SOME CHARACTERIZATION PROBLEMS IN STATISTICS

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1. Introduction

In this paper we shall discuss problems connected with tests of the hypothesis that a theoretical distribution belongs to a given class, for instance, the class of normal distributions, or uniform distribution or Poisson distribution. The statistical data consist of a large number of small samples (see [1]).

2. Reduction to simple hypotheses

Let $(\mathcal{X}, \mathcal{A})$ be a measurable space ($\mathcal{X}$ is a set and $\mathcal{A}$ is a σ-algebra of subsets of $\mathcal{X}$). Let $\mathcal{P}$ be a set of probability distributions defined on $\mathcal{A}$, let $(\mathcal{Y}, \mathcal{B})$ be another measurable space, and let $Y = f(X)$, $X \in \mathcal{X}$, be a measurable mapping of $(\mathcal{X}, \mathcal{A})$ into $(\mathcal{Y}, \mathcal{B})$. With this mapping every distribution P induces on $\mathcal{B}$ a corresponding distribution which we shall denote by Q_P^Y. We will be interested in the mappings (statistics) Y which possess the following two properties:

(1) Q_P^Y is the same for all $P \in \mathcal{P}$; in this case we will simply write $Q_{\mathcal{P}}^Y$.

(2) If for some P' on $\mathcal{A}$ one has $Q_{P'}^Y = Q_{\mathcal{P}}^Y$, then $P' \in \mathcal{P}$.

Sometimes it is expedient to formulate requirement (2) in the weakened form:

(2a) If $P' \in \mathcal{P}' \supset \mathcal{P}$ and $Q_{P'}^Y = Q_{\mathcal{P}}^Y$, then $P' \in \mathcal{P}$. In other words, we can assert in this case only that the equation $Q_{P'}^Y = Q_{\mathcal{P}}^Y$ implies $P' \in \mathcal{P}$ for some a priori restrictions $(P' \in \mathcal{P}')$ on P'.

If Y is a statistic satisfying (1) and (2), then it is clear that the hypothesis that the distribution of X belongs to class $\mathcal{P}$ is equivalent to the hypothesis that the distribution of Y is equal to $Q_{\mathcal{P}}^Y$.

Let us consider some examples. In these examples $(\mathcal{X}, \mathcal{A})$ is an n-dimensional Euclidean space of points $X = (x_1, \cdots, x_n)$ with the σ-algebra of Borel sets. The distributions belonging to $\mathcal{P}$ have a probability density of the form

$$p(x_1, \theta)p(x_2, \theta) \cdots p(x_n, \theta) \tag{2.1}$$

where p is a one-dimensional density and θ a parameter taking values in a parameter space.

EXAMPLE 1 (I. N. Kovalenko [2]). *Translation parameter.* Let $p(x; \theta) = p(x - \theta)$, with $-\infty < \theta < \infty$ (*additive type*). Here obviously it is necessary to take the $(n - 1)$-dimensional statistic $Y = (x_1 - x_n, \cdots, x_{n-1} - x_n)$. Of course, we can take any uniquely invertible function, for example $Y' = (x_1 - \bar{x}, \cdots, x_n - \bar{x})$ where $\bar{x} = (1/n) \sum_1^n x_k$.

In [2] it is shown that for $n \geq 3$, the distribution of Y determines the characteristic function $f(t) = \int_{-\infty}^{\infty} e^{itx} p(x)\, dx$ to within a factor of the form $e^{i\gamma t}$, on every interval where $f(t) \neq 0$. In particular, if $f(t) \neq 0$ for every t, then for $n \geq 3$, the statistic Y satisfies conditions (1) and (2) of section 2. This is also true if $f(t)$ is uniquely determined by its values in some neighborhood of zero (for example, if $f(t)$ is analytic in some neighborhood of zero).

In this paper, for every n there is given a pair of distributions, not belonging to the same additive type, for which the distribution of the statistic Y is the same for samples of size n. In section 4 these results are extended to a sample from a multidimensional population, and in section 5 to the case of a scale parameter.

REMARK. Let us assume that a distribution with density $p(x)$ has four finite moments: $m_1 = 0$, m_2, m_3, m_4, and that $p(x) \leq A$. Let us denote by $F(x)$ the corresponding distribution function and let $G(x)$ be another distribution function such that the distribution of Y is the same for F and G. Then it can be shown that

$$\inf_{\theta} \sup_{x} |G(x) - F(x - \theta)| \leq C(A, m_2, m_3, m_4) \frac{1}{\sqrt{n}}. \tag{2.2}$$

That is, if the sample size n is large, all the additive types corresponding to a given distribution of the statistic Y must be close to each other.

