ON FIXED POINTS OF SEMIGROUPS OF ENDOMORPHISMS OF LINEAR SPACES

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1. Introduction

There are several theorems guaranteeing the existence of fixed points of different classes of transformations in various types of spaces. In the case of locally convex linear spaces and families of linear mappings, two of them, due to Markov-Kakutani and Kakutani himself (quoted as theorems 1 and 2 in this paper), are especially prominent.

The main purpose of this paper is to show a generalization (theorem 3) of the second theorem. However, our method of proof is quite different from that of Kakutani. It appears very strange to the author that this theorem which has a "deterministic" content has a "probabilistic" proof, which is based on a kind of Monte Carlo method. Let us mention that this paper is related to an earlier work of the author [6] on random ergodic theorems and their applications. In particular, theorem 5 from [6] was an intermediate form between the theorem of Kakutani and theorem 3 from this paper. Paper [6] is an announcement (some proofs are only sketched there), and the complete proof was never published. Later the author has observed that the most interesting part of [6], namely theorem 5, can be generalized using a more direct method independent of ergodic theory, and this is presented here.

Theorem 3 from this paper gives, of course, the consequences which the earlier theorem 5 from [6] had. For the sake of completeness we reproduce here, as corollaries from theorem 3, the existence of invariant mean for weakly almost periodic functions (which was conjectured, for example, by K. Deleeuw and I. Glicksberg ([1], pp. 88-89)). For other corollaries, see [6].

2. General assumptions and notation

Let G be a semigroup of endormorphisms of a locally convex linear topological space X (that is, every element $T \in G$ is a linear continuous operator from X into X and superposition is the semigroup operation). Let Q be a convex and closed subset of X which is G-*invariant* (namely, $T(Q) \subseteq Q$ for every $T \in G$).

These assumptions are used throughout this paper; additional assumptions, whenever necessary, will be explicitly given.

DEFINITION. *We will say that the semigroup G is* not contracting *in Q if for every pair of different elements x, y from Q the null vector of X does not belong to the closure of the set $\{Tx - Ty \colon T \in G\}$, or equivalently, there is a continuous seminorm $\|\cdot\|$ (depending on x and y) such that*

$$\inf_{T \in G} \|Tx - Ty\| > 0. \tag{1}$$

The condition formulated above means not only that every T from G is one-to-one on Q, but moreover, that images (under G) of every two points of Q remain distant each from the other.

It is easy to see that every equicontinuous group of endomorphisms of X is not contracting in X. Moreover, the following well-known remark will be useful in this paper. First, for any group G let F_G denote the family of all closed convex symmetric and G-invariant neighborhoods of the null vector 0.

PROPOSITION. *A group G is equicontinuous if and only if F_G is a basis of neighborhoods of* 0.

Now we will quote the two classical theorems concerning the existence of fixed points in linear spaces which we have mentioned in the introduction.

(See [5] and [4] or [2], pp. 456–457. Our version of theorems 1 and 2, adjusted better for our purposes, differs unessentially from the original.)

THEOREM 1 (Markov-Kakutani). *If a semigroup G is commutative and Q is weakly compact, then there exists a fixed point p of G in Q (that is, $Tp = p$ for each $T \in G$).*

THEOREM 2 (Kakutani). *If G is a group equicontinuous on Q and Q is compact, then there is a fixed point of G in Q.*

It is visible how the rejection of the assumption of commutativity in G in theorem 2 is compensed by heavy assumptions of regularity imposed on G and Q.

3. Main result

The main result of this paper is the following generalization of theorem 2.

THEOREM 3. *If a semigroup G is not contracting on Q and Q is weakly compact, then there is a fixed point of G in Q.*

PROOF. First of all, we may always assume that G is finitely generated, for example, by $T_0, T_1, \cdots, T_{m-1}$. In fact, denoting by I_Z the set of all Z-invariant points of Q where Z is a subset of G, it is easy to see that we have $I_G = \bigcap_{G_0} I_{G_0}$ where G_0 runs over all finitely generated subsemigroups of G; moreover, each I_{G_0} is a weakly closed (hence, a weakly compact!) subset of Q, and the family $\{I_{G_0}\}$ is directed by the relation of inclusion $\supseteq$.

