# A NOTE ON MAXIMAL POINTS OF CONVEX SETS IN $\ell_\infty$

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## 1. Introduction

The problem of characterizing maximal points of convex sets often arises in the study of admissible statistical decision procedures, of efficient allocation of economic resources (cf. Koopmans, [4], chapter 1, and references given there), and of mathematical programming (cf. Arrow, Hurwicz, and Uzawa, [2]).

Let $C$ be a convex set in a finite dimensional vector space, partially ordered coordinate-wise (that is, for $x = (x_i)$ and $z = (z_i)$, $x \geq z$ means that $x_i \geq z_i$ for every coordinate $i$). Let $D$ be the set of all strictly positive vectors (namely vectors all of whose coordinates are strictly positive); further, let $B$ be the set of vectors in $C$ that maximize $\sum_i y_i x_i$ for some vector $y = (y_i)$ in $D$. It is obvious that every vector in $B$ is maximal in $C$ with respect to the partial ordering $\leq$. One can also show that every vector that is maximal in $C$ also maximizes $\sum_i y_i x_i$ on $C$ for some nonnegative vector $y$. Arrow, Barankin, and Blackwell [1] showed further that every vector maximal in $C$ is in the (topological) closure of $B$. They also gave an example (in 3 dimensions) in which a vector in the closure of $B$ (and in $C$) is not maximal in $C$.

The purpose of this note is to generalize the Arrow-Barankin-Blackwell result to the case of $\ell_\infty$, the space of bounded sequences topologized by the sup norm. In this generalization, however, the set $C$ is assumed to be compact.

## 2. The theorem

Let $X$ denote $\ell_\infty$, that is, the Banach space of all bounded sequences of real numbers, with the sup norm topology, where the norm of $x = (x_i)$ in $X$ is

$$\|x\| \equiv \sup_i |x_i|. \tag{2.1}$$

For $x$ in $X$, I shall say that $x \geq 0$ if $x_i \geq 0$ for every $i$, and that $x > 0$ if $x \geq 0$ but $x \neq 0$. Also, for $x^1 = (x_i^1)$ and $x^2 = (x_i^2)$ in $X$, I shall say that $x^1 \geq x^2$ if $x^1 - x^2 \geq 0$ (and so on for $x^1 > x^2$).

A point $\hat{x}$ in a subset $C$ of $X$ will be called *maximal in* $C$ if there is no $x$ in $C$ for which $x > \hat{x}$.

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Let $Y$ denote the set of all continuous linear functions on $X$. For any $y$ in $Y$, I shall say that $y \geq 0$ if $y(x) \geq 0$ for all $x \geq 0$ in $X$, and that $y \gg 0$ if $y(x) > 0$ for all $x > 0$. Define

$$S \equiv \{y: y \in Y,\ \|y\| = 1,\ y \geq 0\},$$
$$S^+ \equiv \{y: y \in S,\ y \gg 0\}. \tag{2.2}$$

(Recall that for $y$ in $Y$, $\|y\| \equiv \sup \{|y(x)|: x \in X.\ \|x\| = 1\}$). It shall be understood that $Y$ has the weak* topology, and that the Cartesian product $X \times Y$ has the corresponding product topology.

If $\hat{y} \gg 0$, and $\hat{x}$ maximizes $\hat{y}(x)$ in a subset $C$ of $X$, then $\hat{x}$ is clearly maximal in $C$. On the other hand, if $\hat{x}$ is maximal in a *convex* subset $C$ of $X$, then there is a $\hat{y} \geq 0$ in $Y$ such that $\hat{x}$ maximizes $\hat{y}(x)$ in $C$. (To see this, consider the nonnegative orthant of $X$; this is a convex set with a nonempty interior, and its interior is disjoint from the convex set of all points $(x - \hat{x})$ for which $x$ is in $C$. The hyperplane that separates these two convex sets corresponds to the required $\hat{y}$.) It is easy to see that there can be maximal points in a convex set $C$ that do not maximize any strictly positive continuous linear function on $C$. The following theorem gives information about such points in the case in which $C$ is compact.

THEOREM. *If $\hat{x}$ is maximal in a compact convex subset $C$ of $X$, then there is a $\hat{y}$ in $S$ such that*

(1) *$\hat{x}$ maximizes $\hat{y}(x)$ on $C$, and*

(2) *$(\hat{x}, \hat{y})$ is the limit of a generalized sequence $(x^m, y^m)$ of points in $C \times S^+$ such that for each $m$, $x^m$ is maximal in $C$ and maximizes $y^m(x)$ on $C$.*

LEMMA 1. *Define $f(x, y) \equiv y(x)$; then $f$ is continuous on $X \times S$.*

PROOF. For any $x$, $\bar{x}$ in $X$ and $y$, $\bar{y}$ in $S$,

$$\begin{aligned} |f(x, y) - f(\bar{x}, \bar{y})| &= |y(x - \bar{x}) + y(\bar{x}) - \bar{y}(\bar{x})| \\ &\leq 1 \cdot \|x - \bar{x}\| + |y(\bar{x}) - \bar{y}(\bar{x})|. \end{aligned} \tag{2.3}$$

Hence $\|x - \bar{x}\| < \epsilon/2$ and $|y(\bar{x}) - \bar{y}(\bar{x})| < \epsilon/2$ imply $|f(x, y) - f(\bar{x}, \bar{y})| < \epsilon$, which completes the proof of the lemma.

