

RANDOM SHIFTS OF STATIONARY PROCESSES

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1. Introduction

A variety of results concerning strongly stationary processes with smooth trajectories turns out to be derivable from a theorem, which extends the formula for changing the variable in the differential on the real axis to the case of measure spaces with a one-parameter group of measure preserving transformations.

The paper starts with the statement of three results, which will be shown in the end to be special cases of the main theorem. The middle part consists of the formulation and proof of this main theorem.

EXAMPLE I. Let $x_t, t \in R$ be an E -valued measurable stationary process, that is, a Lebesgue-measurable family of mappings from a measure space $(\Omega, \mathfrak{M}, P)$ into a measure space (E, B) . The σ -algebra \mathfrak{M} is generated by the x_t , and P is a σ -finite measure such that the shift transformations $S_u, u \in R$, leave P unchanged; that is,

$$(1) \quad \begin{aligned} PS_u(x_{t_1} \in B_1, \dots, x_{t_n} \in B_n) &= P(x_{t_1+u} \in B_1, \dots, x_{t_n+u} \in B_n) \\ &= P(x_{t_1} \in B_1, \dots, x_{t_n} \in B_n), \end{aligned}$$

and therefore,

$$(2) \quad PS_u(A) = P(A), \quad \text{for all } A \in \mathfrak{M}.$$

Assume now that E is the real axis and that for almost all $\omega, \omega \in \Omega, x(t, \omega) = x_t(\omega)$ is a differentiable function of t with derivative $\dot{x}(t, \omega)$. Let $t = h(x)$ be a differentiable function such that for almost all ω the graph in $R \times R$ of $t \rightarrow x(t, \omega)$ has exactly one point in common with the graph of $x \rightarrow h(x)$. This common point will be called $(t(\omega), x(t(\omega), \omega))$; $h(x(t(\omega), \omega)) = t(\omega)$. The most trivial example is $h = t^* = \text{const}$. If $\dot{x}(t, \omega)$ is bounded from above for all paths of a process, then every h with sufficiently small positive derivative would do.

Now let h be fixed. We write $(x(\omega), \dot{x}(\omega))$ which is short for $(x(t(\omega), \omega), \dot{x}(t(\omega), \omega))$. We consider the "shift by h " S_h defined as follows: $S_h(\omega)$ belongs to the set $\{x_u \in B\}$ if and only if $x(t(\omega) + u, \omega) \in B$. The result that interests us here is

$$(3) \quad (P)S_h = P(1 - \dot{x}(0, \omega)h'(x(0, \omega)))$$

or

$$(4) \quad (P(1 - \dot{x}(\omega)h'(x(\omega)))^{-1})S_h = P.$$

To clarify the notation, let us spell out the result. The image of P by S_h is absolutely continuous with respect to P , and the Radon-Nikodym density is the above function of $[x(0), \dot{x}(0)]$. This implies the following:

- (a) if the joint distribution of $(x(0, \omega), \dot{x}(0, \omega))$, or equivalently, that of $(x(t, \omega), \dot{x}(t, \omega))$ is $d\mu(x, \dot{x})$, then the joint distribution of $x(t(\omega), \omega), \dot{x}(t(\omega), \omega)$ is

$$(5) \quad (1 - \dot{x} \cdot h'(x)) d\mu(x, \dot{x});$$

- (b) if we associate a function $F(t, x, \dot{x})$ with every P -integrable f on Ω by the relation

$$(6) \quad \mathbf{E}(f|x(t, \omega), \dot{x}(t, \omega)) = F(t, x(t, \omega), \dot{x}(t, \omega)),$$

then

$$(7) \quad \mathbf{E}(f|x(\omega), \dot{x}(\omega)) = F(t(\omega), x(\omega), \dot{x}(\omega)).$$

Notice that $F(t, x, \dot{x})$ is determined by $d\mu(x, \dot{x})$, almost everywhere for every fixed t . Thus, for a fixed h , $F(t(\omega), x(\omega), \dot{x}(\omega))$ is usually nowhere on Ω determined by f .

However, if $f_s(\omega) = f(T_s\omega)$, then we can choose the associated functions F_s so that

$$(8) \quad F_s(t, x, \dot{x}) = F(t + s, x, \dot{x}).$$

The result started above is an oversimplification of the relation that will actually be derived below; namely, the set of all triples s, x, \dot{x} with

$$(9) \quad \mathbf{E}(f_s|x(\omega), \dot{x}(\omega)) \neq F(s + t(\omega), x(\omega), \dot{x}(\omega))$$

is a null set with respect to the product of Lebesgue measure and $d\mu$.

EXAMPLE II. Let x_t again be an E -valued measurable stationary process on $(\Omega, \mathfrak{M}, P)$. Assume that V is a strictly positive function on E such that for almost all ω ,

$$(10) \quad s(a, \omega) = \int_0^a V(x(t, \omega)) dt \text{ is finite and } s(a, \omega) \rightarrow \pm\infty \text{ as } a \rightarrow \pm\Gamma_{\pm\infty}.$$

The function $s(a, \omega)$ thus has a uniquely determined inverse $\tau(s, \omega)$,

$$(11) \quad s = \int_0^{\tau(s, \omega)} V(x(t, \omega)) dt.$$

For every s we consider the "shift by $\tau(s, \omega)$ " T_s defined as follows: $T_s(\omega)$ belongs to $\{x_u \in B\}$ if and only if $x(\tau(s, \omega) + u, \omega) \in B$. Clearly, $T_{s'} \cdot T_{s''} = T_{s'+s''}$. In this case our result on the images of P by the T_s is:

$$(12) \quad y(s, \omega) = x(\tau(s, \omega), \omega), s \in R \text{ is a stationary process on } (\Omega, \mathfrak{M}, P \cdot V(x(0, \omega))). \text{ The shifts of the } y\text{-process are the } T_s.$$

As one may notice, this implies that the stationary distribution of the y -process is $V(x) \cdot d\mu(x)$, if that of the x -process was $d\mu(x)$. The transformation described, when applied to the y -process with $V' = 1/V$, gives back the x -process.

EXAMPLE III. R. A. Dudley has studied families of probability measures

P_x^r , $x = (x_0, x_1, x_2, x_3)$, $v = (v_1, v_2, v_3)$, $|v|^2 < 1$, on the set Ω of all world lines with right-hand side continuous tangents everywhere in relativistic space-time with the following properties.

(i) If ω is the orbit of a mapping from $(-\infty, +\infty)$ into R^4 ,

$$(13) \quad t \rightarrow (x_0 + t, x_1(t, \omega), x_2(t, \omega), x_3(t, \omega)) = (x_0 + t, \vec{x}(t, \omega)),$$

then P_x^r is defined on the σ -algebra $\mathfrak{M}^x = \mathfrak{M}^{x_0}$ generated by the sets $\{\vec{x}(t, \omega) \in B\}$, where B is a Borel set in R^3 and $t \geq 0$.

(ii) For P_x^r -almost all ω , one has

$$(14) \quad (x_0, \vec{x}(0, \omega)) = x \quad \text{and} \quad \frac{d}{dt} \vec{x}(0, \omega) = v$$

“ P_x^r -almost all world lines pass through x with velocity v .”

