GENERALIZED UNIFORM COMPLEX MEASURES IN THE HILBERTIAN METRIC SPACE WITH THEIR APPLICATION TO THE FEYNMAN INTEGRAL

Dedicated to Professor Charles Loewner

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1. Introduction and summary

As we pointed out in the Fourth Berkeley Symposium [4], an infinite dimensional version of the complex measure in the k-space E_k,

$$F_k(dx) = \lambda_k(dx)/(\sqrt{2\pi h i})^k, \qquad \lambda_k = \text{Lebesgue measure} \tag{1}$$

is useful for a mathematical formulation of the Feynman integral [1]; h is a positive constant which is supposed to indicate the Planck constant in its application to quantum mechanics and $\sqrt{z}$, $(z \neq 0)$ denotes the branch for which $-\pi/2 < \arg \sqrt{z} < \pi/2$ throughout this paper. Since neither λ_k nor $(\sqrt{2\pi h i})^k$ has any meaning when $k = \infty$, we cannot directly extend this measure to the infinite dimensional space E_∞ (Hilbert space). Therefore, we shall consider a linear functional $F_k(f)$ induced by the measure F_k in (1):

$$F_k(t) = \int_{E_k} f(x) \frac{\lambda_k(dx)}{(\sqrt{2\pi h i})^k}, \tag{2}$$

and extend this by putting convergent factors as

$$F_k(f) = \lim_{n \to \infty} \int_{E_k} f(x) \exp\left[-\frac{1}{2n}(V^{-1}(x-a), (x-a))\right] \frac{\lambda_k(dx)}{(\sqrt{2\pi h i})^k} \tag{3}$$

where a is any element of E_k and V is a strictly positive-definite symmetric operator. The domain $\mathfrak{D}(F_k)$ of definition of F_k is the space of all Borel measurable functions for which the limit in (3) exists for every (a, V) and has a finite value independent of (a, V). We shall rewrite (3) as

$$F_k(f) = \lim_{n \to \infty} \prod_{\nu=1}^{k} \sqrt{1 + \frac{n v_\nu}{hi}} \int_{E_k} f(x) N(dx\colon a, nV), \tag{3'}$$

where $\{v_\nu\}$ are the eigenvalues of V and $N(dx: a, V)$ denotes the Gauss measure with the mean vector a and the covariance operator V. Since the Gauss measure $N(dx: a, V)$ can be defined in the real Hilbert space E_∞ if V is positive-definite symmetric operator with the sum of eigenvalues $< \infty$ [3], [7], we can define F_∞ by putting $k = \infty$ in (3); notice that the infinite product in (3′) is convergent by virtue of $\sum_\nu v_\nu < \infty$.

Following an important suggestion of L. Gross we shall modify (3′) to make the approximating measures more uniform and define F_k as follows. We introduce a directed semiorder in the class $\mathcal{V}$ of all strictly positive-definite symmetric operators of finite trace by

$$V_1 < V_2 \qquad \text{if and only if} \quad V_2 - V_1 \in \mathcal{V}. \tag{4}$$

DEFINITION. *Denoting with* $\lim_V$ *the limit along this directed system* $\mathcal{V}$, *we shall define* $F_k(f)$ *as follows:*

$$F_k(f) = \lim_V \prod_{\nu=1}^{k} \sqrt{1 + \frac{v_\nu}{hi}} \int_{E_k} f(x) N(dx: a, V), \tag{5}$$

where the domain $\mathfrak{D}(F_k)$ *of definition of* F_k *is the class of all Borel measurable functions for which this limit exists for every* a *and has a finite value independent of* a.

In order to discuss this functional we shall introduce some notions.

Let A be a bounded positive-definite symmetric operator. Then $\sum_n (Ae_n, e_n)$, $\{e_n\}$ being an orthonormal basis, is independent of the special choice of the basis $\{e_n\}$ and is called the *trace* of A, in symbol $\operatorname{Tr} A$. If $\operatorname{Tr} A < \infty$, then A is completely continuous and $\operatorname{Tr} A$ is equal to the sum of all eigenvalues of A.

Let A be a bounded operator. We shall define the trace norm of class α (> 0) of A by

$$\|A\|_\alpha = [\operatorname{Tr}((A^*A)^{\alpha/2})]^{1/\alpha}. \tag{6}$$

If A is symmetric, then $\|A\|_\alpha = [\operatorname{Tr}(|A|^\alpha]^{1/\alpha}$. The uniform norm of A is defined by $\|A\| = \sup_{|x| \leq 1} |Ax|$. It is easy to see that

$$\|A\|_\alpha < \infty \Rightarrow \|A\|_\beta < \infty \qquad \text{if} \quad \alpha < \beta \tag{7.a}$$

and

$$\|A\|_\alpha < \infty \Rightarrow \|A\| < \infty. \tag{7.b}$$

We shall call A a *trace operator* if $\|A\|_1 < \infty$ and call it a *Hilbert-Schmidt operator* if $\|A\|_2 < \infty$. Because of (7.a) we can see that if $\|A\|_\alpha < \infty$ for some $\alpha < 1$, then A is a trace operator, but not vice versa.

If $\|A\|_1 < \infty$, then $\sum (Ae_n, e_n)$ is also convergent for every orthonormal basis $\{e_n\}$ and has a value independent of $\{e_n\}$. This is also called the *trace* of A, in symbol $\operatorname{Tr} A$.

A bounded operator is called *nearly orthogonal* if A is one-to-one and if $\|(A^*A)^{1/2} - I\|_\alpha < \infty$ for some $\alpha < 1$. The first condition is equivalent to the condition that $(A^*A)^{1/2}$ is strictly positive-definite.

A one-to-one transformation from E_k onto itself is called *nearly isometric* if C is expressed as $Cx = a + A \cdot x$, where A is nearly orthogonal.

Writing the eigenvalues of $(A^*A)^{1/2} - I$ as $\{\alpha_\nu\}$, we shall define $J(C)$ by

$$J(C) = \prod_\nu (1 + \alpha_\nu). \tag{8}$$

This is well-defined and does not vanish because A is nearly orthogonal. It is needless to say that every isometric transformation from E_∞ onto E_∞ is nearly isometric. In case $k < \infty$, every one-to-one linear transformation C from E_k onto E_k is nearly isometric and $J(C)$ turns out to be the absolute value of the Jacobian of C.

We are now in a position to state the main properties of F_k.

THEOREM 1. *The linear space $\mathfrak{D}(F_k)$ is invariant under nearly isometric transformations, and we have*

(i) $$f = \alpha_1 f_1 + \alpha_2 f_2 \Rightarrow F_k(f) = \alpha_1 F_k(f_1) + \alpha_2 F_k(f_2);$$

(ii) *if $g(x) = f(Cx)$ and C is nearly isometric, then $F_k(g) = J(C)^{-1}F_k(f)$.*

The functional F_k is not trivial. In other words $\mathfrak{D}(F_k)$ is a fairly large class of functions, as the following theorem shows.

