RANDOM ELEMENTS IN LINEAR SPACES

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1. Introduction

For different applications it is necessary to consider random elements which take values in linear spaces which are not Banach spaces. On the other hand, because from a physical point of view these elements are observed with the help of some "instruments," two spaces have to be considered; the space E in which the random element takes its values, and the space F in which the instruments are defined. The case of linear instruments is particularly important. A mathematical theory of such a situation was proposed by Gelfand, Itô, Minlos, and others. During the past few years this theory was generalized by S. Ahmad [1] and A. Badrikian [2].

All the spaces considered in this paper are real linear spaces and all the topologies are separated, locally convex topologies. By measure, we shall always mean probability measure (that is, positive measure with total mass equal to one).

2. Paired linear spaces and cylinder sets

DEFINITION. *Let E and F be two real linear spaces and let $(x, y) \rightarrow B(x, y)$ be a bilinear form on $E \times F$; we shall say that E and F are paired spaces, by the pairing functional B, if the two following conditions are fulfilled:*

(1) *for every $x \neq 0$ in E, there exists $y \in F$ such that $B(x, y) \neq 0$;*

(2) *for every $y \neq 0$ in F, there exists $x \in E$ such that $B(x, y) \neq 0$.*

For any $y \in F$, let $B_{.y}$ be the linear form $x \rightarrow B(x, y)$ on E, it is clear that $y \rightarrow B_{.y}$ is a linear mapping of F in the algebraic dual space E^* of E. The condition (1) above means that this mapping is injective, and thus it is possible to identify F with its image in E^*, and in the same way E with its image in F^*. When doing so, we write $\langle x, y\rangle$ instead of $B(x, y)$.

DEFINITION. *Let E and F be two linear spaces paired by a bilinear form $(x, y) \rightarrow \langle x, y\rangle$. We call weak topology on E, defined by the duality between E and F, and denote $\sigma(E, F)$, the weakest locally convex topology on E, such that every linear form on E: $x \rightarrow \langle x, y\rangle$, $y \in F$, is continuous.*

The topological dual of E with the topology $\sigma(E, F)$ is F. The weak topology $\sigma(F, E)$ on F is defined in the same way.

The topology $\sigma(E, F)$ is separated; it is defined by the family of seminorms: $x \to |\langle x, y\rangle|$, $y \in F$.

Let G be a finite dimensional linear subspace of F, and let $G^\perp$ be the subspace of E orthogonal to G, that is;

$$G^\perp = \{x\colon x \in E \quad \text{such that} \quad \langle x, y\rangle = 0 \ \forall\, y \in G\}, \tag{2.1}$$
$$= \bigcap_{y \in G} \{x\colon x \in E \quad \text{such that} \quad \langle x, y\rangle = 0\}.$$

The space $G^\perp$ is $\sigma(E, F)$ closed.

The natural homomorphism Π_G of E onto $E/G^\perp$ is the map which associates to $x \in E$ its restriction on G. Moreover, if $G \subset H$ are two finite dimensional subspaces of F, we have the coherency relation $\Pi_G = \Pi_{GH} \circ \Pi_H$ where Π_{GH} is the canonical linear mapping of $E/H^\perp$ onto $E/G^\perp$. Order the set $\mathcal{G}$ by the inclusion relation $G \subset H$ for which it becomes a directed set. The family of spaces $E_G = E/G^\perp$ so indexed is a projective system of linear spaces for the applications Π_{GH}, $G \subset H$. The projective limit set of the sets $(E_G)_{G \in \mathcal{G}}$ is the algebraic dual space of F.

The usual mapping of E in $\varprojlim E_G$ is that mapping which associates to an element of E the linear form that it defines in F. Then $\varprojlim E_G$ is the completion of E for the $\sigma(E, F)$ topology, $\varprojlim E_G = \hat{E}_\sigma$. Instead of considering the family of finite dimensional subspaces of F, it is possible to consider the family $\mathfrak{S}$ of circled, convex subsets of F which are bounded with respect to the finest locally convex topology, $\mathfrak{T}_1$. Every set in $\mathfrak{S}$ is contained in a finite dimensional subspace and bounded in it. A. Badrikian [2] calls "champ d'instruments" every element of $\mathfrak{S}$. These "champs d'instruments" will be provided with the topology induced by the $\mathfrak{T}_1$-topology.

