ON THE DENSITIES OF PROBABILITY MEASURES IN FUNCTIONAL SPACES

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1. Introduction

1.1. Let us assume that there is defined on some probability field $\{\Omega, \mathcal{B}, P\}$ a random process $\xi(t, \omega)$, $t \in E$, where E is some set on the line, and $\omega \in \Omega$. We denote by $\mathbf{F}_E$ the set of all functions, defined on the set E and assuming numerical values. The mapping $\xi(\cdot, \omega)$ carries over the σ-algebra $\mathcal{B}$ on Ω to some σ-algebra $\mathfrak{F}$ of subsets of $\mathbf{F}_E$ and the measure P on $\mathcal{B}$ to a measure μ on $\mathfrak{F}$. The σ-algebra $\mathfrak{F}$ contains at least the sets of the form $\{x(\cdot); x(t_1) < x_1\}$ for $t_1 \in E$ and x_1 real (because $\{\omega; \xi(t_1, \omega) < x_1\} \in \mathcal{B}$) and, consequently, contains all cylinder subsets of the space $\mathbf{F}_E$. If we denote by $\mathfrak{F}_0$ the smallest σ-algebra of subsets of $\mathbf{F}_E$ containing all cylinder subsets of $\mathbf{F}_E$, then $\mathfrak{F}_0 \subset \mathfrak{F}$. As a rule the measure μ on $\mathfrak{F}$ is completely determined by its values on $\mathfrak{F}_0$ ($(\mu, \mathfrak{F})$ is the completion of $(\mu, \mathfrak{F}_0)$). Therefore, it suffices to consider the measure μ on the σ-algebra $\mathfrak{F}_0$, which depends only on the set E and not on the specific form of the process. We shall call the measure μ on $\mathfrak{F}_0$ the measure corresponding to the process $\xi(t, \omega)$. In many problems one can identify the process and the measure, because from the measure μ one can define the probability space $\{\mathbf{F}_E, \mathfrak{F}, \mu\}$, on which the natural mapping $\xi(t, x(\cdot)) = x(t)$ defines a random process to which corresponds the measure μ.

If two probability measures μ_1 and μ_2 are defined on the σ-algebra $\mathfrak{F}_0$, then, as is well-known, μ_2 is said to be *absolutely continuous* with respect to μ_1, if $\mu_2(A) = 0$ for all $A \in \mathfrak{F}_0$ for which $\mu_1(A) = 0$. The absolute continuity of μ_2 with respect to μ_1 is a necessary and sufficient condition for the existence of an $\mathfrak{F}_0$-measurable function $\rho(x)$ such that

$$\mu_2(A) = \int_A \rho(x)\mu_1(dx) \tag{1.1}$$

for all $A \in \mathfrak{F}_0$. This function $\rho(x)$ is called the *density* or the *derivative* of the measure μ_2 with respect to μ_1 and is denoted by $(d\mu_2/d\mu_1)(x)$. If, for some A, $\mu_1(A) = 1$, $\mu_2(A) = 0$, then μ_1 and μ_2 are *mutually singular*.

1.2. In recent times a substantial part of the work in the theory of random processes has been devoted to the solution of the question of the absolute continuity (or the singularity) of measures corresponding to random processes. One can indicate various directions, frequently having important practical interest, in which results on the absolute continuity (or singularity) and density of measures in functional spaces are used. The first results on the form of the density of measures in functional spaces were obtained by Cameron and Martin [1]–[3] in connection with the study of change of variable in the Wiener integral. The Wiener integral can be regarded as an integral with respect to the measure μ_W corresponding to the Wiener process $W(t)$, defined on the interval [0, 1] (that is, a Gaussian process with independent increments; for which $\mathbf{E}W(t) = 0$, $\mathbf{D}W(t) = t$). Then if $f(x)$ is a functional which coincides almost everywhere with an $\mathfrak{F}_0$-measurable functional, $\int f(x)\mu_W(dx)$ is called its Wiener integral. In the aforementioned papers a transformation formula was found for the integral

$$\int f(T^{-1}x)\mu_W(dx) = \int f(x)\mathfrak{D}(x)\mu_W(dx), \tag{1.2}$$

where T is an $\mathfrak{F}_0$-measurable transformation of $\mathbf{F}_{[0,1]}$ into $\mathbf{F}_{[0,1]}$, satisfying certain definite conditions (which it is not necessary to give here), and $\mathfrak{D}(x)$ is a functional, constructed from the transformation T, which generalizes the concept of the Jacobian of a transformation in the finite dimensional case. Formula (1.2) can be rewritten in the following way: let $\mu_W(TA) = \nu(A)$. Then ν is also a probability measure, and

$$\int f(T^{-1}x)\mu_W(dx) = \int f(x)\nu(dx). \tag{1.3}$$

Consequently, $\int f(x)\nu(dx) = \int f(x)\mathfrak{D}(x)\mu_W(dx)$ for all measurable functionals $f(x)$, which is possible only under the condition that $\mathfrak{D}(x) = (d\nu/d\mu_W)(x)$.

Thus, the density of one measure with respect to another can be used as the Jacobian of a transformation in a "change of variable" in integrals with respect to measures in functional spaces.

In the case where the process corresponding to μ_1 has been well-studied, one can use the sole fact that the measure μ_2 is absolutely continuous with respect to μ_1 in order to study which properties of the process corresponding to μ_2 have probability unity. The study of these properties represents one of the important problems of the theory of random processes, and the use of the absolute continuity of measures frequently facilitates its solution. Further, the value of the density $(d\mu_2/d\mu_1)(x)$ makes it possible to reduce the calculation of the mathematical expectations of functionals of one process to the calculation of the mathematical expectations of functionals of the other process. For this, one can use the formula

$$\mathbf{E}f(\xi_2(\cdot)) = \mathbf{E}f(\xi_1(\cdot))\frac{d\mu_2}{d\mu_1}(\xi_1(\cdot)), \tag{1.4}$$

where $\xi_i(\cdot)$ is the process to which the measure μ_i corresponds. The validity of formula (1.4) follows from the relation $\mathbf{E}f(\xi_i(\cdot)) = \int f(x)\mu_i(dx)$. Thus, if we know how to calculate the mean values of characteristics of the process $\xi_1(\cdot)$, we are able with the help of formula (1.4) to find also the mean values of characteristics of the process $\xi_2(\cdot)$.

By means of the density with respect to a measure which is considered as known, one can define other probability measures in a functional space. Thus, we arrive at one of the constructive methods of definition of a random process: defining the process (more precisely, the measure corresponding to the process) by means of a density.

The densities of measures in functional spaces can be used in a natural way in solving statistical problems involving random processes. Suppose, for example, that the problem consists of choosing between two hypotheses concerning the measure corresponding to a random process. Let us assume that the hypothesis H_i, $(i = 0, 1)$ consists of the assertion that this measure coincides with μ_i. Then, in the case where the measure μ_1 is absolutely continuous with respect to μ_0, rejecting the hypothesis H_0 for $(d\mu_1/d\mu_0)(x) > C$ and adopting it for $(d\mu_1/d\mu_0)(x) < C$, we obtain a class of optimal criteria (to every $C > 0$ there corresponds one criterion). To calculate the probability of errors of the first and second kind, we have to know the distribution of $(d\mu_1/d\mu_0)(x)$ under the zero and the one hypothesis. In the case where one has a family of measures μ_α in the functional space and $(d\mu_\alpha/d\mu_{\alpha_0})$ exists for some α_0, then the latter expression can be used to find an estimate of the parameter α by the maximum likelihood method.