Now let us observe that the "mean operator" $S \overset{\mathrm{df}}{=} 1/m\,(T_0 + \cdots + T_{m-1})$ transforms Q into itself so that theorem 1 yields the existence of an S-invariant element $x_0 \in Q$:

$$x_0 = Sx_0 = \frac{1}{m}(T_0 + \cdots + T_{m-1})x_0. \tag{2}$$

It will be shown that formula (2) implies (under our assumption on G and Q) that the element x_0 is T_k-invariant ($0 \leq k \leq m-1$) which will complete the proof.

In view of the definition, there is a continuous seminorm $\|\cdot\|$ on X and a positive number δ such that

(3) if $x_0 \neq T_k x_0$, then $\|T(x_0 - T_k x_0)\| > \delta$ for each $T \in G$ and $k = 0, 1, \cdots, m-1$.

At this point we shall introduce temporarily the following condition.

CONDITION (C). *The seminorm* $\|\cdot\|$ *(from (3)) is a norm, and the pair* $\langle X, \|\cdot\|\rangle$ *is a separable Banach space.*

Let us consider now the expansions of reals $t \in [0, 1]$:

$$t = \sum_{s=1}^{\infty} \frac{\epsilon_s}{m^s}, \qquad \text{where} \quad \epsilon_s = 0, 1, \cdots, m-1. \tag{4}$$

Let $J_{\epsilon_1,\cdots,\epsilon_s}$ denote the elementary interval of all t whose first s digits are $\epsilon_1, \cdots, \epsilon_s$. We will construct an auxiliary countably additive vector-valued measure $y(\cdot)$ defined on Lebesgue measurable subsets of the interval $[0, 1]$ with values in X and such that

$$y(J_{\epsilon_1,\cdots,\epsilon_s}) = m^{-s} T_{\epsilon_1} \cdots T_{\epsilon_s} x_0 \tag{5}$$

where x_0 satisfies (2) and

$$\frac{y(E)}{|E|} \in Q \tag{6}$$

for each Lebesgue measurable $E \subseteq [0, 1]$, whose Lebesgue measure $|E|$ is positive; if $|E| = 0$, then $y(E) = 0$.

To see this, let us first ascertain that there exists a unique measure $y_0(\cdot)$ defined on the field generated by all elementary intervals $J_{\epsilon_1,\cdots,\epsilon_s}$ and satisfying (5). In fact, this follows from formula (2) which clearly implies the additivity law

$$y_0(J_{\epsilon_1,\cdots,\epsilon_s}) = \sum_{k=0}^{m-1} y_0(J_{\epsilon_1,\cdots,\epsilon_s,k}). \tag{7}$$

Since Q is convex, closed, and bounded (in view of its weak compactness), the measure $y_0(\cdot)$ admits the unique extension to a countably additive measure $y(\cdot)$ defined for all Lebesgue measurable subsets of $[0, 1]$. Of course, $y(\cdot)$ is not only absolutely continuous, but it is in some sense Lipschitzian (see (6)).

Using standard arguments, we will show that $y(\cdot)$ has an integral representation $y(E) = \int_E u(t)\, dt$, where the density function $u(t)$, $(0 \leq t \leq 1)$ is strongly measurable, and $u(t) \in Q$ for almost every number $t \in [0, 1]$.

To this purpose it suffices to show that

$$z(t) \overset{\mathrm{df}}{=} y([0, t]) = \int_0^t u(\tau)\, d\tau. \tag{8}$$

Let us observe that

$$\frac{z(t') - z(t)}{t' - t} \in Q \qquad \text{for all} \quad 0 \leq t < t' \leq 1. \tag{9}$$

