ON A CLASS OF TWO-SAMPLE BIVARIATE NONPARAMETRIC TESTS

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1. Introduction

The object of the present investigation is to propose and study a general class of nonparametric tests for the various types of problems that may usually arise in the case of two independent samples with bivariate observations. For this purpose, the concept of permutation tests has been used in the formulation of a class of tests based on appropriate generalized U-statistics, and the theory of permutation distribution of such generalized U-statistics has been developed further.

The advent of the theory of nonparametric methods in multivariate analyses may be regarded to be still in a more or less rudimentary stage, and only a few nonparametric contenders of some standard parametric multivariate procedures are available in the literature. The up-to-date development of distribution-free techniques in this field of research relates specifically to the problem of location in the single, as well as multisample case, and the problem of independence in the single sample case. In this study, I have confined myself to the multisample case only.

The earliest work on this line is the permutation test based on Hotelling's T^2-statistic, proposed and studied by Wald and Wolfowitz [25], in as early as 1944. This test is, however, a strictly value-permutation test and is subject to the usual limitations of this type of tests. Following this, there is a gap of nearly twenty years, during which practically no nonparametric test has evolved in this field. However, very recently, some attention has been paid to the development of nonparametric methods in multivariate multisample analyses.

Some genuine distribution-free tests for location in the bivariate two-sample, as well as p-variate c-sample ($p, c \geq 2$), case have been proposed and studied by Chatterjee and Sen [2], [4]. On the other hand, some tests for the same problem, which are only asymptotically distribution-free, have been considered by Bhapkar [1]. Chatterjee and Sen [3] have also considered some exact distribution-free tests for the two-sample bivariate association problem, and some of these tests have been extended to the c-sample case by Sen [22].

However, all these tests are based on specific forms of test criteria and relate specifically to the problem of location and association. No attempt has yet been

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made to develop a general method of constructing suitable nonparametric tests for the different types of problems that may usually arise in the multivariate multisample case.

The scope of the present investigation has been confined only to the bivariate two-sample case, whereas the more complicated case with p-variates and c-samples $(p, c \geq 2)$ is intended to be considered separately. A class of permutation tests based on appropriate generalized U-statistics has been studied here, and these may be used to test the null hypothesis of permutation invariance against various types of admissible alternatives. The present study not only extends the scope of nonparametric tests to a more varied type of problems in the bivariate two-sample case, but also extends the theory of asymptotic permutation distribution to a more general class of statistics.

The literature on permutation tests relates mostly to the linear permutation statistics of the type considered by Wald and Wolfowitz [25], Noether [19], Hoeffding [11], [12], Dwass [5], [6], Motoo [17], Hájek [8], among others. The class of generalized U-statistics, considered here, is more general than the usual linear permutation statistics. Finally, in the univariate case, a class of multisample permutation tests based on appropriate U-statistics, has been proposed and studied by the present author [21], [22], and the present investigation also extends these findings in the multivariate case.

2. Preliminary notions

Let $X = (X^{(1)}, X^{(2)})$ be a vector-valued random variable, and let the first sample be composed of n_1 independent and identically distributed bivariate random variables (i.i.d.b.r.v.) $X_1, \cdots, X_{n_1}$, distributed according to the bivariate distribution function (cdf) $F_1(x)$, where $x = (x^{(1)}, x^{(2)})$. Similarly, let $Y_1, \cdots, Y_{n_2}$ be n_2 i.i.d.b.r.v., constituting the second sample drawn independently from another distribution with a bivariate cdf $F_2(x)$. Also, let Ω be the set of all pairs of nondegenerate bivariate cdf's, and it is assumed that

$$(F_1, F_2) \in \Omega. \tag{2.1}$$

It may be noted that Ω is the set of all possible types of pairs of bivariate cdf's, and it includes the family of pairs of continuous, or absolutely continuous cdf's, as subsets. Afterwards, some mild restrictions will have to be imposed on Ω, and these will be stated as and when necessary. Let $\mathcal{W}_0$ be a subset of points $(F_1, F_2) \in \Omega$, for which $F_1(x) \equiv F_2(x)$. Our problem is then to test the null hypothesis

$$H_0\colon (F_1, F_2) \in \mathcal{W}_0, \tag{2.2}$$

against various types of admissible alternatives. Since, under the null hypothesis (2.2), the joint distribution of the $N = n_1 + n_2$ observations of the combined sample, remains invariant under any permutation of the coordinated variables, the hypothesis (2.2) may also be termed the hypothesis of permutation invariance.

Now, the alternative hypotheses often relate to differences of location, scale, association pattern, or of some measurable characteristics of the two cdf's (F_1, F_2). In this context, I therefore introduce the role of *estimable parameters* or *regular functionals* (cf. Hoeffding [10], Lehmann [15]), which may be readily employed in the specification of a variety of alternative hypotheses. Also, in the bivariate case, we usually require a vector-valued regular functional to specify completely the alternative hypotheses. Thus, let

$$\boldsymbol{\theta}(F_1, F_2) = (\theta_1(F_1, F_2), \cdots, \theta_p(F_1, F_2)), \qquad p \geq 1; \tag{2.3}$$

be a vector-valued regular functional of the two cdf's (F_1, F_2), and (2.3) is assumed to be estimable, so that $\boldsymbol{\theta}(F_1, F_2)$ exists for all $(F_1, F_2) \in \Omega$.

Now to induce the nonparametric structure of the hypothesis (2.2), it is further assumed that

$$\boldsymbol{\theta}(F_1, F_2) = \boldsymbol{\theta}^0 = (\theta_1^0, \cdots, \theta_p^0) \qquad \text{for} \quad (F_1, F_2) \in \mathcal{W}_0, \tag{2.4}$$

where $\boldsymbol{\theta}^0$ is a vector with known elements.

Now let $\mathcal{W}_\theta$ be a subset of Ω for which $\boldsymbol{\theta}(F_1, F_2) = \boldsymbol{\theta}^0$. Obviously then, $\mathcal{W}_0 \subset \mathcal{W}_\theta$. We are now interested in the set of alternatives

$$H_\theta\colon (F_1, F_2) \in \Omega - \mathcal{W}_\theta \subset \Omega - \mathcal{W}_0, \tag{2.5}$$

that is, $\boldsymbol{\theta}(F_1, F_2) \neq \boldsymbol{\theta}^0$. Since $\boldsymbol{\theta}(F_1, F_2)$ is assumed to be estimable, there exists a vector-valued *kernel* of it, which is denoted by

$$\boldsymbol{\phi} = (\phi_i(X_{\alpha_1}, \cdots, X_{\alpha_{m_{i1}}}, Y_{\beta_1}, \cdots, Y_{\beta_{m_{i2}}}), i = 1, \cdots, p), \tag{2.6}$$

where ϕ_i is symmetric in its first m_{i1} arguments and also in its last m_{i2} arguments, though the roles of these two sets may not be symmetric, and where m_{i1}, m_{i2} are positive integers, for $i = 1, \cdots, p$. The *degree* of $\boldsymbol{\phi}$ is then denoted by

$$\mathbf{m} = \begin{pmatrix} m_{11}, & \cdots, & m_{p1} \\ m_{12}, & \cdots, & m_{p2} \end{pmatrix}. \tag{2.7}$$

It is further assumed that $\phi_1, \cdots, \phi_p$ are all linearly independent. Then, the generalized U-statistic corresponding to $\boldsymbol{\phi}$ is given by

$$\mathbf{U}_N = (U_{N1}, \cdots, U_{Np}), \qquad N = n_1 + n_2, \tag{2.8}$$

where

$$U_{Ni} = \binom{n_1}{m_{i1}}^{-1}\binom{n_2}{m_{i2}}^{-1} \sum_{S_i} \phi_i(X_{\alpha_1}, \cdots, X_{\alpha_{m_{i1}}}, Y_{\beta_1}, \cdots, Y_{\beta_{m_{i2}}}), \tag{2.9}$$

the summation S_i being extended over all possible

$$1 \leq \alpha_1 < \cdots < \alpha_{m_{i1}} \leq n_1; \qquad 1 \leq \beta_1 < \cdots < \beta_{m_{i2}} \leq n_2, \qquad \text{for} \quad i = 1, \cdots, p. \tag{2.10}$$

It is well known that under certain conditions on Ω (cf. Fraser [7], p. 142), $\mathbf{U}_N$ has uniformly the minimum concentration ellipsoid (as well as minimum risk with any convex loss function) among all unbiased estimators of $\boldsymbol{\theta}(F_1, F_2)$. Even when these conditions on Ω do not hold, the U-statistic corresponding to any unbiased estimator of $\boldsymbol{\theta}(F_1, F_2)$ has a concentration ellipsoid which cannot be larger than that of the estimator itself. Thus, it seems reasonable to base a test for H_0 in (2.2) against the set of alternatives in (2.5) on the values of $\mathbf{U}_N$, and the same has been accomplished here through a permutation approach.