Finally, one needs expressions for the densities of measures for calculating the amount of information which is contained in one process $\xi(t)$, concerning another process $\eta(t)$. If $\mu_\xi(dx)$ and $\mu_\eta(dx)$ are the measures corresponding to the processes $\xi(\cdot)$ and $\eta(\cdot)$ respectively, let $\mu_\xi(dx) \times \mu_\eta(dy)$ be the product of these measures in the space $\mathsf{F}_E \times \mathsf{F}_E$. Let $\mu_{\xi,\eta}(dx, dy)$ be the measure in the same space, corresponding to the two-dimensional process $(\xi(\cdot), \eta(\cdot))$. The quantity $\mathcal{I}_{\xi,\eta}$, which is the amount of information in the process $\xi(t)$ concerning $\eta(t)$ (or vice versa) is defined by the formula

$$\mathcal{I}_{\xi,\eta} = \int \log \frac{d\mu_{\xi,\eta}}{d(\mu_\xi \times \mu_\eta)} (x, y)\mu_{\xi,\eta}(dx, dy), \tag{1.5}$$

if $d\mu_{\xi,\eta}/d(\mu_\xi \times \mu_\eta)$ exists. (In this connection, see the monograph of Pinsker ([18], pp. 9–10).)

1.3. Usually, theorems on the absolute continuity of measures and formulas for densities have been proved for certain specific classes of processes. The first papers of Cameron and Martin [1]–[3], as was already mentioned, considered the question of the absolute continuity of the measure, corresponding to the process obtained from the Wiener process by means of a linear or nonlinear transformation, with respect to the Wiener measure. Prohorov [4] proved the absolute continuity of the measure, corresponding to a diffusion process with diffusion coefficient 1 and shift coefficient $a(t, x)$, satisfying certain smoothness conditions, with respect to the Wiener measure, and found the corresponding density. We remark that his result involved the necessity of studying certain sets of probability zero for the diffusion process. More general results for Markov processes were obtained in the works of Skorohod [5], [6] and Girsanov [7]; the former considered processes having a diffusion and a discontinuous part, and the latter considered a class of processes embracing Markovian diffusion processes. The absolute continuity of the measures corresponding to processes with independent increments were considered by Skorohod [8], [9]. In this case we succeeded in finding necessary and sufficient conditions for absolute continuity and in determining the density of one measure with respect to another.

Particularly, many papers have been devoted to the absolute continuity of Gaussian measures. In the case where the Gaussian processes are distinguished only by their mean values, this problem was considered in a paper of Grenander [10]. Gaussian processes of general form were considered in papers of Hájek [11], Feldman [12], [13], Rozanov [14]–[16], and Rao and Varadarajan [17]. A number of papers have considered sufficient conditions for the absolute continuity and singularity of measures corresponding to stationary Gaussian processes (see, in this connection, Pinsker [18], Rozanov [15], [16], and Alekseev [20]; a more complete bibliography of work in this domain is to be found in the review papers of Yaglom [19] and Rozanov [16]).

1.4. In this paper we consider some new results on the absolute continuity of measures and the form of the density of one measure with respect to another. In section 2 we consider some general theorems on the absolute continuity of measures, corresponding to processes which differ only in their mean value. Some results on the structure of the set of admissible mean values (that is, mean values for which the absolute continuity with respect to the measures corresponding to processes with zero mean is preserved) appear in a paper of Pitcher [21]. Section 3 is devoted to absolute continuity for Markov processes. Here we consider continuous processes which are more general than diffusion processes. In section 4 we consider stationary Gaussian processes. The basic aim here is to obtain sufficient conditions for absolute continuity and sufficient conditions for singularity in terms of the spectral densities of the process.

2. On admissible translations for infinitely divisible distributions in a Hilbert space

2.1. In this subsection we consider probability measures in a separable Hilbert space H. We shall assume that every measure is defined on the σ-algebra $\mathcal{B}_H$ consisting of all Borel subsets of H (by virtue of the separability of the space, it is sufficient for this that every halfspace $\{x; (x, a) < \alpha\}$, where a is an arbitrary element of H, and α is a real number, belong to $\mathcal{B}_H$). We mention that with every measurable separable process $x(t)$, defined on a finite interval $[a, b]$, one can associate a measure in a Hilbert space, as long as one of the conditions

$$P\{\sup_{a \leq t \leq b} |x(t)| < \infty\} = 1, \tag{1}$$

$$\int_a^b \mathbf{E}|x(t)|^2\, dt < \infty \tag{2}$$

is fulfilled.

The convolution $\mu_1 * \mu_2$ of two measures which are defined on $\mathcal{B}_H$ can be defined in the usual way. This convolution will be the distribution of a random variable ξ, with values in H, which is equal to the sum $\xi_1 + \xi_2$ of two independent variables, where ξ_k is a variable with values in H, having the distribution μ_k.

A measure μ on $\mathcal{B}_H$ will be called *infinitely divisible*, if for every n one can find a measure $\mu^{(n)}$ such that μ is the n-fold convolution of the measure $\mu^{(n)}$ with itself.

It is convenient to define measures on $\mathcal{B}_H$ by means of their characteristic functionals:

$$(2.1) \qquad \varphi_\mu(z) = \int \exp\{i(z, x)\}\mu(dx), \qquad (z \in H).$$

Under certain conditions the characteristic functional of an infinitely divisible measure μ has the form

$$(2.2) \qquad \varphi_\mu(z) = \exp\left\{i(z, b) - \tfrac{1}{2}(Az, z) + \int \left[e^{i(z,x)} - 1 - \frac{i(z, x)}{1 + (x, x)}\right]\pi(dx)\right\},$$

where $b \in H$, A is a nonnegative definite symmetric operator on H with finite trace, and the measure $\pi(dx)$ is such that the measure

$$(2.3) \qquad \nu(C) = \int_C \frac{(x, x)}{1 + (x, x)}\,\pi(dx)$$

is finite.

Let us denote by T_a the operator of translation in H by a: $T_a x = x + a$, and by μ_a the measure defined by the formula $\mu_a(A) = \mu(T_{-a}A)$ (if μ is the distribution of the variable ξ, then μ_a is the distribution of the variable $\xi + a$). We shall be interested in admissible translations for μ, which name, following Pitcher [21], we give to those a for which μ_a is absolutely continuous with respect to μ. With regard to the measure μ, we shall assume that it is infinitely divisible and has a characteristic functional of the form

$$(2.4) \qquad \varphi_\mu(z) = \exp\left\{\int \left[e^{i(z,x)} - 1 - \frac{i(z, x)}{1 + (x, x)}\right]\pi(dx)\right\}.$$

The fact is that a measure having a characteristic functional of the form (2.2) can be represented as the convolution of a measure with a characteristic functional of the form (2.4) and a Gaussian measure (whose characteristic functional is $\exp\{i(z, b) - \frac{1}{2}(Az, z)\}$), and the set of admissible translations for Gaussian measures has been thoroughly studied (see [10]).

2.2. We shall now prove a general theorem which gives sufficient conditions for a translation by a to be admissible for a measure μ having a characteristic functional of the form (2.4).

THEOREM 1. *Suppose, for $a \in H$, that one can find a sequence of nonnegative functions $g_n(x)$ which are $\mathcal{B}_H$-measurable and which satisfy the conditions*

$$(1) \qquad \lim_{n \to \infty} \int g_n(x) x \pi(dx) = a,$$

(2) $$\lim_{n\to\infty} \int g_n(x)|x|^2\pi(dx) = 0,$$

(3) *there exists a monotone, differential function $\varphi(t)$, convex downwards, defined for $t > 0$ and satisfying the conditions $\varphi(t) > 0$, $\varphi(1) = 1$, $\varphi(ts) \le \varphi(t)\varphi(s)$, $\lim_{t\to\infty} \varphi(t)/t = 0$ and, moreover, for $t < 0$, $\varphi(1 + t) - 1 - \varphi'(1)t = O(t^2)$ and*

(2.5) $$\limsup_{n\to\infty} \int [\varphi(1 + g_n(x)) - 1 - \varphi'(1)g_n(x)]\pi(dx) < \infty.$$

Then μ_a is absolutely continuous with respect to μ.