THEOREM 2. *If $f(x)$ is of the form*

$$f(x) = \exp\left[\frac{i}{2h}|x|^2\right] \int_{E_k} e^{i(x,y)}\mu(dy) \tag{9}$$

where μ is a complex measure of bounded absolute variation defined for all Borel subsets of E_k, then

$$f \in \mathfrak{D}(F_k) \quad \text{and} \quad F_k(f) = \int_{E_k} \exp\left[\frac{h}{2i}|y|^2\right]\mu(dy). \tag{10}$$

Combining this with theorem 1, we can see that all functions $f(Cx)$, f being of the form (9) and C being nearly isometric, and their linear combinations belong to $\mathfrak{D}(F_k)$.

In view of these facts, we shall call the functional F_k a *generalized uniform complex measure* and write it as

$$F_k(f) = \int_{E_k} f(x)F_k(dx). \tag{11}$$

In case $k < \infty$, every Borel measurable function f that is λ_k-summable on E_k belongs to $\mathfrak{D}(F_k)$, and we have (1). Therefore, we call F_k, $(k = 1, 2, \cdots, \infty)$ a generalized uniform measure with *index* $\sqrt{2\pi hi}$.

We shall write $\mathcal{E}_k$ for the class of all functions f of the form in theorem 2. It is easy to see that *$\mathcal{E}_k$ is a linear space invariant under isometric transformations.*

The space $E_{k+\ell}$, $(k, \ell = 1, 2, \cdots, \infty)$ is considered as the Cartesian product of E_k and E_ℓ. Any point x of $E_{k+\ell}$ is written as the pair (y, z), $y \in E_k$, $z \in E_\ell$. Then we can easily prove the following theorem.

THEOREM 3. *Suppose that $f \in \mathcal{E}_{k+\ell}$. Then $f(x) = f(y, z)$ belongs to $\mathcal{E}_k$ as a function of y for each z and $\int f(y, z)F_k(dy)$ belongs to $\mathcal{E}_\ell$ as a function of z. Furthermore,*

(12) $$\int_{E_{k+\ell}} f(x)F_{k+\ell}(dx) = \int_{E_\ell} \left[\int_{E_k} f(y, z)F_k(dy)\right] F_\ell(dz).$$

Similarly, we have

(13) $$\int_{E_{k+\ell}} f(x)F_{k+\ell}(dx) = \int_{E_k} \left[\int_{E_\ell} f(y, z)E_\ell(dz)\right] F_k(dz).$$

If $g \in \mathcal{E}_k$ and $h \in \mathcal{E}_\ell$, then $f(y, z) = g(y)h(z)$ belongs to $\mathcal{E}_{k+\ell}$ and

(14) $$\int_{E_{k+\ell}} f(x)F_{k+\ell}(dx) = \int_{E_k} g(y)F_k(dy) \int_{E_\ell} h(z)F_\ell(dz).$$

A metric space M_k is called *Hilbertian* with dimension k if there exists an isometric mapping Φ from M_k onto E_k. If there are two such mappings Φ_1 and Φ_2, then $\Phi_2\Phi_1^{-1}$ will be an isometric mapping from E_k onto itself. Because of this fact, all notions in E_k invariant under isometric transformations can be transplanted into the Hilbertian metric space M_k: for example, Gaussian measure, nearly isometric transformations, $J(C)$, the function class $\mathcal{E}_k$, the generalized uniform complex measure F_k, and so on.

Theorems 1, 2, and 3 can be restated in terms of the Hilbertian metric space. The statement does not change for theorem 1. Theorem 2 should be stated as follows.

THEOREM 2′. *If $f(x)$ is of the form*

(9′) $$f(x) = \int \exp\left[\frac{i}{2h}\overline{xy}^2\right]\nu(dy), \qquad \overline{xy} = \text{distance } (x, y),$$

then we have

(10′) $$f \in \mathfrak{D}(F_k) \quad \text{and} \quad F_k(f) = \nu(E_k).$$

In order to state theorem 3 for M_k we need some notions. The concept of a linear mapping with a translation from $E_{k+\ell}$ onto E_ℓ is invariant under isometric transformations, so that we can define such a mapping from $M_{k+\ell}$ onto M_k, which we shall call a *linear mapping* from $M_{k+\ell}$ onto M_k. Let C be such a mapping. Then

(15) $$d(y_1, y_2) = \inf \{\overline{x_1x_2}: x_1 \in C^{-1}(y_1), x_2 \in C^{-1}(y_2)\}$$

defines a new metric on M_ℓ, so that there exists a linear mapping T from M_ℓ onto itself such that

(16) $$d(y_1, y_2) = \overline{(Ty_1)(Ty_2)}.$$

We shall call C *normal*, or *nearly normal*, according to whether T is isometric or nearly isometric. If C is nearly normal, then we shall define $J(C)$ to be $J(T)$. There are many T's for a single C, but these notions are independent of the choice of T. Now we shall state theorem 3 for the Hilbertian metric space, omitting the statements about the domain of definition.

THEOREM 3′. *Let C be a normal linear mapping from $M_{k+\ell}$ onto M_ℓ. Then $C^{-1}(y)$ is a k-dimensional Hilbertian metric subspace of $M_{k+\ell}$, and we have, for $f \in \mathcal{E}_{k+\ell}$,*

(17) $$\int_{M_{k+l}} f(x)F_{k+l}(dx) = \int_{M_l} \int_{C^{-1}(y)} f(x)F_k(dx)F_l(dy).$$

Combining theorem 3′ with theorem 2, we shall obtain a slight generalization of theorem 3′.

THEOREM 3″. *Let C be nearly normal from M_{k+l} onto M_l. Then we have*

(18) $$\int_{M_{k+l}} f(x)F_{k+l}(dx) = J(C) \int_{M_l} \int_{C^{-1}(y)} F_k(dx)F_l(dy).$$

Let us now formulate the Feynman integral in terms of F_∞.