It is well known that the dual space $F'_{\mathfrak{T}_1}$ of F with respect to $\mathfrak{T}_1$ is the algebraic dual space of F. It is also the set of linear forms on F whose restrictions to any "champ d'instruments" are continuous. Thus it is the set of linear forms for which small changes of an instrument in a "champ d'instruments" cause small changes. It is possible to generalize the above definition in the following way. Let $(F, \mathfrak{T})$ be a separated, locally convex space, and let $\mathfrak{S}$ be a family of $\mathfrak{T}$-bounded, circled, convex subsets of F; then these sets are called "champs d'instruments." There exists one and only one separated, locally convex topology, $\mathfrak{T}_\mathfrak{S}$, which is the finest topology inducing on every "champ d'instruments" the same topology as that induced by the $\mathfrak{T}$-topology. The dual space $F'_{\mathfrak{T}_\mathfrak{S}}$ of F with respect to $\mathfrak{T}_\mathfrak{S}$ is the set of linear forms on F whose restrictions to every $A \in \mathfrak{S}$ are continuous with respect to the topology induced by the $\mathfrak{T}$-topology.

Definition. *A cylinder set in E is any subset of E of the form $\Pi_G^{-1}(B_G)$ where B_G is a Borel set in E_G. This B_G is called the basis of the cylinder set $\Pi_G^{-1}(B_G)$, and $G^\perp$ is its generating subspace.*

Due to the coherency relation, the set of all cylinder sets constitutes a Boolean algebra $\mathfrak{A}$ of subsets of E. Let $\mathfrak{L}$ be the σ-algebra generated by $\mathfrak{A}$; $\mathfrak{L}$ is the smallest

σ-algebra such that all linear forms on E, which are members of F, are measurable.

Let us now consider the case where E is a separated, locally convex linear space $(E, \mathfrak{T})$, and F is its (topological) dual space $F = (E, \mathfrak{T})' = E'_{\mathfrak{T}} \subset E^*$. The spaces E and F are paired by the canonical bilinear form $\langle x, x' \rangle$. Then it is convenient to consider in E not only the $\mathcal{L}$ σ-algebra but also the following:

(1) the Baire σ-algebra $\mathcal{B}_{\mathfrak{T}}$; that is, the smallest σ-algebra with respect to which all bounded, continuous real functions over E are measurable;

(2) the Borel σ-algebra $\hat{\mathcal{B}}_{\mathfrak{T}}$; that is, the smallest σ-algebra generated by the open sets.

It is obvious that $\mathcal{A} \subset \mathcal{L} \subset \mathcal{B}_{\mathfrak{T}} \subset \hat{\mathcal{B}}_{\mathfrak{T}}$.

The two main problems that arise are finding conditions such that

(1) $\mathcal{L} = \mathcal{B}_{\mathfrak{T}}$ or $\mathcal{L} = \hat{\mathcal{B}}_{\mathfrak{T}}$,

(2) a given subspace E_1 of E is $\mathcal{L}$-measurable.

It is well known [6] that in any separable Banach space $\mathcal{L} = \hat{\mathcal{B}}$. However, other important results are known, especially the following.

Lemma (Prohorov [8]). *Let $(E, \mathfrak{T})$ be a real separated, locally convex linear space. Let K_n be an increasing sequence of compact sets in E, and let $C = \bigcup_{n=1}^{\infty} K_n$. Then $\mathcal{B}_{\mathfrak{T}} \cap C = \mathcal{L} \cap C$.*

If E is the dual of a Fréchet space F (complete, metrizable, locally convex linear space), and if we consider on E the $\sigma(E, F)$ topology $\mathfrak{T}_\sigma$, then $\mathcal{L} = \mathcal{B}_{\mathfrak{T}_\sigma}$. If, in addition, F is separable, then

$$\mathcal{L} = \mathcal{B}_{\mathfrak{T}_\sigma} = \hat{\mathcal{B}}_{\mathfrak{T}_\sigma}. \tag{2.2}$$

Problem (2) was investigated by A. Badrikian [2].