(iii) For every $t \geq 0$ and every A in the σ -algebra generated by the x_u , $u \geq t$, the Markov property is satisfied; that is,

$$(15) \quad P_x^r \left(A \mid \vec{x}_t, \frac{d}{dt} \vec{x}_t \right) = P_y^w \left(\frac{\omega}{\omega} \right) (A)$$

where $y(\omega) = (x_0 + t, \vec{x}(t, \omega))$, $w(\omega) = (d/dt)\vec{x}(t, \omega)$.

(iv) If the action of the orthochronous inhomogeneous Lorentz group on the space Ω of world lines is defined in the obvious pointwise fashion, then the induced mappings on the measures on Ω act on the family $\{P_x^r\}$. “The Lorentz group transforms the P_x^r into each other.”

Dudley’s result is, roughly speaking, that there is a one-to-one correspondence between such families $\{P_x^r\}$ and the infinitely divisible (radial) probability measures on the Lobatchevski space.

It is another application of the main theorem of this paper to show the existence of a σ -finite measure P on Ω to every family $\{P_x^r\}$, which is invariant under all Lorentz transformations and such that the P_x^r are the conditional processes. It turns out that P restricted to the σ -algebra \mathfrak{M}^{x_0} can be defined by an integral: for $A \in \mathfrak{M}^{x_0}$,

$$(16) \quad P(A) = \iint P_{(x_0, \vec{x})}^v(A) d\mu(\vec{x}) (1 - |v|^2)^{-1/2} d\nu(v)$$

where $d\mu$ denotes the 3-dimensional Lebesgue measure and $d\nu$ the Haar measure on the Lobatchevski space. We will not give a full proof of the Lorentz invariance of P in this paper. However it may be of interest to see a link to the examples I and II. From (iii) and (iv) and the strong Markov property, for every P_x^r it follows that the process v with

$$(17) \quad v(\tau, \omega) = \frac{d}{dt} \vec{x}(t(\tau, \omega), \omega)$$

is a process with independent increments on the Lobatchevski space for every P_x^r , if $t(\tau, \omega)$ is determined as

$$(18) \quad \tau = \int_0^{t(\tau, \omega)} \left(1 - \left(\frac{d}{ds} \vec{x}(s, \omega) \right)^2 \right)^{1/2} ds$$

(τ measures the proper time on a world line, a quantity, which does not refer to any specific coordinate frame). Since the Haar measure $d\nu(v)$ is a stationary distribution for the v -process, according to example II,

$$(19) \quad (1 - |v_i|^2)^{-1/2} d\nu(v)$$

is a stationary distribution for the process $(d/dt) \vec{x}(t, \omega)$.

This is roughly the argument for the invariance of P under time shifts; the result of the first example can be used to establish the invariance under homogeneous Lorentz transformations; space shifts act in a trivial way on P . A detailed study of the Lorentz-invariant Markov random functions P will be given in a forthcoming paper. Dudley's work will appear in *Arkiv för Matematik*.

These results can be obtained by application of the following theorem.

THEOREM. *Let $(\Omega, \mathfrak{M}, P)$ be a σ -finite measure space, and $T_t, t \in R$ a one-parameter group of P -preserving transformations of Ω , acting measurably on Ω . Furthermore, let φ be a measurably invertible transformation of Ω such that there exists a real-valued function $t(\omega)$ with the property that $\varphi(\omega)$ belongs to a set $A \in \mathfrak{M}$ if and only if $T_t(\omega)$ belongs to A for $t = t(\omega)$.*

The following regularity assumption on $t(\omega)$ is made. For almost all ω , $t(s, \omega) = t(T_s \omega)$ as a function of s , $s \in R$, has only finitely many jumps on a bounded interval, and between two jumps $t(s, \omega)$ is absolutely continuous. The assertion is that if $v(\omega) = 1 + (d/ds)t(0, \omega)$, then φ maps the measure $P \cdot v$ onto P in the somewhat weakened sense that for every integrable g ,

$$(20) \quad \int g dP = \int g(T_s(\varphi(\omega))) \cdot |v(\omega)| dP$$

for Lebesgue-almost all s .

In sections 2, 3, 4, and 5 we shall study a couple of special cases of the theorem before we establish the proof for the general case in section 6. The considerations in sections 2 and 3 will be needed there. The arguments in sections 4 and 5 taken together are very close to giving a proof of the theorem. The gap lies in the irregularity of the decomposition of Ω into parts, where sections 4 and 5, respectively would apply, if the decomposition were nice. The argument in section 6 seems less direct. Besides, it requires a lot of preparation of topological nature. Since some of those topological concepts are also needed in sections 4 and 5, this preparation is presented after section 3.

In more complicated formulas we shall use a more convenient notation. For a P -integrable function f and for a measurable set A we shall write, $\int f dP = P \diamond f$ and $\int_A dP = P \diamond (A)$.

If f' is a measurable function on (Ω', \mathfrak{M}') and φ a measurable mapping from (Ω, \mathfrak{M}) into (Ω', \mathfrak{M}') , then $\varphi^*(f')$ denotes the \mathfrak{M} -measurable function f defined by $f(\omega) = f'(\varphi(\omega))$. Let $(P)\varphi_*$, or shortly, $(P)\varphi$, denote the image of P on (Ω', \mathfrak{M}') . We have by definition of this image

$$(21) \quad (P)\varphi \diamond f' = P \diamond \varphi^*(f')$$

for every f' for which $\varphi^*(f')$ is P -integrable.

If g is a nonnegative \mathfrak{M} -measurable function; then $P \cdot g$ denotes the measure defined by

$$(22) \quad P \cdot g \diamond (A) = \int_A g \, dP.$$

The assertion of the theorem is, roughly speaking,

$$(23) \quad (P \cdot |v|)\varphi_* = P;$$

the exact statement is, for every P -integrable g and for L -almost every s ,

$$(24) \quad (P \cdot |v|)\varphi_* \diamond T_s^*(g) = P \diamond T_s^*(g) = P \diamond g.$$

Throughout the paper L denotes the Lebesgue measure on the real line.

2. Transformations of the Lebesgue measure

The real axis with Lebesgue measure L is the measure space Ω . The translations form the group of L -preserving transformations $T_t(\omega) = \omega + t$, and φ is a piecewise monotone function and is absolutely continuous on every piece; it attains every real value exactly once. Clearly, $t(\omega) = \varphi(\omega) - \omega$ then satisfies the regularity conditions stated in the formulation of the theorem. Thus, in this special case, the assertion is

$$(25) \quad \left(L \cdot \left| 1 + \frac{d}{d\omega} t \right| \right) \varphi_* = \left(L \cdot \left| \frac{d\varphi}{d\omega} \right| \right) \varphi_* = L.$$

This is just the formula for changing the variable in the measure element dx , usually written as

$$(26) \quad \begin{aligned} |d(\varphi(x))| &= |\varphi'| \cdot dx, \\ \int f(\varphi(x)) \cdot |\varphi'(x)| \, dx &= \int f(y) \, dy. \end{aligned}$$

This formula is usually proven in elementary texts for continuous φ with continuous derivative. In our case φ' is measurable; therefore, by Egorov's theorem, we can find a countable union of intervals with a total length smaller than ϵ , such that φ' is continuous on its complement A_ϵ . We can assume, that A_ϵ contains none of the points of discontinuity of φ . There is a continuous function φ'_ϵ which coincides with φ' on A_ϵ , has constant sign in every interval of monotonicity for φ , and satisfies for every pair $\omega, \omega' \in A_\epsilon$,

$$(27) \quad \int_\omega^{\omega'} \varphi'_\epsilon \, d\omega = \int_\omega^{\omega'} \varphi' \, d\omega = \varphi(\omega') - \varphi(\omega).$$

We will now fix an arbitrary $\omega^* \in A_\epsilon$ and consider the function φ_ϵ , which does not depend on the choice of ω^* ,

$$(28) \quad \varphi_\epsilon(\omega) = \varphi(\omega^*) + \int_{\omega^*}^{\omega} \varphi'_\epsilon \, d\omega.$$

The function φ_ϵ coincides with φ on A_ϵ , has continuous derivative φ'_ϵ , and

defines a one-to-one mapping of the real axis onto itself. The classical formula yields, therefore,

$$(29) \quad (L|\varphi'_\epsilon|)\varphi_\epsilon = L$$

and

$$(30) \quad (L \cdot [A_\epsilon] \cdot |\varphi'_\epsilon|)\varphi = (L \cdot [A_\epsilon] \cdot |\varphi'_\epsilon|)\varphi_\epsilon = L \cdot [A'_\epsilon]$$

if $[A_\epsilon]$ denotes the indicator function of A_ϵ and $[A'_\epsilon]$ that of the φ -image of A_ϵ .