Consider a classical mechanical system of a particle of mass m moving on the real line $-\infty < q < \infty$ where the field of force is given by a potential $u(q)$. The *Lagrangian* of this system is

(19) $$L(q, \dot{q}) = \frac{m}{2} \dot{q}^2 - mu(q)$$

and its action integral $\mathfrak{A}(\gamma)$ along a motion $\gamma = \gamma(\tau)$, $s \leq \tau \leq t$, is given by

(20) $$\mathfrak{A}(\gamma) = \int_s^t L(\gamma(\tau), \gamma'(\tau))\, d\tau = \tfrac{1}{2} \int_s^t m\gamma'(\tau)^2\, d\tau - \int_s^t mu(\gamma(\tau))\, d\tau.$$

Let $\Gamma = \Gamma(t, b|s, a)$ be the space of all motions $\gamma = \gamma(\tau)$, $s \leq \tau \leq t$, starting at $\gamma(s) = a$ and ending with $\gamma(t) = b$ such that the velocity function $\gamma'(\tau)$ is square summable on $s \leq \tau \leq t$. The space Γ is a Hilbertian metric space M_∞ with the metric

(21) $$\overline{\gamma_1\gamma_2}^2 = \int_s^t m(\gamma_1'(\tau) - \gamma_2'(\tau))^2\, d\tau.$$

Thus we can define the generalized measure on F_∞ on Γ. The *Feynman principle of quantization* of this mechanical system is that the function

(22) $$G(t, b|s, a) \equiv \frac{\sqrt{m}}{\sqrt{2\pi hi(t - s)}} \int_\Gamma \exp\left[\frac{i}{h}\, \mathfrak{A}(\gamma)\right] F_\infty(d\gamma)$$

is the *Green function* of the corresponding quantum mechanical system, namely that $\exp[(i/h)\mathfrak{A}(\gamma)]$ belongs to $\mathfrak{D}(F_\infty)$, and the function $G(t, b|s, a)$ defined above is the elementary solution of the *Schrödinger equation*

(23) $$\frac{h}{i}\frac{\partial \varphi}{\partial t} = \frac{h^2}{2m}\frac{\partial^2 \varphi}{\partial q^2} - m\cdot u\cdot \varphi$$

for the quantum mechanical system; the right side of (22) is called the *Feynman integral*. We shall prove this for the following cases.

Case 1: u is the Fourier transform of a complex measure of bounded absolute variation on $(-\infty, \infty)$.

Case 2: $u(q) \equiv c_1 \cdot q$, where c_1 is a real constant.

Case 3: $u(q) \equiv c_2 q^2$, where c_2 is a positive constant.

Let us mention one word about the index $c = \sqrt{2\pi hi}$. As a matter of fact, we can carry out the same argument in the case where $c \neq 0$ and $\mathrm{Re}\, c^2 \geq 0$ by replacing hi in (5) with $c^2/2\pi$, and the case where $c = \sqrt{2\pi}$ turns out to be in close connection with the *Wiener measure*.

In the course of writing this paper we found a gap in our argument based on definition (3′). We also found that we were able to overcome the difficulty by adopting definition (5) suggested by Professor L. Gross, and by using Kuroda's theory on infinite determinants [5] to which Professor T. Kato drew our attention. We would like to express our hearty thanks to them.

2. Properties of the generalized uniform complex measure F_k

We shall start with some preliminary facts.

LEMMA 1. *Suppose that V is a positive-definite symmetric trace operator with the eigenvalues $\{v_\nu\}$ and the eigenvectors $\{e_\nu\}$. Then $\xi_\nu(x) \equiv (x, e_\nu)$, $\nu = 1, 2, \cdots$ are independent random variables on the probability space $(E_k, N(dx\colon a, V))$, each (x, e_ν) having a Gaussian distribution with the mean $a_\nu = (a, e_\nu)$ and the variance v_ν.*

PROOF. It is enough to observe that

$$(24)\qquad \int \exp\left[i \sum_{\nu=1}^{n} z_\nu(x, e_\nu)\right] N(dx, a, V) = \int \exp\,[i(x, \sum z_\nu e_\nu]N(dx, a, V)$$

$$= \exp\,\{i(a, \sum z_\nu e_\nu) - \tfrac{1}{2}(V \sum z_\nu e_\nu, \sum z_\nu e_\nu)\}$$

$$= \exp\,[\sum\,(iz_\nu a_\nu - \tfrac{1}{2}v_\nu z_\nu^2)] = \prod_{\nu=1}^{n} \exp\,(iz_\nu a_\nu - \tfrac{1}{2}v_\nu z_\nu^2).$$

LEMMA 2. *For $-\infty < y < +\infty$ and $\operatorname{Re} \alpha > 0$, one has*

$$(25)\qquad \int_{-\infty}^{\infty} e^{ixy} e^{-(\alpha/2)x^2}\, dx = \sqrt{\frac{2\pi}{\alpha}}\, e^{-y^2/2\alpha}.$$

PROOF. This is well known for $\alpha > 0$. By analytic continuation we can see that it holds for $\operatorname{Re} \alpha > 0$.

LEMMA 3. *Let H be a real or complex Hilbert space and suppose that V_1 and V_2 are bounded symmetric linear operators. If*

$$(26)\qquad |V_1 x|^2 \geq |V_2 x|^2 \geq c|x|^2, \qquad c \text{ is a positive constant,}$$

then

$$(27)\qquad |V_1^{-1}x|^2 \leq |V_2^{-1}x|^2 \leq c^{-1}|x|^2.$$

PROOF. It follows from the assumption that $|V_2V_1^{-1}x|^2 \leq |x|^2$, that is, $\|V_2V_1^{-1}\| \leq 1$, so that $\|V_1^{-1}V_2\| = \|(V_1^{-1}V_2)^*\| = \|V_2V_1^{-1}\| \leq 1$. Thus we have $|V_1^{-1}V_2x| \leq |x|^2$, that is, $|V_1^{-1}x| \leq |V_2^{-1}x|$. It is obvious that $|V_1^{-1}x|^2 \leq c^{-1}|x|^2$.

LEMMA 4. *For any bounded Borel measurable function $f(x)$ defined on E_k, any linear operator $A\colon E_k \to E_k$ and any $b \in E_k$, we have*

$$(28)\qquad \int_{E_k} f(b + Ax)N(dx\colon a, V) = \int_{E_k} f(x)N(dx\colon b + Aa, AVA^*).$$

PROOF. If $f(x)$ is of the form $e^{i(x,y)}$, y being any fixed element in E_k, then this is true by virtue of

$$(29)\qquad \int e^{i(x,y)}N(dx\colon a, V) = \exp\,\{i(a, y) - \tfrac{1}{2}(Vy, y)\}.$$

Taking linear combinations and limits, we can see that it is true in general.

In section 1 we defined $\operatorname{Tr} A$, $\|A\|_\alpha$, and $\|A\|$ for the real separable Hilbert space E_k. We have analogous concepts for the complex separable Hilbert space H_k of dimension k. Any linear operator A from E_k into itself induces an operator $\tilde{A}$ on H_k by $\tilde{A}(x + iy) = Ax + iAy$. It is easy to see $\operatorname{Tr} A = \operatorname{Tr} \tilde{A}$, $\|A\|_\alpha = \|\tilde{A}_\alpha\|$, and $\|A\| = \|\tilde{A}\|$. Therefore, we write A for $\tilde{A}$ without any ambiguity. We summarize some known facts about norms and determinants. Kuroda's paper [5] is to be referred to concerning the definition and the properties of the determinant of a linear operator A such that $\|A - I\|_1 < \infty$.