Lemma. *Let $(F, \mathfrak{T})$ be a separable, locally convex space and let E be its dual space. Let E_1 be a subset of E, which is a union of an increasing sequence of weakly closed, equicontinuous, circled convex subsets V_n of E; then $E_1 \in \mathcal{L}$.*

Proof. Since V_n is equicontinuous, circled, weakly closed, and convex, it is identical to its bipolar set $V_n^{\circ\circ} = (V_n^\circ)^\circ = V_n$. But, as V_n is equicontinuous, V_n° is a neighborhood of zero. Being convex it is identical to the $\mathfrak{T}$-closure of its interior U_n. There exists a sequence of points a_k dense in U_n, and

$$\begin{aligned} V_n = (V_n^\circ)^\circ = U_n^\circ &= \{x \colon x \in E \mid \langle x, x' \rangle \mid \leq 1 \forall x' \in U_n\}, \\ &= \{x \colon x \in E \mid \langle x, a_k \rangle \mid \leq 1 \forall k \in N\}. \end{aligned} \tag{2.3}$$

Theorem. *Let $(F, \mathfrak{T})$ be a separable, locally convex space and E its dual space. Let E_1 be a subspace of E. If there exists on F a metrizable topology $\mathfrak{T}_1$ weaker than $\mathfrak{T}$, compatible with the structure of linear space, and such that E_1 is the dual space of F with respect to $\mathfrak{T}_1$, then E_1 is $\mathcal{L}$-measurable.*

Proof. The space E_1 is the dual $E'_{\mathfrak{T}_1}$ of E; $\mathfrak{T}_1$ being metrizable, there exists a fundamental sequence $\{U_n\}$ of $\mathfrak{T}_1$-neighborhoods of zero ($\mathfrak{T}$-open) such that $\bigcap U_n = \{0\}$. But $U_n^\circ = V_n$ is an equicontinuous, $\sigma(E, F)$-closed, convex circled subset of E, and $E_1 = \bigcup U_n^\circ$. Thus the assumptions of the above lemma are fulfilled.

Application 1 (Vakhania [13]). Let $E = R^N$ be the linear space of all denumerable sequences, $x = \{x_k\}$, of real numbers, and let $F = R_\circ^N$ be the linear space of such sequences where only a finite number of terms are different from zero. Let $\mathfrak{T}$ be the usual inductive limit topology on $R_\circ^N$; $\mathfrak{T}$ is separable and the dual space of $(R_\circ^N, \mathfrak{T})$ is R^N.

Let ℓ^p, $(1 \leq p < +\infty)$ be the usual Banach space of sequences $x \in R^N$ such that

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} < +\infty. \tag{2.4}$$

If $q > 1$, then ℓ^q is the dual space of ℓ^p, with $(1/p) + (1/q) = 1$. If $q = 1$, then ℓ^1 is the dual space of $C_\circ$, the linear space of sequences $x \in R^N$ converging to zero, with $\|x\| = \sup_k |x_k|$.

The set $R_\circ^N$ is dense everywhere in ℓ^p and in $C_\circ$. Let $\mathfrak{T}_1$ be the (metrizable) topology induced in $R_\circ^N$ by ℓ^p (resp. $C_\circ$); the dual space of $(R_\circ^N, \mathfrak{T}_1)$ is ℓ^q (resp. ℓ^1). Then ℓ^q, $(1 \leq q < +\infty)$ is $\mathfrak{L}$-measurable in R^N.

Application 2. *The space of Schwartz distributions.* Let $\mathfrak{D}$ be the space of all infinitely differentiable functions having compact supports in R^n and let $\mathfrak{T}$ be its usual inductive limit topology; $(\mathfrak{D}, \mathfrak{T})$ is separable and its dual space $\mathfrak{D}'$ is the space of Schwartz distributions.

Let $\mathcal{E}$ be the space of all infinitely differentiable functions in R^n, its dual space $\mathcal{E}'$ is the space of Schwartz distributions having compact supports. The space $\mathfrak{D}$ is a subspace of $\mathcal{E}$, and $\mathfrak{D}$ is dense everywhere in $\mathcal{E}$. Let $\mathfrak{T}_1$ be the topology induced on $\mathfrak{D}$ by $\mathcal{E}$. Note that $\mathfrak{T}$ is the topology of the uniform convergence of functions and of their derivatives. The space $(\mathfrak{D}, \mathfrak{T}_1)$ is metrizable and its dual space is $\mathcal{E}'$. Then $\mathcal{E}'$ is a member of the $\mathfrak{L}\sigma$-algebra of $\mathfrak{D}'$: $\mathfrak{L}(\mathfrak{D}')$. On the other hand, $\mathfrak{D}$ is everywhere dense in the Lebesgue space $L^p(dx)$. Since the topology induced by L^p on $\mathfrak{D}$ is metrizable and weaker than $\mathfrak{T}$, it follows that $L^q \in \mathfrak{L}(\mathfrak{D}')$ if $q > 1$.