PROOF. If the conditions of the theorem are satisfied, then one can choose a sequence $\epsilon_n \to 0$ such that

(2.6) $$\int_{|x|>\epsilon_n} g_n(x)x\pi(dx) \to a.$$

Let m_n be a sequence of integers such that

(2.7) $$\lim_{n\to\infty} \frac{1}{m_n}\left(\int_{|x|>\epsilon_n} (1 + g_n(x))\pi(dx)\right)^2 = 0.$$

Let us introduce independent random variables with values in H: $\xi_1^{(n)}, \cdots, \xi_{m_n}^{(n)}$, having the same distribution,

(2.8) $$\mathbf{P}\{\xi_1^{(n)} \in C\} = \frac{1}{m_n}\int_{|x|>\epsilon_n} \chi_C(x)\pi(dx) \qquad \text{if} \quad 0 \notin C,$$

(2.9) $$\mathbf{P}\{\xi_1^{(n)} = 0\} = 1 - \frac{1}{m_n}\int_{|x|>\epsilon_n} \pi(dx).$$

Similarly, $\eta_1^{(n)}, \cdots, \eta_{m_n}^{(n)}$ are independent and identically distributed, with

(2.10) $$\mathbf{P}\{\eta_1^{(n)} \in C\} = \frac{1}{m_n}\int_{|x|>\epsilon_n} (1 + g_n(x))\chi_C(x)\pi(dx) \qquad \text{if} \quad 0 \notin C,$$

(2.11) $$\mathbf{P}\{\eta_1^{(n)} = 0\} = 1 - \frac{1}{m_n}\int_{|x|>\epsilon_n} (1 + g_n(x))\pi(dx).$$

We denote the distribution of $\xi_1^{(n)}$ by $\nu^{(n)}$, of $\eta_1^{(n)}$ by $\tilde{\nu}^{(n)}$, of the variable

(2.12) $$\zeta_n = \xi_1^{(n)} + \cdots + \xi_{m_n}^{(n)} - \int_{|x|>\epsilon_n} \frac{x}{1 + (x, x)}\pi(dx)$$

by $\mu^{(n)}$, and of the variable

(2.13) $$\tilde{\zeta}_n = \eta_1^{(n)} + \cdots + \eta_{m_n}^{(n)} - \int_{|x|>\epsilon_n} \frac{x}{1 + (x, x)}\pi(dx)$$

by $\tilde{\mu}^{(n)}$. It is easy to see that the measure $\mu^{(n)}(\tilde{\mu}^{(n)})$ converges weakly on cylinder sets to the measure $\mu(\mu_a)$, since the characteristic functionals of these measures have the form

$$\varphi_{\mu^{(n)}}(z) = \left(1 + \frac{1}{m_n}\int_{|x|>\epsilon_n}(e^{i(z,x)} - 1)\pi(dx)\right)^{m_n} \times \exp\left\{-i\int_{|x|>\epsilon_n}\frac{(z,x)}{1+(x,x)}\pi(dx)\right\} = \exp\left\{\int_{|x|>\epsilon_n}\left(e^{i(z,x)} - 1 - \frac{i(z,x)}{1+(x,x)}\right)\pi(dx)\right\} + o(1) \tag{2.14}$$

(by virtue of condition (2.7)); in exactly the same way,

$$\varphi_{\tilde{\mu}^{(n)}}(z) = \exp\left\{\int_{|x|>\epsilon_n}\left(e^{i(z,x)} - 1 - \frac{i(z,x)}{1+(x,x)}\right)(1+g_n(x))\pi(dx) + \int_{|x|>\epsilon_n}\frac{(z,x)g_n(x)}{1+(x,x)}\pi(dx)\right\} + o(1), \tag{2.15}$$

and

$$\int_{|x|>\epsilon_n} g_n(x)\left(e^{i(z,x)} - 1 - \frac{i(z,x)}{1+(x,x)}\right)\pi(dx) \to 0, \tag{2.16}$$

and

$$\int g_n(x)\frac{(z,x)}{1+(x,x)}\pi(dx) \to (a,z) \tag{2.17}$$

in view of conditions (1) and (2) of the theorem. By virtue of lemma 3, section 2, chapter 4 of [6],

$$\frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}(\zeta_n) = \mathbf{E}\left(\prod_{k=1}^{m_n}\frac{d\tilde{\nu}^{(n)}}{d\nu^{(n)}}(\xi_k^{(n)})|\zeta_n\right). \tag{2.18}$$

It is easy to compute that

$$\frac{d\tilde{\nu}^{(n)}}{d\nu^{(n)}}(\xi_k^{(n)}) = 1 - \frac{1}{m_n}\int_{|x|>\epsilon_n} g_n(x)\pi(dx)(1 - y_{n,k}) + y_{n,k}g_n(\xi_k^{(n)}) + o\left(\frac{1}{m_n}\right), \tag{2.19}$$

where $y_{n,k} = 1$ if $\xi_k^{(n)} \neq 0$, $y_{n,k} = 0$ if $\xi_k^{(n)} = 0$. Further,

$$\begin{aligned}
\mathbf{E}\varphi\left(\frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}(\zeta_n)\right) &\le \mathbf{E}\varphi\left(\mathbf{E}\left[\prod_{k=1}^{m_n}\frac{d\tilde{\nu}^{(n)}}{d\nu^{(n)}}(\xi_k^{(n)})|\zeta_n\right]\right) \\
&\le \mathbf{E}\varphi\left(\prod_{k=1}^{m_n}\frac{d\tilde{\nu}^{(n)}}{d\nu^{(n)}}(\xi_k^{(n)})\right) \le \prod_{k=1}^{m_n}\mathbf{E}\varphi\left(\frac{d\tilde{\nu}^{(n)}}{d\nu^{(n)}}(\xi_k^{(n)})\right) \\
&= \prod_{k=1}^{m_n}\left(1 + \mathbf{E}\psi\left(\frac{d\tilde{\nu}^{(n)}}{d\nu^{(n)}}(\xi_k^{(n)}) - 1\right)\right) \\
&\le \exp\left\{m_n\mathbf{E}\psi\left(\frac{d\tilde{\nu}^{(n)}}{d\nu^{(n)}}(\xi_k^{(n)}) - 1\right)\right\} \\
&\le \exp\left\{m_n o\left(\frac{1}{m_n}\right) + \int_{|x|>\epsilon_n}\psi(g_n(x))\pi(dx)\right\},
\end{aligned} \tag{2.20}$$

where $\psi(t) = \varphi(1 + t) - \varphi(1) - \varphi'(1)t$. Therefore,

$$\limsup_{n\to\infty} \mathbf{E}\varphi\left(\frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}(\zeta_n)\right) < \infty. \tag{2.21}$$

Further, the inequality

$$\begin{aligned}\mu_a(C) &= \lim_{n\to\infty} \tilde{\mu}^{(n)}(C) = \lim_{n\to\infty} \mathbf{E}\chi_C(\zeta_n)\frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}(\zeta_n) \\ &\leq R \limsup_{n\to\infty} \mu^{(n)}(C) + \limsup_{n\to\infty} \int_{\{x;\ (d\tilde{\mu}^{(n)}/d\mu^{(n)})(x)>R\}} \frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}(x)\mu^{(n)}(dx) \\ &\leq R\mu(C) + \frac{R}{\varphi(R)} \limsup_{n\to\infty} \mathbf{E}\varphi\left(\frac{d\tilde{\mu}^{(n)}}{d\mu^{(n)}}(\zeta_n)\right)\end{aligned} \tag{2.22}$$

is valid for all C in the algebra generated by the cylinder sets, which generates the σ-algebra of measurable sets. If $\mu(C) = 0$, then $\mu_a(C) = O(R/\varphi(R))$ for any R whatsoever; consequently, $\mu_a(C) = 0$, since $R/\varphi(R) \to 0$ as $R \to \infty$.