LEMMA 5. *The following relations hold:*

(30.a) $$\|A\| \leq \|A\|_1,$$

(30.b) $$\|AB\|_1 = \|(AA^*)^{1/2}(B * B)^{1/2}\|_1 \leq \|A\| \, \|B\|_1 \leq \|A\|_1 \|B\|_1,$$

(30.c) $$|\operatorname{Tr} A| = \|A\|_1,$$

(30.d) $$\det A = \lim_{n\to\infty} \det (Ae_p, e_q)^n_{p,q=1},$$

where $\|A - I\|_1 < 1$ *and* $\{e_p\}$ *is any orthonormal basis;*

(30.e) $$\det AB = \det A \det B \quad \textit{if} \quad \|A - I\|_1 < 1 \quad \textit{and} \quad \|B - I\|_1 < 1,$$

(30.f) $$\det (I + D) = \exp\left[\sum_{n=1}^{\infty} \frac{(-1)^n}{r} \operatorname{Tr}(D^n)\right] \quad \textit{if} \quad \|D\|_1 < 1.$$

With these facts in mind we shall discuss the properties of F_k.

Proof of theorem 1. It is clear by the definition that $\mathfrak{D}(F_k)$ is a linear space and F_k is a linear functional on $\mathfrak{D}(F_k)$.

Using lemma 4 and the fact that OVO^* has the same eigenvalues as V for any orthogonal transformation O, we can easily verify the invariance of $\mathfrak{D}(F_k)$ and F_k under translations and orthogonal transformations.

Let C be nearly isometric. Then we have $Cx = a + A \cdot x$, where A is nearly orthogonal. Applying Neumann's decomposition to A, we can write C as

(31) $$Cx = a + BOx$$

where O is orthogonal and B is a strictly positive definite symmetric operator with

(32) $$\|B - I\|_\alpha < \infty \qquad \text{for some } \alpha \text{ such that } 0 < \alpha < 1.$$

Since we have already proved the invariance under translations and orthogonal transformations, it is enough to discuss the case $C = B$ in (31).

Let $\{\beta_\nu\}$ be the eigenvalues of $B - I$. Then we have $\beta_\nu > -1$ and $\sum_\nu |\beta_\nu|^\alpha < \infty$. The second inequality implies that $0 < \gamma_* < 1 + \beta_\nu < \gamma^* < \infty$ with γ_* and γ^* independent of ν. Thus we have

(33) $$\|B^{1/n} - I\|_\alpha \leq \|B - I\|_\alpha < \infty \qquad \text{for every} \quad n = 1, 2, \cdots,$$

and

(34) $$\|B^{-2/n} - I\|_1 = \sum_\nu |(1 + \beta_\nu)^{-2/n} - 1| < 1 \qquad \text{for } n \text{ sufficiently large.}$$

If we can prove our theorem for $C = B^{1/n}$, then we can verify it for $C = B$ by applying $B^{1/n}$ n times and noticing $J(B) = \Pi\,(1 + \beta_\nu) = (\Pi\,(1 + \beta_\nu)^{1/n})^n = J(B^{1/n})^n$. Therefore, in order to prove that

$$f \in \mathfrak{D}(F_k) \Rightarrow f_B \equiv f(Bx) \in \mathfrak{D}(F_k) \quad \text{and} \quad F_k(f_B) = J(B)^{-1}F_k(f), \tag{35}$$

we can assume, in addition to (32), that

$$\|B^{-2} - I\| < 1. \tag{36}$$

Let $\{v_\nu\}$, $\{\beta_\nu\}$, and $\{w_\nu\}$ be the eigenvalues of V, $B - I$, and BVB respectively. By the definition of F_k, we can derive (35) easily from the fact that

$$\lim_V \frac{\prod_\nu \left[1 + \frac{w_\nu}{hi}\right]^{1/2}}{\prod_\nu \left[1 + \frac{v_\nu}{hi}\right]^{1/2}} = \prod_\nu (1 + \beta_\nu), \quad (= \det B). \tag{37}$$

To prove this, we shall consider the complexified Hilbert space $H_k = E_k + iE_k$ and denote the complexification of V and that of B with the same notations as we remarked before.

First, we shall derive the identity

$$\frac{\prod_\nu \left[1 + \frac{tw_\nu}{hi}\right]^{1/2}}{\prod_\nu \left[1 + \frac{tv_\nu}{hi}\right]^{1/2}} = \det B \exp\left\{\frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \operatorname{Tr}(D_t^n)\right\}, \tag{38}$$

where

$$D_t = \left(I + \frac{tV}{hi}\right)^{-1}(B^{-2} - I). \tag{39}$$

Using lemma 7, for $0 \leq t \leq 1$ we have

$$\begin{aligned}
\frac{\prod_\nu \left(1 + \frac{tw_\nu}{hi}\right)}{\prod_\nu \left(1 + \frac{tv_\nu}{hi}\right)} &= \frac{\det\left(I + \frac{tBVB}{hi}\right)}{\det\left(I + \frac{tV}{hi}\right)} = \frac{\det B \det\left(B^{-2} + \frac{tV}{hi}\right)\det B}{\det\left(I + \frac{tV}{hi}\right)} \\
&= (\det B)^2 \det\left[\left(I + \frac{tV}{hi}\right)^{-1}\left(B^{-2} + \frac{tV}{hi}\right)\right] \\
&= (\det B)^2 \det\,[I + D_t] \\
&= (\det B)^2 \exp\left\{\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \operatorname{Tr}(D_t^n)\right\},
\end{aligned} \tag{40}$$

where we should notice

$$\left\|B^{-2} + \frac{tV}{hi} - I\right\|_1 = \|B^{-2} - I\|_1 + \left\|\frac{tV}{hi}\right\|_1 < \infty, \tag{41}$$

$$\|D_t\|_1 = \left\| \left(I + \frac{t^2V^2}{h^2}\right)^{-1/2} |B^{-2} - I| \right\|_1, \qquad \text{by (30.b)}, \tag{42}$$

$$\leq \left\| \left(I + \frac{t^2V^2}{h^2}\right)^{-1/2} \right\| \, \|B^{-2} - I\|_1, \qquad \text{by (30.b)},$$

$$\leq \|B^{-2} - I\|_1 < 1,$$

and so

$$|\mathrm{Tr}\,(D_t^n)| \leq \|D_t^n\|_1 \leq \|D_t\|_1^n \leq \|B^{-2} - I\|_1^n, \qquad \text{by (30.b)}. \tag{43}$$

It follows from (40) that (38) holds with the sign + or − in front of det B. By our convention for $\sqrt{z}$ (section 1), the left side of (38) is continuous in $t \in [0, 1]$. The right side of (38) is also continuous in $t \in [0, 1]$, because the infinite series is convergent uniformly on $0 \leq t \leq 1$ by virtue of (42) and (43). Since the right side of (38) never vanishes, the $\pm$ sign remains unchanged as t moves from 0 to 1. But both sides of (38) are positive at $t = 0$ and so (38) holds for every $t \in [0, 1]$.

Setting $t = 1$, we have

$$\frac{\prod_\nu \left[1 + \frac{w_\nu}{hi}\right]^{1/2}}{\prod_\nu \left[1 + \frac{v_\nu}{hi}\right]^{1/2}} = \det B \exp\left\{\tfrac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{Tr}\,(D(V)^n)\right\}, \tag{44}$$

where $D(V) = (I + V/hi)^{-1}(B^{-2} - I)$.