3. Cylinder set measures and measures

With the above notations let E and F be two paired linear spaces. Let G be a finite dimensional subspace of F and let $E_G = E/G^\perp$.

Gelfand and Minlos call cylinder set measure a projective system of probabilities on the E_G's, $G \in \mathcal{G}$. It is not a measure on $\mathfrak{L}$.

DEFINITION. *A projective system of probabilities (Borelian on E_G) is a system $(P_G)_{G \in \mathcal{G}}$ such that if $G \subset H$ $(H^\perp \subset G^\perp)$, we have $P_G \circ \Pi_{GH}^{-1} = P_H \Leftrightarrow P_G = \Pi_{GH}(P_H)$.*

A projective system of probabilities defines an additive (but in general non-σ-additive) set function on the algebra $\mathfrak{A}$ of the $\mathcal{G}$-cylinder sets of E. The first fundamental problem is to know when it defines a probability on $\mathfrak{A}$, that is when there exists a unique extension on $\mathfrak{L}$. However, for many applications this is not enough. The second problem is the following. If $(E, \mathfrak{T})$ is a topological linear space and F its dual space, when does the projective system of probabilities define a Baire or a Borel probability; that is, when does it define a probability on $\mathcal{B}_{\mathfrak{T}}$ or on $\hat{\mathcal{B}}_{\mathfrak{T}}$?

3.1. *First problem: solution according to Bochner and Kolmogorov.*

THEOREM. *Let E and F be two paired spaces and let $\hat{E}_\sigma = \varprojlim E_G$. Then a projective system of probabilities $(P_G)_{G \in \mathcal{G}}$ on the E_G's defines a measure on the σ-algebra, $\mathcal{L}(\hat{E}_\sigma)$, generated by the cylinder sets of $\hat{E}_\sigma$.*

PROOF. By Bochner's results ([3], p. 120) it is sufficient to prove that the stochastic family $(E_G, \hat{\mathcal{B}}_G, P_G)$ where $\hat{\mathcal{B}}_G$ is the Borelian σ-algebra of E_G, is sequentially maximal; that is, for any increasing sequence $\{G_n\}$, finite or not, of finite dimensional subspaces of F, the natural mapping $\varprojlim E_G = \hat{E}_\sigma \to \varprojlim_n E_{G_n}$ is surjective.

But $\varprojlim_n E_{G_n}$ is the algebraic dual space of $\bigcup G_n$; the natural mapping of $\varprojlim E_G \to \varprojlim_n G_n$ is the mapping which to every linear form on F associates its restriction on $\bigcup G_n$; it is surjective if and only if a linear form on $\bigcup G_n$ may be extended in a linear form on F. Here this is the case.

For the second problem we will use the following condition referred to as the C-condition. Let $(E, \mathfrak{T})$ be a separated, locally convex space and F its dual space. Then a cylinder set measure, that is a projective system of probabilities $(P_G)_{G \in \mathcal{G}}$, satisfies the C-condition if for every $\epsilon > 0$ there exists a compact set K_ϵ in E such that

$$P_G(\Pi_G(K_\epsilon)) > 1 - \epsilon, \qquad \forall G \in \mathcal{G}. \tag{3.1}$$

THEOREM. *If $(E, \mathfrak{T})$ is a separated, locally convex space and F its dual space, and if a projective system of probabilities $(P_G)_{G \in \mathcal{G}}$ satisfies the C-condition, then it defines a unique Borelian probability P on $(E, \mathfrak{T})$ such that*

$$P_G = \Pi_G \cdot P = P \cdot \Pi_G^{-1}, \qquad \forall G \in \mathcal{G}. \tag{3.2}$$

Furthermore, for each $\epsilon > 0$ there exists a compact K_ϵ such that $P(K_\epsilon) > 1 - \epsilon$.

Indeed, it is known that P is a probability measure on $\mathcal{L}(\hat{E}_\sigma)$; let $K(1/n)$ be compact sets in $(E, \mathfrak{T})$, $C = \bigcup_n K(1/n)$ is a compact set of $\hat{E}_\sigma$ and $C \subset E$. Due to the C-condition, P is "pseudo-portée" by C, then from Prohorov results [8] it may be extended to a unique tight Baire measure and to a unique tight Borel measure.