We pool the two samples together into a combined sample of size $N = n_1 + n_2$ and denote these N (paired) observations by

$$\mathbf{Z}_N = (Z_1, \cdots, Z_N), \qquad Z_i = (Z_i^{(1)}, Z_i^{(2)}), \qquad i = 1, \cdots, N, \tag{2.11}$$

where conventionally we let

$$\begin{aligned} Z_i &= X_i, & i &= 1, \cdots, n_1, \\ Z_i &= Y_{i-n_1}, & i &= n_1 + 1, \cdots, N. \end{aligned} \tag{2.12}$$

In what follows, $\mathbf{Z}_N$ will be called the *collection vector*, as it is a collection of N random paired observations. Then, under the null hypothesis (2.2), $\mathbf{Z}_N$ is composed of N i.i.d.b.r.v., and hence, the joint distribution of $\mathbf{Z}_N$ is symmetric in its N arguments. Consequently, under (2.2) and given the collection matrix (2.11), all possible permutations of the coordinates of $\mathbf{Z}_N$ are equally likely, each such permutation having the same conditional probability $1/N!$.

Hence, all possible partitioning of the N variables into two subsets of n_1 and n_2 respectively are equally likely (conditionally), each having the same permutation probability $\binom{N}{n_1}^{-1}$. Since this probability is independent of $\mathbf{Z}_N$ as well as of $(F_1, F_2) \in \mathcal{W}_0$, we may readily use this to formulate various tests based on $\mathbf{U}_N$. Naturally, such a test is strictly distribution-free under the null hypothesis (2.2).

Now the formulation of the critical function $I(\mathbf{U}_N)$ depends evidently on the permutation distribution of $\mathbf{U}_N$. Consequently, we will study first some properties of the permutation distribution of $\mathbf{U}_N$ and later, with the aid of these, proceed further to consider $I(\mathbf{U}_N)$ and its various properties.

3. Permutation distribution of $\mathbf{U}_N$

Let us define first

$$\phi_i^*(Z_{\alpha_1}, \cdots, Z_{\alpha_{m_{i1}+m_{i2}}}) = \frac{1}{(m_{i1} + m_{i2})!} \sum_{S_i^*} \phi_i(Z_{\alpha_1}, \cdots, Z_{\alpha_{m_{i1}+m_{i2}}}), \tag{3.1}$$

where the summation S_i^* extends over all possible $(m_{i1} + m_{i2})!$ permutation of the variables

$$Z_{\alpha_1}, \cdots, Z_{\alpha_{m_{i1}+m_{i2}}} \tag{3.2}$$

in the ordered position of $\phi_i(\cdots)$, for $i = 1, \cdots, p$. Thus,

$$\boldsymbol{\phi}^* = (\phi_1^*, \cdots, \phi_p^*) \tag{3.3}$$

is the symmetric form of $\boldsymbol{\phi}$.

Then extending the idea of Sen [21], we say that $\mathbf{U}_N$ is a type A generalized U-statistic, if $\boldsymbol{\phi}^*$ is nonstochastic for all $Z_{\alpha_1}, \cdots, Z_{\alpha_{m_{i1}+m_{i2}}}$.

In this paper, we will be concerned with type A generalized U-statistics only. It may also be noted that they include as a special case the differences of individual sample U-statistics (cf. Sen [21]). Further, we will also assume that for the given class of generalized U-statistics, the dispersion matrix of

$$N^{1/2}\{\mathbf{U}_N - \boldsymbol{\theta}(F_1, F_2)\} \tag{3.4}$$

(where $\mathbf{U}_N$ and $\boldsymbol{\theta}(F_1, F_2)$ are defined in (2.8) and (2.3) respectively), has asymptotically a positive definite limit (as $N \to \infty$), for all $(F_1, F_2) \in \mathcal{W}_0$ (this limit may, of course, depend on the particular $(F_1, F_2) \in \mathcal{W}_0$).

Now the condition of nonstochasticness of $\boldsymbol{\phi}^*$, along with (2.4) and (3.1), implies that

$$\boldsymbol{\phi}^* = \boldsymbol{\theta}^0 \qquad \text{for all} \quad (F_1, F_2) \in \Omega \quad \text{and all} \quad \mathbf{Z}_N. \tag{3.5}$$

Let us now write

$$U_{Ni}(\mathbf{Z}_N) = \binom{N}{m_{i1} + m_{i2}}^{-1} \sum_{C_i} \phi_i^*(Z_{\alpha_1}, \cdots, Z_{\alpha_{m_{i1}+m_{i2}}}), \tag{3.6}$$

where the summation C_i extends over all possible

$$1 \leq \alpha_1 < \cdots < \alpha_{m_{i1}+m_{i2}} \leq N, \qquad \text{for} \quad i = 1, \cdots, p, \tag{3.7}$$

and let

$$\mathbf{U}_N(\mathbf{Z}_N) = (U_{N1}(\mathbf{Z}_N), \cdots, U_{Np}(\mathbf{Z}_N)). \tag{3.8}$$

Also, let $\mathcal{P}(\mathbf{Z}_N)$ denote the permutation probability distribution generated by the $N!$ permutations of the coordinates of $\mathbf{Z}_N$. It is then readily seen that

$$E\{\mathbf{U}_N | \mathcal{P}(\mathbf{Z}_N)\} = \mathbf{U}_N(\mathbf{Z}_N), \tag{3.9}$$

and hence, from (3.5), (3.6), (3.8) and (3.9), we obtain

$$E\{\mathbf{U}_N | \mathcal{P}\mathbf{Z}_N)\} = \boldsymbol{\theta}^0, \qquad \text{for all} \quad \mathbf{Z}_N \quad \text{and} \quad (F_1, F_2) \in \Omega. \tag{3.10}$$

For the time being, let us assume that

$$E\{\boldsymbol{\phi}' \cdot \boldsymbol{\phi} | \mathcal{P}(Z_N)\} < \infty. \tag{3.11}$$

and later, we will establish certain conditions under which (3.11) holds. Then, let the covariance of ϕ_i and ϕ_j (with respect to $\mathcal{P}(\mathbf{Z}_N)$), when c of the X_α's and d of the Y_β's are common between the two sets of X_α's and the two sets of Y_β's, be denoted by

$$\zeta_{cd}^{(i,j)}(\mathbf{Z}_N), \qquad \begin{array}{l} 0 \leq c \leq \min(m_{i1}, m_{j1}), \\ 0 \leq d \leq \min(m_{i2}, m_{j2}), \end{array} \qquad \text{for} \quad i, j = 1, \cdots, p. \tag{3.12}$$

It may be noted then that all these quantities are random variables, as they depend on the random collection matrix $\mathbf{Z}_N$. Also, let

$$\zeta_{cd}^{(i,j)}(F), \qquad \begin{array}{l} 0 \leq c \leq \min(m_{i1}, m_{j1}), \\ 0 \leq d \leq \min(m_{i2}, m_{j2}), \end{array} \tag{3.13}$$

be the unconditional covariance of ϕ_i and ϕ_j, when $\{X_\alpha\}$, $\{Y_\beta\}$ are i.i.d.b.r.v. distributed according to the cdf $F(x)$, and when c of the X_α's and d of the Y_β's are common between the two sets of X_α's and two sets of Y_β's, for $i = 1, \cdots, p$. Then, it follows by simple algebraic manipulations that

$$\begin{aligned}(3.14)\qquad \sigma_{ij}(\mathbf{Z}_N) &= \operatorname{cov}\{U_{Ni}, U_{Nj}|\mathcal{P}(\mathbf{Z}_N)\}\\ &= \binom{n}{m_{j1}}^{-1}\binom{n_2}{m_{j2}}^{-1}\sum_{c=0}^{m_{j1}}\sum_{d=0}^{m_{j2}}\binom{m_{i1}}{c}\binom{m_{i2}}{d}\\ &\times\binom{n_1-m_{i1}}{m_{j1}-c}\binom{n_2-m_{i2}}{m_{j2}-d}\zeta_{cd}^{(i,j)}(\mathbf{Z}_N),\end{aligned}$$

for all $i, j = 1, \cdots, p$, and

$$\begin{aligned}(3.15)\qquad \sigma_{ij}(F) &= \operatorname{cov}\{U_{Ni}, U_{Nj}|F_1 \equiv F_2 \equiv F\}\\ &= \binom{n_1}{m_{j1}}^{-1}\binom{n_2}{m_{j2}}^{-1}\sum_{c=0}^{m_{j1}}\sum_{d=0}^{m_{j2}}\binom{m_{i1}}{c}\binom{m_{i2}}{d}\\ &\times\binom{n_1-m_{i1}}{m_{j1}-c}\binom{n_2-m_{i2}}{m_{j2}-d}\zeta_{cd}^{(i,j)}(F),\end{aligned}$$

for all $i, j = 1, \cdots, p$.