Since we have, by (43),

$$\left|\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{Tr}\,(D(V)^n)\right| \leq \sum_{n=1}^{\infty} \|D(V)\|_1^n \tag{45}$$

$$\leq \|D(V)\|_1 \sum_{n=1}^{\infty} \|B^{-2} - I\|_1^{n-1}$$

$$\leq \frac{\|D(V)\|_1}{1 - \|B^{-2} - I\|_1},$$

the proof will be completed, if we prove

$$\lim_V \|D(V)\|_1 = 0. \tag{46}$$

Using (30.b), we shall evaluate $D(V)$ obtaining

$$\|D(V)\|_1 = \left\| \left(I + \frac{V^2}{h^2}\right)^{-1/2} |B^{-2} - I| \right\|_1 \tag{47}$$

$$\leq c_1 \left\| \left(I + \frac{V^2}{h^2}\right)^{-1/2} |B - I| \right\|_1 \quad (c_1 \equiv \|B^{-2}(B + I)\| < \infty)$$

$$\leq c_1 c_2 \left\| \left(I + \frac{V^2}{h^2}\right)^{-1/2} |B - I|^{1-\alpha} \right\| \quad (c_2 = \|\,|B - I|^\alpha\|_1 = \|B - I\|_\alpha < \infty).$$

Using the spectral decomposition of V and noticing $2(1 + \lambda^2) \geq (1 + \lambda)^2$, we have

$$\left\|\left(I + \frac{V^2}{h^2}\right)^{-1/2} x\right\| \leq \sqrt{2} \left\|\left(I + \frac{V}{h}\right)^{-1} x\right\|, \tag{48}$$

and so

$$\|D(V)\|_1 < \sqrt{2}\, c_1 c_2 \left\|\left(I + \frac{V}{h}\right)^{-1} |B - I|^{1-\alpha}\right\|. \tag{49}$$

Hence, it follows by lemma 3 that $V > \ell|B - I|$ implies that

$$\|D(V)\|_1 \leq \sqrt{2}\, c_1 c_2 \left\|\left(I + \frac{\ell}{h} |B - I|\right)^{-1} |B - I|^{1-\alpha}\right\|. \tag{50}$$

Using the spectral decomposition of B, we have

$$\|D(V)\|_1 \leq \sqrt{2}\, c_1 c_2 \sup_{\lambda \geq 0} \left(1 + \frac{\ell\lambda}{h}\right)^{-1} \lambda^{1-\alpha} \leq \sqrt{2}\, c_1 c_2 \left(\frac{h}{\ell}\right)^{1-\alpha}. \tag{51}$$

Taking $\ell = \ell(\epsilon)$ large enough, we have $\|D(V)\|_1 < \epsilon$ for $V > \ell(\epsilon)|B - I|$, which proves (46).

PROOF OF THEOREM 2. Setting

$$F(a, V) = \prod_\nu \sqrt{1 + \frac{v_\nu}{hi}} \int_{E_k} \exp\left(\frac{i}{2h} |x|^2\right) \int_{E_k} e^{i(x,y)} \mu(dy) N(dx\colon a, V), \tag{52}$$

and changing the order of integration, we have

$$F(a, V) = \int_{E_k} I(y\colon a, V) \mu(dy), \tag{53}$$

where

$$\begin{aligned} I(y\colon a, V) &= \prod_\nu \sqrt{1 + \frac{v_\nu}{hi}} \int_{E_k} \exp\left[\frac{i}{2h} |x|^2 + i(x, y)\right] N(dx\colon a, V) \\ &= \prod_\nu \sqrt{1 + \frac{v_\nu}{hi}} \int_{E_k} \exp\left[\frac{i}{2h} |x + a|^2 + i(x + a, y)\right] N(dx\colon 0, V). \end{aligned} \tag{54}$$

Let $\{e_\nu\}$ be the eigenvectors of V corresponding to $\{\lambda_\nu\}$ and write x_ν, y_ν, and a_ν for (x, e_ν), (y, e_ν), and (a, e_ν), respectively. Noticing that

$$\exp\left[\frac{i}{2h} |x + a|^2 + i(x + a, y)\right] = \prod_\nu \exp\left[\frac{i}{2h} (x_\nu + a_\nu)^2 + i(x_\nu + a_\nu) y_\nu\right] \tag{55}$$

and using lemmas 1 and 2, we can get

$$I(y\colon a, V) = \exp\left[\frac{h}{2i} \sum_\nu \frac{1}{v_\nu + hi} \left(y_\nu + \frac{a_\nu}{hi}\right)^2\right] \exp\left[\frac{h}{2i} |y|^2\right] \tag{56}$$

by usual computation. Therefore,

$$\begin{aligned} &\left|F(a, V) - \int \exp\left(\frac{h}{2i} |y|^2\right) \mu(dy)\right| \\ &\qquad \leq \int \exp\left[\frac{h}{2i} \sum_\nu \frac{1}{v_\nu + hi} \left(y_\nu + \frac{a_\nu}{h}\right)^2 - 1\right] |\mu|(dy), \end{aligned} \tag{57}$$

where $|\mu|$ is the absolute variation of μ.

Writing $A(y\colon a, V)$ for the inside of the bracket, we have

(58.a) $\quad \operatorname{Re} A(y\colon a, V) \le 0, \quad \text{and so} \quad |e^{A(y:a,V)} - 1| \le 2,$

and

(58.b)
$$|A(y\colon a, V)| \le h \sum_{\nu} \frac{1}{v_\nu + h}\left(y_\nu + \frac{a_\nu}{h}\right)^2, \qquad (\text{by } v_\nu + h \le 2|v_\nu + hi|),$$
$$\le h\left|(V + hI)^{-1/2}\left(y + \frac{a}{h}\right)\right|^2.$$

In order to prove our theorem, it is enough to find $V_0 = V_0(\epsilon)$ for every $\epsilon > 0$ such that the integral in (35) is less than ϵ for every $V > V_0$.

Since $|\mu|$ is a measure with $|\mu|(E_\infty) < \infty$, we have a compact subset K of E_∞ such that

(59)
$$|\mu|(E_\infty - K) < \frac{\epsilon}{4}$$

by a theorem due to Prohorov [6]. Take $\delta > 0$ such that $e^{2\delta h} - 1 < \epsilon/2$, or find a finite set $\{y', y'', \cdots, y^{(n)}\}$ such that every $y \in K$ is within the distance $h^{1/2}\delta$ from some $y^{(\nu)}$, and choose $V_0 \in \mathcal{V}$ such that

(60)
$$\left|V_0^{-1/2}\left(y^{(\nu)} + \frac{a}{h}\right)\right| < \delta, \qquad \nu = 1, 2, \cdots, n.$$

Such a V_0 can be constructed easily.