REMARK. The C-condition may be written in the following way. For every $\epsilon > 0$, there exists a compact set K_ϵ such that for a cylinder set A, $A \supset K_\epsilon$ implies $P(A) > 1 - \epsilon$. Such a set function is called tight measure (Le Cam [5]) or cylindrically concentrated (Schwartz [11]).

THEOREM. *If $(E, \mathfrak{T})$ is a Fréchet space, every Borel tight measure for the $\mathfrak{T}$ topology is a Borel tight measure for the weak topology and conversely.*

Later on, it will be seen that it is not easy to express the C-condition in terms of characteristic functionals. Therefore, it is useful to define a weaker condition:

the $C_\circ$-condition. If $(E, \mathfrak{T})$ is a separated, locally convex space, and F its dual space, we say that a cylinder set measure satisfies the $C_\circ$-condition if for every $\epsilon > 0$ there exists a compact K_ϵ such that

$$P_y(y(K_\epsilon)) > 1 - \epsilon, \qquad \forall y \in F. \tag{3.2}$$

Schwartz [11] calls a set function which satisfies the $C_\circ$-condition "scalarly concentrated."

REMARK 1. Later on we shall need conditions similar to the $C_\circ$-condition resp. C-condition but slightly stronger. Instead of permitting K_ϵ to be any compact set, we shall impose that K_ϵ belongs to a given family $\mathfrak{S}$. Then such conditions will be labeled $C_\circ$-condition resp. C-condition with respect to $\mathfrak{S}$.

REMARK 2. The $C_\circ$-condition was introduced in a particular case by Minlos. Let $(E, \mathfrak{T})$ be a separated, locally convex space, and let F be its dual space provided with the $\sigma(E, F)$ topology. The Minlos continuity condition is equivalent to the $C_\circ$-condition with respect to the family of subsets of E which are convex, equicontinuous, circled, and weakly closed. Note that these sets are weakly compact.

3.2. *Cylinder set measures and random functions.*

THEOREM (Schwartz [11]). *There is a bijective correspondence between cylinder set measures on a separated, locally convex space $(E, \mathfrak{T})$ and real linear random functions on its dual space F.*

If the $C_\circ$-condition is fulfilled, then the associated random function $y \to f(y)$, $(y \in F, f(y)$: random variable) is continuous in probability when F is provided with the topology $\mathfrak{T}$ of uniform convergence on the compact sets of E.

If the C-condition is fulfilled, then the associated random function is almost-surely continuous on $(F, \mathfrak{T})$.

For the statement of Minlos' theorem and related results we shall need the following definition.

Let E_1 and E_2 be two separated, locally convex spaces and let u be a continuous linear mapping from E_1 to E_2. Let A be a cylinder set in E_2 (that is, $A = \Pi_G^{-1}(\Pi_G(A))$, with $\Pi_G(A)$ a Borelian set). Then

$$u^{-1}(A) = u^{-1}[\Pi_G^{-1}(\Pi_G(A))] \tag{3.3}$$

is a cylinder set in E_1. Let P be a cylinder set measure in E_1. We define the *cylinder set measure $u(P)$ induced by P in E_2* by

$$u(P)(A) = P[u^{-1}(A)]. \tag{3.4}$$

Let E_1 and E_2 be two Hilbert spaces. A linear mapping u from E_1 to E_2 is a *nuclear linear mapping* if

$$u(x) = \textstyle\sum_i \lambda_i \langle x, x_i \rangle y_i, \qquad \forall x \in E_1 \tag{3.5}$$

where $x_i \in E_1$, $||x_i|| \leq 1$, and $y_i \in E_2$, $||y_i|| \leq 1$, and λ_i are scalars $\lambda_i \geq 0$ such that $\sum_i \lambda_i < +\infty$. Such mappings are the S-operators of Sazonov [10]. The

Hilbert-Schmidt mappings are defined in the same way. The only difference is that the condition $\sum_i \lambda_i < +\infty$ should be replaced by the condition $\sum_i \lambda_i^2 < +\infty$.

