CAPACITY AND BALAYAGE FOR DECREASING SETS

M. BRELOT
UNIVERSITY OF PARIS

1. Introduction

A general concept of capacity (extracted by Bourbaki from Choquet's theory [8]) which we shall call general real capacity, is a real function (finite or not) $\mathcal{C}(e)$ of a set e in a Hausdorff space, satisfying the following conditions:

(i) $\mathcal{C}(e)$ is increasing; (ii) for any increasing sequence e_n, $\sup \mathcal{C}(e_n) = \mathcal{C}(\bigcup e_n)$; (iii) for any decreasing sequence of compact sets e_n, $\inf \mathcal{C}(e_n) = \mathcal{C}(\bigcap e_n)$.

Taking the value of $\mathcal{C}$ in a complete lattice [9] (with a greatest and a smallest element), these conditions may be considered as defining a "general capacity" and we shall meet interesting examples where it is a function of a variable in a suitable lattice of functions.

There has been no general study of the limit of a capacity for a decreasing sequence of noncompact sets. We shall develop and complete a lecture in the seminar on potential theory (November 1964) and a note [5] by giving some examples from balayage theory (and first from the most classical capacity) where the previous limit is made precise. Among these capacities, one will be a function, in the family (ordered by $\leq$ everywhere) of the superharmonic functions. The famous capacitability theorem for the real capacities holds also for this capacity. We shall even consider directed decreasing, and also increasing, sets of sets, and similar set functions which are not capacities but are given by balayage theory, in an axiomatic frame of potential theory, as well as in the classical case. (General research of Doob is in course and has given or inspired the extensions to directed sets, as it will be mentioned in the text.) After the lecture I developed here, I became aware of connected axiomatic discussions by Fuglede (see *C. R. Acad. Sci. Paris*, October 1965.) For this research and these results, the fine closed sets (that is, closed according to the fine topology) play an essential role and the basic tool is a theorem by Choquet on thinness (lemma 1).

Finally, an application will be made to get a proof of a theorem by Getoor (on a smallest fine closed support of a measure) that was first obtained by probability theory, then proved and generalized by Choquet [11] in an axiomatic way. Another application will generalize some results by using, as Doob proposed, the fine upper semicontinuous functions.

We shall work in pure potential theory without giving any probabilistic interpretation and we shall be able to shorten the redaction by referring to a recent paper [4] containing connected or similar concepts, tools or proofs

2. On capacity in classical potential theory

We recall some classical concepts in a Green space Ω, for instance, a bounded domain of $R^n (n \geq 2)$ (see [6]).

The *fine* topology on Ω is the coarsest one for which the superharmonic functions (locally or in Ω) are continuous; the complementary sets of the "fine" neighborhoods of x_0 are the sets which are thin at x_0 (not containing x_0). If $x_0 \in e$, e is thin if $e \backslash \{x_0\}$ is thin and $\{x_0\}$ not polar. The base $\mathcal{B}_e$ of a set e is the set of the points where e is not thin. It is a G_δ set. If $\tilde{e}$ means the fine closure of e, $\tilde{e} = \mathcal{B}_e \cup$ (set of the fine isolated but polar points of e). The second set is polar. Note that "$e = \tilde{e}$" is equivalent to $e \supset \mathcal{B}_e$ and that $\mathcal{B}_{\tilde{e}} = \mathcal{B}_{\mathcal{B}_e} = \mathcal{B}_e$. If $e = \mathcal{B}_e$, e is said to be a base.

For any real function $\varphi \geq 0$ on Ω, $(R_\varphi^e)_\Omega$ or R_φ^e (called reduced function of φ for e) means the lower envelope (infimum) of all hyperharmonic (that is, superharmonic or $+\infty$) nonnegative functions in Ω, majorizing φ on e. For any real function ψ, $\hat{\psi}$ means the lim inf at every point, that is, $\sup_{\omega \ni x} (\inf_{y \in \omega} \psi(y))$, ($\omega$ neighborhood of x), (regularized function of ψ).

(If ψ is the infimum of a family $\{v_i\}$ of hyperharmonic functions ≥ 0, $\hat{\psi}(x)$ is the limit of $\int \psi \, d\rho_x^\omega$, ($d\rho_x^\omega$ harmonic measure for a domain $\omega \ni x$) when ω decreases in a sequence of intersection $\{x\}$.)

The function $\hat{R}_\varphi^e$ is equal to R_φ^e q.e. (quasi everywhere, that is, except on a polar set) and is the smallest hyperharmonic ≥ 0 function which majorizes φ q.e. on e.

For a superharmonic $V \geq 0$, $\hat{R}_V^e$ is called the balayaged function of V relative to e; it is $\leq V$ everywhere, and equal to V q.e. on e. If $\bar{e}$ is compact, $\hat{R}_1^e$ (called capacitary potential) is the Greenian potential of a measure ≥ 0, whose total amount is the *(outer Greenian) capacity* of e, that may be suitably extended to any e, as a general real capacity.

Note that $\hat{R}_\varphi^e$ is a countably subadditive function of e. It is invariant by changing e in $\tilde{e}$, $\mathcal{B}_e$ or by a polar set; and therefore, the same holds for the Greenian capacity.

LEMMA 1 (Choquet [10]). *The set $\complement \mathcal{B}_e$ of points of Ω where e is thin may be embedded in an open set ω such that $e \cap \omega$ has a Greenian capacity arbitrarily small.*

It is equivalent to the same property of $\complement \mathcal{B}_e \cap \bar{e}$.

It is obvious that the decreasing property (iii) of the capacity does not hold for noncompact sets. For example, in R^n, $(n \geq 3)$, let us consider the set $e_n = \{x, \rho/(1 + (1/n)) < |x| < \rho\}$. The Newtonian capacity is the same as the capacity of the ball of radius ρ, but the intersection of these sets e_n is empty. However, the limit property may be true for sets which are not as restrictive as compact sets.

THEOREM 1. *If e_n is a decreasing sequence of fine closed sets on a compact $K \subset \Omega$, the Greenian (outer) capacity $\mathcal{C}(e_n)$ tends to $\mathcal{C}(\bigcap e_n)$; therefore, for any decreasing $\alpha_n \subset K$*

$$\mathcal{C}(\alpha_n) \rightarrow \mathcal{C}(\bigcap \tilde{\alpha}_n) = \mathcal{C}(\bigcap \mathcal{B}_{\alpha_n}). \tag{2.1}$$

Let us introduce an open set ω_n, containing $\complement \mathcal{B}_{e_n}$ and such that $\mathcal{C}(\omega_n \cap e_n) <$

$\epsilon/2^n$. A point x of $\overline{e_n\backslash\omega_n}$ is never in ω_n; the set e_n is not thin at x (if not, x would be in ω_n). Therefore $x \in \mathcal{B}_{e_n} \subset e_n$; then $x \in e_n\backslash\omega_n$. We conclude that $e_n\backslash\omega_n$ is closed.

