MOMENTS OF CHI AND POWER OF t

J. L. HODGES, JR.
and
E. L. LEHMANN
UNIVERSITY OF CALIFORNIA, BERKELEY

1. Introduction and summary

This paper is concerned with two related computational problems: the precise calculation of central moments of the chi random variable of ν degrees of freedom, and the use of these moments in computing the power curve of the t-test. Whereas the methods are standard and available in various textbooks, the results have at several points been pushed farther than we have seen them elsewhere. We try to provide the formulas and coefficient tables that would be needed by the computer, but make no attempt to review the extensive literature on chi moments and t power.

Table A gives the coefficients required for obtaining the first twelve moments in terms of the expectation ϵ_ν. In section 3 the general term of an asymptotic series for log ϵ_ν is derived, which provides in table C the early coefficients of the series for ϵ_ν itself. Section 5 presents a formula for the first three terms in the series for moments of arbitrary order, supplemented in table E by additional terms for the first twelve moments. With these coefficients it is relatively easy to obtain precise values of the low moments for large ν.

Section 6 presents a series for the power of the t test in terms of chi moments. In favorable cases this method permits the precise computation of an entire power curve. It also leads to a relatively simple normal approximation for t power, accurate when ν is not too small and the significance level is moderate, and suggests an effective method of interpolation in the noncentral t tables.

2. The moments of chi in terms of its expectation

Let χ denote the chi random variable with ν degrees of freedom, and consider its standardized form $S = \chi/\sqrt{\nu}$. It is well known that

$$ES^p = \Gamma\left(\frac{\nu + p}{2}\right) \bigg/ \left(\frac{\nu}{2}\right)^{p/2} \Gamma\left(\frac{\nu}{2}\right), \qquad p = 1, 2, \cdots, \tag{2.1}$$

and that both the original and central moments of S can be expressed in terms of its expectation,

$$ES = \epsilon_\nu = \Gamma\left(\frac{\nu + 1}{2}\right) \bigg/ \sqrt{\frac{\nu}{2}}\, \Gamma\left(\frac{\nu}{2}\right). \tag{2.2}$$

With the partial support of the Office of Naval Research under Contract No. Nonr-222(43).

The expressions for the moments can be written compactly in terms of the products of successive even or odd integers $\pi_q = (\nu + q - 2)(\nu + q - 4)(\nu + q - 6) \cdots$, $q = 2, 3, 4, \cdots$, where the last factor is the smallest integer not smaller than ν. That is, $\pi_2 = \nu$, $\pi_3 = \nu + 1$, $\pi_4 = \nu(\nu + 2)$, $\pi_5 = (\nu + 1)(\nu + 3)$, and so on, and for convenience we define $\pi_0 = \pi_1 = 1$. Application of the gamma recursion formula to (2.1) gives, for $a = 0, 1, 2, \cdots$,

$$\nu^a ES^{2a} = \pi_{2a}, \qquad \nu^a ES^{2a+1} = \epsilon_\nu \pi_{2a+1}. \tag{2.3}$$

These formulas give the even moments exactly. For the odd moments, one may use the six decimal table 35 of ϵ_ν in Pearson and Hartley [7] to obtain six-figure values. Greater precision is available by the methods developed in the next two sections.

Consider now the central moments of S, say $M_p = E(S - \epsilon_\nu)^p$. By expanding the binomial and substituting (2.3), it can be seen that, for $a = 1, 2, 3, \cdots$,

$$\begin{aligned} \nu^{a-1} M_{2a} &= \sum_{i=0}^{a} \epsilon_\nu^{2i} \left[-\binom{2a}{2i-1} \pi_{2a-2i+1} + \binom{2a}{2i} \pi_{2a-2i} \right] \nu^{i-1}, \\ \nu^a M_{2a+1}/\epsilon_\nu &= \sum_{i=0}^{a} \epsilon_\nu^{2i} \left[\binom{2a+1}{2i} \pi_{2a-2i+1} - \binom{2a+1}{2i+1} \pi_{2a-2i} \right] \nu^i, \end{aligned} \tag{2.4}$$

where conventionally we put $\binom{2a}{-1} = 0$. The right sides of (2.4) are polynomials in ϵ_ν^2, whose coefficients are themselves polynomials in ν. The coefficients of these latter polynomials are shown in table A for the first twelve central moments. For example, $M_2 = 1 - \epsilon_\nu^2$ and $\nu M_3/\epsilon_\nu = (1 - 2\nu) + 2\nu\epsilon_\nu^2$.