Since φ is absolutely continuous, the complement of A_ϵ is mapped into a set of measure less than $\delta(\epsilon)$, where $\delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Therefore, $L \cdot [A'_\epsilon]$ increases to L as ϵ decreases to 0. This shows that

$$(31) \quad (L|\varphi'|)\varphi = L.$$

This result has a local counterpart. If φ is a strictly monotone absolutely continuous mapping of an interval I onto an interval I' , then for every function h which vanishes outside I' ,

$$(32) \quad \int h dL = \int h(\varphi(\omega)) \cdot |\varphi'(\omega)| dL.$$

A second special case is well known in fluctuation theory.

3. Measure preserving transformations on denumerable decompositions

The measure space Ω is arbitrary; $\{T_i\}$ is an arbitrary group of measure-preserving transformations; φ is an invertible mapping of Ω onto Ω ; and there exists a decomposition of Ω into denumerably many B_i such that φ is given by a certain T_i on B_i . We shall prove that φ is measure preserving.

The φ -images of the B_i , called B'_i , form a decomposition of Ω . We show for a set A' contained in one of the B'_i , that $(P)\varphi \diamond (A') = P \diamond (A')$. This suffices, since any set is a countable union of such sets.

Set $A = \{\omega: \varphi(\omega) \in A'\}$; by definition of the φ -image of P ,

$$(33) \quad (P)\varphi \diamond (A') = P \diamond (A).$$

For $\omega \in A$, $\varphi(\omega) \in A' \subset B_i$ we have $T_i(\omega) = \varphi(\omega) \in A'$, and since T_i is invertible, only the ω in A satisfy $T_i(\omega) \in A'$. The transformation T_i is measure preserving; thus

$$(34) \quad (P)\varphi \diamond (A') = P \diamond (A) = (P)T_i \diamond (A') = P \diamond (A').$$

There is also a local counterpart of this result. Assume that φ is a mapping into Ω defined on a measurable subset $B \subset \Omega$, and for $\omega \in B$, $\varphi(\omega) = T_{t(\omega)}(\omega)$, where $t(\omega)$ attains on B only denumerably many distinct values in the group. Let g be a positive function which vanishes outside B . Let $h(\omega') = \sum g(\omega)$ where the sum is extended over all ω with $\varphi(\omega) = \omega'$. Assume that $h(\omega')$ is finite for P -almost all ω' . By the argument above we get in this case

$$(35) \quad (P \cdot g)\varphi = P \cdot h.$$

If g is the indicator function of a set B , then we shall speak of $P \cdot g$ as the re-

striction of P to the set B . Some preparations are necessary for the study of more complicated situations.

Let $\{T_u, u \in R\}$ be a one-parameter family of transformations of the measure space $(\Omega, \mathfrak{M}, P)$ which leave P unchanged. An \mathfrak{M} -measurable function f is called continuous, if $f(T_u(\omega))$ is a continuous function of u for every ω . A subset A of Ω is called open, if there exists a function $\alpha(\omega)$ which is strictly positive on A and \mathfrak{M} -measurable, and such that $T_t(\omega) \in A$ for $|t| < \alpha(\omega)$. An open set A is called finite if $P(A) < \infty$; a continuous function is said to have finite support if $\{\omega: f \neq 0\}$ is finite. We call the Borel field generated by the finite open sets \mathfrak{M}_0 . The mapping $T:(u, \omega) \rightarrow T_u(\omega)$ is continuous if Ω carries the topology described and $R \times \Omega$ the obvious product topology.

LEMMA 1. *If P is a measure on (Ω, \mathfrak{M}) , invariant under all $T_u, u \in R$, and Q another measure such that for every continuous P -integrable g with finite support $\int g dP = \int g dQ$, then for every \mathfrak{M} -measurable P -integrable f the set $\{s: \int f(T_s(\omega)) dQ \neq \int f(T_s(\omega)) dP\}$ has Lebesgue measure 0.*

PROOF. If $\int f dP = \int f dQ$ for every continuous f with finite support, then this relation holds also for every continuous integrable function. In fact, approximate f by f_ϵ with

$$(36) \quad f_\epsilon(\omega) = \begin{cases} f(\omega) - \epsilon & \text{for } f(\omega) > \epsilon, \\ 0 & \text{for } |f(\omega)| \leq \epsilon, \\ f(\omega) + \epsilon & \text{for } f(\omega) < -\epsilon. \end{cases}$$

The function f_ϵ is continuous, has finite support $\{\omega: |f(\omega)| > \epsilon\}$, and $|f - f_\epsilon|$ is dominated by $|f|$ and tends monotonely to 0 as ϵ tends to zero. Thus,

$$(37) \quad \int f dP \leftarrow \int f_\epsilon dP = \int f_\epsilon dQ \rightarrow \int f dQ.$$

If g is an arbitrary \mathfrak{M} -measurable P -integrable function, then g_ρ defined by

$$(38) \quad g_\rho(\omega) = \int_{-\infty}^{+\infty} g(T_s(\omega)) \rho(s) ds$$

is continuous for every continuous integrable ρ . Since P is invariant with respect to all T_s ,

$$(39) \quad \int g_\rho dP = \int \rho(s) ds \cdot \int g dP.$$

From our assumption follows

$$(40) \quad \int g_\rho dQ = \int \rho(s) ds \cdot \int g dP,$$

and therefore,

$$(41) \quad \int \left(\int g(T_s(\omega)) dQ - \int g(\omega) dP \right) \rho(s) ds = 0$$

for every integrable continuous ρ . Thus

$$(42) \quad \int g(T_s(\omega)) dQ = \int g(\omega) dP$$

for L -almost all s .