The sets $e_1' = e_1\backslash\omega_1$, $e_2' = e_2\backslash(\omega_1 \cup \omega_2)$, $\cdots$, $e_n' = e_n\backslash(\omega_1 \cup \omega_2 \cdots \cup \omega_n)$ are compact and decreasing, and $\mathcal{C}(e_n') \to \mathcal{C}(\bigcap e_n')$. However,

$$
\begin{aligned}
(2.2) \qquad \mathcal{C}(e_n) &\leq \mathcal{C}(e_n') + \mathcal{C}\left(\bigcup_1^n (\omega_p \cap e_p)\right) \\
&\leq \mathcal{C}(e_n') + \epsilon.
\end{aligned}
$$

Hence, $\mathcal{C}(\bigcap e_n') \leq \mathcal{C}(\bigcap e_n) \leq \lim \mathcal{C}(e_n) \leq \mathcal{C}(\bigcap e_n') + \epsilon$. We conclude that $\mathcal{C}(e_n) \to \mathcal{C}(\bigcap e_n)$.

1. *Similar result with other capacities.* (a) If φ is finite continuous ≥ 0, R_φ^e *is a general capacity,* as a function taking its value in the set of the hyperharmonic nonnegative functions, which is a complete lattice for the natural order.

It is obvious that (i) and (ii) are satisfied. If e_n is compact and decreasing, we may see that $\widehat{\inf \hat{R}_\varphi^{e_n}}$ which is the *infimum in the lattice* of the $\hat{R}_\varphi^{e_n}$, is equal to $\hat{R}_\varphi^{\bigcap e_n}$.

In fact, $R_\varphi^K = \inf_{\omega \supset K} R_\varphi^\omega$, ($K$ compact, ω open) (the proof is easy and is nearly the same as in [2], p. 122). (If a hyperharmonic $v \geq 0$ is $\geq \varphi$ on K, $v \geq \lambda\varphi (0 < \lambda < 1)$ on a neighborhood α of K. Then $v \geq \lambda R_\varphi^\alpha$, $v \geq \lambda \inf_{\omega \supset K} R_\varphi^\omega$, ($\omega$ open), $R_\varphi^K \geq \inf_{\omega \supset K} R_\varphi^\omega$.) Therefore, $R_\varphi^{e_n} \to R_\varphi^{\bigcap e_n}$, $\hat{R}_\varphi^{e_n} \to \hat{R}_\varphi^{\bigcap e_n}$ q.e.; thus

$$
(2.3) \qquad \int \inf \hat{R}_\varphi^{e_n} \, d\rho_{x_0}^{\omega_0} = \int \hat{R}_\varphi^{\bigcap e_n} \, d\rho_{x_0}^{\omega_0},
$$

and hence the desired result.

(b) If m is a measure ≥ 0, which does not charge the polar sets, $\int R_\varphi^e \, dm$ equal to $\int \hat{R}_\varphi^e \, dm$ is a *general real-capacity.*

It is easy to check (i), (ii), and (iii). A particular case is the case of the measure $d\rho_{x_0}^{\omega_0}$ we just met.

(c) Let us recall that, if $e \subset K$ (fixed compact set), $\int \hat{R}_1^e \, d\rho_{x_0}^{\omega_0}$ (fixed ω_0 and x_0) and the outer Greenian capacity of e are simultaneously zero, or the ratio remains between two fixed positive numbers.

Therefore lemma 1 holds for the new capacity $\int \hat{R}_\varphi^e \, d\rho_{x_0}^{\omega_0}$ (consider first the subsets of a K) and also the proof of theorem 1. We are led to the following theorem.

THEOREM 1′. *If e_n is a decreasing sequence of fine closed sets on a compact set K and φ any fixed finite continuous nonnegative function, then*

$$
(2.4) \qquad \widehat{\inf \hat{R}_\varphi^{e_n}} = \hat{R}_\varphi^{\bigcap e_n}
$$

where the first member is the infimum of the capacities $\hat{R}_\varphi^{e_n}$ of e_n in the lattice of the nonnegative hyperharmonic functions (with the natural order).

From $\inf \int \hat{R}_\varphi^{e_n} \, d\rho_{x_0}^{\omega_0} = \int \hat{R}_\varphi^{\bigcap e_n} \, d\rho_{x_0}^{\omega_0}$, we deduce $\int \inf \hat{R}_\varphi^{e_n} \, d\rho_{x_0}^{\omega_0} = - \int \hat{R}_\varphi^{\bigcap e_n} d\rho_{x_0}^{\omega_0}$ and $\widehat{\inf \hat{R}_\varphi^{e_n}} = \hat{R}_\varphi^{\bigcap e_n}$.

COROLLARY. *The general equation* (2.4) *implies the limit property* (2.1) *for the general capacity $\int \hat{R}_\varphi^e \, dm$ of* (b) *when m does not charge polar sets and contains also theorem* 1.

(If u_n decreasing is a potential, harmonic outside a compact set K, the corresponding measure of u_n converges vaguely to the measure associated with $\widehat{\inf u_n}$; or see further theorem 2.)

IMPORTANT REMARK. *For the capacity $\hat{R}^e_\varphi$, the general capacitability theorem holds.*

It holds for the real capacity $\int \hat{R}^e_\varphi \, d\rho^{\omega_0}_{x_0}$, that is, if e is a K-analytic set contained in a K_σ set, $\sup_{K\subset e} \int \hat{R}^K_\varphi \, d\rho^{\omega_0}_{x_0} = \int \hat{R}^e_\varphi \, d\rho^{\omega_0}_{x_0}$; but the first member is equal to $\int \sup_{K\subset e} \hat{R}^K_\varphi \, d\rho^{\omega_0}_{x_0}$, for all x_0, ω_0 and that implies $\widehat{\sup_{K\subset e} \hat{R}^K_\varphi} = \hat{R}^e_\varphi$ (compact set K).

2. *Extension to directed sets* (essentially due to Doob). Without using general unpublished arguments of Doob (and also of the author) that would imply the passage from sequences to directed sets, we shall treat our particular problem by using, according to an idea of Doob, a topological lemma of Choquet (see [2], p. 3 or [6], p. 6).

LEMMA 2. *There exists a finite continuous potential $V_0 > 0$ such that, for any e, the base $\mathcal{B}_e$ is the set $\{x; \hat{R}^e_{V_0}(x) = V_0(x)\}$. Then the condition $\hat{R}^{e_1}_{V_0} \geq \hat{R}^{e_2}_{V_0}$ implies $\mathcal{B}_{e_1} \supset \mathcal{B}_{e_2}$.*

In fact, if $x \in \mathcal{B}_{e_1}$, $\hat{R}^{e_1}_{V_0}(x) = V_0(x) \leq \hat{R}^{e_2}_{V_0}(x)$; therefore there is equality.

LEMMA 3. *Given a decreasing directed set $\{e_i\}$ of fine closed sets $e_i \subset K$, there exists an extracted sequence e_n such that $\mathcal{B}_{\cap e_n} = \mathcal{B}_{\cap e_i} \subset \bigcap e_i$. Any other sequence e_{i_n} such that $e_{i_n} \subset e_n$ has the same property.*

We may choose e_n such that $\widehat{\inf \hat{R}^{e_n}_{V_0}} = \widehat{\inf_i \hat{R}^{e_i}_{V_0}}$; then $\hat{R}^{\cap e_n}_{V_0}$ (first member) is $\leq \hat{R}^{e_i}_{V_0}$, for all i. Therefore, $\mathcal{B}_{\cap e_n} \subset \mathcal{B}_{e_i} \subset e_i$, $\mathcal{B}_{\cap e_n} \subset \bigcap e_i$, and $\mathcal{B}_{\cap e_n} = \mathcal{B}_{\cap e_i} \subset \bigcap e_i$.