Further let Ξ be a denumerable set of linear continuous functionals separating points of X (such a Ξ exists since X is a separable Banach space). Since Q is bounded, in view of (6), the function $\xi z(t)$, $(0 \leq t \leq 1)$ is Lipschitz for each $\xi \in \Xi$. Hence, there is a null set $N \subset [0, 1]$ such that all functions $\xi z(t)$ are differentiable for each $t \notin N$. By weak compactness of Q, the weak derivative $z'(t) = u(t)$ exists for $t \notin N$ and $u(t) \in Q$. Moreover, $u(t)$ is strongly measurable, since it is weakly measurable and the space X is separable. We have $\xi z(t) = \int_0^t \xi u(t)\, dt$ which yields (8). Obviously, the integral in (8) is taken in the Bochner sense as the density function $u(t)$, $(0 \leq t \leq 1)$ is strongly measurable and bounded. By the vector version of the Lebesgue theorem (see [2], p. 217, theorem 8) it follows that the strong derivative of $z(t)$ exists a.e., or more explicitly, for a.e. real $t \in [0, 1]$, the strong limit of $y(J)/J$ exists as $|J| \to 0$ where J denotes an arbitrary interval such that $t \in J$. Taking the expansion (4) of such a "good" t and putting $J = J_{\epsilon_1, \cdots, \epsilon_s}$, in view of (5), we find that the sequence $T_{\epsilon_1} \cdots T_{\epsilon_s} x_0$ is strongly convergent. Hence,

$$\lim_{s \to \infty} \|T_{\epsilon_1} \cdots T_{\epsilon_{s-1}}(x_0 - T_{\epsilon_s} x_0)\| = 0. \tag{10}$$

On the other hand, we know that every digit $k = 0, 1, \cdots, m - 1$ occurs infinitely many times in the m-adic expansion of almost every number t. Therefore, the properties (3) and (10) are compatible only if (3) holds vacuously, that is $x_0 = T_k x_0$ for all $k = 0, 1, \cdots, m - 1$.

Let us point out here that the separability of X is irrelevant. In any case, we can restrict our arguments to the separable subspace generated from x_0 by $T_0, \cdots, T_{m-1}$.

Now to complete the proof of theorem 3 it suffices to show that our additional condition (C) can always be introduced without loss of generality. For this purpose let us take an infinite sequence of continuous seminorms $\{\|\cdot\|_n\}_{n=0}^{\infty}$ for which there exists an increasing sequence of positive integers $p_0 < p_1 < p_2 < \cdots$ such that

$$\|\cdot\|_0 = \|\cdot\| \quad \text{and} \quad \|T_k x\|_n \leq \|x\|_{p_n} \qquad \text{for} \quad k = 0, 1, \cdots, x \in X, \; n = 0, 1, \cdots. \tag{11}$$

The set Q is bounded (being weakly compact); hence $\sup_{x \in Q} \|x\|_n = M_n < \infty$. We denote now by X_1 the linear span of Q endowed with the seminorm

$$|||x||| = \sum_{n=0}^{\infty} \frac{1}{n^2(M_n + 1)} \|x\|_n. \tag{12}$$

Let us consider the factor space $X_2 \overset{\text{df}}{=} X_1/N$ where $N \overset{\text{df}}{=} \{x: x \in X_1 \text{ and } |||x||| = 0\}$, and its completion X_3 with respect to the norm $|||\cdot|||$ (more exactly with respect to the image of the seminorm $|||\cdot|||$ in X_2).

It is not hard to verify that Q/N (the image of Q in X_2) is a weakly compact convex set. On the other hand, some difficulties occur with $T_0, \cdots, T_{m-1}$; they cannot be defined in a natural way in the whole space X_3, but it is easy to see that the transformations T_k/N, $(k = 0, \cdots, m - 1)$ are well-defined on X_2 (since

$T_k(N) \subseteq N$) and are linear on X_2. However they are not necessarily continuous. One can easily observe that only the linearity of T_k and the property (3) were used in the preceding part of the proof. Hence, the condition (C) does not restrict arguments, and this completes the proof of theorem 3.

REMARK. The author realizes that the method used in the proof might appear, for some readers, to be old-fashioned. In fact, instead of Lebesgue measure (see (4)), one can consider product spaces and product measures and then apply a theorem on the convergence of martingales with values in Banach (or more general) spaces. In this way one can prove that (under the assumptions of theorem 3) if $\{T_w\}_{w \in W}$ is a strongly measurable operator function from a probability space W into the semigroup G and

$$\tilde{x} = \int_W T_w \tilde{x}\, dw, \qquad \text{where} \quad \tilde{x} \in Q, \tag{2'}$$

then $\tilde{x} = T_w \tilde{x}$ for almost every $w \in W$.