Also let

$$(3.16)\qquad \begin{aligned}\mathbf{\Sigma}(\mathbf{Z}_N) &= ((\sigma_{ij}(\mathbf{Z}_N)))_{i,j=1,\cdots,p},\\ \mathbf{\Sigma}(F) &= ((\sigma_{ij}(F)))_{i,j=1,\cdots,p}.\end{aligned}$$

Then, we have the following theorems.

THEOREM 3.1. *For any real estimable $\boldsymbol{\theta}(F_1, F_2)$ and for type A generalized U-statistics,*

$$(3.17)\qquad \zeta_{00}^{(i,j)}(\mathbf{Z}_N) = 0 \qquad \text{for all } i, j = 1, \cdots, p \text{ and all } \mathbf{Z}_N.$$

The proof follows more or less on the same line as in Sen ([21], lemma 2.1), and hence is omitted.

THEOREM 3.2. *If (3.11) holds and $(F_1, F_2) \in \mathcal{W}_0$, then*

$$(3.18)\qquad \zeta_{cd}^{(i,j)}(\mathbf{Z}_N) \underset{\text{a.s.}}{\to} \zeta_{cd}^{(i,j)}(F),$$

for all $0 \le c \le \min(m_{i1}, m_{j1})$, $0 \le d \le \min(m_{i2}, m_{j2})$, $i, j = 1, \cdots, p$ (where $F_1 \equiv F_2 \equiv F$). Further, if $\boldsymbol{\phi}$ has finite fourth-order moments and if for the distribution F, the associated order statistic is complete, then $\mathbf{\Sigma}(\mathbf{Z}_N)$ has uniformly (for all $(F_1, F_2) \in \mathcal{W}_0$) the minimum concentration ellipsoid (as well as minimum risk with any convex loss function) among all unbiased estimators of $\mathbf{\Sigma}(F)$.

PROOF. Let us write

$$\begin{aligned}(3.19)\qquad g_{cd}^{(i,j)}&(Z_{\alpha_1}, \cdots, Z_{\alpha m_{i1}+m_{i2}+m_{j1}+m_{j2}-c-d})\\ &= \frac{c!(m_{i1}-c)!(m_{j1}-c)!d!(m_{i2}-d)!(m_{j2}-d)!}{(m_{i1}+m_{i2}+m_{j1}+m_{j2}-c-d)!}\\ &\times \sum{}^* \phi_i(Z_{\alpha_1}, \cdots, Z_{\alpha m_{i1}+m_{i2}})\phi_j(Z_{\beta_1}, \cdots, Z_{\beta m_{j1}+m_{j2}}),\end{aligned}$$

where

(i) $\alpha_\ell = \beta_\ell$ for $\ell = 1, \cdots, c$;

(ii) $\alpha_{m_{i1}+\ell} = \beta_{m_{j1}+\ell}$ for $\ell = 1, \cdots, d$;

(iii) and $\alpha_i \neq \beta_j$ for any other (i, j),

(iv) and where the summation $\sum^*$ extends over every possible choice of α's and β's from $\alpha_1, \cdots, \alpha_{m_{i1}+m_{i2}+m_{j1}+m_{j2}-c-d}$.

It is readily seen that

$$\zeta_{cd}^{(i,j)}(\mathbf{Z}_N) = \binom{N}{m_{i1} + m_{i2} + m_{j1} + m_{j2} - c - d}^{-1} \times \sum g_{cd}^{(i,j)}(Z_{\alpha_1}, \cdots, Z_{\alpha_{m_{i1}+m_{i2}+m_{j1}+m_{j2}-c-d}}) - U_{Ni}(\mathbf{Z}_N)U_{Nj}(\mathbf{Z}_N). \tag{3.20}$$

Thus, $\zeta_{cd}^{(i,j)}(\mathbf{Z}_N) + U_{Ni}(\mathbf{Z}_N)U_{Nj}(\mathbf{Z}_N)$ is again a U-statistic of the N observations $\mathbf{Z}_N$, and hence, using the property of almost sure convergence of U-statistics (cf. Hoeffding [13]), it readily follows that if $E|g_{cd}^{(i,j)}| < \infty$ (which is implied by (3.11)),

$$\zeta_{cd}^{(i,j)}(\mathbf{Z}_N) + U_{Ni}(\mathbf{Z}_N)U_{Nj}(\mathbf{Z}_N) \underset{\text{a.s.}}{\rightarrow} \zeta_{cd}^{(i,j)}(F) + \theta_i^0\theta_j^0. \tag{3.21}$$

Further, from (3.4), (3.9), and (3.10), we have $\mathbf{U}_N(\mathbf{Z}_N) = \boldsymbol{\theta}^0$, for all $\mathbf{Z}_N$, and hence, from (3.21), we obtain

$$\zeta_{cd}^{(i,j)}(\mathbf{Z}_N) \underset{\text{a.s.}}{\rightarrow} \zeta_{cd}^{(i,j)}(F), \tag{3.22}$$

for all $0 \leq c \leq \min(m_{i1}, m_{j1})$; $0 \leq d \leq \min(m_{i2}, m_{j2})$, $i, j = 1, \cdots, p$. Consequently, from (3.14), (3.15), and (3.16), we find that

$$N\boldsymbol{\Sigma}(\mathbf{Z}_N) \underset{\text{a.s.}}{\rightarrow} N\boldsymbol{\Sigma}(F) \qquad \text{for all} \quad (F_1, F_2) \in \mathcal{W}_0. \tag{3.23}$$

Again if $F_1 \equiv F_2 \equiv F$, and if for F, the associated order statistic is complete, it follows from a well-known theorem on U-statistics in the vector case (cf. Fraser [7], p. 142) that a vector-valued U-statistic has uniformly a minimum concentration ellipsoid among all unbiased estimators (having finite second moments) of the same parameter vector. Thus, from (3.20), we obtain after a few algebraic manipulations that $\zeta_{cd}^{(i,j)}(\mathbf{Z}_N)$'s jointly have uniformly the minimum concentration ellipsoid among all unbiased estimators of $\zeta_{cd}^{(i,j)}(F)$'s.

Hence, from (3.14) and (3.15), we directly get that if $\boldsymbol{\phi}$ has finite fourth moments, $\boldsymbol{\Sigma}(\mathbf{Z}_N)$ has uniformly the minimum concentration ellipsoid among all unbiased estimators of $\boldsymbol{\Sigma}(F)$. Hence the theorem.

Let us now consider the properties of $\boldsymbol{\Sigma}(\mathbf{Z}_N)$ when $(F_1, F_2) \notin \mathcal{W}_0$. In this case, no small sample property can be properly studied, and we shall consider here some asymptotic results. The term 'asymptotic' is used in the sense that $N \rightarrow \infty$ subject to

$$n_1/N \rightarrow \lambda : 0 < \lambda < 1. \tag{3.24}$$

Also, let us define

$$\begin{aligned} \bar{F}_N(x) &= (n_1/N)F_1(x) + (n_2/N)F_2(x), \\ \bar{F}(x) &= \lambda F_1(x) + (1 - \lambda)F_2(x). \end{aligned} \tag{3.25}$$

Finally, by virtue of our assumption regarding $\mathbf{U}_N$, we have for $F_1 \equiv F_2 \equiv F$, $(F_1, F_2) \in \mathcal{W}_0$,

$$\lim_{N=\infty} N\boldsymbol{\Sigma}(F) = \boldsymbol{\Gamma}(F), \tag{3.26}$$

where $\boldsymbol{\Gamma}(F)$ is positive definite and finite for all $(F, F) \in \mathcal{W}_0$.