COROLLARY 1. *Let us denote by N_ϵ the set of all elements z, for which there exists a set E, lying in the ball S_ϵ of radius ϵ with center at zero, such that $z = \int_E x\pi(dx)$. Then the set $N_0 = \bigcap_{\epsilon>0} \overline{N}_\epsilon$ (where $\overline{N}_\epsilon$ is the closure of the set N_ϵ) consists of admissible translations for the measure μ.*

To prove this assertion, we choose a sequence ϵ_n which tends to zero monotonically, and sets $E_n \subset S_{\epsilon_n}$ such that $\lim_{n\to\infty} \int_{E_n} x\pi(dx) = a$; moreover, let c_n tend to zero monotonically, but $\sum_{n=1}^{\infty} c_n = +\infty$. Then one can take $g_n(x) = \sum_{m=n}^{k_n} c_m\chi_{E_m}(x)$, where the k_n are such that $\sum_{m=n}^{k_n} c_m \to 1$. It is easy to construct a function $\varphi(t)$ for which condition (3) of the theorem will be fulfilled.

REMARK 1. One can similarly show that if $-a \in N_0$, then a is likewise an admissible translation.

2.3. In this subsection we apply the results obtained to a specific class of infinitely divisible distributions in H—the stable distributions. By this term we shall mean, in analogy with the finite dimensional case, those infinitely divisible distributions for which

$$\pi(A) = \int m(A_r)\frac{dr}{r^{1+\alpha}}, \tag{2.23}$$

where m is some finite measure concentrated on the surface of the ball S_1, and A_r is the set on the surface of S_1 defined by

$$A_r = \{y\colon ry \in A\} \cap \{y\colon |y| = 1\}. \tag{2.24}$$

THEOREM 2. *Suppose that the exponent α in formula (2.23) is not less than 1 and that for given a one can find a sequence of measurable nonnegative functions $h_n(x)$ such that*

$$a = \lim_{n\to\infty} \int h_n(x)xm(dx); \tag{2.25}$$

then a is an admissible translation for the measure μ.

PROOF. Without loss of generality we can assume that the $h_n(x)$ are bounded. Choosing some ϵ_n, we introduce a number $\delta_n(x)$ such that

$$h_n(x) = \int_{\delta_n(x)}^{\epsilon_n} \frac{dr}{r^\alpha}. \tag{2.26}$$

It is obvious that one can choose the sequence ϵ_n so that $\epsilon_{n+1} < \delta_n(x)$. Let $E_n = \{x: \delta_n(x) \leq |x| \leq \epsilon_n\}$. Then $\int_{E_n} x\pi(dx) = \int h_n(x)xm(dx)$. It remains to use corollary 1.

REMARK 2. It is easily seen, that the translations of the form (2.25) constitute a cone in the space H, which we denote by K^+. Let K^- be the cone of vectors having the form (2.25) with negative $h_n(x)$. Then, by virtue of remark 1, the set K^- also consists of admissible translations. Consequently, the set $H_0 = K^- + K^+$, consisting of elements which are representable in the form of a sum $x^- + x^+$, where $x^- \in K^-$, $x^+ \in K^+$, likewise consists of admissible translations.

Let H^- be the orthogonal complement of H_0. Then for $z \in H^-$ we have $\int (z, x)h(x)m(dx) = 0$ for every bounded measurable function $h(x)$, since $\int h(x)xm(dx) \in H_0$. Choosing $h(x) = (z, x)$, we obtain $\int (z, x)^2m(dx) = 0$; hence, for all $c > 0$, $\int_{|x|<c} (z, x)\, \pi(dx) = 0$. It follows from this that $\varphi_\mu(\lambda z) = 1$ for all real λ, that is, $(z, x) = 0$ almost everywhere with respect to the measure μ. On the other hand, $\varphi_{\mu_z}(\lambda z) = e^{i\lambda(z,z)}$, namely $(z, x) = (z, z)$ almost everywhere with respect to the measure μ_z. Consequently, for all $z \in H^-$, $z \neq 0$, the measures μ and μ_z are mutually singular.

3. The absolute continuity of measures corresponding to Markov processes

One can say that for Markov processes of diffusion type, and also Markov processes having continuous, diffusion and jump components, the question of the absolute continuity of measures is basically solved (see [5]–[7]). In the book by Dynkin ([22], pp. 423–425) certain conditions are given which must be satisfied by a random variable ξ, in order that it be the density of a measure corresponding to a homogeneous Markov process of general type. In this section also, only homogeneous processes will be considered. For these processes we shall construct a class of variables which are densities, and we shall consider the relation between characteristics of processes, when the density of the measure of one of the processes with respect to the other is known.

We shall consider a continuous Markov process X: $\{x_t, \zeta, \mathfrak{F}_t, P_x\}$ in some bounded domain G, vanishing on the boundary (that is, ζ, the stopping moment, coincides with the instant at which the boundary of G is reached). We consider the class of additive functionals φ_t, satisfying these conditions:

(1) $\mathbf{E}_x\varphi_t = 0$ for all $t > 0$, $x \in G$,

(2) $\mathbf{E}_x\varphi_t^2$ is measurable in x and $\sup_x \mathbf{E}_x\varphi_t^2 < \infty$,

(3) φ_t is continuous in t with P_x-probability 1 for every $x \in G$.

We shall call the additive functionals satisfying these conditions M-functionals. If A is the infinitesimal operator of the process and $f \in \mathfrak{D}_A$, then an example of an M-functional is the functional

$$\hat{f}_t = f(x_t) - f(x_0) - \int_0^t Af(x_s)\, ds. \tag{3.1}$$

With every pair φ_t, ψ_t of M-functionals one can associate an additive functional $\langle\varphi, \psi\rangle_t$ for which $\mathbf{E}_x\varphi_t\psi_t = \mathbf{E}_x\langle\varphi, \psi\rangle_t$, and the functional $\langle\varphi, \psi\rangle_t$ will be absolutely continuous with respect to the nonnegative functionals $\langle\varphi, \varphi\rangle_t$ and $\langle\psi, \psi\rangle_t$. If $\langle\varphi, \psi\rangle_t = \int_0^t \mathsf{F}(u)\, d\langle\varphi, \varphi\rangle_u$, then $\mathsf{F}(u) = g(x_u)$ and we shall from now on denote the function $g(x)$ by $(\partial\psi/\partial\varphi)(x)$. All these facts are proved in my paper "On the local structure of a continuous Markov process," which will be published shortly.

THEOREM 1. *Let φ_t be an M-functional. Then the quantity* $\exp\{\varphi_\zeta - \frac{1}{2}\langle\varphi, \varphi\rangle_\zeta\}$ *is such that $\tilde{X} = \{x_t, \zeta, \mathfrak{F}_t, \tilde{P}_x\}$, where $\tilde{P}_x(\Lambda) = \mathbf{E}_x\chi_\Lambda(\omega) \exp\{\varphi_\zeta - \frac{1}{2}\langle\varphi, \varphi\rangle_\zeta\}$ will also be a homogeneous Markov process.*

This theorem follows from section 10.9 of Dynkin's book ([22], pp. 424–425), if one takes into account that

$$\mathbf{E}_x \exp\{\varphi_\zeta - \tfrac{1}{2}\langle\varphi, \varphi\rangle_\zeta\} = 1. \tag{3.2}$$

Formula (3.2) can be established by using the relation

$$\exp\{\varphi_\zeta - \tfrac{1}{2}\langle\varphi, \varphi\rangle_\zeta\} = \lim_{n\to\infty} \xi_n, \qquad \xi_n = \prod_{k=0}^{n^2} (1 + \varphi_{k+1/n} - \varphi_{k/n}), \tag{3.3}$$

(the limit is in the sense of convergence in P_x-probability). Further

$$\mathbf{E}_x \prod_{k=0}^{n^2} (1 + \varphi_{k+1/n} - \varphi_{k/n}) = 1, \tag{3.4}$$

and the possibility of passing to the limit under the integral sign is guaranteed by the fact that

$$\begin{aligned} &\mathbf{E}_x \prod_{k=0}^{n^2} (1 + \varphi_{k+1/n} - \varphi_{k/n}) \log \prod_{k=0}^{n^2} (1 + \varphi_{k+1/n} - \varphi_{k/n}) \\ &\quad = O\left(\mathbf{E}_x \prod_{k=0}^{j-1} (1 + \varphi_{k+1/n} - \varphi_{k/n}) \sum_{r=j}^{n^2} (\varphi_{r+1/n} - \varphi_{r/n})^2\right) \\ &\quad = O\left(\mathbf{E}_x \prod_{k=0}^{j-1} (1 + \varphi_{k+1/n} - \varphi_{k/n})\phi(x_{j/n})\right) \leq \sup_x |\phi(x)|, \end{aligned} \tag{3.5}$$

where $\phi(x) = \mathbf{E}_x\varphi_\zeta^2$.