THEOREM 2. *For any decreasing directed set $\{e_i\}$ of fine closed sets e_i in a compact K and for a fixed finite continuous $\varphi \geq 0$,*

$$\widehat{\inf \hat{R}^{e_i}} = \hat{R}^{\cap e_i}_\varphi. \tag{2.5}$$

More extensive, the capacity $\mathcal{C}(e)$, equal to $\hat{R}^e_\varphi$ (in the above lattice), or to $\int R^e_\varphi \, dm$ ($m \geq 0$ not charging the polar sets) or to the Greenian capacity, satisfies the property $\inf \mathcal{C}(e_i) = \mathcal{C}(\bigcap e_i)$.

For any set $e_i \subset K$, this result holds by changing on the right, $\bigcap e_i$ to $\bigcap \tilde{e}_i$ or to $\mathcal{B}_{e_i}$.

We may extract e_n such that $\mathcal{B}_{\cap e_n} \subset \bigcap e_i$ and $\widehat{\inf \hat{R}^{e_i}_\varphi} = \widehat{\inf \hat{R}^{e_n}_\varphi}$. The first condition gives $\hat{R}^{\cap e_n}_\varphi \leq \hat{R}^{\cap e_i}_\varphi$; therefore $\hat{R}^{\cap e_n}_\varphi = \hat{R}^{\cap e_i}_\varphi$. The second one, thanks to theorem 1′, gives $\widehat{\inf \hat{R}^{e_i}_\varphi} = \hat{R}^{\cap e_n}_\varphi = \hat{R}^{\cap e_i}_\varphi$.

In order to complete the proof, let us choose e_n such that $\mathcal{B}_{\cap e_n} = \mathcal{B}_{\cap e_i}$ (which implies $\hat{R}^{\cap e_n}_\varphi = \hat{R}^{\cap e_i}_\varphi$) and $\inf \int \hat{R}^{e_i}_\varphi \, dm = \inf \int \hat{R}^{e_n}_\varphi \, dm$. Hence $\inf \int \hat{R}^{e_i}_\varphi \, dm = \int \hat{R}^{\cap e_n}_\varphi \, dm = \int \hat{R}^{\cap e_i}_\varphi \, dm$.

As for the Greenian capacity, consider ν_i and ν corresponding to the potentials $\hat{R}^{e_i}_1$, $\hat{R}^{\cap e_i}_1$. If α is the measure associated with $\hat{R}^\omega_1$(ω open set $\supset K$),

$$(2.6)\qquad \begin{aligned} \|\nu_i\| &= \int \hat{R}_1^{\omega_i}\, d\nu_i = \int \hat{R}_1^{e_i}\, d\alpha, \\ \inf \|\nu_i\| &= \int \hat{R}_1^{\cap e_i}\, d\alpha = \int \hat{R}_1^{\omega}\, d\nu = \|\nu\|. \end{aligned}$$

Finally note that for any sets e_i, $\bigcap \tilde{e}_i = \bigcap \mathcal{B}_{e_i}$ up to a polar set.

3. *Another extension.*

THEOREM 3. *We cancel the condition $e_i \subset K$ but impose $\varphi \leq V$, a potential. Then with the other hypothesis of theorem 2, we get the same equality*

$$(2.7)\qquad \widehat{\inf \hat{R}_\varphi^{e_i}} = \hat{R}_\varphi^{\cap e_i}$$

and the same limit property of the capacity $\int \hat{R}_\varphi^e\, dm$ (where $m \geq 0$, does not charge the polar sets and satisfies $\int \varphi\, dm < \infty$).

Observe that, for any compact set K,

$$(2.8)\qquad \begin{aligned} \hat{R}_\varphi^{e_i \cap K} &\leq \hat{R}_\varphi^{e_i} \leq \hat{R}_\varphi^{e_i \cap K} + \hat{R}_V^{\complement K}, \\ \hat{R}_\varphi^{\cap e_i \cap K} &\leq \hat{R}_\varphi^{\cap e_i} \leq \hat{R}_\varphi^{\cap e_i \cap K} + \hat{R}_V^{\complement K}. \end{aligned}$$

Therefore, both $\widehat{\inf \hat{R}_\varphi^{e_i}}$ and $\widehat{R_\varphi^{\cap e_i}}$ are between the extreme members of the last inequality. Their integrals with respect to $d\rho_{x_0}^{\omega_0}$ differ by $\int \hat{R}_V^{\complement K}\, d\rho_{x_0}^{\omega_0}$ which may be arbitrarily small. Therefore, these integrals are equal and so are the functions. The same previous inequalities show that

$$(2.9)\qquad \left|\inf \int \hat{R}_\varphi^{e_i}\, dm - \int \hat{R}_\varphi^{\cap e_i}\, dm\right|$$

is arbitrarily small and therefore equal to 0.

COROLLARY. *Lemma 3 extends without the restriction $e_i \subset K$.*

3. More general set functions of the classical balayage theory

1. *The case of increasing sets.* If we consider $\hat{R}_\varphi^e$ for *any* function $\varphi \geq 0$ on Ω, it is not always a capacity (as we shall see later), but the properties (i) and (ii) still hold. Let us examine property (ii) for an *increasing directed set* of sets.

Instead of the fine closed sets e characterized by $e \supset \mathcal{B}_e$ (and that we may call superbasic), we shall now use the sets characterized by $e \subset \mathcal{B}_e$ and called *subbasic* sets (examples are the fine open sets).

LEMMA 4. *For any increasing directed set of sets e_i, there exists an extracted increasing sequence e_n such that $\mathcal{B}_{\cup e_n} \supset \mathcal{B}_{e_i}$ for every i. Therefore, if the e_i are subbasic, $\mathcal{B}_{\cup e_n} \supset \bigcup e_i$, $\mathcal{B}_{\cup e_n} = \mathcal{B}_{\cup e_i}$.*

Let us choose a countable base of relatively compact domains ω_j and a countable dense set $\{x_k\}$. By means of the diagonal process, we may find an increasing sequence e_n such that

$$(3.1)\qquad \sup_n \int \hat{R}_{V_0}^{e_n}\, d\rho_{x_k}^{\omega_j} = \sup_i \int \hat{R}_{V_0}^{e_i}\, d\rho_{x_k}^{\omega_j} \qquad (\text{for all } \omega_j,\ x_k \in \omega_j)$$

where V_0 is the special potential of lemma 2.

Hence $\int \hat{R}_{V_0}^{\cup e_n} d\rho_x^{\omega_j} \geq \int \hat{R}_{V_0}^{e_i} d\rho_x^{\omega_j}$ for every i, first for $x \in \omega_j$ and in the sequence x_k, then for any $x \in \omega_j$. We conclude that $\hat{R}_{V_0}^{\cup e_n} \geq \hat{R}_{V_0}^{e_i}$, $\mathcal{B}_{\cup e_n} \supset \mathcal{B}_{e_i}$, for every i. If the e_i are subbasic, $\mathcal{B}_{\cup e_n} \supset e_i$, $\mathcal{B}_{\cup e_n} \supset \bigcup e_i$, and $\mathcal{B}_{\cup e_n} \supset \mathcal{B}_{\cup e_i}$.