If $|y - y^{(\nu)}| < \delta h$, we have

(61)
$$\left|(V_0 + hI)^{-1/2}\left(y + \frac{a}{h}\right)\right| \le \left|(V_0 + hI)^{-1/2}\left(y^{(\nu)} + \frac{a}{h}\right)\right| + |(V_0 + hI)^{-1/2}(y - y^{(\nu)})|$$
$$\le \left|V_0^{-1/2}\left(y^{(\nu)} + \frac{a}{h}\right)\right| + h^{-1/2}|y - y^{(\nu)}| < 2\delta.$$

Hence, we have

(62)
$$\left|(V_0 + hI)^{-1/2}\left(y + \frac{a}{h}\right)\right| \le 2\delta \qquad \text{for} \quad y \in K.$$

Therefore,

(63)
$$\left|(V + hI)^{-1/2}\left(y + \frac{a}{h}\right)\right| \le 2\delta \qquad \text{for} \quad y \in K \quad \text{and} \quad V > V_0$$

by lemma 3, observing that $V > V_0$ implies

(64)
$$(Vx, x) + h(x, x) \ge (V_0x, x) + h(x, x) \ge h(x, x),$$
$$|(V + hI)^{1/2}x|^2 \ge |(V_0 + hI)^{1/2}x|^2 \ge h|x|^2.$$

Using (58.a) and (63), we have, for $V > V_0$,

(65)
$$\int |e^{A(y:a,V)} - 1|\mu(dy)$$
$$\le \int_K + \int_{E_\infty - K} \le (e^{2\delta h} - 1)\mu(K) + 2\mu(E_\infty - K) \le (e^{2\delta h} - 1) + \frac{\epsilon}{2} < \epsilon,$$

which completes the proof.

Since theorems 3, 3′, and 3″ can be easily proved, we shall omit the proof.

3. Application to the Feynman integral

In section 1 we formulated the Feynman integral in terms of our generalized measure F_k. We shall now carry out the computation in the three cases mentioned there.

Let $L_0^2[s, t]$ be the space of all square summable functions $x(\tau)$ on the interval $s \leq \tau \leq t$,

$$\int_s^t x(\tau)\, d\tau = 0. \tag{66}$$

$L_0^2[s, t]$ is a Hilbert space E_∞ with the usual norm in the space $L^2[s, t]$. Recalling the definition of the metric in Γ in section 1, we can see that

$$\begin{aligned} &\Phi: \Gamma \to L_0^2[s, t], \\ &(\Phi\gamma)(\tau) = \sqrt{m}\left[\gamma'(\tau) - \frac{b-a}{t-s}\right], \qquad s \leq \tau < t \end{aligned} \tag{67}$$

defines an isometric mapping from Γ onto $L_0^2[s, t]$. Introducing $e_{s,t,\tau} \in L_0^2[s, t]$ by

$$e_{s,t,\tau}(\sigma) = \begin{cases} \dfrac{t-\tau}{t-s}, & s \leq \sigma \leq \tau \leq t, \\ -\dfrac{\tau - s}{t-s}, & s \leq \tau \leq \sigma \leq t, \end{cases} \tag{68}$$

for $x \in L_0^2[s, t]$, we have the following relations:

$$\int_s^\tau x(\sigma)\, d\sigma = (e_{s,t,\sigma}, x), \tag{69.a}$$

$$\gamma(\tau) = \gamma_0(\tau) + \frac{1}{\sqrt{m}}(e_{s,t,\tau}, x), \qquad \gamma_0(\tau) = \frac{(t-\tau)a + (\tau - s)b}{t-s}, \tag{69.b}$$

$$\int_s^t m\gamma'(\tau)^2\, d\tau = |x|^2 + \frac{m(b-a)^2}{t-s}. \tag{69.c}$$

Therefore,

$$\mathcal{A}(\gamma) = \tfrac{1}{2}|x|^2 + \tfrac{1}{2}\frac{m(b-a)^2}{t-s} - \int_s^t m\cdot u\left[\gamma_0(\tau) + \frac{1}{\sqrt{m}}(e_{s,t,\tau}, x)\right] d\tau. \tag{69.d}$$

Let $\mathfrak{M}_\infty$ denote the class of complex measures of bounded absolute variation on $L_0^2[s, t]$ and $\mathfrak{F}\mathfrak{M}_\infty$ the class of all functions

$$g(x) = \mathfrak{F}\mu(x) \equiv \int_{L_0^2[s,t]} e^{i(x,y)}\mu(dy), \qquad \mu \in \mathfrak{M}_\infty \tag{70}$$

where $\mathfrak{F}$ denotes the Fourier transform. The following fact will be useful here:

$$g \in \mathfrak{F}\mathfrak{M} \Rightarrow \varphi(g) \in \mathfrak{F}\mathfrak{M}_\infty \qquad \text{for every entire function } \varphi; \tag{71}$$

in fact, if $g = \mathfrak{F}\mu$ and $\varphi(\xi) = \sum \alpha_n \xi^n$, then $\nu = \sum \alpha_n \mu^{*n}$, ($\mu^{*n} = n$ times convolution of μ) converges in the norm of total absolute variation and $\varphi(t) = \mathfrak{F}\cdot\nu$.

Case 1. Assume that $u(q) = \int_{-\infty}^{\infty} e^{iq\xi}\theta(d\xi)$, *where* θ *is a complex measure of bounded absolute variation on* $(-\infty, \infty)$.

First we shall prove that

$$\exp\left(\frac{i}{h}\,\mathfrak{A}(\gamma)\right) \in \mathcal{E}_\infty. \tag{72}$$

According to theorem 2, it is sufficient to prove that

$$\exp\left\{-\frac{im}{h}\int_s^t u\left[\gamma_0(\tau) + \frac{1}{\sqrt{m}}\,(e_{s,t,\tau}, x)\right] d\tau\right\} \in \mathfrak{F}\mathfrak{M}_\infty. \tag{73}$$

By virtue of (71), it is also enough to show that

$$\int_s^t u\left[\gamma_0(\tau) + \frac{1}{\sqrt{m}}\,(e_{s,t,\tau}, x)\right] d\tau \in \mathfrak{F}\mathfrak{M}_\infty. \tag{74}$$

The left side is

$$\int_s^t \int_{-\infty}^{\infty} \exp\left[i\left(\frac{\xi}{\sqrt{m}}\,e_{s,t,\tau}, x\right)\right] e^{i\xi\gamma_0(\tau)}\theta(d\xi)\cdot d\tau, \tag{75}$$

and so it is the Fourier transform of the measure $\mu \in \mathfrak{M}_\infty$,

$$\mu(\cdot) = \int_s^t \int_{-\infty}^{\infty} \delta\left(\cdot, \frac{\xi}{\sqrt{m}}\,e_{s,t,\tau}\right) e^{i\xi\gamma_0(\tau)}\theta(d\xi)\, d\tau, \tag{76}$$

$\delta(\cdot, y)$ being the δ-measure concentrated at y. Therefore, we shall have (74).