THEOREM 3.3. *If ϕ has finite fourth-order moments for all $(F_1, F_2) \in \Omega$, and if* (3.24) *holds,*

$$\zeta_{cd}^{(i,j)}(Z_N) \xrightarrow{P} \zeta_{cd}^{(i,j)}(\bar{F}), \tag{3.27}$$

for all $0 \leq c \leq \min(m_{i1}, m_{j1})$; $0 \leq d \leq \min(m_{i2}, m_{j2})$, $i, j = 1, \cdots, p$; *where $\bar{F}$ has been defined in* (3.25). *Further, if* (3.26) *holds for both $\mathbf{U}_N$ and $N\boldsymbol{\Sigma}(\mathbf{Z}_N)$, and the associate order statistic is complete, then $N\boldsymbol{\Sigma}(\mathbf{Z}_N)$ is asymptotically the minimum concentration ellipsoid estimator of $\boldsymbol{\Gamma}(\bar{F})$, for all $(F_1, F_2) \in \Omega$.*

PROOF. In an earlier paper [23], it has been shown by the present author that a pooled sample U-statistic converges in probability to the associated regular functional of the cdf $\bar{F}$, (defined in (3.24)), when F_1, F_2 are not identical. From this result, it readily follows that if ϕ has finite fourth-order moments for all $(F_1, F_2) \in \Omega$ and (3.24) holds, then

$$\zeta_{cd}^{(i,j)}(\mathbf{Z}_N) \xrightarrow{P} \zeta_{cd}^{(i,j)}(\bar{F}), \tag{3.28}$$

for all c, d, i, and j. Hence, the first part of the theorem.

In the same paper, it has also been shown that if the variance of a pooled sample U-statistic multiplied by the pooled sample size has a nonzero finite asymptotic limit (under $F_1 \equiv F_2 \equiv \bar{F}$), then the pooled sample U-statistic will asymptotically be the minimum variance unbiased estimate of the regular functional of the cdf $\bar{F}$, for all $(F_1, F_2) \in \Omega$.

The same result can be extended in a more or less straightforward manner to vector-valued U-statistics, and the asymptotic minimum variance unbiasedness can then be generalized to asymptotic minimum concentration ellipsoid unbiasedness. In our case, $N\boldsymbol{\Sigma}(\bar{F})$, by virtue of (3.26), is asymptotically equal to $\boldsymbol{\Gamma}(\bar{F})$, which is positive definite for all $(F_1, F_2) \in \Omega$. Hence, it follows from (3.14), (3.15), (3.16), and (3.26) that for all $(F_1, F_2) \in \Omega$,

$$N\boldsymbol{\Sigma}(\mathbf{Z}_N) \xrightarrow{P} \boldsymbol{\Gamma}(\bar{F}). \tag{3.29}$$

Finally, it follows from (3.14) and (3.16) that $N\boldsymbol{\Sigma}(\mathbf{Z}_N)$ is a $p \times p$ matrix, whose elements are linear functions of a set of pooled sample U-statistics. Since, these U-statistics are also assumed to satisfy a condition similar to (3.26), we readily get from (3.14), (3.15), and the discussion made above that $N\boldsymbol{\Sigma}(\mathbf{Z}_N)$ is asymptotically the minimum concentration ellipsoid unbiased estimator of $\boldsymbol{\Gamma}(\bar{F})$, for all $(F_1, F_2) \in \Omega$. Hence the theorem.

Thus the permutation covariance matrix $\boldsymbol{\Sigma}(\mathbf{Z}_N)$ possesses some convergence properties in both the situations when $(F_1, F_2) \in \mathcal{W}_0$ and $(F_1, F_2) \notin \mathcal{W}_0$.

THEOREM 3.4. *If $(F_1, F_2) \in \mathcal{W}_0$ and ϕ has finite moments of the order $2 + \delta$ $(\delta > 0)$, or if $(F_1, F_2) \in \Omega - \mathcal{W}_0$ and ϕ has finite fourth-order moments, then subject to* (3.26), (3.30) $$\mathcal{L}(N^{1/2}[\mathbf{U}_N - \boldsymbol{\theta}^0]) \xrightarrow{P} \mathbf{N}(\mathbf{0}, \boldsymbol{\Gamma}(\overline{F})),$$

where $\mathcal{L}$ stands for the convergence in distribution generated by the permutation probability function $\mathcal{P}(\mathbf{Z}_N)$, $\mathbf{N}$ for the p-variate normal distribution, and $\mathbf{0}$ for a null p-vector. Further, with respect to the same permutation probability measure

(3.31) $$\mathcal{L}([\mathbf{U}_N - \boldsymbol{\theta}^0](\boldsymbol{\Sigma}(\mathbf{Z}_N))^{-1}[\mathbf{U}_N - \boldsymbol{\theta}^0]') \xrightarrow{P} \chi_p^2,$$

where χ_p^2 has the chi square distribution with p degrees of freedom.

Proof. Let us define for each $i(= 1, \cdots, p)$,

(3.32) $$\begin{aligned}\phi_{i(10)}(Z_\alpha) &= E\{\phi(Z_\alpha, Z_{\alpha_2}, \cdots, Z_{\alpha m_{i1}+m_{i2}})|\mathcal{P}(\mathbf{Z}_N)\} \\ &= (N - 1_{Pm_{i1}+m_{i2}-1})^{-1} \cdot \sum_{S_{i\alpha}^*} \phi(Z_\alpha, Z_{\alpha_2}, \cdots, Z_{\alpha m_{i1}+m_{i2}}),\end{aligned}$$

where the summation $S_{i\alpha}^*$ extends over all possible $\alpha_2 \neq \cdots \neq \alpha_{m_{i1}+m_{i2}} = 1, \cdots, N\ (\neq \alpha)$, and

(3.33) $$\begin{aligned}\phi_{i(01)}(Z_\alpha) &= E\{\phi(Z_{\alpha_2}, \cdots, Z_{\alpha m_{i1}+m_{i2}}, Z_\alpha)|\mathcal{P}(\mathbf{Z}_N)\} \\ &= (N - 1_{Pm_{i1}+m_{i2}-1})^{-1} \sum_{S_{i\alpha}^*} \phi(Z_{\alpha_2}, \cdots, Z_{\alpha m_{i1}+m_{i2}}, Z_\alpha).\end{aligned}$$

Also, let

(3.34) $$\begin{aligned}V_{Ni} = (m_{i1}/n_1) &\sum_{j=1}^{n_1} \{\phi_{i(10)}(X_j) - \theta_i^0\} \\ &+ (m_{i2}/n_2) \sum_{j=1}^{n_2} \{\phi_{i(01)}(Y_j) - \theta_i^0\}, \qquad i = 1, \cdots, p,\end{aligned}$$

(3.35) $$\mathbf{V}_N = (V_{N1}, \cdots, V_{Np}).$$

It then follows from the results of Nandi and Sen [18] and of Sen [21], with direct extension to the vector case, that if ϕ has finite second-order moments, then with respect to the permutation probability measure $\mathcal{P}(\mathbf{Z}_N)$,

(3.36) $$N^{1/2}\{[\mathbf{U}_N - \boldsymbol{\theta}^0] - \mathbf{V}_N\} \xrightarrow{P} 0.$$

We will now show that, under (3.26), $N^{1/2}\mathbf{V}_N$ has a permutation distribution which is asymptotically a p-variate normal one. For this it is sufficient to show that if $\boldsymbol{\delta} = (\delta_1, \cdots, \delta_p)$ is any real nonnull vector, then $N^{1/2}(\boldsymbol{\delta}\mathbf{V}_N')$ has asymptotically a normal permutation distribution. If we now write

(3.37) $$\begin{aligned}g_N(Z_\alpha|\boldsymbol{\delta}) = \sum_{i=1}^{p} \delta_i\{&m_{i1}[\phi_{i(10)}(Z_\alpha) - \theta_i^0] \\ &- \frac{n_1}{n_2} m_{i2}[\phi_{i(01)}(Z_\alpha) - \theta_i^0]\}, \qquad \alpha = 1, \cdots, N;\end{aligned}$$

then using (3.9), (3.10), (3.37), (3.38), and (3.39), we have, after some essentially simple steps,