THEOREM 4. *Let $\{e_i\}$ be an increasing directed set of subbasic sets and φ any function ≥ 0 on Ω. Then*

$$\sup_i \hat{R}_\varphi^{e_i} = \hat{R}_\varphi^{\cup e_i}. \tag{3.2}$$

We may extract an increasing sequence e_n such that $\mathcal{B}_{\cup e_n} = \mathcal{B}_{\cup e_i}$ and that for a countable system (ω_j, x_k) as above,

$$\sup_n \int \hat{R}_\varphi^{e_n} d\rho_{x_k}^{\omega_j} = \sup_i \int \hat{R}_\varphi^{e_i} d\rho_{x_k}^{\omega_j}. \tag{3.3}$$

Then

$$\int \hat{R}_\varphi^{\cup e_n} d\rho_{x_k}^{\omega_j} = \int \sup_i \hat{R}_\varphi^{e_i} d\rho_{x_k}^{\omega_j}, \qquad (x_k \in \omega_j, \forall j, \forall i) \tag{3.4}$$

because of (ii) and of the lower semicontinuity of $\hat{R}_\varphi^{e_i}$. The same holds by changing x_k to any $x \in \omega_j$. Hence, $\hat{R}_\varphi^{\cup e_n} = \sup_i \hat{R}_\varphi^{e_i}$. Since $\bigcup e_n$ and $\bigcup e_i$ have same base, the first member is $\hat{R}_\varphi^{\cup e_i}$.

2. *General balayaged potential and decreasing sets.*

LEMMA 5. *If e_n is fine closed, decreasing and $y \notin \bigcap e_n$,*

$$\widehat{\inf \hat{R}_{G_y}^{e_n}} = \hat{R}_{G_y}^{\cap e_n}. \tag{3.5}$$

If σ is an open neighborhood of y, $e_{n_0} \not\ni y$, $n > n_0$, then

$$\begin{aligned} \hat{R}_{G_y}^{e_n \setminus \sigma} &\leq \hat{R}_{G_y}^{e_n} \leq \hat{R}_{G_y}^{e_n \setminus \sigma} + \hat{R}_{G_y}^{e_{n_0} \cap \sigma}, \\ \hat{R}_{G_y}^{(\cap e_n) \setminus \sigma} &\leq \hat{R}_{G_y}^{\cap e_n} \leq \hat{R}_{G_y}^{(\cap e_n) \setminus \sigma} + \hat{R}_{G_y}^{e_{n_0} \cap \sigma}. \end{aligned} \tag{3.6}$$

By changing G_y to a finite continuous function equal to G_y outside σ (in order to use theorem 1′), we see that $\widehat{\inf \hat{R}_{G_y}^{e_n}}$ and $\hat{R}_{G_y}^{\cap e_n}$ lie between $\hat{R}_{G_y}^{\cap e_n \setminus \sigma}$ and $\hat{R}_{G_y}^{\cap e_n \setminus \sigma} + \hat{R}_{G_y}^{e_{n_0} \cap \sigma}$. Hence

$$\int \widehat{\inf \hat{R}_{G_y}^{e_n}} d\rho_{x_0}^{\omega_0} - \int \hat{R}_{G_y}^{\cap e_n} d\rho_{x_0}^{\omega_0} \leq \int \hat{R}_{G_y}^{e_{n_0} \cap \sigma} d\rho_{x_0}^{\omega_0}; \tag{3.7}$$

the second member is arbitrarily small for a suitable σ because e_{n_0} is thin at y. The first member is therefore null for any (ω_0, x_0). That is equivalent to the desired lemma.

THEOREM 5. *Given a decreasing sequence of sets e_n and a potential V of a measure $\mu \geq 0$, let us introduce the potentials V_1, V_2 of the restrictions μ_1, μ_2 of μ on $\bigcap \mathcal{B}_{e_n}$ and $\complement(\bigcap \mathcal{B}_{e_n})$. Then*

$$\widehat{\inf \hat{R}_V^{e_n}} = V_1 + \hat{R}_{V_2}^{\cap \mathcal{B}_{e_n}}. \tag{3.8}$$

(In this equality $\bigcap \mathcal{B}_{e_n}$ may be replaced by $\bigcap \tilde{e}_n$ which differs from it by a polar set.)

We may suppose e_n is a base (the inequality (3.8) remains unchanged if e_n is replaced by its base). Since $V(x) = \int G_y(x)\, d\mu(y)$, one has

$$(3.9) \qquad \hat{R}_V^{e_n}(x) = \int \hat{R}_{G_y}^{e_n}(x)\, d\mu(y) = \int \hat{R}_{G_y}^{e_n}(x)\, d\mu_1(y) + \int \hat{R}_{G_y}^{e_n}(x)\, d\mu_2(y).$$

Now $\hat{R}_{G_y}^{e_n} = G_y$ when $y \in \bigcap e_n$ (e_n is not thin at y), and the first term on the right is $V_1(x)$. Further,

$$(3.10) \qquad \begin{aligned} \int \hat{R}_V^{e_n}(x)\, d\rho_{x_0}^{\omega_0}(x) &= V_1(x_0) + \int \left(\int \hat{R}_{G_y}^{e_n}(x)\, d\rho_{x_0}^{\omega_0}(x)\right) d\mu_2(y), \\ \lim \int \hat{R}_V^{e_n}\, d\rho_{x_0}^{\omega_0} &= V_1(x_0) + \int \left(\int \hat{R}_{G_y}^{\bigcap e_n}(x)\, d\rho_{x_0}^{\omega_0}(x)\right) d\mu_2(y) \\ \int \lim \hat{R}_V^{e_n}\, d\rho_{x_0}^{\omega_0} &= V_1(x_0) + \int \left(\int \hat{R}_{G_y}^{\bigcap e_n}(x)\, d\mu_2(y)\right) d\rho_{x_0}^{\omega_0} \\ &= V_1(x_0) + \int \hat{R}_{V_2}^{\bigcap e_n}\, d\rho_{x_0}^{\omega_0}. \end{aligned}$$

Hence the theorem, by using a decreasing $\omega_0 \ni x_0$.

Another form of theorem 5. Let us introduce the restriction μ_b of μ to the base of $\bigcap \mathcal{B}_{e_n}$ and the corresponding potential V_b, then the positive measures $\mu' = \mu_1 - \mu_b$, $\mu'' = \mu_2 + \mu_b$ and the corresponding potentials V', V''. Then,

$$(3.11) \qquad \widehat{\inf \hat{R}_V^{e_n}} = V' + R_{V''}^{\bigcap \mathcal{B}_{e_n}}.$$

In fact, $V_1 = V' + V_b$, $V'' = V_2 + V_b$, $\hat{R}_{V''}^{\bigcap \mathcal{B}_{e_n}} = \hat{R}_{V_2}^{\bigcap \mathcal{B}_{e_n}} + \hat{R}_{V_b}^{\bigcap \mathcal{B}_{e_n}}$, and $V_b = \hat{R}_{V_b}^{\bigcap \mathcal{B}_{e_n}}$. The second member of (3.11) is $V' + \hat{R}_{V_b}^{\bigcap \mathcal{B}_{e_n}} + \hat{R}_{V_2}^{\bigcap \mathcal{B}_{e_n}}$ equal to the second member of (3.8).