Although table A makes it possible to compute M_p, for $2 \le p \le 12$, once ϵ_ν is known, it should be noted that there is a loss of significant figures as p and ν increase. Since the coefficients increase with p, the number of correct decimals in M_p decreases. Further, since M_p tends to 0 as ν increases (see section 5), the number of significant figures may fall off rapidly. For example, M_8 at $\nu = 10$ is not even reliable to one figure if one starts with a six-decimal value of ϵ_{10}.

Thus it may be necessary when using (2.4) to have ϵ_ν correct to many decimal places. In the two following sections we give expansions that provide precise values of ϵ_ν for large ν. For small ν, one may use the following forms of (2.2) which are convenient for use with a table of log factorials:

$$\epsilon_{2a} = \frac{\sqrt{\pi a}(2a)!}{2^{2a}(a!)^2}, \qquad \epsilon_{2a+1} = \frac{2^{2a+1/2}(a!)^2}{\sqrt{\pi}(2a+1)(2a)!}. \tag{2.5}$$

3. An asymptotic series for log ϵ_ν

An asymptotic series of the form

$$\log \epsilon_\nu = \frac{L_1}{\nu} + \frac{L_2}{\nu^2} + \frac{L_3}{\nu^3} + \cdots \tag{3.1}$$

can be developed from the series for $\log \Gamma(p)$ in terms of the Bernoulli numbers

TABLE A

COEFFICIENTS OF THE POLYNOMIALS FOR CENTRAL MOMENTS

	1						ϵ_ν^2					
	1	ν	ν^2	ν^3	ν^4	ν^5	1	ν	ν^2	ν^3	ν^4	ν^5
M_2	1						−1					
$\nu M_3/\epsilon_\nu$	1	−2						2				
νM_4	2	1					−4	2				
$\nu^2 M_5/\epsilon_\nu$	3	−6	−4					10	0			
$\nu^2 M_6$	8	6	1				−18	6	9			
$\nu^3 M_7/\epsilon_\nu$	15	−33	−33	−6				63	14	−14		
$\nu^3 M_8$	48	44	12	1			−120	40	96	20		
$\nu^4 M_9/\epsilon_\nu$	105	−256	−310	−92	−8			540	156	−180	−48	
$\nu^4 M_{10}$	384	400	140	20	1		−1050	400	1120	380	35	
$\nu^5 M_{11}/\epsilon_\nu$	945	−2535	−3450	−1310	−195	−10		5775	1760	−2530	−1100	−110
$\nu^5 M_{12}$	3840	4384	1800	340	30	1	−11340	5076	15000	6480	1020	54

	ϵ_ν^4					ϵ_ν^6				ϵ_ν^8			ϵ_ν^{10}		ϵ_ν^{12}
	ν	ν^2	ν^3	ν^4	ν^5	ν^2	ν^3	ν^4	ν^5	ν^3	ν^4	ν^5	ν^4	ν^5	ν^5
νM_4	−3														
$\nu^2 M_5/\epsilon_\nu$		4													
$\nu^2 M_6$	−20	−5				−5									
$\nu^3 M_7/\epsilon_\nu$		35	14				6								
$\nu^3 M_8$	−168	−84	14			−56	−28			−7					
$\nu^4 M_9/\epsilon_\nu$		378	252	0			84	48			8				
$\nu^4 M_{10}$	−1800	−1080	180	90		−756	−588	−42		−120	−75		−9		
$\nu^5 M_{11}/\epsilon_\nu$		4950	3894	198	−132		1386	1188	132		165	110		10	
$\nu^5 M_{12}$	−23100	−14960	2860	2420	275	−11880	−10824	−1584	132	−2376	−2178	−297	−220	−154	−11