EXAMPLE 2 (A. A. Zinger, Yu. V. Linnik [3], [4]). Let $\theta = (a, \sigma)$, $-\infty < a < \infty$, $\sigma > 0$, and let

$$p(x, \theta) = \frac{1}{\sigma} \varphi\left(\frac{x - a}{\sigma}\right) \tag{2.3}$$

where φ is a normal (0, 1) density. Here it is natural to take the $(n - 2)$-dimensional statistic $Y = (y_1, \cdots, y_n)$, $y_k = (x_k - \bar{x})/s$, where $s^2 = \sum_{k=1}^{n} (x_k - \bar{x})^2$, $s > 0$. The sum of the components y_k of the vector Y is equal to zero, and the sum of their squares is unity. Thus the distribution of Y is concentrated on an $(n - 2)$-dimensional sphere $\sum y_k = 0$, $\sum y_k^2 = 1$.

It is known [1], [3] that for $p(x, \theta)$ defined by formula (2.3) the distribution of Y is uniform on this sphere. In [3] it is shown that for $n \geq 6$, the statistic Y possesses properties (1) and (2) of section 2; that is, from uniformity of the distribution of Y on the corresponding sphere it follows that the x's are normally distributed. This result is extended to distributions different from the normal in section 6.

It is clear for both examples cited that the choice of the statistic Y is based on considerations of invariance. Namely, there exists a group $\mathcal{G}$ of one-to-one (or almost one-to-one) mappings of the sample space onto itself ($X = (x_1, \cdots, x_n) \to (x_1 - a, \cdots, x_n - a)$ in the first example and $X \to ((x_1 - a)/\sigma), \cdots, (x_n - a)/\sigma))$ in the second) having the property that distributions of "random elements" X and gX, $g \in \mathcal{G}$, simultaneously belong to or do not belong to $\mathcal{P}$. In addition, for any two distributions P_1 and P_2 there exists $g \in \mathcal{G}$ such that for every $\mathcal{A}$, $P_2(\mathcal{A}) = P_1(g\mathcal{A})$.

In this case it is natural to take for Y a maximal invariant of the group $\mathcal{G}$.

Obviously, for $P \in \mathcal{P}$, Y possesses property (1) of section 2. The question of when Y possesses property (2) is related to a number of very difficult questions of analytic statistics. (See below for the problem of characterization of multidimensional distributions; see also [5]).

EXAMPLE 3 (L. N. Bolshev [6]). In the case when $x_1, x_2, \cdots, x_n$, take only values $0, 1, 2, \cdots$ with probabilities $p(x, \theta) = (\theta^x/x!)e^{-\theta}$ with $\theta > 0$, the considerations of invariance appear useless. Another approach based on utilization of sufficient statistics appears suitable. This approach will be discussed in detail in another paper.

3. Multidimensional location parameter

Now we shall consider the case of a family $\mathcal{P}$, given by formula (2.1) under the assumption that the x_j are ℓ-dimensional vectors: $x_j = (x_j^{(1)}, \cdots, x_j^{(\ell)})$ and

$$p(x, \theta) = p(x - \theta) \tag{3.1}$$

where it is known beforehand that θ lies in a k-dimensional subspace π_k of the space R^ℓ. Without loss of generality we shall suppose that π_k is defined by the relations

$$\theta_{k+1} = \theta_{k+2} = \cdots = \theta_\ell = 0. \tag{3.2}$$

As usual, we say that the density p satisfies the condition of Cramér if the integral $\int_{R^\ell} e^{(h,x)}p(x)\,dx$ is finite for all h lying in some neighborhood of zero of the space R^ℓ.

THEOREM 1. *Let $X = (x_1, \cdots, x_n)$ be a sample from the distribution* (3.1) *with conditions* (3.2). *We let $x'_j = (x_j^{(1)}, \cdots, x_j^{(k)}, 0, \cdots, 0)$. Then the statistic $Y = (Y_1, Y_2)$ where $Y_1 = x_1 - x'_3$, $Y_2 = x_2 - x'_3$, satisfies conditions* (1) *and* (2) *of section* 2.