LEMMA 2. Let $(\Omega, \mathfrak{M}, P)$ and $(\Omega', \mathfrak{M}', P')$ be σ -finite measure spaces where Ω' is assumed to be a topological space and \mathfrak{M}' the σ -algebra generated by the continuous functions. Let φ and φ_n , $n = 1, 2, \dots$ be measurable mappings from Ω into Ω' and $(P)\varphi = P'$. We assume that

$$(43) \quad \varphi_n(\omega) \rightarrow \varphi(\omega) \quad \text{for } P\text{-almost all } \omega.$$

For every positive continuous bounded f' , for which $(P)\varphi \diamond f' < \infty$ and $(P)\varphi_n \diamond f < \infty$ for all sufficiently large n , one has

$$(44) \quad (P)\varphi_n \diamond f' \rightarrow (P)\varphi \diamond f'.$$

PROOF. The functions $\varphi_n^*(f') \rightarrow \varphi^*(f')$ P -almost everywhere, and

$$(45) \quad \sup |\varphi_n^*(f')| \leq \sup |f'|.$$

If χ is the indicator function of a set in \mathfrak{M} with finite P -measure, then we have by the dominated convergence theorem

$$(46) \quad P \cdot \chi \diamond \varphi_n^*(f') \rightarrow P\chi \diamond \varphi^*(f'),$$

$$(47) \quad \begin{aligned} P \cdot \chi \diamond \varphi^*(f') &\leq \liminf_{n \rightarrow \infty} P \cdot \chi \diamond \varphi_n^*(f') \\ &= \limsup_{n \rightarrow \infty} P \cdot \chi \diamond \varphi_n^*(f') \leq P \diamond \varphi^*(f'). \end{aligned}$$

If $P \diamond \varphi_n^*(f') < \infty$, then χ_0 can be chosen such that for every indicator function χ , $\chi \geq \chi_0$, the integral $P \cdot \chi \diamond \varphi_n^*(f')$ differs from $P \diamond \varphi_n^*(f')$ arbitrarily little. Therefore, the relations

$$(48) \quad \limsup_{n \rightarrow \infty} P \diamond \varphi_n^*(f') \leq P \diamond \varphi^*(f'),$$

$$(49) \quad P \cdot \chi \diamond \varphi^*(f') \leq \liminf_{n \rightarrow \infty} P \diamond \varphi_n^*(f')$$

hold for all indicator functions χ with $P \diamond \chi < \infty$. Since P is σ -finite, we have

$$(50) \quad \lim_{n \rightarrow \infty} (P)\varphi_n \diamond f' = (P)\varphi \diamond f'.$$

LEMMA 3. Let $\{T_u, u \in R\}$ be as above, A an open set, and $B \subset \Omega$ such that for P -almost every ω , $\{u: T_u(\omega) \in B \cap A\}$ has Lebesgue measure 0. Then $P \diamond (B \cap A) = 0$.

PROOF. Let $A_\delta = \{\omega: T_s(\omega) \in A \text{ for all } s \text{ with } |s| < \delta\}$. We determine the measure of

$$(51) \quad \{(u, \omega): u \in (-\delta, +\delta), T_u(\omega) \in B, \omega \in A\}$$

in the measure space $(R \times \Omega, B \times \mathfrak{M}, L \times P)$. Here B denotes the σ -algebra of all Lebesgue-measurable subsets of R , and L the Lebesgue measure.

By Fubini's theorem we obtain, on the one hand,

$$(52) \quad \int_{-\delta}^{+\delta} ((P)T_u \diamond (B \cap A_\delta)) du = 2\delta \cdot P \diamond (B \cap A_\delta).$$

On the other hand,

$$(53) \quad (L \times P) \diamond \{(u, \omega) : u \in (-\delta, +\delta), T_u(\omega) \in B, \omega \in A_\delta\} \\ \leq \int_A (L \diamond \{\omega : u \in (-\delta, +\delta), T_u(\omega) \in B\}) dP = 0.$$

Therefore, $P \diamond (B \cap A_\delta) = 0$ for all δ , and since A is open, $P \diamond (B \cap A) = 0$.

REMARKS. For an \mathfrak{M} -measurable f the following properties are equivalent:

- (a) $f = 0$, P -almost everywhere,
- (b) for P -almost every ω , $L \{u : f(T_u(\omega)) \neq 0\} = 0$,
- (c) $T^*(f) = 0$, $L \times P$ -almost everywhere.

EXAMPLE. Lemma 3 applies to the situation in our theorem as follows. The set of all ω where $\varphi(\omega)$ is discontinuous has P -measure zero. First, the mapping T of $R \times \Omega$ into Ω sending (u, ω) into $T_u(\omega)$ is continuous, if we have on $R \times \Omega$ the product of the topology on Ω and the usual one on R , and if the topology on Ω is as described above. The mapping τ from Ω into $R \times \Omega$ sending ω into $(t(\omega), \omega)$ is discontinuous in $B = \{\omega : t(T_s(\omega)) \text{ is discontinuous for } s = 0\}$. We have assumed that $\{s : t(T_s(\omega)) \text{ is discontinuous}\}$ is denumerable for P -almost every ω . The lemma tells us that $P(B) = 0$. We shall now show that φ maps null sets into null sets. Even more is true: if $\int_B |v(\omega)| dP = 0$ and $B' = \{\omega' : \varphi^{-1}(\omega') \in B\}$, then $P \diamond (B') = 0$.

LEMMA 4. *With the notation and the assumptions of the theorem the following holds: P is absolutely continuous with respect to $(P|v|\varphi)_*$, that is, $\int f(\varphi(\omega))|v(\omega)| dP = 0$ for an \mathfrak{M} -measurable positive f implies $\int f(\omega) dP = 0$.*

PROOF. Consider the mappings ψ and T from $R \times \Omega$ into Ω :

$$(54) \quad T : (u, \omega) \rightarrow T_u(\omega), \\ \psi : (u, \omega) \rightarrow (u + t(T_u\omega), \omega).$$

We have $T^* \circ \varphi^*(f) = \psi^* \circ T^*(f)$ for every f , and for an f as described above, $L \times P$ -almost everywhere,

$$(55) \quad 0 = T^*(|v|\varphi^*(f)) = |T^*(v)|T^* \circ \varphi^*(f) = |T^*(v)|\psi^* \circ T^*(f).$$

Therefore, $\psi^* \circ T^*(f) = 0$ holds $(L \times P)|T^*(v)|$ -almost everywhere.

Since $((L \times P)|T^*(v)|)\psi = L \times P$, as will be shown later on, we have $T^*(f) = 0$, $(L \times P)$ -almost everywhere, or $f = 0$, P -almost everywhere; q.e.d.

REMARK. Let

$$(56) \quad D'_\epsilon = \{\omega' : t(T_u(\varphi^{-1}(\omega'))) \text{ has a discontinuity for some } |u| \leq \epsilon\}.$$

Using lemmas 3 and 4, we conclude that if $(P|v|\varphi)_*$ restricted to $\Omega \setminus D'_\epsilon$ coincides with (P) restricted to $\Omega \setminus D'_\epsilon$ for every ϵ , then $(P|v|\varphi)_* = P$ on Ω .

PROOF. The support of the charge distribution $(P|v|\varphi)_* - P$ is contained in the closed set $\bigcap_{\epsilon > 0} D'_\epsilon = \{\omega : t(T_u(\varphi^{-1}(\omega))) \text{ is discontinuous in } u = 0\} \equiv \{\omega : \varphi^{-1}(\omega) \in B\}$.