Since the ξ_n are such that $\mathbf{E}\xi_n = 1$ and $\mathbf{E}\xi_n \log \xi_n \leq \sup_x |\phi(x)|$, then $\mathbf{E}_x \lim_{n\to\infty} \xi_n = \lim_{n\to\infty} \mathbf{E}_x\xi_n = 1$. Thus, formula (3.2), and with it theorem 1, are proved.

We shall now study the relation between characteristics of the process $X = \{x_t, \zeta, \mathfrak{F}_t, P_x\}$ and the process $\tilde{X} = \{x_t, \zeta, \mathfrak{F}_t, \tilde{P}_x\}$. To this end we shall consider the resolvent $\tilde{R}_\lambda$ of the process $\tilde{X}$:

$$\tilde{R}_\lambda f(x) = \tilde{\mathbf{E}}_x \int_0^\infty e^{-\lambda t} f(x_t)\, dt, \tag{3.6}$$

where $\tilde{\mathbf{E}}_x$ is the expectation with respect to the measure $\tilde{P}_x$. We observe that

(3.7) $$\mathbf{E}[\exp\{\varphi_\zeta - \tfrac{1}{2}\langle\varphi, \varphi\rangle_\zeta\}|\mathfrak{F}_t] = \exp\{\varphi_t - \tfrac{1}{2}\langle\varphi, \varphi\rangle_t\}.$$

Therefore

(3.8) $$\begin{aligned}\tilde{R}_\lambda f(x) &= \mathbf{E}_x \int_0^\infty e^{-\lambda t} f(x_t) \exp\{\varphi_\zeta - \tfrac{1}{2}\langle\varphi, \varphi\rangle_\zeta\}\, dt \\ &= \mathbf{E}_x \int_0^\infty \exp\{-\lambda t + \varphi_t - \tfrac{1}{2}\langle\varphi, \varphi\rangle_t\} f(x_t)\, dt.\end{aligned}$$

We further use the easily deduced relation

(3.9) $$e^{\alpha_t} = 1 + \int_0^t e^{\alpha_u}\, d\varphi_u,$$

where $\alpha_t = \varphi_t - \frac{1}{2}\langle\varphi, \varphi\rangle_t$. Denoting by R_λ the resolvent of the process X, we obtain

(3.10) $$\begin{aligned}\tilde{R}_\lambda f(x) &= R_\lambda f(x) + \mathbf{E}_x \int_0^\infty e^{-\lambda t} f(x_t) \int_0^t e^{\alpha_u}\, d\varphi_u \\ &= R_\lambda f(x) + \mathbf{E}_x \int_0^\infty e^{-\lambda u + \alpha_u} \left[\int_u^\infty f(x_t) e^{-\lambda t}\, dt\right] d\varphi_u \\ &= R_\lambda f(x) + \mathbf{E}_x \int_0^\infty e^{-\lambda u + \alpha_u} R_\lambda f(x_{u+0})\, d\varphi_u.\end{aligned}$$

We note that for an M-functional, $\mathbf{E}_x \int_0^\infty g(u)\, d\varphi_u = 0$ for every $\mathfrak{F}_t$-measurable function $g(t)$. The meaning of

(3.11) $$\int_0^\infty e^{-\lambda u + \alpha_u} R_\lambda f(x_{u+0})\, d\varphi_u$$

is that in the Stieltjes sums, $R_\lambda f(x_u)$ is taken at the point u_{k+1}, if the increment $\varphi_{u_{k+1}} - \varphi_{u_k}$ is considered. Therefore, the last integral reduces to the integral

(3.12) $$\int_0^\infty e^{-\lambda u + \alpha_u}\, dR_\lambda f(x_u)\, d\varphi_u,$$

which coincides with the integral

(3.13) $$\int_0^\infty e^{-\lambda u + \alpha_u}\, d\langle \widehat{R_\lambda f}, \varphi\rangle_u,$$

where $\widehat{R_\lambda f}$ is the M-functional defined by the relation

(3.14) $$\widehat{R_\lambda f}_u = R_\lambda f(x_u) - R_\lambda f(x_0) - \int_0^u A R_\lambda f(x_s)\, ds.$$

A further transformation of the integral in question, using the "partial derivatives" of an M-functional, leads to the following expression for $\tilde{R}_\lambda f$:

(3.15) $$\begin{aligned}\tilde{R}_\lambda f &= R_\lambda f + \mathbf{E}_x \int_0^\infty e^{-\lambda u + \alpha_u} \frac{\partial \widehat{R_\lambda f}}{\partial \varphi}(x_u)\, d\langle\varphi, \varphi\rangle_u \\ &= R_\lambda f + \tilde{\mathbf{E}}_x \int_0^\infty e^{-\lambda u} \frac{\partial \widehat{R_\lambda f}}{\partial \varphi}(x_u)\, d\langle\varphi, \varphi\rangle_u.\end{aligned}$$

We put $R_\lambda f = g$. It is easily verified that by virtue of the absolute continuity of the measure $\tilde{P}_x$ with respect to the measure P_x, the limits

$$(3.16)\qquad \langle\varphi, \varphi\rangle_u = \lim_{n\to\infty} \sum_{k/n<u} (\varphi_{k+1/n} - \varphi_{k/n})^2 = \lim_{n\to\infty} \sum_{k/n<u} (\alpha_{k+1/n} - \alpha_{k/n})^2$$

exist in $\tilde{P}_x$-measure, coinciding with the limits in P_x-measure, as do also the limits

$$(3.17)\qquad \langle\hat{g}, \varphi\rangle_u = \lim_{n\to\infty} \sum_{k/n<u} (\varphi_{k+1/n} - \varphi_{k/n})(g(x_{k+1/n}) - g(x_{k/n})).$$

Consequently, the definition of the derivative $\partial\hat{g}/\partial\varphi$ does not depend upon which measure, P_x or $\tilde{P}_x$, we are considering. Following the indicated substitutions, we obtain

$$(3.18)\qquad \tilde{R}_\lambda(\lambda g - Ag) = g + \tilde{\mathbf{E}}_x \int_0^\infty e^{-\lambda u} \frac{\partial\hat{g}}{\partial\varphi}\, d\langle\varphi, \varphi\rangle_u,$$

or

$$(3.19)\qquad \tilde{R}_\lambda(Ag) = \lambda\tilde{R}_\lambda g - g - \tilde{\mathbf{E}}_x \int_0^\infty e^{-\lambda u} \frac{\partial\hat{g}}{\partial\varphi}\, d\langle\varphi, \varphi\rangle_u.$$

Since $\lambda\tilde{R}_\lambda g - g = \tilde{A}\tilde{R}_\lambda g$, the last formula assumes the form

$$(3.20)\qquad \tilde{R}_\lambda(Ag) = \tilde{A}\tilde{R}_\lambda g - \mathbf{E}_x \int_0^\infty e^{-\lambda u} \frac{\partial\hat{g}}{\partial\varphi}\, d\langle\varphi, \varphi\rangle_u.$$

Formula (3.20) can be regarded as an equation defining Ag, if $\tilde{R}_\lambda$ and $\tilde{A}$ are considered as known.

