi_q) \to 0$ as p and $q \to +\infty$. A function φ from A to $\hat{\mathfrak{L}}$ is said to be *integrable* if there is a Δ-Cauchy sequence of $\mathfrak{A}$-simple $\{\varphi_p\}$ from A to $\hat{\mathfrak{L}}$ (which is said to *determine* φ) converging in measure to φ. Its *integral* is $\int \varphi \, d\nu = \lim_{p\to\infty} \int \varphi_p \, d\nu$. According to ([16], lemma 16, p. 111), this limit exists and is independent of the sequence chosen to determine φ. For two integrable functions φ, ψ from A to $\hat{\mathfrak{L}}$, let $\Delta(\varphi, \psi) = \int |\varphi(a) - \psi(a)| \, d\nu(a)$.

Actually we will study integrable functions from A to $\mathfrak{L}$, and it is essential to know whether such a function can be determined by a sequence of $\mathfrak{A}$-simple functions from A to $\mathfrak{L}$ (rather than from A to $\hat{\mathfrak{L}}$). The answer is given by the following assertion.

(6.1) *If φ is an integrable function from A to $\mathfrak{L}$, then there is a sequence $\{\varphi_p\}$ of $\mathfrak{A}$-simple functions from A to $\mathfrak{L}$ determining φ.*

PROOF. Since φ is integrable, there is a sequence $\{\psi_p\}$ of $\mathfrak{A}$-simple functions from A to $\hat{\mathfrak{L}}$ determining φ. According to ([16], corollary 3, p. 145), there is a subsequence $\{\psi'_p\}$ of $\{\psi_p\}$ converging ν-uniformly to φ. Thus for every p, there is $E_p \in \mathfrak{A}$ contained in E_{p-1} such that $\nu(E_p) < 1/p$ and $\{\psi'_n\}$ converges uniformly to φ on $A\backslash E_p$. Hence, there is q_p such that

$$|\varphi(a) - \psi'_{q_p}(a)| < \frac{1}{p} \quad \text{on} \quad A\backslash E_p. \tag{i}$$

Consider a set F of the finite partition of $A\backslash E_p$ associated with ψ'_{q_p}, denoting the value of ψ'_{q_p} in F by X_{pF}, which is a point of $\hat{\mathfrak{L}}$ whose distance to the cone $\mathfrak{L}$ is, according to (i), smaller than $1/p$. Select a point Y_{pF} in $\mathfrak{L}$ such that

$$|X_{pF} - Y_{pF}| < \frac{1}{p}. \tag{ii}$$

Now define φ_p as the function from A to $\mathfrak{L}$ that, on each set F of the finite partition of $A\backslash E_p$, takes the value Y_{pF} and, on E_p, takes the value θ. We will prove that the sequence $\{\varphi_p\}$ of $\mathfrak{A}$-simple functions from A to $\mathfrak{L}$ determines φ.

Because of (i) and (ii), for every p, one has

$$|\varphi(a) - \varphi_p(a)| < \frac{2}{p} \quad \text{on} \quad A\backslash E_p. \tag{iii}$$

Therefore, $\{\varphi_p\}$ converges to φ ν-uniformly, hence also in measure. There remains to prove that $\{\varphi_p\}$ is Δ-Cauchy. Assume that $p < q$. Then

$$\begin{aligned}\Delta(\varphi_p, \varphi_q) = \int_A |\varphi_p(a) - \varphi_q(a)| \, d\nu(a) = \int_{A\backslash E_p} |\varphi_p(a) - \varphi_q(a)| \, d\nu(a) \\ + \int_{E_p\backslash E_q} |\varphi_p(a) - \varphi_q(a)| \, d\nu(a) + \int_{E_q} |\varphi_p(a) - \varphi_q(a)| \, d\nu(a).\end{aligned} \tag{15}$$