(see Cramér [1], p. 129). Substitution of the latter series into (2.2), with p replaced successively by $\nu/2$ and $(\nu + 1)/2$, gives

$$\log \epsilon_\nu = -\tfrac{1}{2} + \frac{\nu}{2}\log\left(1 + \frac{1}{\nu}\right) + \sum_{j=1}^{\infty} \frac{2^{2j-1}B_{2j}}{2j(2j-1)\nu^{2j-1}}\left[\left(1 + \frac{1}{\nu}\right)^{-2j+1} - 1\right]. \tag{3.2}$$

On expanding the logarithm, it is seen that the first two terms on the right side of (3.2) contribute to L_k the amount

$$\frac{(-1)^k}{2(k+1)} = \frac{(-1)^{k+1}}{2k(k+1)}\left[\binom{k+1}{0}B_0 + \binom{k+1}{1}2B_1\right], \tag{3.3}$$

where we have used $B_0 = 1$, $B_2 = -\frac{1}{2}$. On expanding the binomials, one sees that the third term contributes to L_k the amount

$$\sum_{2j+m-1=k} (-1)^m \frac{2^{2j-1}B_{2j}}{2j(2j-1)} - \binom{2j+m-2}{m}. \tag{3.4}$$

In this sum, m runs over the limits $1 \leq m \leq k - 1$, with $m + k + 1$ even. Thus $(-1)^m = (-1)^{k+1}$. Since $B_3 = B_5 = \cdots = 0$, we may allow m to run over all integers from 1 to $k - 1$. Rearranging the coefficients, and changing variable to $k - m + 1 = q$, (3.4) may be combined with (3.3) to give

$$L_k = \frac{(-1)^{k+1}}{2k(k+1)} \sum_{q=0}^{k} \binom{k+1}{q} 2^q B_q. \tag{3.5}$$

This formula may be simplified with the aid of the identities (see Miller [4], p. 90; Nörlund [6], p. 22)

$$\sum_{q=0}^{k+1} \binom{k+1}{q} 2^q B_q = 2^{k+1}B_{k+1}\left(\tfrac{1}{2}\right) = -2(2^k - 1)B_{k+1}, \tag{3.6}$$

to yield $L_k = (-1)^k(2^{k+1} - 1)B_{k+1}/k(k+1)$. That is, $L_2 = L_4 = \cdots = 0$, whereas for $b = 1, 2, 3, \cdots$,

$$L_{2b-1} = -(4^b - 1)B_{2b}/(2b-1)(2b). \tag{3.7}$$

The Bernoulli numbers have been extensively tabulated (see Peters [8], table 8), making it possible to give explicitly as many terms of (3.1) as could be needed. Table B shows the first twelve nonzero coefficients as fractions in lowest terms.

For higher coefficients there is an effective approximate formula. From the bound on B_{2b} (see [4], p. 101) it is seen that

$$L_{2b-1} \doteq (-1)^{b+1}2(2b-2)!/\pi^{2b} \tag{3.8}$$

with a relative error not greater than 2^{-2b+1}. At L_{23} this approximation is already correct to 11 figures. Thus, to the terms provided by table B one may add additional terms computed recursively by

$$\frac{L_{2b+1}}{\nu^{2b+1}} \doteq -2b(2b-1)\left(\frac{L_{2b-1}}{\nu^{2b-1}}\right)\left(\frac{1}{(\pi\nu)^2}\right). \tag{3.9}$$

TABLE B

COEFFICIENTS L_{2b-1} OF THE SERIES FOR $\log \epsilon_\nu$

b	Numerator	Denominator
1	−1	4
2	1	24
3	−1	20
4	17	112
5	−31	36
6	691	88
7	−5 461	52
8	929 569	480
9	−3 202 291	68
10	221 930 581	152
11	−4 722 116 521	84
12	968 383 680 827	368