COROLLARY. *For a fixed potential V, $\hat{R}_V^e$ and $\int \hat{R}_V^e\, dm$ ($m \geq 0$, $\neq 0$ and not charging polar sets), are capacities if and only if the measure associated to V does not charge polar sets.*

In this case, (3.11) proves the capacity property because $V' = 0$. Conversely, if $\mu(e) \neq 0$ for a compact polar set e, let us form e_n, compact nonpolar, decreasing with $\bigcap e_n = e$. According to (3.8), $\widehat{\inf \hat{R}_V^{e_n}} = V_1 > 0$, but $\hat{R}_V^{\bigcap e_n} = 0$. Therefore $\hat{R}_V^e$ and $\int \hat{R}_V^e\, dm$ are not capacities.

3. *Extension to directed decreasing sets.*

LEMMA 6. *If $\{u_i\}$ is a family of nonnegative superharmonic functions which forms with the specific order a decreasing directed set, then $\widehat{\inf u_i}$* (inf *in the natural order) is the infimum (greatest lower bound) of $\{u_i\}$, according to the specific order (as well as to the natural order) in the complete lattice of the nonnegative superharmonic functions.*

Let us recall that in the specific order $v \succ w$ means that $v = w +$ a nonnegative superharmonic function (see [2]).

Let us consider first a sequence u_n (decreasing for the specific order) and let $v_n = u_n - \inf_n u_n$ be $+\infty$ where the difference is undetermined. We shall prove that v_n is nearly hyperharmonic, that is $v_n(x_0) \geq \int v_n\, d\rho_{x_0}^{\omega_0}$ for every regular domain ω_p, and every $x_0 \in \omega_0$. We start from $u_n(x) - u_p(x) \geq \int (u_n - u_p)\, d\rho_x^{\omega_0}$, $n < p$ for points x where u_n is finite, as an easy consequence of $u_n \succ u_p$ (specifically).

Hence $u_n(x) - \inf_p u_p(x) \geq \int (u_n - \inf u_p)\, d\rho_x^{\omega_0}$ and the desired property follows.

By considering $\int v_n\, d\rho_x^{\delta_k}$ for decreasing δ_k, $(\bigcap \delta_k = \{x\})$, we get for the limit, a superharmonic function, equal to $u_n - \widehat{\inf_n u_n}$ (for the points where u_n is finite). Hence $u_n \succ \widehat{\inf u_n}$ and the desired result follows.

For any directed set $\{u_i\}$, we extract a decreasing sequence u_{i_n} such that $\widehat{\inf_i u_i} = \widehat{\inf_n u_{i_n}}$. Since we may choose u_{i_n} containing any fixed u_j, we see that $u_j \succ \widehat{\inf_i u_i}$.

REMARK. For potentials u_i, specifically ordered, the associated measures μ_i (in the ordered space of the Radon measures) have the same order as the potentials. Therefore, in the lemma, $\widehat{\inf u_i}$ is the potential of the inf of the set of the corresponding measures μ_i.

THEOREM 6. *Given a decreasing directed set of sets $\{e_i\}$ and a potential V of a measure $\mu \geq 0$, let us introduce the restricted measures μ_i, ν_i on $\mathcal{B}_{e_i}$ and $\complement\mathcal{B}_{e_i}$, their potentials V_i, W_i, and the potentials $V_1 = \widehat{\inf V_i}$ and $V_2 = \sup W_i$ which are the potentials of the measures μ_1, μ_2 defined as* inf *and* sup *of the ordered sets $\{\mu_i\}$ and $\{\nu_i\}$. Then*

$$\widehat{\inf \hat{R}_V^{e_i}} = V_1 + \hat{R}_{V_2}^{\bigcap \mathcal{B}_{e_i}} \tag{3.12}$$

(where the set $\bigcap \mathcal{B}_{e_i}$ of this equality may be replaced by $\bigcap \tilde{e}_i$ which differs by a polar set).

Using once more the idea of Doob, we may extract a decreasing sequence e_{i_n} such that base of $\bigcap \mathcal{B}_{e_{i_n}}$ = base of $\bigcap \mathcal{B}_{e_i}$ and $\widehat{\inf_n V_{i_n}} = \widehat{\inf_i V_i}$. This implies $\sup W_{i_n} = \sup W_i$. The restrictions of μ on $\bigcap \mathcal{B}_{e_{i_n}}$ and $\complement(\bigcap \mathcal{B}_{e_{i_n}})$ are the inf and sup of the sets of measures $\{\mu_{i_n}\}$, $\{\nu_{i_n}\}$, and the corresponding potentials are $\widehat{\inf V_{i_n}}$ (lemma 6) and $\sup W_{i_n}$. According to theorem 5,

$$\widehat{\inf \hat{R}_V^{e_{i_n}}} = \widehat{\inf V_{i_n}} + \hat{R}_{\sup W_{i_n}}^{\bigcap \mathcal{B}_{e_{i_n}}}. \tag{3.13}$$

Hence theorem 6 is proved.

Another form. As above, we introduce the restriction μ_b of μ to the base of $\bigcap \mathcal{B}_{e_i}$, and the potentials V', V'' of $\mu_1 - \mu_b$, $\mu_2 + \mu_b$. Then

$$\widehat{\inf \hat{R}_V^{e_i}} = V' + \hat{R}_{V''}^{\bigcap \mathcal{B}_{e_i}}. \tag{3.12'}$$

REMARK. If μ does not charge polar sets $V' = 0$, $V'' = V$, because μ_1 is the restriction of μ to $\bigcap \mathcal{B}_{e_{i_n}}$, and therefore to its base also, and thus μ_1 is exactly μ_b.

4. *On the set function* $\int \hat{R}_V^e\, dm$ *(positive measure m).* Theorem 5 implies with the same notations that

$$\lim \int \hat{R}_V^{e_n}\, dm = \int V_1\, dm + \int \hat{R}_{V_2}^{\cap \mathcal{B}_{e_n}}\, dm, \tag{3.14}$$

but we cannot do the same with directed sets. However, the proof of theorem 6 gives

$$\inf \int \hat{R}_V^{e_{i_n}}\, dm = \int V_1\, dm + \int \hat{R}_{V_2}^{\cap \mathcal{B}_{e_{i_n}}}\, dm. \tag{3.15}$$

It was possible to choose e_{i_n} such that it also satisfies

$$\inf \int \hat{R}_V^{e_{i_n}}\, dm = \inf \int \hat{R}_V^{e_i}\, dm. \tag{3.16}$$

Hence,

$$\inf \hat{R}_V^{e_i}\, dm = \int V_1\, dm + \int \hat{R}_{V_2}^{\cap \mathcal{B}_{e_i}}\, dm. \tag{3.17}$$

5. *General balayaged superharmonic function.* Let us study now $\hat{R}_V^e$ for any superharmonic function $V \geq 0$. As V = potential V_0 + nonnegative harmonic function h, $\hat{R}_V^e = \hat{R}_{V_0}^e + \hat{R}_h^e$, the limits or infima corresponding to a directed decreasing set of sets $\{e_i\}$ satisfy

$$\widehat{\inf \hat{R}_V^{e_i}} = \widehat{\inf \hat{R}_{V_0}^{e_i}} + \widehat{\inf \hat{R}_h^{e_i}}, \tag{3.18}$$

and we have only to study $\hat{R}_h^e$ (h harmonic > 0).