Let E be a separated, locally convex space and u an open, circled convex neighborhood of zero; for every $x \in E$ let

$$||x||_u = \inf \{\lambda : \lambda \in R^+; x \in \lambda u\}. \tag{3.6}$$

Then $||x||_u$ is a seminorm in E; let E_u be the associated normed space (quotient of E by the subspace such that $||x||_u = 0$). Its completion is the Banach space $\hat{E}_u$.

If $u \subset v$, $||x||_u \geq ||x||_v$, let Π_{uv} be the canonical mapping from E_u to E_v extended by continuity from $\hat{E}_u$ to $\hat{E}_v$.

If $\hat{E}_u$ is a Hilbert space, u is called a prehilbertian neighborhood (Schwartz).

A *nuclear space* is a separated, locally convex space such that there exists a fundamental system $\mathfrak{J}$ of neighborhoods of zero which are prehilbertian neighborhoods such that for every $u \in \mathfrak{J}$ there exists $v \in \mathfrak{J}$, $v \subset u$, such that the mapping Π_{uv} from $\hat{E}_v$ to $\hat{E}_u$ is a nuclear linear mapping.

EXAMPLES. The space $\mathfrak{D}$, $\mathfrak{D}'$ (Schwartz distributions space), R^N and R_0^N, defined above, are nuclear spaces.

MINLOS' FUNDAMENTAL LEMMA. *If E_1 and E_2 are two Hilbert spaces, and if u is a nuclear linear mapping from E_1 to E_2, and if P is a cylinder set measure in E_1 satisfying the C_0-condition with respect to the family $\mathfrak{S}$ of the closed balls, which are weakly compact in E_1, then $u(P)$ satisfies the C-condition and defines a Borel tight measure in E_2* (cf. Gelfand [4], p. 429 or Schwartz [11]).

REMARK. In this lemma the condition "u is a nuclear mapping" may be replaced by the condition "u is a Hilbert-Schmidt mapping."

The following theorem was proved by Minlos. If E is the strong dual space of a denumerably normed nuclear space, then every cylinder set measure in E which satisfies the Minlos continuity condition (that is, the C_0-condition with respect to the equicontinuous, convex, circled, weakly closed subsets of E) satisfies the C-condition with respect to the same family of sets, and thus defines a Borel tight measure.

The following extension is given by Schwartz [11].

THEOREM. *Let $(E, \mathfrak{T})$ be a separated, locally convex space. Assume that its dual space E'_{cc} is nuclear when provided with the topology of uniform convergence on the compact convex sets of E. Then every cylinder set measure in E satisfying the C_0-condition with respect to the compact convex sets of E, satisfies the C-condition with respect to the same family of sets.*

Minlos' theorem is a particular case of Schwartz's theorem. Indeed, if E is the dual space of a denumerably normed nuclear space, or more generally of a Fréchet nuclear space $(F, \mathfrak{T})$, then E is a nuclear space when it is provided with the strong topology (uniform convergence on bounded sets of F). Therefore the topology $\mathfrak{T}$ is the strong topology on F considered as the dual space of E. But E being a Montel space (since it is nuclear and complete), the strong topology on F is the E'_{cc} topology.

4. Characteristic functional

Let $(E, \mathfrak{T})$ be a separated, locally convex space and let $F = E'$ be its dual space. Let P be a cylinder set measure on E. By analogy with the usual definition, it would be natural to call characteristic functional of P, the function of $y \in F$ formally defined by

$$\varphi(y) = \int_E e^{i\langle x,y\rangle} P(dx). \tag{4.1}$$

But since, in general, P does not define a measure concentrated on E, this formula needs some explanation. Any given $y \in F$ defines a continuous linear mapping $x \to \langle x, y\rangle$ from E to R. The image $y(P)$ of P is a measure P_y on R such that the usual characteristic functional is

$$\int_R e^{i\alpha t} P_y(dt) = \mathfrak{F}(P_y)(\alpha). \tag{4.2}$$

DEFINITION. *The characteristic functional—or Fourier transform—of a cylinder set measure P on $(E, \mathfrak{T})$ is the function defined on F by*

$$\varphi(y) = \int_R e^{it} P_y(dt) = \mathfrak{F}(P_y)(1). \tag{4.3}$$

The function $\varphi(y)$ has the following properties (Prohorov [8]):

(1) it is positive-definite, that is, for any n, any $y_1, \cdots, y_n \in F$, and any complex numbers $c_1, \cdots, c_n$,

$$\sum_{i,j=1}^{n} \varphi(y_i = y_j) c_i \bar{c}_j \geq 0; \tag{4.4}$$

(2) for every fixed $y \in F$ the function $\varphi(ty)$ of a real argument t is continuous;

(3) $\varphi(0) = 1$.