The series (3.1) is asymptotic, not convergent, and terms should not be added beyond the point where they begin to increase. However, it is very effective when ν is not too small. As an illustration, table I shows the computation of $\log \epsilon_{40}$ to

TABLE I

CALCULATION OF $\log \epsilon_{40}$

k	$L_k/(40)^k$
1	−. 006 250
3	651 041 666 6667
5	— 488 281 2500
7	926 4265
9	−3 2849
11	187
13	−2

$\log \epsilon_{40}$ = −. 006 249 349 445 691 4232
ϵ_{40} = . 993 770 137 124 628 880

18 decimal places. The conversion from $\log \epsilon_{40}$ to ϵ_{40} is immediate with the aid of the 18-decimal table of e^{-x} in National Bureau of Standards [5]. The value of ϵ_{40} may be substituted into (2.4) to find central moments of S. For example, M_8 at $\nu = 40$ is

$$(85008 + 1435080\epsilon_{40}^2 + 754880\epsilon_{40}^4 - 1881600\epsilon_{40}^6 - 448000\epsilon_{40}^8)/64000 = .0^5\,258\,683\,675\,02. \tag{3.10}$$

If all the terms in table B are used together with (3.9), $\log \epsilon_{40}$ may be obtained to about 40 decimals. For conversion of $\log \epsilon_\nu$ to ϵ_ν if more than 18 decimals are wanted, the following method may be used. First find a 10-place value of ϵ_ν from the exponential table, and use the method of continued fractions to get a close rational approximation m/n to ϵ_ν, where m and n are four-digit integers. Using table 13 of Peters [8], one can find $\eta = \log \epsilon_\nu - \log m + \log n$, to 48 decimals if desired. Since η is small, only a few terms of $\epsilon_\nu = m(1 + \eta + \frac{1}{2}\eta^2 + \cdots)/n$ will be needed. The same technique gives ϵ_ν^{2i} from $2i \log \epsilon_\nu$ for use in (2.4).

4. An asymptotic series for ϵ_ν

The series (3.1) for $\log \epsilon_\nu$ can be converted into a series

$$\epsilon_\nu = 1 + \frac{E_1}{\nu} + \frac{E_2}{\nu^2} + \cdots \tag{4.1}$$

by formally expanding exp ($\log \epsilon_\nu$). The coefficients are

$$E_k = L_k + \frac{1}{2!} \textstyle\sum_2 L_{i_1} L_{i_2} + \frac{1}{3!} \sum_3 L_{i_1} L_{i_2} L_{i_3} + \cdots , \tag{4.2}$$

where $\sum_r$ is taken over all $i_1, i_2, \cdots, i_r \geq 1$ having $i_1 + i_2 + \cdots + i_r = k$. We have computed the first twelve coefficients. In order to simplify the denominators, table C gives $4^k E_k$ as a fraction in lowest terms, rather than E_k itself. That is, these are the coefficients for the series in $(4\nu)^{-k}$:

TABLE C

COEFFICIENTS OF THE SERIES FOR ϵ_ν

k	$4^k E_k$		k	E_k
	Numerator	Denominator		
1	−1	1	13	−104. 762 957 37
2	1	2	14	26. 614 715 5
3	5	2	15	1 933. 225 106
4	−21	8	16	−488. 764 02
5	−399	8	17	−47 030. 779 9
6	869	16	18	11 855. 436
7	39 325	16	19	1 458 576. 31
8	−334 477	128	20	−336 973. 7
9	−28 717 403	128	21	−56 169 531
10	59 697 183	256		
11	8 400 372 435	256		
12	−34 429 291 905	1024		

$$\epsilon_\nu = 1 - \frac{1}{(4\nu)} + \frac{1}{2(4\nu)^2} + \frac{5}{2(4\nu)^3} - \frac{21}{8(4\nu)^4} - \cdots . \tag{4.3}$$

The exact coefficients rapidly become cumbersome, and table C gives the coefficients from E_{13} to E_{21} in decimal form.