We proved already that $P \diamond (B) = 0$, and from lemma 4 it follows that $P \diamond (\bigcap_{\epsilon > 0} D'_\epsilon) = 0$. Therefore,

$$(57) \quad (P \cdot |v|\varphi)_* \diamond \bigcap_{\epsilon > 0} D'_\epsilon = 0 = (P) \diamond \bigcap_{\epsilon > 0} D'_\epsilon.$$

4. Further approximations

We proceed now to more complicated special cases of the theorem. We add an assumption which is one less restrictive than that in section 3, namely, $v(\omega) = 1$, P -almost everywhere. In this case we can approximate φ by mappings φ_n of the type considered in section 3. The φ_n are now not quite invertible:

$$(58) \quad \varphi_n(\omega) = T_{t_n(\omega)}(\omega),$$

where $2^n t_n(\omega)$ is the largest integer less than or equal to $2^n t(\omega)$. The function t_n takes only denumerably many values. Therefore, $(P)\varphi_n = P \cdot h_n$ where $h_n(\omega')$ is the number of ω with $\varphi_n(\omega) = \omega'$. We show, that $h_n(\omega') = 1$ if $t(T_u(\varphi^{-1}(\omega)))$ is continuous for $|u| \leq 2^{-n}$. Therefore the signed measure $(P)\varphi_n - P$ has support contained in the set $\{\omega: t(T_u(\varphi^{-1}(\omega))) \text{ is discontinuous for some } |u| < 2^{-n}\}$. On the other hand, $(P)\varphi_n \rightarrow (P)\varphi$ in the weak sense on $\Omega \setminus \cap D_n$. In fact, $\varphi_n(\omega) \rightarrow \varphi(\omega)$ and $(P)\varphi_n \diamond (\Omega \setminus D'_n) \cap B < \infty$ if B is a finite open set. The convergence relations $(P)\varphi \leftarrow (P)\varphi_n = P \cdot h_n \rightarrow P$ weakly on $\Omega \setminus \cap D'_n$ imply $(P)\varphi \diamond T_s(g) = P \diamond g$ for every P -integrable g and for L -almost all s , as proved in lemma 1.

The proof for $h_n(\omega') = 1$ for $\omega' \in \Omega \setminus D'_n$ is as follows. If $\varphi(\omega) = \omega'$ and if $t(T_u(\omega))$ has no jump for a $u \in (0, 2^{-n})$, then there exists an s , $0 \leq s < 2^{-n}$ such that $\varphi_n(T_s(\omega)) = \omega'$. If $\varphi_n(\omega^*) = \omega'$ and $t(u, \omega^*)$ has no jump for a $u \in (-2^{-n}, 0)$, then there exists an $s \in (-2^{-n}, 0)$ such that $\varphi(T_s(\omega^*)) = \omega'$.

In any case, to every $\omega' \notin D'_n$ there exists a unique ω with $\varphi_n(\omega) = \omega'$. Thus, $h_n(\omega') = 1$.

The argument actually gives somewhat more. If Ω^* is an open part of Ω such that $v(\omega^*) = 1$ for $\omega^* \in \Omega^*$, then the restriction of $(P)\varphi$ to the φ -image of Ω^* equals the restriction of P to this set.

5. The use of local cross-sections

To the assumptions in the formulation of the theorem, we now add one which contradicts the one in section 4 and certainly is very unnatural for the problem. It guarantees, however, in a very nice way the existence of local cross-sections through the orbits $\{T_s(\omega); s \in R\}$ everywhere and allows the application of the argument of section 2 through the use of Fubini's theorem. We assume that

$$(59) \quad \frac{d}{ds} t |(s, \omega)| = |v(T_s(\omega)) - 1| \geq \rho^{-1} \quad \text{for all } \omega, s,$$

where ρ is an arbitrary but fixed number. For convenience, we assume that $\rho > 1$. The set $\Omega \setminus D'_{\epsilon(\rho+1)}$ can be covered by the φ -images of denumerably many sets of the form

$$(60) \quad A_{t_0} = \{\omega: |t(\omega) - t_0| < \epsilon\} \cap \{\omega: t(T_u\omega) \text{ is continuous for } |u| < (\rho+1)\epsilon\} \\ \cap \{\omega: \omega = T_u(\omega_0) \text{ for an } \omega_0 \text{ with } t(\omega_0) = t_0 \text{ and } |u| < (\rho-1)\epsilon\}.$$

We study P on sets of the form

$$(61) \quad B_{t_0} = \{\omega: |t(\omega) - t_0| < \epsilon\} \cap \{\omega: t(T_u\omega) = t_0 \text{ for some } |u| < \rho \cdot \epsilon\} \\ \cap \{\omega: t(T_u\omega) \text{ is continuous for } |u| < \rho \cdot \epsilon\}.$$

For an ω in B there exists exactly one ω_0 and exactly one s , $|s| \leq \epsilon \cdot \rho$, such that $\omega = T_s(\omega_0)$ and $t(\omega) = t_0$. We write $\omega_0 = p(\omega)$, $s = q(\omega)$.

The pair (p, q) maps B in a one-to-one fashion into the Cartesian product $C \times R$, where C is the "cross-section" $C = \{\omega: t(\omega) = t_0\}$. If $\omega, \omega' \in B$ and $\omega' = T_s(\omega)$ for a $s \in (-\epsilon\rho, +\epsilon\rho)$, then $p(\omega) = p(\omega')$ and $q(\omega') = q(\omega) + s$.

Furthermore, B is open, so that for every $\omega \in B$ the set $\{s: T_s(\omega) \in B\}$ is an open interval, whose length is by the way smaller than $2\epsilon\rho$. Put

$$(62) \quad B' = \{(\omega_0, s): T_s(\omega_0) \in B\}, \\ A' = \{(\omega_0, s): T_s(\omega_0) \in A\}.$$

The mapping $i = (p, q)$ establishes an isomorphism of B and B' , which is continuous, if the topology on B' is the product of the discrete topology on C with the usual one on R .

Let B'_δ be the set of all (ω_0, s) such that $(\omega_0, s + u) \in B'$ for all $|u| < \delta$. For a function g with support in B'_δ the functions g_u defined by $g_u(\omega_0, s) = g(\omega_0, s + u)$ has support in B' . We look at the i -image of P and find $(P \cdot \chi_B)i \diamond g_u = (P \cdot \chi_B)i \diamond g$, since $i^*(g_u) = T_u^* \circ i^*(g)$. With the abbreviation $(P)i = \tilde{P}$, this can be written as

$$(63) \quad \int_B g(\omega_0, s) d\tilde{P} = \int_B g(\omega_0, s + u) d\tilde{P} \quad \text{for all } |u| < \delta.$$

From this equation, which holds for every δ and every g satisfying the corresponding conditions on the support, we are going to derive that the restriction of \tilde{P} to B' is the restriction of a product measure, whose second factor is a Lebesgue measure, and whose first factor Q is a certain measure on the cross-section $C = \{\omega: t(\omega) = t_0\}$,

$$(64) \quad (P \cdot \chi_B)i = (Q \times L) \cdot \chi_{B'}.$$

In fact, if f defined on C has support contained in the set of ω_0 's such that $(\omega_0, a) \in B'_\delta$ and $(\omega_0, b) \in B'_\delta$ for a certain pair $a < b$, then F defined on B' by $F(\omega_0, s) = f(\omega_0) \cdot h(s)$ has support in B' if we assume h to be a Lebesgue-integrable function with support in $(a - \delta, b + \delta)$. Fix f and consider the linear functional

$$(65) \quad L(h) = \int F d\tilde{P} = \int f(\omega_0) \cdot h(s) d\tilde{P}(\omega_0, s).$$

If the support of h is contained in (a, b) , then the support of $h_u, h_u(s) = h(s + u)$ is contained in $(a - \delta, b + \delta)$ for all $|u| < \delta$. We conclude from the relation above that $L(h_u) = L(h)$ for $|u| \leq \delta$, since Lebesgue measure is the only translation invariant measure on an interval

$$(66) \quad \int F d\tilde{P} = \int f(\omega_0)h(s) d\tilde{P} = \int h(s) ds \cdot \int f(\omega_0) dQ$$

with a certain measure Q .