PROOF. Let $t = (t^{(1)}, \cdots, t^{(\ell)})$, $\tau = (\tau^{(1)}, \cdots, \tau^{(\ell)})$, and let t' and τ' be defined in terms of t and τ in the same way that x'_j is defined in terms of x_j. Let $f(t) = Ee^{i(t,x_j)}$.

We note first of all that $Y_1 - Y_2 = x_1 - x_2$, and therefore the characteristic function of $x_1 - x_2$, that is $|f(t)|^2$, is uniquely defined by the distribution of the statistic Y. Now let f_1 and f_2 be two different characteristic functions of the x's, constituting a solution of the problem. Then the characteristic function of Y is equal to

$$E \exp [i(t, x_2 - x'_3) + i(\tau, x_2 - x'_3)] = f_u(t)f_u(\tau)\overline{f_u(t' + \tau')}, \quad u = 1, 2. \tag{3.3}$$

We take $\delta > 0$ so small that in a δ-neighborhood of zero, the functions $f_u(t)$, $u = 1, 2$, do not vanish. In what follows we will assume that t, τ and $t + \tau$ lie in this neighborhood. For these t and τ the principal value $A_u(t)$ of the argument of the function $f(t)$ satisfies the equation

$$A_u(t) + A_u(\tau) - A_u(t' + \tau') = \text{given function}. \tag{3.4}$$

Let us consider the corresponding homogeneous equation

$$a(t) + a(\tau) - a(t' + \tau') = 0. \tag{3.5}$$

We are interested in its real continuous solutions with $a(0) = 0$ (actually, from the assumption of the theorem it follows that $A_u(t)$ is infinitely differentiable in the neighborhood of zero which we are considering, and therefore it can be assumed that $a(t)$ is infinitely differentiable). We have $a(t') + a(\tau') = a(t' + \tau')$. Therefore,

$$a(t') = \sum_{j=1}^{k} \gamma_j t^{(j)} \tag{3.6}$$

Further, from $a(t) + a(\tau') = a(t' + \tau')$, it follows that $a(t) = a(t')$. In such a way, in the neighborhood of zero which we are considering

$$A_1(t) - A_2(t) = \sum_{j=1}^{k} \gamma_j t^{(j)} \tag{3.7}$$

and

$$f_1(t) = f_2(t) \exp \left\{ i \sum_{j=1}^{k} \gamma_j t^{(j)} \right\}. \tag{3.8}$$

Because of the analyticity of f_1 and f_2, this equation holds for all values of t.

REMARK. If $f(t) \neq 0$ for every t, then equation (3.8) is obtained without the condition stated in theorem 1.

4. Scale parameter in a multidimensional population

Now we shall consider the case of a family $\mathcal{P}$, given by formula (2.1), under the assumption that x_j is an ℓ-dimensional vector and

$$p(x, \theta) = p\left(\frac{x}{\theta}\right)\frac{1}{\theta^{\ell}}. \tag{4.1}$$

Let $X = (x_1, x_2, \cdots, x_n)$ be a sample from the distribution (4.1) $x_j = (x_j^{(1)}, \cdots, x_j^{(\ell)})$. The distribution of the 2 ℓ-dimensional vectors

$$V_j = (\ln |x_j^{(1)}|, \cdots, \ln |x_j^{(\ell)}|, \quad \operatorname{sign} x_j^{(1)}, \cdots, \operatorname{sign} x_j^{(\ell)}) \tag{4.2}$$

belongs to the 2 ℓ-dimensional additive type with density

$$q(v, \hat{\theta}) = q(v^{(1)} - \hat{\theta}, \cdots, v^{(\ell)} - \hat{\theta}, \quad v^{(\ell+1)}, \cdots, v^{(2\ell)}) \tag{4.3}$$

where $\hat{\theta} = \ln \theta$. The following theorem is easily derived from the result of the preceding section.

THEOREM 2. *Assume that $p(x)$ is bounded and satisfies Cramér's condition. Then the statistic $Y = (V_1 - V_3', V_2 - V_3')$, where V_3' is defined in terms of V_3 according to the rule of theorem* 1 *(with replacement of ℓ by 2ℓ and k by ℓ), possesses properties* (1) *and* (2).

The proof consists of verifying that the distribution of V_j satisfies the conditions of theorem 1.