We shall use the compact Martin space $\hat{\Omega}$, the compact Martin boundary Δ, and the part Δ_1 (a G_δ set) of the "minimal" points X, corresponding to the minimal harmonic functions equal to 1 at a fixed point y_0 and denoted by $K_X(y)$ (see [1] and chiefly L. Naïm [16]). We know that a set $e \subset \Omega$ is said to be thin at $X \in \Delta_1$ if the function $\hat{R}_{K_X}^e$ is different from K_X (and thus a potential). The set on Δ_1 where e is not thin is a G_δ set; any harmonic function $h \geq 0$ has a unique Martin representation

$$h(y) = \int K_X(y)\, d\mu(X) \tag{3.19}$$

where μ is a measure ≥ 0 on Δ, charging only Δ_1 (associated measure).

THEOREM 7. *Given a decreasing sequence of sets $e_n \subset \Omega$ and a harmonic function $h > 0$ (associated measure μ), let us introduce the set $A_n \subset \Delta_1$ where e_n is not thin, the restrictions μ_1, μ_2 of μ to $\cap A_n$ and to $C_\Delta \cap A_n$ and the corresponding harmonic functions h_1, h_2 (according to the representation formula* (3.19)). *Then*

$$\widehat{\inf \hat{R}_h^{e_n}} = h_1 + \hat{R}_{h_2}^{\cap \bar{e}_n} \tag{3.20}$$

(*where* $\cap \bar{e}_n$ *may be replaced by* $\cap \mathcal{B}_{e_n}$).

We know that, from (3.19), $\hat{R}_h^{e_n}(y) = \int \hat{R}_{K_X}^{e_n}(y)\, d\mu(X)$, the partial integral for $d\mu_1$ is equal to h_1. By taking a $d\rho_{x_0}^{\omega_0}$ mean, we get

$$\int \hat{R}_h^{e_n}(y)\, d\rho_{x_0}^{\omega_0} = h_1(x_0) + \int \left(\int R_{K_X}^{e_n}(y)\, d\rho_{x_0}^{\omega_0}(y)\right) d\mu_2(X). \tag{3.21}$$

For a fixed $X \in \complement(\bigcap A_n)$, e_n is thin at X for $n \geq n_X$ large enough, and $\hat{R}_{K_X}^{e_{n_X}}$ is a locally bounded potential in Ω (whose corresponding measure does not charge polar sets). Moreover, the balayaged functions relative to e_n, $\tilde{e}_n$, or $\bigcap \tilde{e}_n$, $(n \geq n_X)$ are the same respectively for K_X and $\hat{R}_{K_X}^{e_{n_X}}$.

Hence, according to theorem 5 and formula (3.11),

$$\lim \int \hat{R}_{K_X}^{e_n} \, d\rho_{x_0}^{\omega_0} = \int \hat{R}_{K_X}^{\bigcap \tilde{e}_n} \, d\rho_{x_0}^{\omega_0}, \qquad (X \in \complement \bigcap A_n). \tag{3.22}$$

Thus

$$\lim \int \hat{R}_h^{e_n}(y) \, d\rho_{x_0}^{\omega_0} = h_1(x_0) + \int \left(\int \hat{R}_{K_X}^{\bigcap \tilde{e}_n}(y) \, d\mu_2(X)\right) d\rho_{x_0}^{\omega_0}, \tag{3.23}$$

$$\int \lim \hat{R}_h^{e_n}(y) \, d\rho_{x_0}^{\omega_0} = h_1(x_0) + \int \hat{R}_{h_2}^{\bigcap \tilde{e}_n} \, d\rho_{x_0}^{\omega_0}. \tag{3.24}$$

The desired formula (3.20) follows immediately.

The difficult point is the limit property (3.22) that we may prove by using only theorem 1′ as follows.

Let us introduce an increasing sequence of compact sets K_p such that $K_p \subset \mathring{K}_{p+1}$ = interior of K_{p+1}, $\bigcup K_p = \Omega$ and denote $e_n'^p = e_n \cap K_p$, and then $\tilde{e}_n'^p \subset \tilde{e}_n \cap K_{p+1}$.

Now, using the abbreviation $[e]$ for $\int \hat{R}_{K_X}^{e} \, d\rho_{x_0}^{\omega_0}$, we have $[e_n] \leq [e_n'^p] + [e_n \backslash K_p]$ and $[\bigcap \tilde{e}_n \cap K_{p+1}] \leq [\bigcap \tilde{e}_n]$. The function $\hat{R}_{K_X}^{e_n \backslash K_p}$ is majorized for $n \geq n_X$ by $\hat{R}_V^{\tilde{e}_{n_X} \backslash K_p}$ where $V = \hat{R}_{K_X}^{\tilde{e}_{n_X}}$ (potential). Its $d\rho_{x_0}^{\omega_0}$ mean is majorized by $\int \hat{R}_V^{\complement K_p} \, d\rho_{x_0}^{\omega_0}$ which is less than ϵ_{p_0} for p greater than a suitable p_0.

Hence $\inf [e_n] \leq [\bigcap_n \tilde{e}_n'^p] + \epsilon_{p_0} \leq [\bigcap \tilde{e}_n \cap K_{p+1}] + \epsilon_{p_0} \leq [\bigcap \tilde{e}_n] + \epsilon_{p_0}$. But $\inf [e_n] = \inf [\tilde{e}_n] \geq [\bigcap \tilde{e}_n]$. Hence $\inf [e_n] = [\bigcap \tilde{e}_n]$ which is (3.22).

THEOREM 8. *Given a decreasing directed set of sets* $\{e_i\}$ *and a harmonic function* $h > 0$ *with associated measure* $\mu \geq 0$, *consider the restricted measures* μ_i, ν_i *of* μ *on* A_i *(set of* Δ_1 *where* e_i *is not thin) and* $\complement_\Delta A_i$, *whose corresponding harmonic functions are denoted* h_1^i, h_2^i *and the harmonic envelopes* $h_1 = \inf h_1^i$, $h_2 = \sup h_2^i$. *Then*

$$\widehat{\inf \hat{R}_h^{e_i}} = h_1 + \hat{R}_{h_2}^{\bigcap \tilde{e}_i} \tag{3.25}$$

(where the set $\bigcap \tilde{e}_i$ *may be replaced by* $\bigcap \mathcal{B}_{e_i}$*).*

As above (theorem 6) we extract e_{i_n} decreasing such that (i) base $\bigcap \mathcal{B}_{e_{i_n}}$ = base $\bigcap \mathcal{B}_{e_i}$, (ii) $\inf h_1^{i_n} = \inf h_1^i$, which implies $\sup h_2^{i_n} = \sup h_2^i$; (iii) the corresponding equalities for the associated measures.