We do not know a simple closed form for E_k, but can give a fairly simple expression for an approximation for large k. Because of the rapid increase of $|L_{2b-1}|$ (see (3.8)), the sum $\sum_r$ of (4.2) will be dominated by those terms

$L_{i_1}L_{i_2} \cdots L_{i_r}$ where one i_j is a large odd integer. Fix an odd n, $k > n > \frac{1}{2}k$, and collect the terms of (4.2) containing the factor L_n. They are

$$L_n\left\{\frac{1}{2!}2L_{k-n} + \frac{1}{3!}3\sum_{i_1+i_2=k-n} L_{i_1}L_{i_2} + \cdots\right\} = E_{k-n}L_n. \tag{4.4}$$

The form of the next most important terms, whose factor L_n with largest subscript has $n \leq \frac{1}{2}k$, depends on k mod 4. The resulting approximations

$$\begin{aligned} E_{4c} &= E_1L_{4c-1} + E_3L_{4c-3} + \cdots + E_{2c-1}L_{2c+1} + \tfrac{1}{64}L_{2c-1}^2, \\ E_{4c+1} &= L_{4c+1} + E_2L_{4c-1} + \cdots + E_{2c}L_{2c+1} + \tfrac{5}{256}L_{2c-1}^2, \\ E_{4c+2} &= E_1L_{4c+1} + E_3L_{4c-1} + \cdots + E_{2c-1}L_{2c+3} + \tfrac{1}{2}L_{2c+1}^2 + \tfrac{1}{32}L_{2c-1}L_{2c+1}, \\ E_{4c+3} &= L_{4c+3} + E_2L_{4c+1} + \cdots + E_{2c}L_{2c+3} - \tfrac{1}{8}L_{2c+1}^2, \end{aligned} \tag{4.5}$$

are good to about 8 figures at the limit of table C. Substitution of (3.8) into (4.5) gives the relations

$$\begin{aligned} E_{2b} &\sim -\tfrac{1}{4}L_{2b-1}\left\{1 - \frac{5\pi^2}{32(2b-3)(2b-2)} + 0(b^{-4})\right\}, \\ E_{2b+1} &\sim L_{2b+1}\left\{1 - \frac{\pi^2}{32(2b-1)(2b)} + 0(b^{-4})\right\}. \end{aligned} \tag{4.6}$$

When computing a precise value of ϵ_ν for large ν by series, one has a choice of (3.1) or (4.1). The latter has the advantage of giving ϵ_ν directly, but the former requires only half as many terms, and with the coefficients here provided is able to give greater precision. For example, (4.1) and table C provides at most 27 decimals in ϵ_{40}, compared with 40 decimals available for log ϵ_{40}. As shown in table II, the direct series requires 14 terms to produce the 18-decimal value of

TABLE II

CALCULATION OF ϵ_{40}

k	$E_k/(40)^k$
0	1.
1	−. 006 250
2	19 531 250
3	610 351 562 5000
4	−4 005 432 1289
5	−475 645 0653
6	3 237 2773
7	915 6065
8	−6 0841
9	−3 2648
10	212
11	187
12	−1
13	−2
ϵ_{40} =	.993 770 137 124 628 880

ϵ_{40} that table I gives with 7 terms. While the L-series is computationally preferable, we shall need the E-series for the development of the following sections.

5. Asymptotic series for M_p

As remarked in section 2, the computation of the central moments M_p by the method of that section becomes awkward when ν and p are large, and we now develop asymptotic series for M_p. Two approaches are used. One may substitute the expansion (4.1) into (2.4), and this method was used to compute the supple-