On $A\backslash E_p$, by (iii), $|\varphi_p(a) - \varphi_q(a)| < 4/p$; hence,

(iv) $$\int_{A \backslash E_p} |\varphi_p(a) - \varphi_q(a)|\, d\nu(a) < \frac{4}{p}\, \nu(A).$$

On $E_p \backslash E_q$, $\varphi_p(a) = \theta$; therefore, $|\varphi_p(a) - \varphi_q(a)| = |\varphi_q(a)| \leq |\varphi(a)| + |\varphi_q(a) - \varphi(a)| \leq |\varphi(a)| + 2/q$, hence

(v) $$\int_{E_p \backslash E_q} |\varphi_p(a) - \varphi_q(a)|\, d\nu(a) \leq \int_{E_p \backslash E_q} |\varphi(a)|\, d\nu(a) + \frac{2}{q}\, \nu(A).$$

On E_q, $\varphi_p(a) = \varphi_q(a) = \theta$; hence,

(vi) $$\int_{E_q} |\varphi_p(a) - \varphi_q(a)|\, d\nu(a) = 0.$$

From (iv), (v), and (vi),

(vii) $$\Delta(\varphi_p, \varphi_q) < \frac{4}{p}\, \nu(A) + \frac{2}{q}\, \nu(A) + \int_{E_p \backslash E_q} |\varphi(a)|\, d\nu(a).$$

When p and $q \to +\infty$, $\nu(E_p \backslash E_q)$, which is less than $1/p$, converges to zero. Therefore, according to ([16], theorem 20.b, p. 114), the last term in (vii) converges to zero. Consequently, so does $\Delta(\varphi_p, \varphi_q)$, Q.E.D.

The next result introduces an important convexity inequality:

(6.2) *if φ, ψ are integrable functions from A to $\mathfrak{L}$, determined by the sequences $\{\varphi_p\}$, $\{\psi_p\}$ of $\mathfrak{A}$-simple functions from A to $\mathfrak{L}$, then the function $\rho[\varphi(a), \psi(a)]$ is integrable and determined by the sequence $\{\rho[\varphi_p(a), \psi_p(a)]\}$ and $\rho[\int \varphi\, d\nu, \int \psi\, d\nu] \leq \int \rho[\varphi(a), \psi(a)]\, d\nu(a)$.*

Proof. According to (2.3), for every $a \in A$,

(16) $$|\rho[\varphi(a), \psi(a)] - \rho[\varphi_p(a), \psi_p(a)]| \leq \delta[\varphi(a), \varphi_p(a)] + \delta[\psi(a), \psi_p(a)].$$

Therefore, given $\epsilon > 0$, $\{a \in A|\, |\rho[\varphi(a), \psi(a)] - \rho[\varphi_p(a), \psi_p(a)]| > \epsilon\}$ is contained in the union of $\{a \in A | \delta[\varphi(a), \varphi_p(a)] > \epsilon/2\}$, and $\{a \in A | \delta[\psi(a), \psi_p(a)] > \epsilon/2\}$. This inclusion relation establishes that $\rho[\varphi_p(a), \psi_p(a)]$ converges in measure to $\rho[\varphi(a), \psi(a)]$. Moreover,

(17) $$\int |\rho[\varphi_p(a), \psi_p(a)] - \rho[\varphi_q(a), \psi_q(a)]|\, d\nu(a) \leq \Delta(\varphi_p, \varphi_q) + \Delta(\psi_p, \psi_q),$$

which establishes that $\{\rho[\varphi_p(a), \psi_p(a)]\}$ is Δ-Cauchy.

According to (5.3), for every p,

(18) $$\rho\left[\int \varphi_p\, d\nu, \int \psi_p\, d\nu\right] \leq \int \rho[\varphi_p(a), \psi_p(a)]\, d\nu(a).$$

When $p \to +\infty$, the left-hand side converges to $\rho[\int \varphi\, d\nu, \int \psi\, d\nu]$ by continuity of ρ, Q.E.D.