Using theorem 2 we can see that

$$\begin{aligned} G(t, b|s, a) &= \frac{\sqrt{m}}{\sqrt{2\pi hi(t-s)}} \int_\Gamma \exp\left(\frac{i}{h}\,\mathfrak{A}(\gamma)\right) F_\infty(d\gamma) \\ &= \frac{\sqrt{m}}{\sqrt{2\pi hi(t-s)}} \exp\left[-\frac{m(b-a)^2}{2hi(t-s)}\right] \sum_n \frac{1}{n!}\left(-\frac{im}{h}\right)^n \int_s^t \cdots \int_s^t Q_n\, d\tau_1 \cdots d\tau_n \end{aligned} \tag{77}$$

where

$$\begin{aligned} Q_n &= Q_n(\tau_1, \tau_2, \cdots, \tau_n; t, b, s, a) \\ &= \int_{-\infty}^{\infty} \cdots \int \exp\left\{i\sum_{\nu=1}^n \gamma_0(\tau_\nu)\xi_\nu + \frac{h}{2i}\int_s^t \left|\sum_{\nu=1}^n \frac{\xi_\nu}{\sqrt{m}}\, e_{s,t,\tau_\nu}(\sigma)\right|^2 d\sigma\right\}\theta(d\xi_1)\cdots\theta(d\xi_n). \end{aligned} \tag{78}$$

If $t - s$ is small, then

$$\begin{aligned} &G(t, b|s, a) \\ &= \frac{\sqrt{m}}{\sqrt{2\pi hi(t-s)}} \exp\left[-\frac{m(b-a)^2}{2hi(t-s)}\right]\left[1 - \frac{im}{h}\,(t-s)\,\frac{u(a)+u(b)}{2} + o(t-s)\right] \end{aligned} \tag{79}$$

where o does not depend on (a, b) as long as they stay in a compact set. Hence, it follows that if φ is a C_2-function with compact support, we have

$$\lim_{t\downarrow s} \frac{1}{t-s}\left[\int G(t, b|s, a)\varphi(a)\, da - \varphi(b)\right] = \frac{hi}{2m}\,\varphi''(b) - \frac{im}{h}\,u(b)\varphi(b). \tag{80}$$

We have a *composition rule*

$$G(t, b|s, a) = \int_{-\infty}^{\infty} G(t, b|u, c)G(u, c|s, a)\, dc \tag{81}$$

where the integral is to be understood in an improper sense, namely

$$\sqrt{2\pi hi} \int_{E_1} G(t, b|u, c)G(u, c|s, a)F_1(dc) \tag{82}$$

or

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} G(t, b|u, c)G(u, c|s, a)e^{-c^2/2n}\, dc. \tag{83}$$

The composition rule can be written in terms of F_∞:

$$\begin{aligned}\int_{\Gamma(t,b|s,a)} \exp\left(\frac{i}{h}\,\mathfrak{A}(\gamma)\right) F_\infty(d\gamma) \\ = \sqrt{\frac{m(t-s)}{(u-s)(t-u)}} \int_{E_1} \int_{\Gamma(t,b|u,c)} \exp\left(\frac{i}{h}\,\mathfrak{A}(\gamma_1)\right) F_\infty(d\gamma_1) \\ \times \int_{\Gamma(u,c|s,a)} \exp\left(\frac{i}{h}\,\mathfrak{A}(\gamma_2)\right) F_\infty(d\gamma_2)F_1(dc),\end{aligned} \tag{81'}$$

which we can get by applying theorem 3″ first to

$$C\colon \Gamma \to E_1; \qquad C(\gamma) = \gamma(u); \qquad J(C) = \sqrt{\frac{m(t-s)}{(u-s)(t-u)}}, \tag{84}$$

and then to

$$\begin{aligned} &C_1\colon \Gamma_c \equiv \{\gamma \in \Gamma : \gamma(u) = c\} \to \Gamma(u, c|s, a); \\ &C_1(\gamma) = \text{restriction of } \gamma \text{ onto } [s, u], \qquad J(C_1) = 1. \end{aligned} \tag{85}$$

Case 2. This case, $u(q) = c\cdot q$, $-\infty < c < \infty$, is not included in the first case, because $u(q)$ is unbounded. By a simple computation, we have

$$\mathfrak{A}(\gamma) = \tfrac{1}{2}|x|^2 + \tfrac{1}{2}\,\frac{m(b-a)^2}{t-s} - \tfrac{1}{2}mc(a+b)(t-s) - \sqrt{m}\, c(y, x), \tag{86}$$

where $y(\sigma) \equiv (1/2)(t+s) - \sigma$. Therefore, it is obvious that $\exp\{(i/h)\mathfrak{A}(\gamma)\} \in \mathcal{E}$, and we get

$$\begin{aligned} &G(t, b|s, a) \\ &= \frac{\sqrt{m}}{\sqrt{2\pi hi(t-s)}} \int_\Gamma \exp\left(\frac{i}{h}\,\mathfrak{A}(\gamma)\right) F_\infty(d\gamma) \\ &= \frac{\sqrt{m}}{\sqrt{2\pi hi(t-s)}} \exp\left\{-\frac{m}{2hi}\left[\frac{(b-a)^2}{t-s} - c(t-s)(a+b) - \frac{c^2}{12}\,(t-s)^3\right]\right\} \end{aligned} \tag{87}$$

observing that

$$|y|^2 = \int_s^t \left(\frac{t+s}{2} - \sigma\right)^2 d\sigma = \tfrac{1}{12}\,(t-s)^3. \tag{88}$$

Case 3. In the case where $u(q) = cq^2$, $c > 0$, we can prove that

$$e^{\frac{i}{\hbar}\mathfrak{A}(\gamma)} = f(Ax), \qquad \in \mathcal{E}_\infty; \qquad A \text{ is nearly isometric,} \tag{89}$$

and so this belongs to $\mathfrak{D}(F_\infty)$.