5. One-dimensional linear type

We return to the one-dimensional case analogous to that considered in example 2, section 2. Let $\theta = (a, b)$, $-\infty < a < \infty$, $b > 0$ and

$$p(x, \theta) = \frac{1}{b} p\left(\frac{x - a}{b}\right). \tag{5.1}$$

We will call a type symmetric if it is possible to choose the function p to be even. Let $\bar{x}$, s, and y_k keep the same meaning as in example 2, section 2. Let us denote by $\mathcal{P}'$ the family of distributions (2.1) which corresponds to symmetric types.

THEOREM 3. *If p is symmetric and bounded and satisfies Cramér's condition, then for $n \geq 6$, the statistic*

$$Y^* = \left[\left(\frac{y_4 - y_3}{y_2 - y_1}\right)^2, \left(\frac{y_6 - y_5}{y_2 - y_1}\right)^2\right] \tag{5.2}$$

possesses properties (1) *and* (2a) *of section* 2.

PROOF. We have

$$Y_1^* = \left(\frac{y_4 - y_3}{y_2 - y_1}\right)^2 = \left(\frac{x_4 - x_3}{x_2 - x_1}\right)^2 \tag{5.3}$$

and an analogous equality for the second component Y_2^* of the vector Y^*. Let p' be a symmetric density, different from p and such that $Q_{p'}^{Y^*} = Q_p^{Y^*}$. From the fact that p satisfies Cramér's condition and is bounded, it follows easily that $\ln Y_1^*$ and at the same time $\ln (x_n - x_1)^2$ satisfy Cramér's condition (both for p and for p'). To the sample of size 3 made up of the variables $\ln (x_2 - x_1)^2$, $\ln (x_4 - x_3)^2$, $\ln (x_6 - x_5)^2$, one can apply what was said in the remark on example 1, section 2. Consequently, the distribution of Y^* determines the distribution of $\ln (x_2 - x_1)^2$ to within a translation parameter, and the distribution of $(x_2 - x_1)^2$ to within a scale parameter. Since the variable $x_2 - x_1$ is symmetrically distributed, its distribution also is determined to within a scale parameter. We note that thus far we have not made use anywhere of the symmetry of p'. If, for example, p is normal, then the distribution of $x_2 - x_1$ under p' is normal, and by Cramér's theorem x_1 is normal. In the general case, for a symmetric density p', the distribution of x_1 is uniquely determined except for a translation parameter by the distribution of $x_2 - x_1$. The theorem is proved.

Without the assumptions of symmetry, the formulation must be changed.

THEOREM 4. *Assume that p is bounded and satisfies Cramér's condition. Then for $n \geq 9$, the statistic $Y^{**} = (Y_1^{**}, Y_2^{**})$ where*

$$\begin{aligned} Y_1^{**} &= \left(\ln\left|\frac{y_3 - y_1}{y_9 - y_7}\right|, \ln\left|\frac{y_2 - y_1}{y_8 - y_7}\right|, \operatorname{sign}(y_3 - y_1), \operatorname{sign}(y_2 - y_1)\right), \\ Y_2^{**} &= \left(\ln\left|\frac{y_6 - y_4}{y_9 - y_7}\right|, \ln\left|\frac{y_5 - y_4}{y_8 - y_7}\right|, \operatorname{sign}(y_6 - y_4), \operatorname{sign}(y_5 - y_4)\right) \end{aligned} \tag{5.4}$$

possesses properties (1) *and* (2), *section* 2.

PROOF. The distribution of the vector $(x_3 - x_1, x_2 - x_1)$ belongs to the multiplicative type. Using a sample of size 3, namely $(x_3 - x_1, x_2 - x_1)$, $(x_6 - x_4, x_5 - x_4)$, $(x_9 - x_7, x_8 - x_7)$, this multiplicative type is determined uniquely by the distribution of the statistic mentioned in the formulation of the theorem. Knowing the distribution of $(x_3 - x_1, x_2 - x_1)$, we determine the additive type of the distribution of x_1. The theorem is proved.