It is now possible to obtain extensions of the Lebesgue dominated convergence theorem to the case of correspondences.

(6.3) *Let φ be an integrable function from A to $\mathfrak{L}$ and $\{\varphi_p\}$ be a sequence of integrable functions from A to $\mathfrak{L}$ such that $\rho[\varphi_p(a), \varphi(a)] \to 0$ (resp. $\rho[\varphi(a), \varphi_p(a)] \to 0$) almost everywhere. If there is an integrable real-*

valued function f on A such that, for every p, $\rho[\varphi_p(a), \theta] \leq f(a)$ (resp. $\rho[\theta, \varphi_p(a)] \leq f(a)$) almost everywhere, then $\rho[\int \varphi_p\, d\nu, \int \varphi\, d\nu] \to 0$ (resp. $\rho[\int \varphi\, d\nu, \int \varphi_p\, d\nu] \to 0$).

PROOF. According to (6.2), for every p, the functions $\rho[\varphi_p(a), \varphi(a)]$ and $\rho[\varphi(a), \varphi_p(a)]$ are integrable. By the Lebesgue dominated convergence theorem ([16], p. 151), $\int \rho[\varphi_p(a), \varphi(a)]\, d\nu(a) \to 0$ (resp. $\int \rho[\varphi(a), \varphi_p(a)]\, d\nu(a) \to 0$). An application of the convexity inequality of (6.2) completes the proof.

As an immediate corollary of (6.3) we have the following proposition.

(6.4) *Let T be a metric space and let ψ be a function from $A \times T$ to $\mathfrak{L}$ that is integrable in a for every $x \in T$. If for almost every a, ψ is an upper semicontinuous (resp. lower semicontinuous) correspondence from T to S at a certain point x_0 of T and there is an integrable real-valued function f on A such that, for every $x \in T$, one has $\rho[\psi(a, x), \theta] \leq f(a)$ (resp. $\rho[\theta, \psi(a, x)] \leq f(a)$) almost everywhere, then the correspondence $\int \psi(a, x)\, d\nu(a)$ from T to S is upper semicontinuous (resp. lower semicontinuous) at x_0.*

PROOF. Let $\{x_p\}$ be a sequence of points of T converging to x_0. It suffices to let $\varphi(a) = \psi(a, x_0)$ and $\varphi_p(a) = \psi(a, x_p)$ to obtain the situation described by (6.3), Q.E.D.

Another concept of integral for a correspondence φ from A to S has been used by all the authors mentioned in the introduction who have treated this subject, with the exception of G. B. Price [34]; namely $\int^0 \varphi\, d\nu = \{z \in S | z = \int f\, d\nu$ for some integrable function f from A to S such that $f(a) \in \varphi(a)$ for every $a \in A\}$.

We will first prove that for an integrable function from A to $\mathfrak{L}$, this new concept does not differ from the concept introduced at the beginning of section 6.

(6.5) *If φ is an integrable function from A to $\mathfrak{L}$, then $\displaystyle\int^0 \varphi\, d\nu = \int \varphi\, d\nu$.*

PROOF. (1) Let z be an arbitrary point of $\int^0 \varphi\, d\nu$. There is an integrable function f from A to S such that $f(a) \in \varphi(a)$ for every a and $z = \int f\, d\nu$. For every a, one has $\rho[f(a), \varphi(a)] = 0$. Therefore, by (6.2), $\rho[\int f\, d\nu, \int \varphi\, d\nu] \leq \int \rho[f(a), \varphi(a)]\, d\nu(a) = 0$. Hence, $z \in \int \varphi\, d\nu$. Thus we have proved that

$$\int^0 \varphi\, d\nu \subset \int \varphi\, d\nu. \tag{19}$$

(2) Conversely, let z be an arbitrary point of $\int \varphi\, d\nu$. There is a sequence $\{\varphi_p\}$ of $\mathfrak{A}$-simple functions from A to $\mathfrak{L}$ determining φ. When $p \to +\infty$, $\int \varphi_p\, d\nu \to \int \varphi\, d\nu$. Therefore, for every p, there is z_p in $\int \varphi_p\, d\nu$ such that $z_p \to z$. In other words, for every p, there is an $\mathfrak{A}$-simple function f_p from A to S such that $f_p(a) \in \varphi_p(a)$ for every a and $z_p = \int f_p\, d\nu$. It will now be proved that