TABLE D

THREE COEFFICIENTS OF SERIES FOR M_p

a	Even moments			Odd moments		
	P_{2a}	$6C_{2a}^{(1)}$	$540C_{2a}^{(2)}$	P_{2a+1}	$90C_{2a+1}^{(1)}$	$7560C_{2a+1}^{(2)}$
1	1	−3	−270	1	45	−12 285
2	3	−6	135	10	−27	−21 357
3	15	−5	1 575	105	−99	−15 471
4	105	4	2 826	1 260	−151	7 893
5	945	25	1 080	17 325	−163	41 847
6	10 395	62	−7 255	270 270	−115	70 095
7	135 135	119	−24 955	4 729 725	13	69 173
8	2 027 025	200	−51 580	91 891 800	241	12 929
9	34 459 425	309	−80 274	1 964 187 225	589	−120 757
10	654 729 075	450	−93 765	45 831 035 250	1 077	−341 013

TABLE E

SUPPLEMENTARY COEFFICIENTS FOR LOW-ORDER MOMENTS

a	Even moments					
	$7560C_{2a}^{(3)}$	$1680C_{2a}^{(4)}$	$480C_{2a}^{(5)}$	$64C_{2a}^{(6)}$	$128C_{2a}^{(7)}$	$128C_{2a}^{(8)}$
1	4 725	4 830	−1 590	−2 372	5 165	110 123
2	28 350	−3 570	−23 100	1 809	144 646	
3	13 230	−65 870	−11 005	58 669		
4	−142 632	−113 043	224 852			
5	−440 073	130 643				
6	−610 858					

a	Odd moments				
	$1008C_{2a+1}^{(3)}$	$5760C_{2a+1}^{(4)}$	$1280C_{2a+1}^{(5)}$	$1024C_{2a+1}^{(6)}$	$2048C_{2a+1}^{(7)}$
1	−4 725	54 675	87 015	−122 101	−3 371 095
2	819	185 715	−7 921	−737 925	
3	11 349	123 687	−329 165		
4	17 229	−329 729			
5	5 687				

mentary coefficients shown in table E. It is, however, applicable only for small p, and another approach is used to find expressions valid for any p, leading to the coefficients shown in table D. In what follows, we retain only enough terms to make the method clear, although more terms were carried in deriving the values shown in table D.

The chi density $f_X(x) \sim x^{\nu-1}e^{-(1/2)x^2}$ may be expanded about $\sqrt{\nu - 1}$ by making the substitution $\chi = \sqrt{\nu - 1} + (Y/\sqrt{2})$. To simplify the notation we write $\sqrt{2(\nu - 1)} = u$, and find

$$\log f_Y(y) = \text{const.} - \frac{y^2}{2} + \frac{y^3}{2\cdot 3u} - \frac{y^4}{2\cdot 4u^2} + \frac{y^5}{2\cdot 5u^3} - \cdots . \tag{5.1}$$

The expansion is, of course, not valid over the entire range $-u < y < \infty$. However, over the interval $|y| < \log \nu$ the remainder after the term in u^{-j} is of smaller order than u^{-j} uniformly in y. Furthermore, the contribution to any moment of Y outside this interval is negligible. This may be seen by comparing the distribution of Y with that of a normal random variable D having $ED = 0$ and var $D = \frac{1}{2}$. It is easy to check that the contribution of $|d| > \log \nu$ to any moment of D is of smaller order than any inverse power of ν. From the fact that

$$\frac{d}{dy} \log \frac{f_Y(y)}{f_D(y)} = \frac{u}{2}\left\{\left(1 + \frac{y}{u}\right)^{-1} - 1\right\} \tag{5.2}$$

decreases monotonely in y, it can be seen that the tails of Y are even less important than those of D. Accordingly, if in our expansions we retain all terms of a given order in ν, the resulting moment series will be valid to that order.

It is notationally convenient to express the series in terms of $\mu = 2\nu$, and we write $\sqrt{\mu} = v$, noting that

$$u^{-j} = v^{-j}\left(1 - \frac{2}{v^2}\right)^{-1/2}. \tag{5.3}$$

Substitution of this expression in (5.1) gives

$$\log f_Y(y) = \text{const.} - \frac{y^2}{2} + \frac{y^3}{6v} - \frac{y^4}{8v^2} + \frac{5y^3 + 3y^5}{30v^3} + \cdots , \tag{5.4}$$

where in general the coefficient of v^{-j} is a polynomial in y of degree $j + 2$ with alternate terms vanishing.