The set B' is open and the conclusion holds for every $\delta > 0$. Thus, the measures $\tilde{P} \cdot \chi_{B'}$ and $(Q \times L)\chi_{B'}$ coincide on the σ -algebra generated by the functions of the form $F(\omega_0, s) = f(\omega_0) \cdot h(s)$, which have the property that $i^*(F)$ is \mathfrak{M} -measurable. This σ -algebra includes the one generated by the continuous functions.

After this study of P near the cross-section C with the help of the isomorphism $i = (p, q)$, we look closer at its φ -image, on the one hand, and its T_{t_0} -image on the other. We put

$$\begin{aligned}
 i^{-1}(\omega_0, s) &= \omega \in B, \\
 V(\omega_0, s) &= v(\omega) = v(T_s \omega_0), \\
 (67) \quad g(\omega_0, s) &= \int_0^s v(\omega_0, u) \, du = s + t(T_s \omega_0) - t(\omega_0), \\
 \psi(\omega_0, s) &= T_{t_0} \circ i^{-1}(\omega_0, s), \\
 \chi(\omega_0, s) &= \varphi \circ i^{-1}(\omega_0, s),
 \end{aligned}$$

and have

$$(68) \quad \chi(\omega_0, s) = T_{g(\omega_0, s)}(\varphi(\omega_0)) = \psi(\omega_0, g(\omega_0, s)).$$

In fact,

$$(69) \quad t(\omega) - t_0 = \int_0^{q(\omega)} \frac{dt}{du} (T_u \omega_0) \, du = -q(\omega) + \int_0^{q(\omega)} v(\omega_0, u) \, du,$$

$$\begin{aligned}
 (70) \quad \varphi(\omega) &= T_{t(\omega)}(\omega) = T_{t(\omega)-t_0} \circ T_{t_0}(\omega) \\
 &= T_{t(\omega)-t_0} \circ T_{q(\omega)} \circ T_{t_0}(p(\omega)) \\
 &= T_{g(\omega_0, s)} \circ T_{t_0}(p(\omega)).
 \end{aligned}$$

For every fixed ω_0 , $g(\omega_0, s)$ is an absolutely continuous monotone function on the interval of all s for which $(\omega_0, s) \in B'$. We proved in section 2

$$(71) \quad \int h(g(\omega_0, s)) |v(\omega_0, s)| \, ds = \int h(u) \, du$$

if g maps in a one-to-one fashion onto an interval containing the support of h .

Remark now that if h is a function on Ω such that $\varphi^*(h)$ has support in $A \subset B$, then $T_{t_0}^*(h)$ has support in B . In fact, the φ -image ω' of an $\omega \in A$, has the form $\omega' = T_{t_0+s}(\omega)$ with a certain s , $|s| < \epsilon$; since $\omega = T_u(\omega_0)$ with $|u| < (\rho - 1) \cdot \epsilon$ we have $\omega' = T_{t_0+s+u}(\omega_0)$ and $|s+u| < \rho \cdot \epsilon$; thus $\omega' = T_{t_0}(\omega'')$ with a $\omega'' \in B$. This proves $\{\omega'' : h(T_{t_0} \omega'') \neq 0\} \subset B$, if $\{\omega' : h(\varphi(\omega')) \neq 0\} \subset A$.

Since denumerably many sets of the type A are sufficient to cover Ω up to a null set, in order to prove $(P|v|)\varphi = P$, it suffices to show that

$$(72) \quad (P \cdot |v|) \diamond \varphi^*(h) = P \diamond T_{t_0}^*(h)$$

for all functions h of the type just described.

We have

$$\begin{aligned}
(73) \quad (P \cdot |v|) \diamond \varphi^*(h) &= \int_{\mathbf{B}} h(\varphi(\omega)|v(\omega))| dP \\
&= \int_{\mathbf{B}'} h(\chi(\omega_0, s)) \cdot |v(\omega_0, s)| dQ(\omega_0) ds \\
&= \int_{\mathbf{C}} \int_{\mathbf{R}} h(\psi(\omega_0, g(\omega_0, s))) \cdot |v(\omega_0, s)| ds dQ(\omega_0) \\
&= \int_{\mathbf{C}} \int_{\mathbf{R}} h(\psi(\omega_0, u)) du dQ \\
&= \int_{\mathbf{B}'} h(\psi(\omega_0, s)) d\tilde{P}(\omega_0, s) \\
&= (P)i \diamond \psi^*(h) \\
&= P \diamond T_{i_0}^*(h).
\end{aligned}$$

Thus $(P|v|)\varphi$ and P coincide on the σ -algebra generated by the continuous functions.

The argument given in this section shows that in the most general case for which the theorem was formulated, the restriction of $P|v|$ to the part of Ω where $v \neq 1$ is mapped by φ onto a restriction of P . After having studied case III, one would perhaps not expect that the part of Ω where $v = 1$ should make necessary all of the somewhat indirect reasoning to be given now. However, I was unable to combine the arguments in sections 4 and 5 properly.

6. The general theorem

In the general case we approximate, as we did in section 4, the mapping φ by mappings φ_n ; $\varphi_n = T_{t_n(\omega)}(\omega)$ and $t_n(\omega)$ is a measurable function attaining only denumerably many values. According to section 3, to such a mapping φ_n there corresponds a real function h_n on Ω , $h_n(\omega') = \sum |v(\omega)|$ where the sum is extended over all ω with $\varphi_n(\omega) = \omega'$, for which the equality $(P \cdot |v|)\varphi_n = P \cdot h_n$ holds.

The proof that $(P \cdot |v|)\varphi_n$ tends to $(P \cdot |v|)\varphi$ in the weak sense, will be an easy consequence of lemma 2. The convergence $P \cdot h_n \rightarrow P$ will be established indirectly. It will be shown that it is equivalent to the convergence $(L \times P)H_n \rightarrow L \times P$ where the function H_n on $R \times \Omega$ is $T^*(h_n)$. On the other hand, this convergence will be shown to be equivalent to $((L \times P)T^*(|v|))\psi_n \rightarrow (L \times P)$, where ψ_n is the mapping sending (u, ω) into $(u + t_n(T_u\omega), \omega)$; this ψ_n clearly satisfies $T(\psi_n(u, \omega)) = \varphi_n(T_u\omega)$, and $(L \times P)T^*(|v|)\psi_n \rightarrow ((L \times P)T^*(|v|))\psi$ follows again easily from lemma 2. Also $((L \times P) \cdot T^*(|v|))\psi = L \times P$ will be derived from the result in section 2. It is convenient to arrange the conclusions in a different order.

(a) Let ψ be a one-to-one mapping of the real axis satisfying the continuity properties listed in section 2. We proved there that for every Lebesgue-integrable function g , $\int g(\psi(u))|\psi'(u)| du = \int g(s) ds$. On the other hand, with $t(u) = \psi(u) - u$ and $\psi_n(u) = u + t_n(u)$ we have $(P \cdot |\psi'|)\psi_n = P \cdot H_n$ with $H_n(u) = \sum |\psi'(s)|$ where the sum is extended over all s with $\psi_n(s) = u$.