(i) *in $L_1(A, \mathfrak{A}, \nu, S)$ the set $\{f_p\}$ is weakly sequentially conditionally compact.*

(2.a) Since $|\varphi_p(a)| = \delta[\theta, \varphi_p(a)]$, $f_p(a) \in \varphi_p(a)$ implies $|f_p(a)| \leq |\varphi_p(a)|$. Hence $\int |f_p(a)|\, d\nu(a) \leq \int |\varphi_p(a)|\, d\nu(a)$. According to ([16], lemma 18, p. 113), the sequence $\{|\varphi_p(\cdot)|\}$ determines $|\varphi(\cdot)|$. Therefore, when $p \to +\infty$, $\int |\varphi_p(a)|\, d\nu(a) \to \int |\varphi(a)|\, d\nu(a)$. Consequently,

(ii) *the numbers* $\int |f_p(a)|\, d\nu(a)$ *form a bounded set.*

(2.b) For every p and every set $E \in \mathcal{A}$, the point $\int_E f_p\, d\nu$ belongs to $\int_E \varphi_p\, d\nu$. Hence, $|\int_E f_p\, d\nu| \leq |\int_E \varphi_p\, d\nu| \leq \int_E |\varphi_p(a)|\, d\nu(a)$. According to the last assertion of ([16], lemma 18, p. 113), $\int_A |\varphi_p(a) - \varphi(a)|\, d\nu(a) \to 0$. Thus by ([16], theorem 6 (ii), p. 122), $\lim_{\nu(E)\to 0} \int_E |\varphi_p(a)|\, d\nu(a) = 0$ uniformly in p. Therefore,

(iii) $\lim_{\nu(E)\to 0} \int_E f_p\, d\nu = 0$ *uniformly in* p.

Because of the second assertion of ([16], corollary 11, p. 294), (ii) and (iii) establish (i).

Now (i) implies that there is a subsequence $\{f'_p\}$ of $\{f_p\}$ such that $\{f'_p\}$ converges weakly to an integrable function f from A to S. Since the integral of a function depends linearly and continuously on that function, $\int_A f'_p\, d\nu \to \int_A f\, d\nu$. Consequently, $z = \int_A f\, d\nu$.

The symbol $\{\varphi'_p\}$ will denote the subsequence of $\{\varphi_p\}$ corresponding to the subsequence $\{f'_p\}$. Let ϵ be an arbitrary positive real number. Choose q such that, for every $p \geq q$, $\int |\varphi'_p(a) - \varphi(a)|\, d\nu(a) \leq \epsilon$. According to ([16], corollary 14, p. 422), there is a convex combination g of the f'_p (with $p \geq q$), $g = \sum_{i=1}^m \alpha_i f'_{j_i}$ where $\sum_{i=1}^m \alpha_i = 1$, and for every $i = 1, \cdots, m$, $\alpha_i \geq 0$, $j_i \geq q$ such that

(iv) $$\int |g(a) - f(a)|\, d\nu(a) \leq \epsilon.$$

For every $a \in A$, $\varphi(a)$ is convex; hence, by (5.3),

(v) $$\rho[g(a), \varphi(a)] \leq \sum_{i=1}^{m} \alpha_i \rho[f'_{j_i}(a), \varphi(a)].$$