Then theorem 7 leads to the desired result (3.25).

COROLLARY. *For any measure* $m \geq 0$ *which does not charge the polar sets and for the previous* e_i, h, h_1, h_2,

$$\inf \int \hat{R}_h^{e_i} \, dm = \int h_1 \, dm + \int \hat{R}_{h_2}^{\bigcap \tilde{e}_i} \, dm. \tag{3.26}$$

The proof is the same as in subsection 4.

4. Extensions to the axiomatic theory of harmonic functions

We shall adapt the theory of Brelot [2] continued by Mrs. Hervé [14] and Gowrisankaran [13]. (For previous works of Tautz, Doob, and for further improvements or extensions of Bauer, Boboc-Constantinescu-Cornea, Mokobodski, Loeb, see a course on the general subject given in Montreal by Brelot [7].) With the essential reference to [2], *we suppose* on the fundamental space with a *countable base*, a sheaf of harmonic functions satisfying *axioms* 1, 2, 3, and D and the *existence of a potential* >0.

These conditions imply axiom 3′, the main convergence theorem on superharmonic functions and (even without D) a Riesz-Martin integral representation. The definitions of R_φ^e and $\hat{R}_\varphi^e$ remain the same.

The fine topology on Ω is the coarsest topology finer than the given one on Ω, which makes continuous the local superharmonic functions. A set e is thin at x_0 if $\complement e \setminus \{x_0\}$ is a fine neighborhood of x_0 and if, moreover, $\{x_0\}$ is not polar when $x_0 \in e$. There exists a finite continuous potential V_0 such that for any $e \subset \Omega$, the base $\mathcal{B}_e$, of points at which the set is not thin, is defined by $\hat{R}_{V_0}^e = V_0$.

The properties of $\hat{R}_\varphi^e$ for increasing e may be developed exactly as in section 3. The study of decreasing sets needs some explanations and after the first properties some new hypothesis.

As previously, $\hat{R}_\varphi^e$ (φ, finite continuous ≥ 0) is a general capacity (with values in the naturally ordered complete lattice of the nonnegative hyperharmonic functions) and $\int \hat{R}_\varphi^e \, dm$ is a general real capacity, for any measure $m \geq 0$ which does not charge the polar sets (for instance, the harmonic measure $d\rho_{x_0}^{\omega_0}$). It does not seem very useful to generalize the ordinary Greenian capacity. (This could be defined if V_0 is a superharmonic finite continuous (≥ 0) function and $e \subset \bar{e} \subset \Omega$, as the total amount of the measure corresponding to $\hat{R}_{V_0}^e$ in a Riesz representation.)

However, we need a suitable extension of Choquet's lemma. By adapting the original proof, I already established (see [3], [4]) that the set $\complement\mathcal{B}_e$ of the points where e is thin may be embedded in an open set ω such that $\int \hat{R}_{V_0}^{\omega \cap e} \, d\rho_{x_0}^{\omega_0}$ (V_0 finite continuous potential, x_0, ω_0 fixed) is arbitrarily small. This is equivalent to the condition: $\widehat{\inf_\omega \hat{R}_{V_0}^{\omega \cap e}} = 0$ (for all neighborhoods ω of $\complement\mathcal{B}_e$ or of $\complement\mathcal{B}_e \cap \bar{e}$).

THEOREM 9. *All properties of $\hat{R}_\varphi^e$, $\int \hat{R}_\varphi^e \, dm$ given in theorems* 1′, 2, *and* 3, *extend without any modification.*

Without further hypothesis, *we may extend theorems* 7 *and* 8. Let us introduce the cone of the positive harmonic functions and the base β of this cone formed by the functions equal to 1 at $y_0 \in \Omega$. The extreme elements of β (the generalized Martin's minimal functions equal to 1 at y_0) form a G_δ-set Δ_1. Any harmonic function $h \geq 0$ is represented by

$$h(y) = \int u(y) \, d\mu(u), \qquad \mu \in \beta' \tag{4.1}$$

where μ is a unique measure on β, charging only Δ_1. According to Mrs. Hervé ([14], theorem 28, 2),

$$\hat{R}_h^e(y) = \int \hat{R}_u^e(y)\, d\mu(u). \tag{4.2}$$

THEOREM 10. *Change K_X to an element minimal at $u \in \Delta_1$. Then theorems 7 and 8 and their corollary extend exactly.*

For (3.22) use the second proof and note that the set A_i of the points where e_i is not thin (on Δ_1) is the intersection with Δ_1 of a K_σ set of β (Gowrisankaran [13]).

1. *Further extensions with additional hypotheses.* We shall complete the hypothesis by α, β, γ:

(α) the potentials with the same point-support are proportional;

(β) there exists a base of completely determinating domains.

Recall that a domain $\omega \subset \bar{\omega} \subset \Omega$ is completely determinating if, for any superharmonic function $v \geq 0$ in Ω (or equivalently any potential), $R_v^{\complement\omega} = v$. We shall determine the potential p_y of support $\{y\}$ by the condition $\int p_y(x)\, d\rho_{x_0}^\delta(x) = 1$, where δ is a fixed completely determinating domain. This determines also the adjoint harmonic and superharmonic functions, according to the theory of Mrs. Hervé [14]. We recall that $x \rightarrow p_y(x)$ is continuous, and even finite continuous at y when $\{y\}$ is not polar ($\hat{R}_{V_0}^{\{y\}}$ is $\leq V_0$ and equal to V_0 at y (not polar), and therefore continuous; p_y is proportional); $y \rightarrow p_y(x)$ is an adjoint potential denoted $p_x^*(y)$. We shall use the fundamental Riesz representation of any potential v: $v(x) = \int p_y(x)\, d\mu(y)$, μ being a unique measure ≥ 0 on Ω.

(γ) At every point thinness and adjoint thinness are identical.

THEOREM 11. *The statements of theorems 5 and 6 are still valid in the present case.*

The developments of the subsections 2, 3, and 4 of section 3 still hold by changing G_y to p_y. The only point which needs explanations is in the proof of lemma 5, where we have to see that

$$\int \hat{R}_{p_y}^{e_n \cap \sigma}(x)\, d\rho_{x_0}^{\omega_0}(x) \tag{4.3}$$

is arbitrarily small for a suitable open neighborhood σ($y \notin e_{n_0}$ and e_{n_0} is fine closed).

Let us study $\inf_\sigma \hat{R}_{p_y}^{e \cap \sigma}$ for a thin set e at $y \notin e$. If $\{y\}$ is nonpolar and p_y is locally bounded, the strong thinness [7] implies that $\inf_\sigma \hat{R}_{p_y}^{e\cap\sigma}(y) = 0$, then $\widehat{\inf_\sigma \hat{R}_{p_y}^{e\cap\sigma}} = 0$ (this is also a consequence of theorems 1′–9). If $\{y\}$ is polar, $\hat{R}_{p_y}^{e\cap\sigma}(x) = R_{p^*_x}^{*e\cap\sigma}(y)$ (Mrs. Hervé), e which is adjoint-thin, is also strongly adjoint-thin and $\inf_\sigma R_{p^*_x}^{*e\cap\sigma}(y) = 0$ for $x \neq y$; then $\inf_\sigma \hat{R}_{p_y}^{e\cap\sigma}(x) = 0$, and finally $\widehat{\inf \hat{R}_{p_y}^{e\cap\sigma}} = 0$, as previously. Hence we have the desired property.