The measures $(P|\psi'|)\psi_n$ converge to $(P|\psi')\psi$ in the weak sense. In fact, the conditions of lemma 2 are satisfied: $\psi_n(u) \rightarrow \psi(u)$ for every u , and for an arbitrary interval the $(P \cdot |\psi'|)\psi_n$ measure differs from the $(P \cdot |\psi'|)\psi$ measure, that is, the Lebesgue measure by at most 2^{-n} . We have, therefore, for every continuous function g with compact support

$$(74) \quad \int g(\psi_n(u)) \cdot |\psi'(u)| \, du \rightarrow \int g(\psi(u)) \cdot |\psi'(u)| \, du = \int g(s) \, ds,$$

or equivalently,

$$(75) \quad P \cdot H_n \diamond g = (P \cdot |\psi'|)\psi_n \diamond g \rightarrow (P \cdot |\psi'|)\psi \diamond g = P \diamond g.$$

(b) Let φ be a mapping satisfying the conditions of the theorem, and let t, t_n, φ_n be defined as above. We define the mappings ψ, ψ_n of $R \times \Omega$ into itself as

$$(76) \quad \begin{aligned} \psi(u, \omega) &= (u + t(T_u\omega), \omega), \\ \psi_n(u, \omega) &= (u + t_n(T_u\omega), \omega). \end{aligned}$$

For P -almost every ω the mapping $u \rightarrow u + t(T_u\omega)$ satisfies the conditions discussed in section 2. In fact, it is one-to-one and onto. Further, $u + t(T_u\omega) = v + t(T_v\omega)$ implies

$$(77) \quad \varphi(T_u\omega) = \varphi(T_v\omega), \quad T_u(\omega) = T_v(\omega), \quad t(T_u\omega) = t(T_v\omega),$$

and finally $u - v = -t(T_u\omega) + t(T_v\omega) = 0$. For a given s , we find u such that $s = u + t(T_u\omega)$. If $\varphi^{-1}(T_s\omega) = T_v(\omega)$, then

$$(78) \quad T_{v+t(T_v\omega)}(\omega) = T_s(\omega),$$

and for $u = v - (v + t(T_v\omega) - s)$ we have

$$(79) \quad T_u(\omega) = T_v \circ T_{-(v+t(T_v\omega))+s}(\omega) = T_v(\omega),$$

and therefore, $t(T_u\omega) = t(T_v\omega)$. Obviously,

$$(80) \quad s = v + t(T_v\omega) - (v + t(T_v\omega) - s) = t(T_v\omega) + u = u + t(T_u\omega).$$

The continuity assumption on φ stated in the premises of the theorem implies, according to (a), that for P -almost every ω ,

$$(81) \quad ((L \times \delta_\omega) \cdot |v|)\psi = L \times \delta_\omega.$$

Here $L \times \delta_\omega$ denotes the product of the Lebesgue measure and the unit measure concentrated in $\omega \in \Omega$:

$$(82) \quad V(u, \omega) = \frac{d}{du} (u + t(T_u\omega)) = v(T_u\omega).$$

Let g be a $B \times \mathfrak{M}$ -measurable function such that $g(s, \omega) = 0$ if either $|s| > L$ or $\omega \notin A$; here L is an arbitrary constant and A an \mathfrak{M} -measurable subset of Ω with finite P -measure. We can then apply Fubini's theorem

$$(83) \quad \begin{aligned} (L \times P) \diamond g &= \int dP (L \times \delta_\omega) \diamond g \\ &= \int dP ((L \times \delta_\omega)|v|)\psi \diamond g = ((L \times P)|v|)\psi \diamond g. \end{aligned}$$

Thus we have proved

$$(84) \quad ((L \times P)|v|)\psi = L \times P \quad \text{on } B \times \mathfrak{M}.$$

(c) By the argument in (a), with the definition of $H_n(s, \omega)$ as given there, we have for P -almost every ω ,

$$(85) \quad (L \times \delta_\omega)H_n = ((L \times \delta_\omega)|v|)\psi_n \rightarrow L \times \delta_\omega \quad \text{weakly.}$$

By Fubini's theorem and the dominated convergence theorem we obtain the following: if g is a $B \times \mathfrak{M}$ -measurable bounded function, which is partially continuous in the first argument and has finite support in the sense described in (b), then

$$(86) \quad (L \times P)H_n \diamond g \rightarrow (L \times P) \diamond g.$$

In fact, in

$$(87) \quad \int dP ((L \times \delta_\omega)H_n \diamond g)$$

the integrand converges almost everywhere, vanishes outside a fixed set A of finite measure, and is bounded by an integrable function. If $0 \leq g \leq C$, then

$$(88) \quad 0 \leq (L \times \delta_\omega)H_n \diamond g = ((L \times \delta_\omega)|v|)\psi_n \diamond g \leq (L + 2^{-n})C.$$

(d) From the property of $(L \times P) \cdot H_n$ just proved, a property of $P \cdot h_n$ will be derived. Namely, H_n depends on $T_u(\omega)$ rather than on the pair (u, ω) , that is, $H_n = T^*(h_n)$. In fact,

$$(89) \quad \begin{aligned} h_n(T_u\omega) &= \sum |v(T_s\omega)| \text{ summed over all } s \text{ with } \varphi_n(T_s\omega) = T_u(\omega), \\ H_n(u, \omega) &= \sum |v(s, \omega)| \text{ summed over all } s \text{ with } \psi_n(s, \omega) = (u, \omega). \end{aligned}$$

Since $T(\psi_n(s, \omega)) = \varphi_n(T_s\omega)$, we have to show that the number of different $T_s\omega$ with $\varphi_n(T_s\omega) = T_u\omega$ is equal to the number of different s with the property that $\psi_n(s, \omega) = (u, \omega)$.

(α) Let $T_{s_i}(\omega)$ be pairwise unequal and all $\varphi_n(T_{s_i}\omega)$ be equal, $i = 1, 2, \dots, k$. We construct pairwise different s'_i such that $T_{s'_i}(\omega) = T_{s_i}(\omega)$ and all $s'_i + t_n(T_{s'_i}\omega)$ are equal. For simplicity of notation we write $t(u, \omega)$ for $t(T_u\omega)$. Put

$$(90) \quad s'_i = s_i + (s_1 - s_i + t_n(s_1, \omega) - t_n(s_i, \omega)).$$

Then we have $T_{s'_i}(\omega) = T_{s_i}(\omega)$, since

$$(91) \quad T_{s_1 - s_i + t_n(s_1, \omega) - t_n(s_i, \omega)}(\omega) = \omega.$$

Therefore,

$$(92) \quad t_n(s'_i, \omega) = t_n(s_i, \omega) \quad \text{and} \quad s'_i = s_1 + t_n(s_1, \omega) - t_n(s'_i, \omega).$$

(β) Let the $s_i + t_n(s_i, \omega)$ all be equal and let the s_i pairwise unequal $i = 1, 2, \dots, k$. We show that the $T_{s_i}(\omega)$ are pairwise unequal. In fact $T_{s_i}(\omega) = T_{s_k}(\omega)$ implies $t_n(s_i, \omega) = t_n(s_k, \omega)$, and $0 = t_n(s_i, \omega) - t_n(s_k, \omega) = s_k - s_i$; therefore $i = k$. However, the φ_n -images of the $T_{s_i}(\omega)$ coincide.

Statement (α) shows that $h_n(T_u\omega) \leq H_n(u, \omega)$ and (β) shows that $h_n(T_u\omega) \geq H_n(u, \omega)$.