On the other hand, for every a and every p, $f'_p(a) \in \varphi'_p(a)$; hence, $\rho[f'_p(a), \varphi(a)] \leq \rho[\varphi'_p(a), \varphi(a)]$. Therefore, for every $p \geq q$, $\int \rho[f'_p(a), \varphi(a)]\, d\nu(a) \leq \int \rho[\varphi'_p(a), \varphi(a)]\, d\nu(a) \leq \int \delta[\varphi'_p(a), \varphi(a)]\, d\nu(a) \leq \epsilon$. Consequently, by (v),

(20) $$\int \rho[g(a), \varphi(a)]\, d\nu(a) \leq \epsilon.$$

However, (iv) states that

(21) $$\int \rho[f(a), g(a)]\, d\nu(a) \leq \epsilon.$$

Combining the last two inequalities and using (2.2), one obtains

(22) $$\int \rho[f(a), \varphi(a)]\, d\nu(a) \leq 2\epsilon.$$

Since this holds for every $\epsilon > 0$,

(23) $$\int \rho[f(a), \varphi(a)]\, d\nu(a) = 0.$$

Therefore, $\rho[f(a), \varphi(a)] = 0$ almost everywhere; hence, $f(a) \in \varphi(a)$ almost everywhere.

In summary, given an arbitrary point z in $\int \varphi\, d\nu$, there is an integrable function f from A to S such that $z = \int f\, d\nu$ and $f(a) \in \varphi(a)$ for every a outside a null set A_0. Let f_0 be a function that coincides with f on $A \backslash A_0$ and such that $f_0(a) \in \varphi(a)$ for every $a \in A$. Since f_0 is integrable and $z = \int f_0\, d\nu$, we have proved that

$$\int \varphi\, d\nu \subset \int^0 \varphi\, d\nu, \tag{24}$$

Q.E.D.

7. Integrable compact-valued correspondences

Henceforth S is a finite-dimensional normed real vector space.

Let φ be an $\mathfrak{A}$-measurable compact-valued correspondence from A to S. We wish to define and to study the integral of φ. According to (3.3), A can be partitioned into an atomless part A_0 and a countable family of atoms $\{A_1, A_2, \cdots\}$. It is therefore sufficient to define the integral of φ over each A_i, $(i = 0, 1, \cdots)$ and then to define the integral of φ over A as the sum of the countable family of subsets of S so obtained. The definition of $\int_{A_i} \varphi\, d\nu$ for $i > 0$ is immediate, for, by (4.1), there is an atom A_i' contained in A_i and on which φ is constant. Let X_i be the nonempty, compact subset of S that is the value of φ on A_i'. Since $A_i \backslash A_i'$ is null, $\int_{A_i} \varphi\, d\nu = \nu(A_i) X_i$ necessarily.

The integration-theoretic difficulties of the study of $\int_A \varphi\, d\nu$ are therefore those of the study of $\int_{A_0} \varphi\, d\nu$, the integral of φ over the atomless part of A. For this reason we will assume from now on that ν is atomless.

We first prove a generalization due to H. Richter [36] of a result established by D. Blackwell [7], [8] for the case of a constant correspondence.

(7.1) *If ν is atomless and φ is a correspondence from A to S, then $\int^0 \varphi\, d\nu$ is convex.*

Proof. Let z_1, z_2 be two points of $\int^0 \varphi\, d\nu$. There are two integrable functions f_1, f_2 from A to S such that $z_1 = \int f_1\, d\nu$, $z_2 = \int f_2\, d\nu$ and, for every $a \in A$, $f_1(a)$ and $f_2(a)$ belong to $\varphi(a)$. Consider the measure μ with values in $S \times S$ defined on $\mathfrak{A}$ by $\mu(E) = (\int_E f_1\, d\nu, \int_E f_2\, d\nu)$ for each $E \in \mathfrak{A}$.