Applying (69.d) to our case, we have

$$\begin{aligned}\mathfrak{A}(\gamma) &= \tfrac{1}{2}|x|^2 + \tfrac{1}{2}\frac{m(b-a)^2}{t-s} - \int_s^t mc\left[\gamma_0(\tau) + \frac{1}{\sqrt{m}}(e_{s,t,\tau}, x)\right]^2 d\tau, \\ &= \tfrac{1}{2}|x|^2 + \tfrac{1}{2}\frac{m(b-a)^2}{t-s} - mcI_1 - 2\sqrt{m}\,cI_2 - cI_3,\end{aligned} \tag{90}$$

where

$$I_1 = \int_s^t \gamma_0(\tau)^2\, d\tau = \tfrac{1}{3}(t-s)(a^2+ab+b^2), \tag{91}$$

$$I_2 = \int_s^t \gamma_0(\tau)(e_{s,t,\tau}, x)\, d\tau = \int_s^t \gamma_0(\tau)\int_s^\tau x(\sigma)\, d\sigma\, d\tau, \tag{92}$$

$$I_3 = \int_s^t (e_{s,t,\tau}, x)^2\, d\tau = \int_s^t \left(\int_s^\tau x(\sigma)\, d\sigma\right)^2 d\tau. \tag{93}$$

Let us set

$$C_n(\tau) = \sqrt{\frac{2}{t-s}}\cos\frac{n\pi(\tau-s)}{t-s}, \qquad n = 0, 1, 2, \cdots, \tag{94}$$

$$S_n(\tau) = \sqrt{\frac{2}{t-s}}\sin\frac{n\pi(\tau-s)}{t-s}, \qquad n = 1, 2, \cdots. \tag{95}$$

Then each of $\{C_n(\tau), n = 0, 1, 2, \cdots\}$ and $\{S_n(\tau), n = 1, 2, \cdots\}$ is a complete orthonormal system in $L^2[s, t]$ and $\{C_n(\tau), n = 1, 2, \cdots\}$ is a complete orthonormal system in $L_0^2[s, t]$. Expanding x as

$$x(\tau) = \sum_{n=1}^{\infty} (x, C_n)\cdot C_n(\tau), \tag{96}$$

we can express I_2 and I_3 as follows.

$$I_2 = (y, x), \qquad y = \sum_{n=1}^{\infty} \frac{\sqrt{2}(t-s)^{3/2}(a + (-1)^{n-1}b)}{n^2\pi^2}\cdot C_n, \tag{97}$$

$$I_3 = (Bx, x), \qquad Bx = \sum_{n=1}^{\infty}\frac{(t-s)^2}{n^2\pi^2}(x, C_n)\cdot C_n. \tag{98}$$

Thus we have

$$\begin{aligned}\mathfrak{A}(\gamma) &= \tfrac{1}{2}|x|^2 + \tfrac{1}{2}\frac{m(b-a)^2}{t-s} - \frac{mc}{3}(t-s)(a^2+ab+b^2) \\ &\quad - 2\sqrt{m}\,c(y, x) - c(Bx, x) = \tfrac{1}{2}|Ax|^2 + \tfrac{1}{2}\frac{m(b-a)^2}{t-s} \\ &\quad - \frac{mc}{3}(t-s)(a^2+ab+b^2) + c(z, Ax),\end{aligned} \tag{99}$$

where

$$Ax = (I - 2cB)^{1/2}x = \sum_{n=1}^{\infty}\left(1 - \frac{2c(t-s)^2}{n^2\pi^2}\right)^{1/2}(x, C_n)\cdot C_n, \tag{100}$$

(101)
$$z = (I - 2cB)^{-1/2}y = \sum_{n=1}^{\infty}\left(1 - \frac{2c(t-s)^2}{n^2\pi^2}\right)^{-1/2}\frac{\sqrt{2}(t-s)^{3/2}(a + (-1)^{n-1}b)}{n^2\pi^2}C_n,$$

as far as $t - s < \pi/\sqrt{2c}$. Since we have

(102)
$$(\|A - I\|_\alpha)^\alpha = \mathrm{Tr}\,[|A - I|^\alpha] = \sum_{n=1}^{\infty}\left|\left(1 - \frac{2c(t-s)^2}{n^2\pi^2}\right)^{-1/2} - 1\right|^\alpha \sim \sum_{n=1}^{\infty} n^{-2\alpha},$$

we get $\|A - I\|_\alpha < \infty$ for every $\alpha > 1/2$. Therefore, it is obvious $\|A - I\|_{2/3} < \infty$, which shows that A is nearly isometric because A is a bounded symmetric operator. By theorems 1 and 2 we have $\exp((i/h)\mathfrak{A}(\gamma) \in \mathfrak{D}(F_\infty))$ and

(103)
$$\begin{aligned} G(t, b|s, a) &= \frac{\sqrt{m}}{\sqrt{2\pi hi(t-s)}}\int_\Gamma \exp\left(\frac{i}{h}\mathfrak{A}(\gamma)F_\infty(d\gamma)\right) \\ &= \frac{\sqrt{m}}{\sqrt{2\pi hi(t-s)}} J(A)^{-1} \\ &\quad\times \exp\left\{\frac{i}{2h}\frac{m(b-a)^2}{t-s} - \frac{imc}{3h}(t-s)(a^2 + ab + b^2) + \frac{h}{2i}\left|\frac{ic}{h}\right|^2 |z|^2\right\}. \end{aligned}$$

Using

(104)
$$\frac{\sin z}{z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right),$$

we have

(105)
$$J(A) = \prod_n\left(1 - \frac{2c(t-s)}{n^2\pi^2}\right)^{1/2} = \sqrt{\frac{\sin[\sqrt{2c}(t-s)]}{\sqrt{2c}(t-s)}}.$$

Observing that

(106)
$$\begin{aligned} |z|^2 &= \sum_{n=1}^{\infty}\left(1 - \frac{2c(t-s)^2}{n^2\pi^2}\right)^{-1}\frac{2(t-s)^3(a + (-1)^{n-1}b)^2}{n^4\pi^4} \\ &= \frac{t-s}{c}\sum_{n=1}^{\infty}\left(\frac{1}{n^2\pi^2 - 2c(t-s)^2} - \frac{1}{n^2\pi^2}\right)(a^2 + b^2) \\ &\quad + 2\,\frac{t-s}{c}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n-1}}{n^2\pi^2 - 2c(t-s)^2} - \frac{(-1)^{n-1}}{n^2\pi^2}\right)ab \end{aligned}$$

and using

(107)
$$\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2 - x^2} = \frac{1}{2x}\left(\frac{1}{x} - \cot x\right),$$

(108)
$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2\pi^2 - x^2} = \frac{1}{2x}\left(\frac{1}{\sin x} - \frac{1}{x}\right),$$

(109)
$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12},$$

we have

$$(110)\quad |z|^2 = \frac{t-s}{c}\left[\frac{1}{2\sqrt{2c}(t-s)}\left(\frac{1}{\sqrt{2c}(t-s)}\cot\left(\sqrt{2c}(t-s)\right)\right) - \tfrac{1}{6}\right](a^2+b^2)$$
$$+ 2\,\frac{t-s}{c}\left[\frac{1}{2\sqrt{c}(t-s)}\left(\frac{1}{\sin\,[2c(t-s)]} - \frac{1}{\sqrt{2c}(t-s)}\right) - \tfrac{1}{12}\right]ab.$$

Putting (105) and (110) in (103), we get

$$(111)\quad G(t, b|s, a)$$
$$= \sqrt{\frac{m\sqrt{2c}}{2\pi hi\sin\,[\sqrt{2c}(t-s)]}}\exp\left\{\frac{im\sqrt{2c}}{2h}\cdot\frac{(a^2+b^2)\cos\,[\sqrt{2c}(t-s)] - 2ab}{\sin\,[\sqrt{2c}(t-s)]}\right\}$$

for $t - s < \pi/\sqrt{2c}$.

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