Since we are interested in the central moments of S, we shift the origin by $Z = Y - EY = Y - v\epsilon_\nu + u = Y - (1/2v) + (5/8v^3) + \cdots$, finding

$$\log f_Z(z) = \text{const.} - \frac{z^2}{2} + \frac{-3z + z^3}{6v} + \frac{-1 + 2z^2 - z^4}{8v^2} + \cdots , \tag{5.5}$$

where the coefficient of v^{-j} is a polynomial in z of degree $j + 2$ with alternate terms vanishing. Application of the exponential transformation gives

$$f_Z(z) = C\varphi(z)\left\{1 + \frac{-3z + z^3}{6v} + \frac{-9 + 27z^2 - 15z^4 + z^6}{72v^2} + \cdots\right\} \tag{5.6}$$

where the coefficient of v^{-j} is a polynomial in z of degree $3j$ with alternate terms vanishing. Finally, C may be determined to make the probability equal 1, giving

$$f_Z(z) = \varphi(z)\left\{1 + \frac{-3z + z^3}{6v} + \frac{3 + 27z^2 - 15z^4 + z^6}{72v^2} + \cdots\right\}, \tag{5.7}$$

where the coefficient of v^{-j} has the same structure as before.

From (5.7) it is now easy to express the moments of Z in terms of the even moments of the standard normal distribution, say

$$N_{2a} = \int_{-\infty}^{\infty} z^{2a}\varphi(z)\,dz = 1\cdot 3\cdot 5\cdot \cdots \cdot (2a-1). \tag{5.8}$$

In the even case,

$$EZ^{2a} = N_{2a} + \frac{1}{72\mu}(3N_{2a} + 27N_{2a+2} - 15N_{2a+4} + N_{2a+6}) + \cdots, \tag{5.9}$$

where in general the coefficient of μ^{-k} is a linear combination of $N_{2a}, N_{2a+2}, \cdots, N_{2a+6k}$. If we factor out N_{2a} and note that $M_{2a} = EZ^{2a}/\mu^a$, it appears that

$$M_{2a} = \frac{P_{2a}}{\mu^a}\left\{1 + \frac{C_{2a}^{(1)}}{\mu} + \frac{C_{2a}^{(2)}}{\mu^2} + \cdots\right\} \tag{5.10}$$

where $P_{2a} = N_{2a}$ and where $C_{2a}^{(k)}$ is a polynomial in a of degree $3k$. The first two of these polynomials are

$$\begin{aligned} 18C_{2a}^{(1)} &= 2a^3 - 6a^2 - 5a, \\ 9720C_{2a}^{(2)} &= 20a^6 - 300a^5 + 512a^4 + 3708a^3 - 4753a^2 - 4047a. \end{aligned} \tag{5.11}$$

The odd case may be handled similarly, giving

$$M_{2a+1} = \frac{P_{2a+1}}{\mu^{a+1}}\left\{1 + \frac{C_{2a+1}^{(1)}}{\mu} + \frac{C_{2a+1}^{(2)}}{\mu^2} + \cdots\right\}, \tag{5.12}$$

where $P_{2a+1} = aN_{2a+2}/3$ and where $C_{2a+1}^{(k)}$ is again a polynomial in a of degree $3k$. We find

$$\begin{aligned} 270C_{2a+1}^{(1)} &= 10a^3 - 60a^2 - 106a + 291, \\ 68040C_{2a+1}^{(2)} &= 28a^6 - 588a^5 + 1372a^4 + 18480a^3 \\ &\quad - 33377a^2 - 114993a + 18513. \end{aligned} \tag{5.13}$$

Table D gives the values of these coefficients for the moments up to order 21, while as mentioned earlier table E provides certain additional coefficients for moments up to order 12. The use of these coefficients is illustrated in table III for M_8 at $\nu = 40$. As always, when using asymptotic series, one must be guided by the rate of decrease of the successive terms in judging the resulting accuracy. The numbers in table III suggest that the value of M_8 should be good to about eight figures. Comparison with the direct calculation (3.10) shows an error of 2 in the ninth figure. In this case, the series computation is much easier than the direct calculation, which requires one to carry about 17 decimals in $\epsilon_{40}^2, \cdots, \epsilon_{40}^8$.