The measure μ is easily seen to be atomless. Suppose that E is an atom for μ. Then $\mu(E) \neq 0$ implies $\nu(E) > 0$. By (3.2) applied to ν, E can be partitioned into E_1 and E_1' belonging to $\mathfrak{A}$ such that $\nu(E_1) = \nu(E_1') = (1/2)\nu(E)$ and $\mu(E_1) = \mu(E)$, $\mu(E_1') = 0$. Repeating this construction one obtains a sequence $\{E_p\}$ with the property that for every p, $E_p \in \mathfrak{A}$, $E_{p+1} \subset E_p$, $\nu(E_p) = (1/2^p)\nu(E)$, and $\mu(E_p) = \mu(E)$. Let $F = \bigcap_{p=1}^{\infty} E_p$. Then $\nu(F) = 0$ and $\mu(F) = \mu(E) \neq 0$, a contradiction.

Observe now that $\mu(\varnothing) = (0, 0)$ and $\mu(A) = (z_1, z_2)$. Applying Lyapunov's theorem (3.2) to μ, one establishes that for any real number $\alpha \in [0, 1]$, there is $E \in \mathfrak{A}$ such that $\mu(E) = (\alpha z_1, \alpha z_2)$. Therefore, $\mu(A \backslash E) = [(1 - \alpha)z_1, (1 - \alpha)z_2]$.

Define the function f as coinciding with f_1 on E and with f_2 on $A\backslash E$. Since $\int f\,d\nu = \alpha z_1 + (1-\alpha)z_2$, the point $\alpha z_1 + (1-\alpha)z_2$ belongs to $\int^0 \varphi\,d\nu$, Q.E.D.

Let φ be a compact-valued correspondence from A to S. We denote by $\dot\varphi$ the correspondence taking for its value $\dot\varphi(a)$, the convex hull of $\varphi(a)$, for each $a \in A$. Since S is finite-dimensional, according to a classical result (see, for example, [20], theorem 10, p. 22), $\dot\varphi(a)$ is a *compact*, convex subset of S. Therefore, $\dot\varphi$ can be considered as a function from A to $\mathfrak{L}$. Our last proposition asserts that if ν is atomless and $\dot\varphi$ is integrable, then $\int^0 \varphi\,d\nu$ coincides with $\int \dot\varphi\,d\nu$, which is already known by (6.5) to coincide with $\int^0 \dot\varphi\,d\nu$. It is therefore legitimate to introduce the following definitions.

In the case of a finite-dimensional S and of an atomless ν, a compact-valued correspondence φ from A to S is said to be integrable if $\dot\varphi$ is integrable and its integral is then defined by $\int \varphi\,d\nu = \int \dot\varphi\,d\nu$.

(7.2) *Let φ be a compact-valued correspondence from A to S. If ν is atomless and $\dot\varphi$ is integrable, then $\int^0 \varphi\,d\nu = \int \dot\varphi\,d\nu$.*

PROOF. First, $\int^0 \varphi\,d\nu \subset \int^0 \dot\varphi\,d\nu = \int \dot\varphi\,d\nu$. The inclusion is obvious since, for every $a \in A$, $\varphi(a) \subset \dot\varphi(a)$. The equality is the assertion of (6.5). Thus

$$\int^0 \varphi\,d\nu \subset \int \dot\varphi\,d\nu. \tag{25}$$

Conversely, let z be a point in the boundary of $\int \dot\varphi\,d\nu$. It will be shown that z belongs to $\int^0 \varphi\,d\nu$. Since by (7.1) the latter set is convex, this will establish that $\int \dot\varphi\,d\nu \subset \int^0 \varphi\,d\nu$.

The proof is by induction on the dimension of the space S, the theorem being trivially true if the dimension of S is zero.

The relation $z \in \int \dot\varphi\,d\nu$ implies by (6.5) that there is an integrable function f from A to S such that $z = \int f\,d\nu$, and, for every $a \in A$, $f(a) \in \dot\varphi(a)$.