6. Property of stability

We shall consider now the question of continuity of the correspondence $\mathcal{P} \Leftrightarrow Q_{\mathcal{P}}^{Y}$ assuming that the statistic Y satisfies conditions (1) and (2). In order to avoid unwieldy formulas, we will consider at first the case of the one-dimensional additive type

$$p(x, \theta) = p(x - \theta), \qquad -\infty < \theta < \infty, \tag{6.1}$$

and the one-dimensional linear type

$$p(x, \theta) = \frac{1}{b} p\left(\frac{x - a}{b}\right). \tag{6.2}$$

Let us recall the concept of convergence of types. One says (see [7]) that a sequence of types $T^{(N)}$ converges weakly to type T ($T^{(N)} \Rightarrow T$) if there exist $\mathfrak{F}^{(N)} \in T^{(N)}$ converging weakly to $\mathfrak{F} \in T$.

In the case of linear types, one usually considers convergence *to proper types only.*

Let $p^{(N)}$ and p be probability densities. Assume that $T(p^{(N)}) \Rightarrow T(p)$. Then from the property of weak convergence, it follows immediately that $Q_p^Y(N) \Rightarrow Q_p^Y$ where $Y = (x_1 - x_n, \cdots, x_{n-1} - x_n)$ for the case (6.1), and

$$Y = \left(\frac{x_1 - \bar{x}}{s}, \cdots, \frac{x_n - \bar{x}}{s}\right) \tag{6.3}$$

for the case (6.2). The reciprocal assertion gives the following theorem.

THEOREM 5. *For the situation described by* (6.1) *or* (6.2), *suppose that*

$$Q_p^Y(N) \Rightarrow Q_p^Y, \tag{6.4}$$

where the type $T(p)$ is uniquely determined by p. Then the sequence of types generated by $p^{(N)}$ converges to the type generated by p: $T(p^{(N)}) \Rightarrow T(p)$. For the case of linear types we assume in addition that we have a sample size $n \geq 4$.

PROOF. A. *Additive type.* From the convergence (6.4) follows, as is easily seen, that $|f^{(N)}(t)|^2 \rightarrow |f(t)|^2$, from which follows (see [8]) "shift compactness" of $p^{(N)}$. (This means that for appropriately chosen θ_N the sequence of distributions with densities $p^{(N)}(x - \theta_N)$ is weakly compact.) Now if the distributions with densities $p^{(N_k)}(x - \theta_{N_k})$ converge weakly to the limit distribution with density p', then

$$Q_{p'}^Y \Leftarrow Q_p^Y(N_k) \Rightarrow Q_p^Y, \tag{6.5}$$

from which we obtain $p'(x) = p(x - \theta)$.

B. *Linear type.* Let $x_j^{(N)}, y_j^{(N)}, \cdots$ be values of x, y, and so on, with distributions generated by $p^{(N)}$. From the convergence (6.4) follows convergence of the distributions of

$$Y^{(N)*} = \ln\left(\frac{x_4^{(N)} - x_3^{(N)}}{x_2^{(N)} - x_1^{(N)}}\right)^2 \tag{6.6}$$

to the distribution of

$$Y^* = \ln\left(\frac{x_4 - x_3}{x_2 - x_1}\right)^2. \tag{6.7}$$

From this follows, as is easily seen, the "shift-compactness" of the distributions of $\ln (x_2^{(N)} - x_1^{(N)})^2$. We shall take now an arbitrary sequence $N_k \uparrow$ of natural numbers and choose from it a subsequence M_k for which the distributions of

$$\ln \left(\frac{x_2^{(M_k)} - x_1^{(M_k)}}{b_k} \right)^2, \qquad b_k > 0, \tag{6.8}$$

converge weakly to a limit distribution. Then the distributions of

$$\frac{x_2^{(M_k)}}{b_k} - \frac{x_1^{(M_k)}}{b_k} \tag{6.9}$$

also form a weakly convergent sequence, and the sequence of distributions of $(x_1^{(M_k)})/b_k$ is "shift-compact." From this it is obvious that the convergence (6.4) implies relative compactness of the sequence of types $T(p^{(N)})$. The proof can now be completed in the same way as in part A.

7. Characterization of multidimensional linear types

We shall say that the distributions of random vectors x and y belong to the same type if there exists a nonsingular matrix A and vector b such that

$$gx = Ax + b \tag{7.1}$$

has the same distribution as y.

Let $p(x)$ be any ℓ-dimensional density and $\theta = (A, b)$. We denote by $\mathcal{P} = T(p) = \{p_\theta\}$ the linear type generated in the obvious manner by the density p.