LEMMA 2. *For any $p \gg 0$ in $Y$, define*

$$S_p \equiv \{y: y \in S,\ y \geq p\}; \tag{2.4}$$

*then for every $p \gg 0$ in $Y$, $S_p$ is convex and compact.*

PROOF. The set $S_p$ is immediately seen to be convex, as the intersection of two convex sets, $S$ and $\{y: y \in Y.\ y \geq p\}$. Note that the latter set is also closed. The set $S$ can also be characterized as $\{y: y \in Y, y \geq 0,\ y(e) = 1\}$, where $e \equiv (1, 1, \cdots, \text{etc.} \cdots)$, and is therefore clearly closed. Thus $S$ is a closed subset of the unit sphere in $Y$, which, by Alaoglu's theorem, is compact in the weak* topology; hence, $S$ is compact, and therefore also $S_p$.

LEMMA 3. *If $y(\bar{x}) \geq 0$ for every $y$ in $S^+$, then $\bar{x} \geq 0$.*

PROOF. Suppose that $\bar{x} = (\bar{x}_i)$ and that for some $k$, $\bar{x}_k < 0$. Let

$$q_k \equiv \frac{\|\bar{x}\| - (\frac{1}{2})\,\bar{x}_k}{\|\bar{x}\| - \bar{x}_k}, \tag{2.5}$$

let $q_j$ $(j \neq k)$ be any sequence of positive numbers such that

$$\sum_{j \neq k} q_j = 1 - q_k, \tag{2.6}$$

and define $q(x) \equiv \sum_i q_i x_i$. It is easy to verify that $q \gg 0$, $\|q\| = 1$, and $q(\bar{x}) < 0$, which completes the proof of the lemma.

PROOF OF THE THEOREM. The point $\hat{x}$ is maximal in the compact convex set $C$ if and only if 0 is maximal in the compact convex set $C - \{\hat{x}\}$; hence, without loss of generality we may take $\hat{x} = 0$.

By lemmas 1 and 2, for every $p \gg 0$ in $Y$, the hypotheses of a minimax theorem of Ky Fan (cf. [3], p. 121) are satisfied for the function $f$ defined on $C \times S_p$. Hence, there exist $x^p$ in $C$ and $y^p$ in $S_p$ such that, for all $x$ in $C$ and $y$ in $S_p$,

$$y(x^p) \geq y^p(x^p) \geq y^p(x). \tag{2.7}$$

In particular, since 0 is in $C$,

$$y^p(x^p) \geq 0. \tag{2.8}$$

Let $D$ be the set of all $p \gg 0$ in $Y$. The family $\mathfrak{N} \equiv \{(x^p, y^p): p \in D\}$ is a net if $D$ is directed by $\leq$. It was noted in the proof of lemma 2 that $S$ is compact; hence, $\mathfrak{N}$ has a cluster point, say $(\bar{x}, \hat{y})$, in $C \times S$, and a subnet, say $\mathfrak{M}$, of $\mathfrak{N}$ converges to $(\bar{x}, \hat{y})$. Note that for every $(x^p, y^p)$ in $\mathfrak{M}$, inequality (2.7) implies that $x^p$ maximizes $y^p(x)$ on $C$, and therefore (since $y^p \gg 0$), $x^p$ is maximal in $C$.

I now show that $\bar{x} = 0$. For every $y$ in $S^+$ and $p$ in $Y$ such that $0 \ll p \leq y$, we have $y$ in $S_p$, and hence, by (2.7) and (2.8), $y(x^p) \geq 0$; hence, by continuity, $y(\bar{x}) \geq 0$. In other words, for every $y$ in $S^+$, $y(\bar{x}) \geq 0$. It follows by lemma 3 that $\bar{x} \geq 0$. Since 0 is maximal in $C$, $\bar{x} = 0$.

To complete the proof, it suffices to show that the maximum of $\hat{y}(x)$ on $C$ is 0. From (2.7), for every $p \gg 0$ in $Y$ and every $x$ in $C$,

$$f[(x - x^p), y^p] \leq 0. \tag{2.9}$$

Hence, by the continuity of $f$ (lemma 1), $f(x, \hat{y}) \leq 0$.

Every continuous linear function $y$ on $X$ can be represented as an integral with respect to a finitely additive, finite, measure on the integers. In particular, it can be represented in the form

$$y(x) = \sum_{i < \infty} y_i x_i + y_\infty(x), \tag{2.10}$$

where $\sum_{i<\infty} |y_i| < \infty$, and $y_\infty$ is a continuous linear function such that $y_\infty(x) = 0$ for every $x$ with only a finite number of nonzero coordinates. From this representation, it is clear that $y \gg 0$ if and only if, in (2.10), $y_i > 0$ for every $i < \infty$.

It is an open question whether the theorem can be sharpened by replacing the set $S^+$ by the set of continuous linear functions of the form (2.10) with $y \gg 0$, $y_\infty = 0$, and $\sum_{i<\infty} y_i = 1$. It is also not known whether the condition that $C$ be compact can be dispensed with.

## REFERENCES

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[4] T. C. Koopmans, *Three Essays on the State of Economic Science*, New York, McGraw-Hill, 1957.