Conversely, every functional φ on F satisfying conditions (1), (2), and (3) is the characteristic functional of a cylinder set measure on E, uniquely defined by φ.

A family $\{P_\alpha\}$ of cylinder set measures is said to satisfy the uniform $C_\circ$-condition [resp. the uniform C-condition] with respect to a given family $\mathfrak{S}$ if for every $\epsilon > 0$ there exists $K_\epsilon \in \mathfrak{S}$ such that for all $y \in F$ and all α's, one has

$$P_{\alpha,y}(y(K_\epsilon)) > 1 - \epsilon, \text{ [resp. } \forall G \in \mathcal{G}, \forall \alpha, P_{\alpha,G}(\Pi_G(K_\epsilon)) > 1 - \epsilon]. \tag{4.5}$$

THEOREM. *In order that the P's satisfy the uniform $C_\circ$-condition with respect to $\mathfrak{S}$, it is necessary and sufficient that their Fourier transforms be uniformly equicontinuous on F provided with the topology $\mathfrak{T}$ of uniform convergence on the sets of $\mathfrak{S}$.*

COROLLARY. *The cylinder set measure P satisfies the $C_\circ$-condition with respect to $\mathfrak{S}$ if and only if its Fourier transform is $\mathfrak{T}$-continuous.*

This result and the Schwartz theorem allow us to obtain a theorem similar to the classical Bochner theorem for random variables.

THEOREM. *Let $(E, \mathfrak{T})$ be a separated, locally convex space such that $F = E'_{cc}$ be nuclear. Then every continuous, positive-definite functional φ on F, such that $\varphi(0) = 1$, is the Fourier transform of a Borel tight measure.*

REMARK. We get a similar proposition by replacing "continuous functional" with "equicontinuous set of functionals" $\{\varphi_\alpha\}$, and "tight measure" by "tight set $\{P_\alpha\}$"; that is, the P_α satisfying the uniform C-condition.

THEOREM (Schwartz). *Let P_y be a cylinder set measure in $(E, \mathfrak{T})$. In order that P satisfies the C_0-condition with respect to $\mathfrak{S}$, it is necessary and sufficient that the mapping $y \to P_y$ from F, with the topology of uniform convergence on the sets of $\mathfrak{S}$, to the space of probabilities on R with the topology of weak convergence be continuous.*

A condition that for φ to be a characteristic functional of a tight measure was given in [5] as follows:

(1) φ is positive-definite and $\varphi(0) = 1$;

(2) for every $\epsilon > 0$ there exists a compact K_ϵ of $(E, \mathfrak{T})$ and a number $\delta > 0$ such that for any n, any $y_1, \cdots y_n \in F$, and any complex numbers $c_1, \cdots, c_n$, the relations

$$\left|\sum_{i=1}^{n} c_j e^{i\langle x, y_k\rangle}\right| \leq 1 \qquad \text{for every} \quad x \in E, \tag{4.6}$$

$$\left|\sum_{j=1}^{n} c_j e^{i\langle x, y_k\rangle}\right| < \delta \qquad \text{for every} \quad x \in K_\epsilon, \tag{4.7}$$

imply that $|\sum_{j=1}^{n} c_j\varphi(y_j)| < \epsilon$.

With Sazonov we call S-topology the topology in which a basis of neighborhoods of zero is given by the sets

$$U_s(0) = \{x: x \in H, (Sx, x) < 1\} \tag{4.8}$$

where S is a nuclear linear operator (S-operator).

THEOREM (Sazonov [10]). *Let H be a Hilbert space and let φ be a positive-definite functional on H such that $\varphi(0) = 1$. Then φ is the characteristic functional of a tight measure on H if and only if it is continuous for the S-topology.*

Prohorov and Sazonov [9] proved that in a Hilbert space there does not exist any topology for which the equicontinuity of a family of characteristic functionals is equivalent to the weak relative compactness of the corresponding Borel tight measures. Let $(E, \mathfrak{T})$ be a separated, locally convex space such that $F = E'_{cc}$ is nuclear. The uniform C_0-condition which is equivalent to the equicontinuity of characteristic functionals is, in this case, equivalent to the uniform C-condition (Schwartz's theorem above). Then if a family of characteristic functionals are uniformly equicontinuous, the corresponding measures are uniformly tight. From a theorem of Le Cam [5] a set of cylinder set measures satisfying the uniform C-condition is weakly relatively compact. Thus, in this case, the equicontinuity of characteristic functionals implies (but is not equivalent to) the weak relative compactness of the corresponding measures.