All presently known results on characterization of multidimensional distributions have been obtained under the assumption that the distributions considered belong to the class $\mathcal{P}'$, defined in the following manner. The distribution of an ℓ-dimensional random vector x belongs to class $\mathcal{P}'$ if in some coordinate system its components are independent. The group $\mathcal{G}$ of all transformations (7.1) induces a group $\hat{\mathcal{G}}$ of transformations $\hat{g}X = (gx_1, \cdots, gx_n)$ in the $n\ell$-dimensional space of vectors $X = (x_1, \cdots, x_n)$. A maximal invariant Y of the group $\hat{\mathcal{G}}$ can be expressed in terms of the determinants

(7.2)

$$\Delta_{i_1}, \cdots, {}_{i_\ell} = [x_{i_1} - \bar{x}, \cdots, x_{i_\ell} - \bar{x}], \qquad \text{where} \quad \bar{x} = (1/n)(x_1 + \cdots + x_n)$$

and where $[z_1, \cdots, z_\ell]$ denotes the volume of the oriented parallelepiped constructed on the vectors $z_1, \cdots, z_\ell$.

Let us assume that the sample size $n \geq 6\ell$. We shall take vectors $z_j = x_{2j} - x_{2j-1}$ with components $z_j^{(k)}$, $k = 1, 2, \cdots, \ell$. Let

$$\delta_k = [z_{k\ell+1}, \cdots, z_{k\ell+\ell}]^2, \qquad k = 0, 1, 2 \tag{7.3}$$

$$\hat{Y}_1 = \ln \frac{\delta_1}{\delta_0}, \qquad \hat{Y}_2 = \ln \frac{\delta_2}{\delta_0}, \tag{7.4}$$

$$\hat{Y} = (\hat{Y}_1, \hat{Y}_2). \tag{7.5}$$

It is clear that $\hat{Y}$ is a function of a maximal invariant Y of the group $\hat{\mathcal{G}}$. The following theorem (see [5]) holds.

THEOREM 6. *If a density p satisfies Cramér's condition and if it is bounded, then the statistic Y possesses properties* (1) *and* (2a) *with respect to the class* $\mathcal{P}''$ *of distributions of random vectors x which can be transformed into vectors with independent, identically distributed, symmetrical components by a transformation of the form* (7.1).

The proof of this theorem is based on a lemma which has independent interest.

LEMMA. *Let* $V_j^{(i)}$, $(i, j = 1, \cdots, \ell)$ *be independent random variables with the same distribution function* $V(x)$, *and let* $W_j^{(i)}$, $(i, j = 1, \cdots, \ell)$ *also be independent and have a distribution function* $W(x)$. *If all moments of* $V(x)$ *exist and the distribution of the determinant* $\Delta = \det \|V_j^{(i)}\|$ *coincides with the distribution of the determinant* $\delta = \det \|W_j^{(i)}\|$, *then* $V = W$.

8. Application to testing hypothesis

The classical method of testing the hypothesis that the distribution of a sample belongs to a given parametric family (2.1) consists in the construction, based on the results of observations, of an estimate θ^* for θ and in the subsequent test of the significance of the deviation of the empirical distribution from the theoretical with $\theta = \theta^*$. Another statement of the problem will interest us.

A large number s of small samples $X_1, \cdots, X_s$ of sizes $n_1, \cdots, n_s$ respectively is given. The null hypothesis H_0 is that for every j the distribution of X_j is in the family (2.1). If there exist statistics $Y_1, \cdots, Y_s$; $Y_j = f_j(X_j)$, satisfying properties (1) and (2), then the composite hypothesis H_0 is replaced by the simple hypothesis H_0': for every j the distribution of Y_j is equal to Q^{Y_j}. Let the dimensionality of the statistic Y_j be equal to m_j. With the proper transformation one can translate Y_j into z_j, $Y_j = \psi_j(z_j)$, where z_j has a uniform distribution on the unit cube in m_j-dimensional Euclidean space. This transforms the hypothesis H_0' into the equivalent hypothesis H_0'': the components of the $(m_1 + \cdots + m_s)$-dimensional vector $Z = (z_1, \cdots, z_s)$ are independent and uniformly distributed on the interval $[0, 1]$. In this way one can give a *standard form* to the hypothesis H_0. Of course, the first question which arises in connection with such transformations concerns the form taken by the alternative hypotheses. From this point of view the transformations mentioned must be "sufficiently smooth" so that they transform the "alternatives close" to H_0 into the "alternatives close" to H_0''.

For now we shall postpone the corresponding analysis.

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