Now $\int \dot\varphi\,d\nu$ is convex. Therefore, if the dimension of S is not zero, there is a linear form g on S, not vanishing everywhere, such that

$$g(z) = \max_{y \in \int \dot\varphi\,d\nu} g(y). \tag{i}$$

Let $\psi(a) = \{x \in \varphi(a) | g(x) = \max_{y \in \varphi(a)} g(y)\}$. Consequently,

$$\dot\psi(a) = \{x \in \dot\varphi(a) | g(x) = \max_{y \in \dot\varphi(a)} g(y)\}. \tag{26}$$

Applying (4.5) to the $\mathfrak{A}^*$-measurable compact-valued correspondence $\dot\varphi$ from A to S and to the real-valued function g which is continuous on S and independent of a, one establishes that the correspondence $\dot\psi$ is $\mathfrak{A}^*$-measurable on A. Moreover, $\dot\psi(a) \subset \dot\varphi(a)$ implies $|\dot\psi(a)| \leq |\dot\varphi(a)|$ for the norm in $\mathfrak{L}$. Therefore, according to ([16], theorem 22.b, p. 117), $\dot\psi$ is integrable.

By a new application of (6.5), one obtains $\int^0 \dot\psi\,d\nu = \int \dot\psi\,d\nu$. Since the latter set, hence the former, is not empty, there is an integrable function f' from A to S such that $f'(a) \in \dot\psi(a)$ for every $a \in A$. By ([16], theorem 19.c, p. 113), $g(f(a))$

and $g(f'(a))$ are integrable, $\int g(f(a))\, d\nu(a) = g(\int f\, d\nu)$, and $\int g(f'(a))\, d\nu(a) = g(\int f'\, d\nu)$. On the other hand,

$$\text{(ii)} \qquad g(f(a)) \leq g(f'(a)) \qquad \textit{for every} \quad a \in A,$$

whereas by (i),

$$\text{(iii)} \qquad g\left(\int f\, d\nu\right) \geq g\left(\int f'\, d\nu\right)$$

because $\int f\, d\nu = z$ and $\int f'\, d\nu$ belongs to $\int^0 \dot{\varphi}\, d\nu = \int \dot{\varphi}\, d\nu$. Inequalities (ii) and (iii) together establish that $g(f(a)) = g(f'(a))$ almost everywhere, that is

$$\text{(37)} \qquad f(a) \in \dot{\psi}(a) \qquad \text{almost everywhere.}$$

Consider now $\zeta(a) = \psi(a) - \{f(a)\}$. According to ([16], theorem 19.a, p. 113), $\dot{\zeta}(a) = \dot{\psi}(a) - \{f(a)\}$ is integrable. Furthermore, for almost every a, $x \in \dot{\zeta}(a)$ implies $g(x) = 0$, while $0 \in \dot{\zeta}(a)$ almost everywhere. The latter relation implies

$$\text{(iv)} \qquad 0 \in \int^0 \dot{\zeta}\, d\nu = \int \dot{\zeta}\, d\nu.$$

Let H be the hyperplane $\{z \in S | g(z) = 0\}$. For almost every a, $\dot{\zeta}(a) \subset H$. Thus, by the induction hypothesis, $\int^0 \zeta\, d\nu = \int \dot{\zeta}\, d\nu$. By (iv), $0 \in \int^0 \zeta\, d\nu$. Therefore, there is an integrable function h from A to S such that $\int h\, d\nu = 0$ and, for every $a \in A$, $h(a) \in \zeta(a)$.

However, $h(a) \in \zeta(a)$ means that $f(a) + h(a) \in \psi(a)$. In conclusion, $f + h$ is integrable, for every $a \in A$ one has $[f(a) + h(a)] \in \psi(a)$, and $\int (f + h)\, d\nu = z$. Consequently, $z \in \int^0 \psi\, d\nu \subset \int^0 \varphi\, d\nu$, Q.E.D.

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