Let us note the case where $g \in \mathfrak{D}_{\tilde{A}}$. (Since $g \in \mathfrak{D}_A$, then by our assumption $g \in \mathfrak{D}_A \cap \mathfrak{D}_{\tilde{A}}$; however, the possibility that $\mathfrak{D}_A \cap \mathfrak{D}_{\tilde{A}}$ is empty is not ruled out.) Then

$$(3.21)\qquad \tilde{A}g - Ag = \tilde{R}_\lambda^{-1}\tilde{\mathbf{E}}_x \int_0^\infty e^{-\lambda u} \frac{\partial\hat{g}}{\partial\varphi}\, d\langle\varphi, \varphi\rangle_u.$$

If $\langle\varphi, \varphi\rangle_u = \int_0^u V(x_s)\, ds$, then

$$(3.22)\qquad \tilde{R}_\lambda^{-1}\tilde{\mathbf{E}}_x \int_0^\infty e^{-\lambda u} \frac{\partial\hat{g}}{\partial\varphi}(x_u)V(x_u)\, du = \frac{\partial\hat{g}}{\partial\varphi}(x)V(x),$$

and so

$$(3.23)\qquad \tilde{A}g - Ag = \frac{\partial\hat{g}}{\partial\varphi}(x)V(x).$$

Let

$$(3.24)\qquad \tilde{A} = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i\, \partial x_j} + \sum_i b_i(x) \frac{\partial}{\partial x_i}, \qquad A = \sum_{i,j} a_{ij}(x) \frac{\partial^2}{\partial x_i\, \partial x_j}$$

be the infinitesimal operators of a process in n-dimensional space. In order that formula (3.23) hold, one has to take $\varphi_t = \int_0^t \sum_i c_i(x_s)\, dx_s^{(i)}$, where the $x_s^{(i)}$ are the components of the vector x_s. Then

$$\langle \varphi, \varphi \rangle_t = \int_0^t \sum_{i,j} a_{ij}(x_s)c_i(x_s)c_j(x_s)\, ds, \tag{3.25}$$

and the $c_i(x)$ are determined from the system of equations

$$\sum_{i=1}^{n} a_{ij}(x)c_i(x) = b_j(x), \tag{3.26}$$

since

$$\langle \mathfrak{g}, \varphi \rangle_t = \int_0^t \sum_{i,j} a_{ij}(x_s)c_i(x_s) \frac{\partial g}{\partial x_j}(x_s)\, ds. \tag{3.27}$$

Thus, in the case of diffusion processes we obtain the well-known result: if $\tilde{P}$ and P are the measures corresponding to the diffusion processes with diffusion matrix $\|a_{ij}(x)\|$ and shift vectors $(b_1(x), \cdots, b_n(x))$ and 0, respectively, and the processes are considered on the interval $[0, T]$, then

$$\frac{d\tilde{P}}{dP}(x) = \exp\left\{\int_0^T \sum_i c_i(x_s)\, dx_s^{(i)} - \tfrac{1}{2}\int_0^T \sum_{i,j} a_{ij}(x_s)c_i(x_s)c_j(x_s)\, ds\right\}, \tag{3.28}$$

where the $c_i(x)$ are determined from the system (3.26).

It is easy to see that in the case where the system (3.26) does not have solutions, P and $\tilde{P}$ are mutually singular.

4. The absolute continuity of measures, corresponding to stationary Gaussian processes

In this section we consider certain conditions which are sufficient for the absolute continuity or for the singularity of measures, corresponding to stationary Gaussian processes, under the assumption that the given processes have spectral densities. These conditions will be expressed in terms of the spectral density. The papers [13], [15], [16], [20] are devoted to various conditions for the absolute continuity or singularity of measures. The conditions presented below are generalizations of these. Since the measures corresponding to Gaussian processes can only be either mutually absolutely continuous or mutually singular (see [11]), we shall, in what follows, for brevity use the terms "equivalence" for absolute continuity and "orthogonality" for singularity, as is more often done in the literature on stationary Gaussian processes.

4.1. *Sufficient conditions for absolute continuity.* We shall assume that the stationary Gaussian processes $\xi_1(t)$ and $\xi_2(t)$ are such that $\mathbf{E}\xi_i(t) = 0$; $R_i(t)$ is the correlation function of the process $\xi_i(t)$. We denote by μ_i^T the measure corresponding to the process $\xi_i(t)$ considered over $[0, T]$. The fundamental theorem which we shall use in studying sufficient conditions for equivalence is the following.

THEOREM 1. *Suppose there exists a measurable integrable function $c(t, s)$ on $[0, T] \times [0, T]$, for which*

$$R_2(t-s) - R_1(t-s) = \int_0^T c(t, u) R_1(u-s)\, du; \qquad t, s \in [0, T] \tag{4.1}$$

and

$$\int_0^T \int_0^T c(t, u)^2\, dt\, du < \infty. \tag{4.2}$$

Then μ_1^T and μ_2^T are equivalent.

A proof of this theorem can be obtained by using the general conditions for the absolute continuity of Gaussian measures, stated, for example, in [17].

Let us consider the case where $R_1(t) = e^{-a|t|}$. Then, after differentiating twice with respect to s, the integral equation (4.1) can be rewritten as the relation

$$\frac{\partial^2}{\partial s^2}[R_2(t-s) - R_1(t-s)] = a^2[R_2(t-s) - R_1(t-s)] - 2a\, c(t, s), \tag{4.3}$$

and the condition (4.2) can be rewritten as

$$\int_0^T \int_0^T [a^2(R_2(t-s) - R_1(t-s)) - (R_2(t-s) - R_1(t-s))_{ss}]^2\, dt\, ds < \infty. \tag{4.4}$$

If we use the representation of the correlation functions in terms of the spectral densities $R_k(t) = \int e^{i\lambda t} f_k(\lambda)\, d\lambda$, then the condition (4.4) can be rewritten in the form

$$\frac{4}{\pi^2} \iint \frac{\sin^2 \frac{\lambda - \mu}{2} T}{(\lambda - \mu)^2} \frac{f_2(\lambda) - f_1(\lambda)}{f_1(\lambda)} \frac{f_2(\mu) - f_1(\mu)}{f_1(\mu)}\, d\lambda\, d\mu < \infty. \tag{4.5}$$

Condition (4.5) is a sufficient condition for the absolute continuity of μ_1^T and μ_2^T, if $f_1(\lambda) = 1/\pi(a^2 + \lambda^2)$.

Using this result, we can obtain various sufficient conditions for the equivalence of μ_1^T and μ_2^T for all $T > 0$ for a larger class of processes.

THEOREM 2. *Suppose there exists an entire analytic function $g(\lambda)$ of finite exponential type, real for real λ, and such that the following conditions are satisfied:*

(1) $\int_{-\infty}^{\infty} g(\lambda)^2\, d\lambda < \infty$,

(2) $\lim_{|\lambda| \to \infty} \dfrac{\lambda^2 (f_1(\lambda) - f_2(\lambda))}{g(\lambda)^2} = 0$,

(3) $\int_{-\infty}^{\infty} \dfrac{\lambda^2 (f_1(\lambda) - f_2(\lambda))^2}{g(\lambda)^4}\, d\lambda < \infty$,

(4) *for some $a > 0$ and $C > 0$,* $\dfrac{f_1(\lambda) + f_2(\lambda)}{2} - \dfrac{g(\lambda)^2}{\pi(a^2 + \lambda^2)} \geq 0$ *for all real λ for which $|\lambda| > C$,*

(5) *for some* $m > 0$, $\liminf_{|\lambda| \to \infty} |f_k(\lambda)|/g(\lambda)^m > 0$.