Nevertheless, there is a case where some sort of equivalence occurs; it is the following one.

If E is a nuclear Fréchet space, then it is reflexive, its strong dual is nuclear, and it has the E'_{cc}-topology. In this case the sequential relative compactness for a family of measures on E is equivalent to the uniform C-condition. Thus the equicontinuity of characteristic functionals is equivalent to the sequential relative compactness of the corresponding measures. Prohorov and Sazonov [9] proved that this is the only case where such an equivalence exists.

5. Gaussian measures

DEFINITION. *Let $(E, \mathfrak{T})$ be a separated, locally convex space. A cylinder set measure P on E is called a Gaussian measure if the projective system $\{P_G\}$ consists of Gaussian measures.*

It is easy to prove that P is a Gaussian measure if and only if there exists a nondegenerate scalar product $\mathfrak{B}(y_1, y_2)$ on F and

$$P_G(Y) = \frac{1}{(2\pi)^{n/2}} \int_Y e^{-\frac{1}{2}\mathfrak{B}(y,y)}\, dy, \qquad Y \subset E_G \tag{5.1}$$

where dy is the Lebesgue measure corresponding to the scalar product induced by $\mathfrak{B}$ on G (we identify G and E_G) (cf. Gelfand and Vilenkin [4], p. 337). If E is a Hilbert space ($F \cong E$), we take for $\mathfrak{B}(y_1, y_2)$ the scalar product of E.

If $(F, \mathfrak{T})$ is a separated, locally convex space and $E = F'_{\mathfrak{T}}$, the Gaussian measure satisfies the C_0-condition if and only if $\mathfrak{B}(y_1, y_2)$ is continuous.

The use of Gaussian measures allows the construction of counter-examples.

(1) A Gaussian measure on a Hilbert space does not satisfy the C-condition.

(2) If E_1 and E_2 are two Hilbert spaces and if u is a linear mapping from E_1 to E_2, which is not a Hilbert-Schmidt one, then there exists a cylinder set measure P in E_1 (for example, the Gaussian measure) such that the image $u(P)$ is not a measure.

(3) Let $(E, \mathfrak{T})$ be a separated, locally convex space. Suppose that every cylinder set measure in E which satisfies the C_0-condition with respect to a family $\mathfrak{S}$ of subsets of E satisfies the C-condition with respect to $\mathfrak{S}$. Then $E'^{\mathfrak{S}}$, the dual space of E, provided with the topology of uniform convergence on the sets of $\mathfrak{S}$, is nuclear.

(4) Let $(E, \mathfrak{T})$ be a quasi-complete, separated, locally convex space and $F = E'_{cc}$. If every positive-definite, continuous functional φ such that $\varphi(0) = 1$, on E'_{cc} is the Fourier transform of a Borel tight measure, then E'_{cc} is nuclear.

Application to the ℓ^p spaces (Vakhania [12], [13]). It has been shown that ℓ^p is a Borelian set in R^N; R^N is a nuclear Fréchet space (it is not denumerably normed), and its dual space R^N_0 is nuclear.

(1) Every positive-definite, continuous functional φ defined on R^N_0 and such that $\varphi(0) = 1$, is the Fourier transform of a probability measure in R^N.

(2) In order that $\varphi(y)$, $y \in \ell^q$, $(1 < q < +\infty)$ be the Fourier transform of a probability measure on ℓ^p, it is necessary and sufficient that:

(a) φ is positive-definite, $\varphi(0) = 1$;

(b) φ is continuous with respect to the norm topology in ℓ^q;

(c) $\lim_{n\to\infty} \sum_{k=n}^{\infty} E_p|x_k|^p = 0$;

(d) if P is a Borelian Gaussian measure in ℓ^p, $(1 < p < +\infty)$, then

$$\int_{\ell^p} ||x||^t P(dx) < \infty \qquad \text{for all} \quad t \text{ in } R^+. \tag{5.2}$$

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