(3.38) $$\boldsymbol{\delta}\mathbf{V}_N' = \sum_{\alpha=1}^{N} C_{N\alpha}g_N(Z_\alpha|\boldsymbol{\delta}),$$

where

$$n_1 C_{N\alpha} = \begin{cases} 1, & \text{if } Z_\alpha \text{ belongs to the first sample,} \\ 0, & \text{otherwise; for } \alpha = 1, \cdots, N. \end{cases} \tag{3.39}$$

Now, we apply Wald-Wolfowitz-Noether-Hoeffding-Dwass-Motoo-Hâjek theorem on the asymptotic permutation distribution of linear permutation statistics to our particular case of $\boldsymbol{\delta}\mathbf{V}'_N$, defined in (3.38). For this, it appears to be sufficient to show that $\{C_{N\alpha}\}$ satisfies the condition

$$\frac{(1/N) \sum_{\alpha=1}^{N} (C_{N\alpha} - 1/N)^r}{\left\{(1/N) \sum_{\alpha=1}^{N} (C_{N\alpha} - 1/N)^2\right\}^{r/2}} = 0(1) \qquad \text{for} \quad r = 3, 4, \cdots; \tag{3.40}$$

and $\{g_n(Z_\alpha|\boldsymbol{\delta})\}$ satisfies in probability, the condition

$$\lim_{N=\infty} \frac{\sum_{\alpha=1}^{N} |g_N(Z_\alpha|\boldsymbol{\delta})|^r}{\left\{\sum_{\alpha=1}^{N} |g_N(Z_\alpha|\boldsymbol{\delta})|^2\right\}^{r/2}} = 0, \qquad \text{for some} \quad r > 2. \tag{3.41}$$

Since n_1 of the $C_{N\alpha}$'s are equal to $1/n_1$ and the rest equal to 0, it is easily seen that if (3.24) holds, (3.40) also holds. Further, if $\boldsymbol{\phi}$ has finite moments of order $2 + \delta(\delta > 0)$, it is then readily seen that $g_N(Z_\alpha|\boldsymbol{\delta})$ has also a finite moment of order $2 + \delta$, uniformly in N. Proceeding, then, precisely on the same line as in Sen ([21], (3.6), (3.7)), we get that

$$\frac{1}{N} \sum_{\alpha=1}^{N} |g_N(Z_\alpha|\boldsymbol{\delta})|^r = 0_p(1), \tag{3.42}$$

for any given $r > 2$. Further, extending the results of Sen ([21], (2.23)) to the vector case in a more or less straightforward manner, it can be shown by following the lines of Nandi and Sen ([18], (3.10)) and using the results of Sen [23], that under (3.24),

$$\frac{1}{N} \sum_{\alpha=1}^{N} [g_N(Z_\alpha|\boldsymbol{\delta})]^2 \xrightarrow{P} \frac{1}{1-\lambda} \sum_{i=1}^{p} \sum_{j=1}^{p} \delta_i \delta_j \{m_{i1} m_{j1} \zeta_{10}^{(i,j)}(\overline{F}) + m_{i2} m_{j2} \zeta_{01}^{(i,j)}(\overline{F})\}, \tag{3.43}$$

provided either $(F_1, F_2) \in \mathcal{W}_0$ and $\boldsymbol{\phi}$ has finite second-order moments or $(F_1, F_2) \in \Omega - \mathcal{W}_0$ and $\boldsymbol{\phi}$ has finite fourth-order moments. Since, by (3.26), $\boldsymbol{\Gamma}(\overline{F})$ is positive definite, we get from (3.43) that the right-hand side of it is essentially positive for any nonnull $\boldsymbol{\delta}$. Consequently, from (3.42) and (3.43) we get that (3.41) holds, in probability. Hence, $N^{1/2}(\boldsymbol{\delta}\mathbf{V}'_N)$ has asymptotically, in probability, a normal distribution with mean zero and a finite variance for all nonnull $\boldsymbol{\delta}$. Thus, $N^{1/2}\mathbf{V}_N$ has asymptotically, in probability, a p-variate normal permutation distribution. The first part of the theorem then follows readily from (3.36) and the preceding two theorems.

To prove the second part of the theorem, we note that by virtue of theorems 3.2 and 3.3, under the conditions stated in the theorem,

$$N\boldsymbol{\Sigma}(\mathbf{Z}_N) \xrightarrow{P} \boldsymbol{\Gamma}(\overline{F}), \tag{3.44}$$

for all $(F_1, F_2) \in \Omega$, and hence,

$$[\mathbf{U}_N - \boldsymbol{\theta}^0](\boldsymbol{\Sigma}(\mathbf{Z}_N))^{-1}[\mathbf{U}_N - \boldsymbol{\theta}^0]' \tag{3.45}$$
$$\mathbf{P}\{N^{1/2}[\mathbf{U}_N - \boldsymbol{\theta}^0]\}(\boldsymbol{\Gamma}(\overline{F}))^{-1}\{N^{1/2}[\mathbf{U}_N - \boldsymbol{\theta}^0]'\} = S_N \text{ (say)}.$$

Now by a well-known theorem (cf. Sverdrup [24]) on the limiting distribution of a continuous function of random variables, and from the distribution theory of quadratic forms of multinormal distributions, it follows from the first part of the theorem that S_N has asymptotically a chi square distribution with p degrees of freedom. Consequently, we get from (3.45) that

$$\mathcal{L}([\mathbf{U}_N - \boldsymbol{\theta}^0](\boldsymbol{\Sigma}(\mathbf{Z}_N))^{-1}[\mathbf{U}_N - \boldsymbol{\theta}^0]') \xrightarrow{P} \chi^2_p. \tag{3.46}$$

Hence, the theorem.

With these theorems, we will now proceed to consider our desired class of permutation tests.

4. The permutation test procedure

In the preceding two sections, the rationality of using $\mathbf{U}_N$ in the formulation of the tests as well as some properties of the permutation distribution of $\mathbf{U}_N$ have been discussed. Now, we are in a position to construct a suitable test function $I(\mathbf{U}_N)$ for testing the null hypothesis (2.2) against the set of alternatives (2.5). Since $I(\mathbf{U}_N)$ associates with each $\mathbf{U}_N$ a probability of rejecting H_0 in (2.2), and as this probability is determined by the permutation distribution function of $\mathbf{U}_N$ (conditioned on $\mathbf{Z}_N$), it follows readily that $I(\mathbf{U}_N)$ possesses the property of S-structure of tests (cf. Lehmann and Stein [16]). Consequently, it is a strictly distribution-free test.

Now $\mathbf{U}_N$ assumes values on a p-dimensional lattice, and conditioned on a given $\mathbf{Z}_N$, the number of points on this lattice is equal to $\binom{N}{n_1}^p$; though $\mathbf{U}_N$ can assume only $\binom{N}{n_1}$ values out of these, and at the remaining $\binom{N}{n_1}^p - \binom{N}{n_1}$ points, the permutation probability is zero. The permutation center of gravity of these mass points on the p-dimensional lattice is the point $\boldsymbol{\theta}^0$, and if (2.2) actually holds, then the permutation distribution will have a dense cluster around $\boldsymbol{\theta}^0$. Thus, we are to demarcate a set of points of this lattice, which will constitute the critical region. In small samples, all possible $\binom{N}{n_1}$ partitionings may be considered, and this set of points may be isolated. However, this procedure becomes prohibitively laborious as the sample sizes increase. In large samples, we are thus faced with the problem of using some suitable function of $\mathbf{U}_N$ as the test statistic and in approximating the permutation distribution of this statistics by some simple form.