Then for all $T > 0$, μ_1^T *and* μ_2^T *are equivalent.*

PROOF. Let us first assume that for all real λ

$$\frac{\pi(a^2 + \lambda^2)|f_1(\lambda) - f_2(\lambda)|}{2g(\lambda)^2} < 1 \tag{4.6}$$

and that condition (4) is fulfilled for all real λ. We introduce the spectral densities

$$\varphi_1(\lambda) = \frac{1}{\pi(a^2 + \lambda^2)}\left[1 - \frac{(f_1(\lambda) - f_2(\lambda))\pi(a^2 + \lambda^2)}{2g(\lambda)^2}\right], \tag{4.7}$$

$$\varphi_2(\lambda) = \frac{1}{\pi(a^2 + \lambda^2)}\left[1 + \frac{(f_1(\lambda) - f_2(\lambda))\pi(a^2 + \lambda^2)}{2g(\lambda)^2}\right], \tag{4.8}$$

$$\psi(\lambda) = \frac{f_1(\lambda) + f_2(\lambda)}{2} - \frac{g(\lambda)^2}{\pi(a^2 + \lambda^2)}. \tag{4.9}$$

If $\mu_{\varphi_1}^T$ and $\mu_{\varphi_2}^T$ are the measures corresponding to processes with spectral densities $\varphi_1(\lambda)$ and $\varphi_2(\lambda)$, then these measures will be equivalent to the measure μ_a^T, corresponding to a process with spectral density $1/\pi(a^2 + \lambda^2)$ (on the basis of the foregoing) and thus also equivalent to each other for any T. We now observe that

$$g(\lambda) = \int_{-k}^{k} e^{i\lambda t} d(t)\, dt, \qquad \text{where} \quad \int_{-k}^{k} |d(t)|^2\, dt < \infty, \tag{4.10}$$

and k is the exponential type of the function $g(\lambda)$. Let $\xi_1(t)$ and $\xi_2(t)$ be processes with spectral densities $\varphi_1(\lambda)$ and $\varphi_2(\lambda)$, and $\eta(t)$ a process with spectral density $\psi(\lambda)$. It is easy to see that the process

$$\xi_k(t) = \int_{-k}^{k} d(s)\xi_k(t - s + k)\, ds + \eta(t) \tag{4.11}$$

has spectral density $f_k(\lambda)$. Further, the values of the process $\xi_k(t)$ over $[0, T]$ are completely determined by the values of $\xi_k(t)$ over $[0, T + 2k]$ and the values of $\eta(t)$ over $[0, T]$. Since the processes $\xi_k(t)$ are obtained by means of the same transformation from processes to which correspond equivalent measures, then the measures μ_1^T and μ_2^T will be equivalent for all T.

The complete proof of the theorem follows from the fact that a change in spectral densities which satisfy condition (5) on any finite interval does not disturb the equivalence of the measures (this is easily deduced from the part of the theorem already proved).

COROLLARY. *Suppose that the spectral densities* $f_1(\lambda)$ *and* $f_2(\lambda)$ *satisfy the following conditions: for sufficiently large* λ, *for certain* C_1 *and* C_2,

$$C_1 \le f_k(\lambda)|\lambda|^\alpha \le C_2 \tag{4.12}$$

and

$$\int \frac{|f_1(\lambda) - f_2(\lambda)|^2}{1 + |\lambda|^{2\alpha}}\, d\lambda < \infty, \quad \text{and} \quad \lim_{|\lambda| \to \infty} \frac{|f_1(\lambda) - f_2(\lambda)|}{1 + |\lambda|^\alpha} = 0. \tag{4.13}$$

Then μ_1^T *and* μ_2^T *are equivalent for all* T.

The proof of this assertion follows, for $\alpha > 3$, from theorem 2, if we put

$$g(\lambda) = \int \left(\frac{\sin(\lambda - \mu)}{\lambda - \mu}\right)^m \sqrt{\frac{1 + \mu^2}{1 + |\mu|^\alpha}}\, d\mu, \qquad m > \alpha. \tag{4.14}$$

This entire function is of exponential type m, and satisfies all the conditions of theorem 2. If $\alpha < 3$ (but necessarily $\alpha > 1$), then we consider processes $\tilde{\xi}_k(t)$ with spectral densities $\tilde{f}_k(\lambda) = f_k(\lambda)(1 + \lambda^2)^{-1}$; for these processes, now, $\alpha > 3$ and the equivalence of the corresponding measures follows from the foregoing. The processes $\xi_k(t) = \tilde{\xi}_k(t) + \tilde{\xi}'_k(t)$ will have spectral densities $f_k(\lambda)$, and the measures corresponding to them will also be equivalent (since they are obtained by the same transformation on equivalent measures).

4.2. *Sufficient conditions for orthogonality.* To derive conditions for orthogonality, we shall use the following theorem.

THEOREM 3. *If $\xi_1(t)$ and $\xi_2(t)$ are Gaussian processes on $[0, T]$ and μ_1^T and μ_2^T are the measures corresponding to them, then for the orthogonality of μ_1^T and μ_2^T it is necessary and sufficient that there exist a sequence of positive definite functions $g_n(t, s)$ on $[0, T] \times [0, T]$ such that*

$$\limsup_{n\to\infty} \frac{\left|\mathbf{E} \int_0^T \int_0^T g_n(t, s)[\xi_2(t)\xi_2(s) - \xi_1(t)\xi_1(s)]\, dt\, ds\right|}{\mathbf{D} \int_0^T \int_0^T g_n(t, s)\xi_1(t)\xi_1(s)\, dt\, ds} > 0 \tag{4.15}$$

and

$$\lim_{n\to\infty} \mathbf{D} \int_0^T \int_0^T g_n(t, s)\xi_1(t)\xi_1(s)\, dt\, ds = +\infty. \tag{4.16}$$

The necessity of the conditions of the theorem follows from general conditions for absolute continuity (see [11], [12], [14], [17]). The sufficiency follows from the fact that the mapping of $\mathbf{F}_{[0,T]}$ into the space of sequences given by

$$x(t) \to \left\{\frac{\int_0^T \int_0^T g_n(t, s)x(t)x(s)\, dt\, ds - \mathbf{E} \int_0^T \int_0^T g_n(t, s)\xi_1(t)\xi_1(s)\, dt\, ds}{\mathbf{D} \int_0^T \int_0^T g_n(t, s)\xi_1(t)\xi_1(s)\, dt\, ds}\right\} \tag{4.17}$$

carries the measures μ_1^T and μ_2^T into singular measures: since the denominator of this fraction tends to infinity, then it converges to zero in μ_1^T-probability, but not in μ_2^T-probability.

A consequence of this theorem is the following.