Obviously (2) is a very particular case of (2′), but it was sufficient for our proof.

5. Applications

Applying theorem 3 we will prove (see [6]) the following theorem.

THEOREM 4. *If a function (real or complex) $f(\cdot)$ defined on an abstract group U is weakly almost periodic (that is, the left shifts $\{f(u\ \cdot)\}_{u \in U}$ form a conditionally weakly compact set in the space $B(U)$ of bounded functions on U), then*

(1) *there exists a left mean value M of f satisfying the condition that for each $\epsilon > 0$ there exists a finite system of numbers c_i ($c_i 0$ and $\sum_i c_i = 1$) and elements $u_i \in U$ such that*

$$\Big|\sum_i c_i f(u_i u) - M\Big| < \epsilon \qquad \text{for all} \quad u \in U; \tag{13}$$

(2) *there exists a right mean value of f satisfying an analogous condition;*

(3) *all right and left means are equal (and their common value will be denoted by $M(f)$ or $M_u(f(u))$. The functional M is linear, nonnegative, normalized, left and right invariant.*

PROOF. In theorem 3 let $X = B(U)$, let $G =$ the group of left shifts, and let $Q =$ the closed convex hull of all left shifts of a given w.a.p. function f. We obtain (1) since left and right almost-periodicity are equivalent (this fact is proved by A. Grothendieck [3], proposition 7). Finally, (3) follows from (1) and (2) in a well-known way.

The next theorem gives the unicity of invariant vectors.

THEOREM 5. *If G is an equicontinuous group of endomorphisms of a locally compact linear space X and if $O_G(x)$ denotes the* convex G-orbit *of an element $x \in X$ (that is, the convex and closed hull of all vectors of the form Tx where T runs over G), then*

(i) *If $O_G(x)$ is weakly compact, then there exists exactly one G-invariant element in $O_G(X)$ (it will be denoted by Mx);*

(ii) *the set X_0 of all vectors $x \in X$ such that $O_G(x)$ is weakly compact forms a closed subspace of X;*

(iii) *the operator M (defined in* (i)) *has the following properties on X_0: it is linear continuous and $TM = MT = M^2 = M$.*

PROOF. (i) The existence of such a vector follows from theorem 3. For the proof of uniqueness we remark that the function of T given by $\xi(Tx)$ (ξ is a linear functional over the space X) is a right w.a.p. function on the group G. Consequently, we may write

$$\underset{T}{M}\, \xi(Tx) = \underset{T}{M}\, \xi(TSx) \qquad \text{for each} \quad S \in G, \tag{14}$$

and further,

$$\underset{T}{M}\, \xi(Tx) = \underset{T}{M}\, \xi(Ty) \qquad \text{for each} \quad y \in O_G(x). \tag{15}$$

Finally, for each pair of G-invariant vectors $y_1, y_2 \in O_G(x)$, we have

$$\xi(y_1) = \underset{T}{M}\, \xi(Ty_1) = \underset{T}{M}\, \xi(Ty_2) = \xi(y_2) \tag{16}$$

for each functional ξ. Hence $y_1 = y_2$.

(ii) The set X_0 is linear since

$$O_G(c_1x_1 + c_2x_2) = c_1O_G(x_1) + c_2O_G(x_2). \tag{17}$$

Also X_0 is closed since for every neighborhood $V \in F_G$ (see proposition 1), we have $O_G(x_1) - O_G(x_2) \subset V$, provided $x_1 - x_2 \in V$.

(iii) The proof is obvious and identical to the proof in the case of strong almost-periodicity.

6. Problems

Finally, we want to formulate a conjecture concerning a nonlinear case. Let Q be a metric compact space, and let us assume that a semigroup G of continuous mappings of Q into Q satisfies the following condition of *noncontractibility:*

CONDITION (P). *For each pair of different points $x, y \in Q$ we have $\inf_{T \in G} \rho(Tx, Ty) > 0$, where ρ is a metric on Q.*

Must there exist a probability measure μ on the field of Borel subsets of Q such that $\mu(T^{-1}(E)) = \mu(E)$ for each $T \in G$ and each Borel set $E \subseteq Q$?

At the end we repeat once more the following question: is it possible to eliminate the randomization method from the proof of theorem 3?

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