Now using the usual concept of distance in the multivariate normal distribution, and noting that $\mathbf{U}_N$ has asymptotically, in probability, a multi-normal permutation distribution, it seems quite reasonable to propose the following test statistic,

$$T_N = [\mathbf{U}_N - \boldsymbol{\theta}^0](\boldsymbol{\Sigma}(\mathbf{Z}_N))^{-1}[\mathbf{U}_N - \boldsymbol{\theta}^0]', \tag{4.1}$$

and to reject (2.2) for large values of T_N. Thus, we consider the following test function:

$$\begin{aligned} I(\mathbf{U}_N) &= 1, &&\text{if} \quad T_N > T_{N,\epsilon}(\mathbf{Z}_N),\\ I(\mathbf{U}_N) &= a_\epsilon(\mathbf{Z}_N), &&\text{if} \quad T_N = T_{N,\epsilon}(\mathbf{Z}_N),\\ I(\mathbf{U}_N) &= 0, &&\text{if} \quad T_N < T_{N,\epsilon}(\mathbf{Z}_N), \end{aligned} \tag{4.2}$$

where $T_{N,\epsilon}(\mathbf{Z}_N)$ and $a_\epsilon(\mathbf{Z}_N)$ are so chosen that

$$E\{I(\mathbf{U}_N)|\mathcal{P}(\mathbf{Z}_N)\} = \epsilon, \qquad 0 < \epsilon < 1; \tag{4.3}$$

ϵ being the given level of significance. It then readily follows that

$$E\{I(\mathbf{U}_N)|(F_1, F_2) \in \mathcal{W}_0\} = \epsilon, \tag{4.4}$$

so that the test (4.2) has exactly the size ϵ. In small samples, the values of $T_{N\epsilon}(\mathbf{Z}_N)$ and $a_\epsilon(\mathbf{Z}_N)$ can be found out using the permutation distribution of $\mathbf{U}_N$, whereas in large samples, we have by virtue of theorem 3.4 that as N increases, subject to (3.24),

$$a_\epsilon(\mathbf{Z}_N) \xrightarrow{P} 0 \quad \text{and} \quad T_{N,\epsilon}(\mathbf{Z}_N) \xrightarrow{P} \chi^2_{p,\epsilon}, \tag{4.5}$$

where $\chi^2_{p,\epsilon}$ is the $100(1 - \epsilon)\%$ point of a χ^2 distribution with p degrees of freedom. Hence, asymptotically, the test (4.2) reduces to

$$\begin{aligned} I(\mathbf{U}_N) &= 1, \quad \text{if} \quad T_N \geq \chi^2_{p,\epsilon},\\ I(\mathbf{U}_N) &= 0, \quad \text{if} \quad T_N < \chi^2_{p,\epsilon}. \end{aligned} \tag{4.6}$$

Equation (4.6) will be termed the asymptotic permutation test and (4.2), the exact permutation test.

THEOREM 4.1. *If $\boldsymbol{\phi}$ has finite fourth-order moments for all $(F_1, F_2) \in \Omega$ and if* (3.26) *holds, then the permutation test $I(\mathbf{U}_N)$ in* (4.2) *or* (4.6) *is consistent against the set of alternatives $H:(F_1, F_2) \in \Omega - \mathcal{W}_\theta$.*

PROOF. It follows from the well-known properties of generalized U-statistics that for $(F_1, F_2) \in \Omega - \mathcal{W}_\theta$,

$$\mathbf{U}_N - \boldsymbol{\theta}^0 \xrightarrow{P} \boldsymbol{\theta}(F_1, F_2) - \boldsymbol{\theta}^0 = \boldsymbol{\xi}(F_1, F_2), \tag{4.7}$$

where $\boldsymbol{\xi}(F_1, F_2)$ is nonnull for all $(F_1, F_2) \in \Omega - \mathcal{W}_\theta$. Also, it follows from our theorem 3.3 that under the stated conditions

$$N\boldsymbol{\Sigma}(\mathbf{Z}_N) \xrightarrow{P} \boldsymbol{\Gamma}(\overline{F}), \tag{4.8}$$

where $\boldsymbol{\Gamma}(\overline{F})$ is positive definite. Consequently, from (4.1), (4.7), and (4.8), we obtain for $(F_1, F_2) \in \Omega - \mathcal{W}_\theta$,

$$T_N/N \xrightarrow{P} \xi(F_1, F_2)(\boldsymbol{\Gamma}(\bar{F}))^{-1}\xi'(F_1, F_2) > 0, \tag{4.9}$$

and thus,

$$\lim_{N=\infty} P\{T_N \geq \chi^2_{p,\epsilon} | (F_1, F_2) \in \Omega - \mathcal{W}_\theta\} = 1. \tag{4.10}$$

Hence the theorem.

Let now $N\boldsymbol{\Sigma}(F)$ be defined as in (3.16), and let $N\hat{\boldsymbol{\Sigma}}(F)$ be any consistent estimate of $N\boldsymbol{\Sigma}(F)$. Then, we consider a statistic T_N^* of the form

$$T_N^* = [\mathbf{U}_N - \boldsymbol{\theta}^0](\hat{\boldsymbol{\Sigma}}(F))^{-1}[\mathbf{U}_N - \boldsymbol{\theta}^0]'. \tag{4.11}$$

It may be noted that T_N^* can be easily shown to have asymptotically, under H_0 in (2.2), a χ^2 distribution with p degrees of freedom. Consequently, an asymptotically distribution-free test for H_0 in (2.2) may be based on T_N^*, using the following test function:

$$\begin{array}{lll} \text{if} & T_N^* \geq \chi^2_{p,\epsilon}, & \text{reject } H_0 \text{ in (2.2)}, \\ \text{if} & T_N^* < \chi^2_{p,\xi}, & \text{accept } H_0 \text{ in (2.2)}. \end{array} \tag{4.12}$$

This type of test has been proposed by Bhapkar [1] for the location problem only. We will term this test an asymptotic unconditional test. It then follows from our results in the preceding two sections that under H_0 in (2.2),

$$T_N \overset{P}{\sim} T_N^*. \tag{4.13}$$

In the next section we will consider some further relations between T_N and T_N^*, and here we only note that the consistency of T_N^*-test follows similarly as in theorem 4.1.

5. Asymptotic power properties of the tests

For studying the asymptotic power properties of the test (4.2), (4.6), and (4.12), we require to study the asymptotic nonnull distribution of T_N and T_N^*, defined in (4.1) and (4.11) respectively. First, these have to be considered for some sequence of alternative specifications for which the power will lie in the open interval (0, 1), and second, we are to consider the unconditional distributions of T_N and T_N^*, as the same will be required to study the power.

Thus, we assume that the two cdf's $F_1(x)$ and $F_2(x)$ are replaced by two sequences $\{F_{1N}(x)\}$ and $\{F_{2N}(x)\}$ of cdf's, each converging to a common cdf $F(x)$ as $N \to \infty$, in such a manner that

$$H_N\colon \boldsymbol{\theta}(F_{1N}, F_{2N}) = \boldsymbol{\theta}^0 + N^{-1/2}\boldsymbol{\lambda}, \tag{5.1}$$

where $\boldsymbol{\lambda}$ is a p-vector with finite elements, and it is assumed to be nonnull. Then we have the following results.

THEOREM 5.1. *Under the sequence of alternatives* $\{H_N\}$,

$$\mathcal{L}(T_N) \to \chi^2_{p,\Delta}, \tag{5.2}$$

where $\chi^2_{p,\Delta}$ *has the noncentral* χ^2 *distribution with* p *degrees of freedom and the noncentrality parameter*

$$\Delta = \boldsymbol{\lambda}(\boldsymbol{\Gamma}(F))^{-1}\boldsymbol{\lambda}', \tag{5.3}$$

provided (3.24) *and* (3.26) *hold, and* ϕ *has finite fourth-order moments for all* $(F_1, F_2) \in \Omega$.

PROOF. It is well known (cf. Fraser [7]) that whatever be $(F_1, F_2) \in \Omega$, under the stated regularity conditions $N^{1/2}\{\mathbf{U}_N - \boldsymbol{\theta}(F_{1N}, F_{2N})\}$ has asymptotically a multinormal distribution with a nonsingular dispersion matrix, as (3.26) holds. Using now the results of Sen [23], it can be readily seen that under $\{H_N\}$, the dispersion matrix of $N^{1/2}\{\mathbf{U}_N - \boldsymbol{\theta}(F_{1N}, F_{2N})\}$ converges to $\boldsymbol{\Gamma}(F)$, defined in (3.26). Consequently, it follows that

$$[N^{1/2}\{\mathbf{U}_N - \boldsymbol{\theta}^0\}(\boldsymbol{\Gamma}(F))^{-1}N^{1/2}\{\mathbf{U}_N - \boldsymbol{\theta}^0\}'] = S_N^* \text{ (say)}, \tag{5.4}$$

has asymptotically a noncentral χ^2 distribution with p degrees of freedom and the noncentrality parameter Δ, defined in (5.3). It also follows from theorem 3.3 and condition (5.1) that under $\{H_N\}$,

$$|N\boldsymbol{\Sigma}(\mathbf{Z}_N) - \boldsymbol{\Gamma}(\overline{F}_N)| \xrightarrow{P} 0; \qquad \overline{F}_N = \frac{n_1}{N}F_{1N} + \frac{n_2}{N}F_{2N}; \tag{5.5}$$

and from (5.1) we obtain, using the results of Sen [23], that under $\{H_N\}$ and (3.24)

$$\boldsymbol{\Gamma}(\overline{F}_N) \to \boldsymbol{\Gamma}(F) \quad \text{as} \quad N \to \infty. \tag{5.6}$$

Consequently, we get from (4.1), (5.4), (5.5), and (5.6) that

$$T_N \overset{P}{\sim} S_N^*, \qquad \mathcal{L}(T_N) \xrightarrow{P} \mathcal{L}(S_N^*) \to \chi^2_{p,\Delta}. \tag{5.7}$$

Hence the theorem.