REMARK. All the previous hypotheses are satisfied in the case of the solutions of linear homogeneous elliptic partial differential equations of second order, at least with suitably smooth coefficients [14].

5. Application to a theorem of Getoor

Getoor [12] in probability theory gave results which imply, for instance in R^n, that there exists for any measure μ (for example ≥ 0) which does not charge polar sets, a smallest fine closed support. (Getoor introduced a restriction, on the Borel structure of the supports. This is immaterial. P. A. Meyer [15] developed further the probabilistic theory of Getoor and found again some results of the present paper.) Proofs are easy in pure potential theory as consequences of the previous study.

(a) In the classical case, consider the Greenian potential V of $\mu \geq 0$ not charging polar sets and the fine closed supports e_i of μ which form a directed decreasing set. We shall prove that $\bigcap e_i$ is the smallest fine closed support and is a base.

Theorem 6 (remark) gives $\widehat{\inf \hat{R}_V^{e_i}} = \hat{R}_V^{\cap e_i}$. We may use the characteristic property of the balayage that, for a potential, W, $\hat{R}_W^e = W$ if and only if $\mathcal{B}_e$ is a support of the measure associated with W. As $\mu(\complement e_i) = 0$, $e_i \backslash \mathcal{B}_{e_i}$ polar, then $\mu(\complement \mathcal{B}_{e_i}) = 0$, $\hat{R}_V^{e_i} = V$, and $\widehat{\inf \hat{R}_V^{e_i}} = V$. Now $\hat{R}_V^{\cap e_i} = V$ implies that $\mathcal{B}_{\cap e_i}$ is a support, and that $\bigcap e_i$ also; $\bigcap e_i$ is the smallest fine closed support and is its own base.

(b) Instead of some extension of this proof to a general axiomatic case, it is better to give another and simpler one, in Doob's frame of arguments.

THEOREM 12. *We suppose that axioms* 1, 2, 3, and D, *hold and that there is a countable base and a potential* >0. *We consider on* Ω *any Radon measure* μ *(not charging polar sets) and the fine closed supports* e_i. *Then* $\bigcap e_i$ *is the smallest fine closed support, and it is a base.*

It is sufficient to use the axiomatic form of lemma 3 (extension in corollary of theorem 3). There exists a decreasing sequence e_{i_n} such that $\mathcal{B}_{\cap e_{i_n}} = \mathcal{B}_{\cap e_i}$. Now $\bigcap e_{i_n}$ is a fine closed support; its base is also a fine closed support, and thus so is $\bigcap e_i$.

6. Some extensions of the theory

A new idea of Doob, which is in process of development, is the systematic use of functions instead of sets and of fine upper semicontinuous functions (a particular case of which is the indicator of a fine closed set). This led me to some results such as the following which illustrate this kind of generalization.

For $\varphi \geq 0$ in the fundamental space (in classical or axiomatic potential theory), let $R_\varphi = R_\varphi^\Omega$. We recall that $\hat{R}_\varphi = \inf v$, for all nonnegative hyperharmonic functions $v \geq \varphi$ quasi everywhere on Ω (even in axiomatic theory but with axioms 1, 2, 3, and D, countable base).

THEOREM 13. *In classical theory and Greenian space* Ω, *let* $\{\varphi_i\}$ *be a decreasing directed set of fine upper semicontinuous functions, satisfying* $0 \leq \varphi_i \leq V$ *(for all* i*),* $V > 0$ *potential of a measure which does not charge the set* $\{x, V(x) = +\infty\}$. *Then*

$$\widehat{\inf \hat{R}_{\varphi_i}} = \hat{R}_{\inf \varphi_i}. \tag{6.1}$$

If $\psi \leq V$ is superharmonic, ≥ 0 and $\inf_i \varphi_i$, for $\epsilon > 0$ let

$$e_i = \{x; \varphi_i \geq \psi + \epsilon V\}; \tag{6.2}$$

then $\varphi_i \leq \psi + \epsilon V + \hat{R}_V^{e_i}$, q.e. (because on e_i, $\hat{R}_V^{e_i} = V$ q.e.). Therefore, $\hat{R}_{\varphi_i} \leq \psi + \epsilon V + \hat{R}_V^{e_i}$, but e_i is fine closed and $\bigcap e_i$ is a subset of $\{x, V(x) = +\infty\}$. Thanks to theorem 6 (formula (3.12′) and the remark), $\widehat{\inf \hat{R}_V^{e_i}} = 0$. Hence

$$\begin{aligned} \widehat{\inf \hat{R}_{\varphi_i}} &\leq \psi + \epsilon V \\ &\leq \psi \\ &\leq R_{\inf \varphi_i} \\ &\leq \hat{R}_{\inf \varphi_i}. \end{aligned} \tag{6.3}$$

The inverse inequality is obvious.

1. *Extension in axiomatic theory.*

THEOREM 14. (a) *With the axioms* 1, 2, 3, *and* D, *a countable base and a finite continuous potential* V_0, *the decreasing directed set of fine upper semicontinuous* φ_i *such that* $0 \leq \varphi_i \leq V_0$, satisfies

$$\widehat{\inf \hat{R}_{\varphi_i}} = \hat{R}_{\inf \varphi_i}. \tag{6.4}$$

(b) *With the supplementary hypothesis* α, β, γ *of subsection* 1 *of section* 4, *theorem* 13 *still holds.*

The proof is the same. Part (a) needs only the axiomatic extension of theorem 3. It is obvious that, if θ_i is the indicator of a fine closed set e_i

$$\hat{R}_\varphi^{e_i} = \hat{R}_{\varphi\theta_i}, \qquad \hat{R}_\varphi^{\bigcap e_i} = \hat{R}_{\inf \varphi\theta_i}. \tag{6.5}$$

If φ is finite continuous and $\leq V$, a potential, then $\varphi \leq \hat{R}_\varphi$, which is a finite continuous potential, and the axiomatic extension of theorem 3 is contained in theorem 14, (a) (therefore equivalent to it).

Without giving here further developments, let us only observe that we cannot say that this theorem 14 (a) contains the extension of Choquet's lemma; in other words, that the property

$$\widehat{\inf_\omega \hat{R}_{V_0}^{e\bigcap\omega}} = \hat{R}_{V_0}^{\bigcap \widetilde{e\bigcap\omega}} \tag{6.6}$$

implies $\inf_\omega \hat{R}_{V_0}^{e\bigcap\omega} = 0$ (ω open set $\supset \complement \mathcal{B}_e$) because we do not know if $\bigcap \widetilde{e \bigcap \omega}$ is polar. However, this is an easy consequence of the property that for a fine closed set α and $x \in \alpha$, there exists a compact set $K \subset \alpha$ such that $\alpha - K$ is thin at x; but the only proof I see for that uses Choquet's lemma.

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