THEOREM 4. *Let $\mu_{f_1}^T, \mu_{f_2}^T, \mu_{f_3}^T$ be the measures corresponding to stationary Gaussian processes, with spectral densities $f_1(\lambda), f_2(\lambda), f_3(\lambda)$, on $[0, T]$. If $f_1(\lambda) \le f_2(\lambda) \le f_3(\lambda)$, then the orthogonality of $\mu_{f_1}^T$ and $\mu_{f_2}^T$ implies that of $\mu_{f_1}^T$ and $\mu_{f_3}^T$.*

PROOF. Let $\xi_1(t)$, $\eta(t)$, $\zeta(t)$ be independent Gaussian random processes with spectral densities $f_1(\lambda)$, $f_2(\lambda) - f_1(\lambda)$, $f_3(\lambda) - f_2(\lambda)$. Then the process $\xi_2(t) = \xi_1(t) + \eta(t)$ will have spectral density $f_2(\lambda)$, and the process $\xi_3(t) = \xi_2(t) + \zeta(t)$ will have spectral density $f_3(\lambda)$. Let $\mu_{f_1}^T$ and $\mu_{f_2}^T$ be orthogonal, and let the sequence of positive definite functions $g_n(t, s)$ be such that conditions (4.15) and (4.16) are fulfilled. It is easy to compute that for the case at hand

$$(4.18)\qquad \mathbf{E}\int_0^T\int_0^T g_n(t,s)\xi_2(t)\xi_2(s)\,dt\,ds - \mathbf{E}\int_0^T\int_0^T g_n(t,s)\xi_1(t)\xi_1(s)\,dt\,ds = \mathbf{E}\int_0^T\int_0^T g_n(t,s)\eta(t)\eta(s)\,dt\,ds,$$

and

$$(4.19)\qquad \mathbf{E}\int_0^T\int_0^T g_n(t,s)\xi_3(t)\xi_3(s)\,dt\,ds - \mathbf{E}\int_0^T\int_0^T g_n(t,s)\xi_1(t)\xi_1(s)\,dt\,ds$$
$$= \mathbf{E}\int_0^T\int_0^T g_n(t,s)[\eta(t)\eta(s) + \zeta(t)\zeta(s)]\,dt\,ds$$
$$\geq \mathbf{E}\int_0^T\int_0^T g_n(t,s)\eta(t)\eta(s)\,dt\,ds \geq 0.$$

Therefore, from (4.15) and (4.16), the very same inequalities will hold in the present case for $\xi_3(t)$ and $\xi_1(t)$. It remains to use the sufficiency condition of theorem 3.

The theorem just proved enables us to solve the question of the singularity of the measures $\mu_{f_1}^T$ and $\mu_{f_2}^T$, using the singularity of the measures $\mu_{\tilde f_1}^T$ and $\mu_{\tilde f_2}^T$, if $f_1 \leq \tilde f_1 \leq \tilde f_2 \leq f_2$ and the spectral densities $\tilde f_k$ have a simpler form.

The following result is of a more specific nature.

THEOREM 5. *Suppose the spectral densities $f_1(\lambda)$ and $f_2(\lambda)$ satisfy the conditions*

(1) $f_1(\lambda) \leq f_2(\lambda)$,

(2) *there exists a sequence of even entire analytic functions $\Delta_n(\lambda)$ of exponential type not exceeding τ, satisfying these conditions:*

(a) $0 \leq \Delta_n(\lambda) \leq \dfrac{(f_2(\lambda) - f_1(\lambda))^{1/2}}{f_1(\lambda)}$ *for real* λ,

(b) $\int_{-\infty}^{\infty} \Delta_n^2(\lambda)\,d\lambda < \infty$ *for all* n,

(c) $\lim_{n\to\infty} \int_{-\infty}^{\infty} \Delta_n^2(\lambda)\Delta_n^2(\mu)f(\lambda)f(\mu)\dfrac{\sin^4\frac{a}{2}(\lambda-\mu)}{(\lambda-\mu)^4}\,d\lambda\,d\mu = +\infty.$

Then for $T > 2a + 2\tau$, $\mu_{f_1}^T$ and $\mu_{f_2}^T$ are orthogonal.

PROOF. It follows from the conditions of the theorem that

$$(4.20)\qquad \Delta_n(\lambda) = \int e^{i\lambda s} g_n(s)\,ds,$$

where $g_n(s)$ is different from zero only on $[-\tau, \tau]$ and $\int_{-\infty}^{\infty} |g_n(s)|^2\,ds < \infty$. We introduce the function

$$(4.21)\qquad G_n(t,s) = \int_{-a}^{a} g_n(s-u)g_n(t+u)[a - |u|]\,du.$$

An elementary calculation shows that

$$(4.22)\qquad \iint G_n(t,s)e^{i\lambda t + i\mu s}\,dt\,ds = \Delta_n(\lambda)\,\Delta_n(\mu)\,\frac{4\sin^2\frac{a}{2}(\lambda-\mu)}{(\lambda-\mu)^2},$$

(the last formula shows that $G_n(t, s)$ is positive definite). Using (4.22), we can calculate that for $T = 2a + 2\tau$

$$\mathbf{E} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} G_n(t, s)[\xi_2(t)\xi_2(s) - \xi_1(t)\xi_1(s)]\, dt\, ds = a^2 \int [f_2(\lambda) - f_1(\lambda)]\, \Delta_n^2(\lambda)\, d\lambda, \tag{4.23}$$

and

$$\mathbf{D} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} G_n(t, s)\xi_1(t)\xi_1(s)\, dt\, ds = 32 \iint \Delta_n^2(\lambda)\, \Delta_n^2(\mu) f_1(\lambda) f_1(\mu) \frac{\sin^4 \frac{a}{2}(\lambda - \mu)}{(\lambda - \mu)^4}\, d\lambda\, d\mu. \tag{4.24}$$

Thus, by virtue of the hypotheses of the theorem,

$$\mathbf{D} \iint G_n(t, s)\xi_1(t)\xi_1(s)\, dt\, ds \to +\infty. \tag{4.25}$$

Since

$$\begin{aligned} &\iint \Delta_n^2(\lambda)\, \Delta_n^2(\mu) f_1(\lambda) f_1(\mu) \frac{\sin^4 \frac{a}{2}(\lambda - \mu)}{(\lambda - \mu)^4}\, d\lambda\, d\mu \\ &\quad \le \iint \Delta_n^4(\lambda) f_1^2(\lambda) \frac{\sin^4 \frac{a}{2}(\lambda - \mu)}{(\lambda - \mu)^4}\, d\lambda\, d\mu \\ &\quad = \int \Delta_n^4(\lambda) f_1^2(\lambda)\, d\lambda \int \frac{\sin^4 \frac{a}{2}\mu}{\mu^4}\, d\mu \\ &\quad \le \int \frac{\sin^4 \frac{a}{2}\mu}{\mu^4}\, d\mu \int [f_2(\lambda) - f_1(\lambda)]\, \Delta_n^2(\lambda)\, d\lambda, \end{aligned} \tag{4.26}$$

the limit (4.15) is in this case not less than

$$\frac{a^2}{32 \int \frac{\sin^4 \frac{a}{2}\mu}{\mu^4}\, d\mu}. \tag{4.27}$$

Thus the conditions of theorem 3 are fulfilled, and the theorem is proved.

COROLLARY. *Suppose that*

(1) $f_2(\lambda) \ge f_1(\lambda)$;

(2) *for some* $A < 1$, *some* B *and* $m > 0$, *the inequality*

$$|\varphi(\lambda + h) - \varphi(\lambda)| \le [A\varphi(\lambda) + B](1 + h^{2m}) \tag{4.28}$$

holds, where

(4.29) $$\varphi(\lambda) = \frac{\sqrt{f_2(\lambda) - f_1(\lambda)}}{f_1(\lambda)};$$

(3) $\varphi(\lambda) \to \infty$ *and* $\int_{-\infty}^{\infty} \frac{[f_2(\lambda) - f_1(\lambda)]^2}{f_1^2(\lambda)} d\lambda = \infty$;

(4) *there exists an entire analytic function* $g(\lambda)$ *of finite exponential type such that* $f_k(\lambda) > |g(\lambda)|^2$ *for sufficiently large* $|\lambda|$.

Then for all $T > 0$, $\mu_{f_1}^T$ *and* $\mu_{f_2}^T$ *are orthogonal.*

For the proof, take

(4.30) $$\Delta_n(\lambda) = kC_n \int_{-n}^{n} \varphi(h) \frac{\sin^{2m+2}\epsilon(\lambda - h)}{(\lambda - h)^{2m+2}} dh,$$

(4.31) $$C_n = \left(\int_{-n}^{n} \frac{\sin^{2m+2}\epsilon\lambda}{\lambda^{2m+2}} d\lambda\right)^{-1},$$

where k is sufficiently small, and apply theorem 5.

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