THEOREM 5.2. *Under* $\{H_N\}$, $T_N \overset{P}{\sim} T_N^*$.

PROOF. Since $N\hat{\boldsymbol{\Sigma}}(F)$ estimates $N\boldsymbol{\Sigma}(F)$, and as under $\{H_N\}$ the dispersion matrix of $N^{1/2}\{\mathbf{U}_N - \boldsymbol{\theta}(F_{1N}, F_{2N})\}$ converges to $\boldsymbol{\Gamma}(F)$, which is also the limiting form of $N\boldsymbol{\Sigma}(F)$, it follows from (5.4) and a well-known convergence theorem that T_N^* is asymptotically equivalent to S_N^*, for the sequence of alternatives $\{H_N\}$.

Hence, from (5.7), we obtain under $\{H_N\}$,

$$T_N \overset{P}{\sim} T_N^* \sim S_N^*. \tag{5.8}$$

Hence the theorem.

Thus, it follows from a well-known result by Hoeffding ([12], p. 172) that the permutation test based on T_N and the asymptotically distribution-free test based on T_N^* are asymptotically power equivalent for the sequence of alternatives $\{H_N\}$.

The asymptotic power efficiency of the test based on T_N with respect to any other rival test can only be properly studied and made independent of $\boldsymbol{\lambda}$ in $\{H_N\}$, if the other test criterion has also (under $\{H_N\}$) a noncentral χ^2 distribution with the same degrees of freedom and the two noncentrality parameters are proportional for all $\boldsymbol{\lambda}$. Usually, these two tests may have noncentral χ^2 distributions with the same degrees of freedom, but their noncentrality parameters are not generally proportional (for all $\boldsymbol{\lambda}$). Thus, in general, the power efficiency depends on $\boldsymbol{\lambda}$, and in such a case, either one has to show that for all $\boldsymbol{\lambda}$, one of the two noncentrality parameters is at least as large as the other, or, one has to compute the supremum and infimum of the ratio of the two noncentrality parameters (with respect to $\boldsymbol{\lambda}$) and study the bounds for the asymptotic efficiency. The usual concept of Pitman efficiency is, generally, not adaptable in the multivariate case.

6. Illustrations and applications

Now, we will consider the two-sample bivariate location, scale, and association problem and study suitable nonparametric tests based on our results in the preceding sections. Let us first consider the location problem.

Here, let Ω be the set of all pairs of bivariate distributions, which are nondegenerate in the sense that the grade correlation of either of the cdf's is bounded away from ± 1. These two cdf's may be continuous or they may also be purely discrete distributions. Let us then define (with the same notations as in earlier sections)

$$\theta_i(F_1, F_2) = P\{X^{(i)} < Y^{(i)}\} + \tfrac{1}{2}P\{X^{(i)} = Y^{(i)}\}, \qquad i = 1, 2; \tag{6.1}$$
$$\boldsymbol{\theta}(F_1, F_2) = (\theta_1(F_1, F_2), \theta_2(F_1, F_2)).$$

Then for $(F_1, F_2) \in \mathcal{W}_0$, $\boldsymbol{\theta}(F_1, F_2) = (\frac{1}{2}, \frac{1}{2})$, whereas if for at least one of the two variates, the first sample observations are stochastically larger or smaller than the second sample observations, $\boldsymbol{\theta}(F_1, F_2) \neq (\frac{1}{2}, \frac{1}{2})$. Moreover, if we let

$$F_2(\mathbf{x}) = F_1(\mathbf{x} + \boldsymbol{\delta}), \qquad \text{where} \quad \boldsymbol{\delta} = (\delta_1, \delta_2), \tag{6.2}$$

then for nonnull $\boldsymbol{\delta}$, it is easily shown that $\boldsymbol{\theta}(F_1, F_2) \neq (\frac{1}{2}, \frac{1}{2})$.

Thus, for the location problem, we may use a permutation test based on the individual variate Wilcoxon-Mann-Whitney statistics, being compounded together as in (4.1). This follows more or less on the same line as in Chatterjee and Sen [2], with further generalizations to cover the case of discrete bivariate distributions too. Thus, we let

$$\begin{aligned} \mathbf{U}_N &= (U_{N1}, U_{N2}), \\ U_{Ni} &= \frac{1}{n_1 n_2} \sum_{\alpha=1}^{n_1} \sum_{\beta=1}^{n_2} \phi(X_\alpha^{(i)}, Y_\beta^{(i)}), \qquad i = 1, 2; \end{aligned} \tag{6.3}$$

where

$$\begin{aligned} \phi(a, b) &= 1 \quad \text{if} \quad a < b, \\ \phi(a, b) &= \tfrac{1}{2} \quad \text{if} \quad a = b, \\ \phi(a, b) &= 0 \quad \text{if} \quad a > b. \end{aligned} \tag{6.4}$$

Also among the N values of $Z_\alpha^{(i)}$, N_{ij} are equal to

$$Z_{\alpha j}^{(i)},\ j = 1, \cdots, k_{iN}, \qquad i = 1, 2, \tag{6.5}$$

where $N_{ij} \geq 0$. It is then easily shown that

$$V\{U_{Ni}|\mathcal{P}(\mathbf{Z}_N)\} = \frac{N^2}{n_1 n_2 (N-1)} \left\{\frac{1}{12}\left[1 - \sum_{j=1}^{k_{iN}} (N_{ij}/N)^3\right]\right\}, \qquad i = 1, 2; \tag{6.6}$$

and the permutation covariance of U_{N1}, U_{N2} is the rank correlation between $\{Z_\alpha^{(1)}, Z_\alpha^{(2)}; \alpha = 1, \cdots, N\}$, when both the sets of observations contain ties, and the expression for the same is available in Kendall ([14], p. 38). Once these are obtained, we can define T_N as in (4.1) and proceed similarly as in Chatterjee and Sen [2]. This test thus generalizes Putter's [20] Wilcoxon test to the bivariate case, and also Chatterjee and Sen's [2] test to the more general case of any pair of nondegenerate bivariate cdf's.

Let us next consider the scale problem. Extending the idea of Lehmann [15] to the bivariate case, let us define

$$\begin{aligned} \theta_i(F_1, F_2) &= P\{|X_\alpha^{(i)} - X_\beta^{(i)}| < |Y_\gamma^{(i)} - Y_\delta^{(i)}|\} \\ &\quad + \tfrac{1}{2}P\{|X_\alpha^{(i)} - X_\beta^{(i)}| = |Y_\gamma^{(i)} - Y_\delta^{(i)}|\}, \qquad i = 1, 2; \\ \boldsymbol{\theta}(F_1, F_2) &= (\theta_1(F_1, F_2), \theta_2(F_1, F_2)). \end{aligned} \tag{6.7}$$

Here also, for $(F_1, F_2) \in \mathcal{W}_0$, $\boldsymbol{\theta}(F_1, F_2) = (\frac{1}{2}, \frac{1}{2})$, while for any heterogeneity of scales, $\boldsymbol{\theta}(F_1, F_2) \neq (\frac{1}{2}, \frac{1}{2})$. Thus, if we define

$$\begin{aligned} \phi(a, b; c, d) &= 1, \quad \text{if} \quad |a - b| < |c - d|, \\ \phi(a, b; c, d) &= \tfrac{1}{2}, \quad \text{if} \quad |a - b| = |c - d|, \\ \phi(a, b; c, d) &= 0, \quad \text{if} \quad |a - b| > |c - d|, \end{aligned} \tag{6.8}$$

and write

$$\begin{aligned} U_{Ni} &= \binom{n_1}{2}^{-1}\binom{n_2}{2}^{-1} \sum \phi(X_\alpha^{(i)}, X_\beta^{(i)}; Y_\gamma^{(i)}, Y_\delta^{(i)}), \qquad i = 1, 2; \\ \mathbf{U}_N &= (U_{N1}, U_{N2}); \end{aligned} \tag{6.9}$$

an appropriate permutation test may be based on $\mathbf{U}_N$. Let us also define

$$g_N(Z_\alpha^{(i)}) = \frac{2}{(N-1)(N-2)(N-3)} \sum_{S_\alpha} \phi(Z_\alpha^{(i)}, Z_\beta^{(i)}, Z_\gamma^{(i)}, Z_\delta^{(i)}), \tag{6.10}$$

where the summation S_α extends over all possible choices of distinct β, γ, δ which are not equal to α, and where $\alpha = 1, \cdots, N$, $i = 1, 2$. Finally, let

$$\alpha_{Nij} = \frac{1}{N} \sum_{\alpha=1}^{N} g_N(Z_\alpha^{(i)}) g_N(Z_\alpha^{(j)}) - \tfrac{1}{4}, \qquad i, j = 1, 2. \tag{6.11}$$