(e) Consider a function on $R \times \Omega$ of the form $F \cdot G$ where $F(u, \omega) = f(T_u \omega)$ with a bounded continuous function f on Ω with finite support, and $G(u, \omega) = g(u)$ with a continuous function g on R with compact support. The function $F \cdot G$ satisfies the conditions which guarantee, according to (e), the convergence

$$(93) \quad (L \times P)H_n \diamond F \cdot G \rightarrow (L \times P) \diamond F \cdot G.$$

On one hand,

$$(94) \quad (L \times P) \diamond F \cdot G = \int_R du g(u) \cdot \int_{\Omega} f(T_u \omega) dP = \int_R du g(u) \cdot \int_{\Omega} f dP;$$

on the other hand,

$$(95) \quad \begin{aligned} (L \times P)H_n \diamond F \cdot G &= \int_R du g(u) \cdot \int_{\Omega} h_n(T_u \omega) \cdot f(T_u \omega) dP \\ &= \int_R du g(u) \int_{\Omega} f \cdot h_n dP. \end{aligned}$$

Therefore, $P \cdot h_n \diamond f \rightarrow P \diamond f$ for every continuous function f with finite support. For such functions we have

$$(96) \quad P \cdot h_n \diamond f = (P|v|)\varphi_n \diamond f \rightarrow (P \cdot |v|)\varphi \diamond f.$$

Lemma 1 yields the following conclusion: for an arbitrary P -integrable function g , the equality

$$(97) \quad (P \cdot |v|)\varphi \diamond T_s^*(g) = P \diamond g$$

holds for L -almost every s . The theorem is thus completely proved.

7. Applications

We now show how the theorem applies to the situations sketched in the beginning. Let $(\Omega, \mathfrak{M}, P, x_t, E, B)$ be a strongly stationary measurable process with values in E ; the shifts are called T_t . We assume h to be a real-valued B -measurable function on E such that

- (1) $h(x(t, \omega))$ is a continuous function of t for P -almost every ω ,
- (2) $h(x(t, \omega)) - t = g(t, \omega)$ vanishes for exactly one value $t(\omega)$, P -almost surely.

It will be seen that (1) and (2) imply that $h(x(t, \omega)) - t$ is monotone. We assume, furthermore, that

- (3) $g(t, \omega)$ maps Lebesgue measure into a measure absolutely continuous with respect to Lebesgue measure $L \rightarrow L \cdot \gamma(t, \omega)$.

Then "the shift by h ," S_h , defined by the property that $x_s(S_h(\omega)) \in B$ if and only if $x_{s+t(\omega)}(\omega) \in B$, satisfies the assumptions required, and $(P \cdot |v|)S_h = P$ for $|v(\omega)| = \gamma(0, \omega)$.

PROOF. The solution $t(\omega)$ of the equation $h(x(t, \omega)) - t = 0$ is clearly measurable, since the family $\{x_t\}$ is measurable. The operation S_h is invertible: $x_s(S_h^{-1}(\omega')) \in B$ if and only if $x_{s-h(x(0, \omega'))}(\omega') \in B$. Conditions (1) and (2) imply

that $t(T_s\omega)$ is continuous as a function of s for P -almost every ω . Further, $0 = t(T_s\omega) - h(x(s + t(T_s\omega), \omega))$ identically in s . We do not indicate the dependence of ω for the sake of brevity. Writing

$$(98) \quad \begin{aligned} \sigma(s) &= s + t(T_s\omega) \\ g(t) &= h(x(t, \omega)) - t, \end{aligned}$$

we have $g(\sigma(s)) = s$ identically in s .

Since S_h is invertible, σ is a one-to-one mapping of the real axis; this is shown in section 6 (b). The inverse g of σ is thus monotone. Condition (3) is obviously equivalent to the condition that σ is absolutely continuous. The theorem therefore applies.

To calculate $\sigma'(0) = v$ we notice that $(L \cdot v)\sigma = L$ as shown in section 2, and $(L)g = (L \cdot v)\sigma \circ g = L \cdot v$. Therefore, v is the value of the Radon-Nikodym density of the image measure by the mapping g . Roughly, $(P|\gamma(0, \omega)|)S_h = P$; precisely, $(P|\gamma(0, \omega)|)S_h \diamond T_s^*(f) = P \diamond f$ holds for almost all s if f is an arbitrary P -integrable function. Moreover, if $g(t, \omega)$ is absolutely continuous, then $\gamma(t, \omega) = (g(t, \omega))^{-1}$. If the process is real-valued, and if in addition the trajectories $x(t, \omega)$ are absolutely continuous, then

$$(99) \quad \gamma(t, \omega) = (h'(x(t, \omega)) \cdot \dot{x}(t, \omega) - 1)^{-1}.$$

We discuss how the theorem applies to the second example. Here $(\Omega, \mathfrak{R}, P, x_t, E, B)$ is again a stationary process, and V is a real-valued function on E , such that for P -almost every ω , $\{s: V(x_s(\omega)) \leq 0\}$ is a Lebesgue null set. For every fixed s then $\tau(s, \omega)$ is P -almost everywhere determined by

$$(100) \quad s = \int_0^{\tau(s, \omega)} V(x(u, \omega)) du.$$

We show that the "shift by $\tau(s, \omega)$," φ_s , satisfies the assumptions of the theorem. The function φ_s is defined by the following relations: for every real u and every $B \in \mathfrak{B}$,

$$(101) \quad x_u(\varphi_s(\omega)) \in B \text{ if and only if } x_{u+\tau(s, \omega)}(\omega) \in B.$$

Clearly, $\varphi_s \circ \varphi_t = \varphi_{s+t}$; in particular, $\varphi_s^{-1} = \varphi_{-s}$. We now fix s and study the behavior of $\tau(s, T_u\omega)$ (or $t(u, \omega)$ for short) as a function of u . We have identically in u the following relation:

$$(102) \quad \int_0^{t(0, \omega)} V(x(s, \omega)) ds = \int_0^{t(u, \omega)} V(x(s, T_u\omega)) ds = \int_u^{u+t(u, \omega)} V(x(s, \omega)) ds.$$

We suppress the argument ω and write

$$(103) \quad \begin{aligned} f(u) &= \int_0^u V(x(s, \omega)) ds, \\ \sigma(u) &= u + t(u, \omega). \end{aligned}$$

Both f and σ are monotone functions of u . The function f is absolutely continuous and has the property that it maps a set of Lebesgue measure 0 into a Le-

besgue null set. This follows from the assumption that $V(x(s, \omega))$ vanishes on a null set only. The inverse function of f , $f^{-1} = h$, is therefore absolutely continuous.

We have identically in u ,

$$(104) \quad \begin{aligned} f(\sigma(u)) - f(u) &= s, \\ \sigma(u) &= h(s + f(u)). \end{aligned}$$

Since h and f are absolutely continuous and s is a constant, σ is absolutely continuous. The theorem is applicable, and

$$(105) \quad f'(\sigma(u)) \cdot \sigma'(u) - f'(u) = 0,$$

$$(106) \quad v(\omega) \cdot V(x(t(0, \omega), \omega)) = v(\omega) \cdot V(x(\varphi_s(\omega))) = V(x(0, \omega)),$$

$$(107) \quad \left(P \cdot \frac{V(x(0, \omega))}{V(x(0, \varphi_s \omega))} \right) \varphi_s = P,$$

$$(108) \quad (P \cdot V(x(0, \omega))) \varphi_s = P \cdot V(x(0, \omega)).$$

This completes the argument.

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