It is then easily shown (cf. Sen [21] for the univariate case) that neglecting terms of the order N^{-2},

$$\operatorname{cov}(U_{Ni} U_{NJ}|\mathcal{P}(\mathbf{Z}_N)) = \frac{4N}{n_1 n_2} \alpha_{Nij} + 0(N^{-2}), \qquad \text{for} \quad i, j = 1, 2. \tag{6.12}$$

Thus, neglecting terms of the order N^{-2}, we may construct T_N as in (4.1) and proceed as in section 4. This test is thus an extension of Lehmann's [15] test to the bivariate as well as discrete type of cdf's case.

Finally, let us consider the association problem. Extending the notion of Hoeffding [9] to cover also the case of bivariate discrete distributions, let us define

$$\theta(F_1, F_2) = \theta(F_1) - \theta(F_2), \tag{6.13}$$

where

$$\begin{aligned}\theta(F_1) = P\{&\text{sign}\,(X_\alpha^{(1)} - X_\beta^{(1)})\,\text{sign}\,(X_\alpha^{(2)} - X_\beta^{(2)}) > 0\} \\ &+ \tfrac{1}{2}P\{(X_\alpha^{(1)} - X_\beta^{(1)})(X_\alpha^{(2)} - X_\beta^{(2)}) = 0\},\end{aligned} \tag{6.14}$$

and $\theta(F_2)$ is defined precisely on the same line with Y_α and Y_β. It may be noted that $\theta(F)$ may be treated as the probability of concordance in the general case, and it is related with another well-known measure of correlation, namely rank correlation τ (cf. Kendall [14]) by means of the simple relation

$$\tau(F) = 2\theta(F) - 1. \tag{6.15}$$

Thus for $(F_1, F_2) \in \mathcal{W}_0$, $\theta(F_1, F_2) = 0$, and it also implies that $\tau(F_1) = \tau(F_2)$.

Now, if we define for two vectors $\mathbf{a}$ and $\mathbf{b}$

$$\begin{aligned}\phi(\mathbf{a}, \mathbf{b}) &= 1 \quad \text{if} \quad (a^{(1)} - b^{(1)})(a^{(2)} - b^{(2)}) > 0, \\ \phi(\mathbf{a}, \mathbf{b}) &= \tfrac{1}{2} \quad \text{if} \quad (a^{(1)} - b^{(1)})(a^{(2)} - b^{(2)}) = 0, \\ \phi(\mathbf{a}, \mathbf{b}) &= 0 \quad \text{if} \quad (a^{(1)} - b^{(1)})(a^{(2)} - b^{(2)}) < 0,\end{aligned} \tag{6.16}$$

and

$$U_N = \binom{n_1}{2}^{-1} \sum \phi(X_\alpha, X_\beta) - \binom{n_2}{2}^{-1} \sum \phi(Y_\alpha, Y_\beta), \tag{6.17}$$

then the permutation test is based on U_N.

This test has been considered by Chatterjee and Sen [3] in the case of continuous cdf's, while Sen [22] has also extended the test to the c-sample as well as discrete case. Hence, this is not considered in detail.

In this paper, we have not considered the asymptotic power efficiency aspect of the tests, discussed above. It may be noted that for bivariate continuous cdf's, Chatterjee and Sen [2] have considered the asymptotic power efficiency of their location test with respect to Hotelling's T^2-test. The asymptotic power efficiency of the association test is also under investigation. The details of this aspect of the tests is being kept pending for a further investigation.

REFERENCES

[1] V. P. Bhapkar, "Some non-parametric tests for the multivariate several sample location problem," *Inst. Statist. Univ. North Carolina*, Mimeo. Ser. No. 415 (1965).

[2] S. K. Chatterjee and P. K. Sen, "Non-parametric tests for the bivariate two sample location problem," *Calcutta Statist. Assoc. Bull.*, Vol. 13 (1964), pp. 18–58.

[3] ———, "Some non-parametric tests for the two sample bivariate association problem," *Calcutta Statist. Assoc. Bull.*, Vol. 14 (1965), pp. 14–35.

[4] ———, "Non-parametric tests for the multisample multivariate location problem," *Ann. Math. Statist.*, submitted.

[5] M. Dwass, "On the asymptotic theory of certain rank-order statistics," *Ann. Math. Statist.*, Vol. 24 (1953), pp. 303–306.
[6] ———, "On the asymptotic normality of some statistics used in non-parametric tests," *Ann. Math. Statist.*, Vol. 26 (1955), pp. 334–339.
[7] D. A. S. Fraser, *Nonparametric Methods in Statistics*, New York, Wiley, 1957.
[8] J. Hájek, "Some extensions of the Wald-Wolfowitz-Noether theorem," *Ann. Math. Statist.*, Vol. 32 (1961), pp. 506–523.
[9] W. Hoeffding, "On the distribution of the rank correlation when the variates are not independent," *Biometrika*, Vol. 34 (1947), pp. 183–196.
[10] ———, "A class of statistics with asymptotically normal distributions," *Ann. Math. Statist.*, Vol. 19 (1948), pp. 293–325.
[11] ———, "A combinatorial central limit theorem," *Ann. Math. Statist.*, Vol. 22 (1951), pp. 558–566.
[12] ———, "The large sample power of tests based on permutation of observations," *Ann. Math. Statist.*, Vol. 23 (1952), pp. 169–192.
[13] ———, "The strong law of large numbers for *U*-statistics," *Inst. Statist. Univ. North Carolina*, Mimeo. Ser. No. 302 (1962).
[14] M. G. Kendall, *Rank Correlation Methods*, New York, Hafner, 1955.
[15] E. L. Lehmann, "Consistency and unbiasedness of certain non-parametric tests," *Ann. Math. Statist.*, Vol. 22 (1957), pp. 165–179.
[16] E. L. Lehmann and C. Stein, "On the theory of some non-parametric hypotheses," *Ann. Math. Statist.*, Vol. 20 (1949), pp. 28–45.
[17] M. Motoo, "On Hoeffding's combinatorial central limit theorem," *Ann. Inst. Statist. Math.*, Vol. 8 (1957), pp. 145–154.
[18] H. Nandi and P. K. Sen, "On the properties of *U*-statistics when the observations are not independent. Part two: Unbiased estimation of the parameters of a finite population," *Calcutta Statist. Assoc. Bull.*, Vol. 12 (1963), pp. 124–148.
[19] G. E. Noether, "On a theorem by Wald and Wolfowitz," *Ann. Math. Statist.*, Vol. 20 (1949), pp. 455–458.
[20] J. Putter, "The treatment of ties in some non-parametric tests," *Ann. Math. Statist.*, Vol. 26 (1955), pp. 368–386.
[21] P. K. Sen, "On some permutation tests based on *U*-statistics," *Calcutta Statist. Assoc. Bull.*, Vol. 14 (1965), pp. 106–126.
[22] ———, "On some multisample permutation tests based on a class of *U*-statistics," *J. Amer. Statist. Assoc.*, submitted.
[23] ———, "*U*-statistics and combination of independent estimates of regular functionals," *Ann. Math. Statist.*, submitted.
[24] E. Sverdrup, "The limit distribution of a continuous function of random variables," *Skand. Aktuarietidskr.*, Vol. 35 (1952), pp. 1–10.
[25] A. Wald and J. Wolfowitz, "Statistical tests based on permutation of observations," *Ann. Math. Statist.*, Vol. 15 (1944), pp. 368–372.