TABLE III

CALCULATION OF M_8 AT $\nu = 40$ BY SERIES

k	$C_8^{(k)}/(80)^k$
0	1.000 000 00
1	8 333 33
2	817 71
3	−36 85
4	−1 64
5	14
	1.009 112 69

$$\times \frac{105}{(80)^4} = .0^5\,258\,683\,673 = M_8$$

6. Power of the t-test in terms of chi moments

The power of the t-test with ν degrees of freedom can be expressed as an asymptotic series involving the moments M_p of $\chi/\sqrt{\nu}$. In some cases, this series is an effective way of computing precise values of t power. It also throws some light on approximations and on the problem of interpolation in the noncentral t tables.

Let X have the standard normal distribution, so that $(X + \delta)/S$ has the noncentral t distribution with ν degrees of freedom and noncentrality parameter δ. For both one- and two-sided t-tests, the power can be expressed in terms of the quantity $\beta = P((X + \delta/S) < t) = E\Phi(tS - \delta)$ where Φ is the standard normal cumulative. If we write $w = t\epsilon_\nu - \delta$ and expand $\Phi(tS - \delta) = \Phi(w + t(S - \epsilon_\nu))$ about w, we have the asymptotic series

$$(6.1) \qquad \beta = \Phi(w) + T_2 + T_3 + \cdots, \qquad T_p = \frac{t^p}{p!} M_p \varphi^{(p-1)}(w).$$

The term T_p is a product of three factors, each depending on only one of the variables t, ν, w on which β depends. We have just seen how M_p can be computed with considerable precision for low values of p. The factor $t^p/p!$ offers no problems. The normal derivatives are extensively tabled in Harvard Computation Laboratory [2], and this table can, if necessary, be supplemented by the expressions

$$(6.2) \qquad \varphi'(w) = -w\varphi(w), \qquad \varphi''(w) = (w^2 - 1)\varphi(w),$$
$$\varphi'''(w) = -(w^3 - 3w)\varphi(w), \cdots$$

of normal derivatives in terms of Hermite polynomials. Thus, it is feasible to compute a number of terms of (6.1).

To gain an appreciation of the circumstances in which such a computation will be effective, note that $|\varphi^{(p-1)}(w)|$, and hence $|T_p|$, is bounded in w for any given p, ν, and t. Therefore, for those values of ν and t for which (6.1) works, it should do so for all w and hence for all δ. If ν is not too small, the order of magnitude of M_p will be given by the first term of the series (5.10) or (5.12). With this approximation,

$$(6.3)\quad \begin{aligned} \max_w |T_{2a}| &\sim \left(\frac{t^2}{\mu}\right)^a R_{2a}, & R_{2a} &= \max_w |\varphi^{(2a-1)}(w)|/a!2^a, \\ \max_w |T_{2a+1}| &\sim \frac{1}{t}\left(\frac{t^2}{\mu}\right)^{a+1} R_{2a+1}, & R_{2a+1} &= \max_w |\varphi^{(2a)}(w)|/3(a-1)!2^a. \end{aligned}$$

Some values of R_p are shown in table F. For both the even and odd cases, these values change slowly enough so that, roughly speaking, the series of maxima of the terms in (6.1) are like geometric series with ratio t^2/μ. This indicates that (6.1) will work to the extent that t^2/μ is less than one. Furthermore, the entries in table F serve to indicate about how many terms of (6.1) will be required in any given case.

TABLE F

FACTORS FOR MAXIMA OF $|T_p|$

a	R_{2a}	R_{2a+1}
1	.1210	.0665
2	.0688	.0997
3	.0481	.1247
4	.0369	.1454
5	.0300	.1636
6	.0252	.1800
7	.0218	.1950
8	.0192	.2089
9	.0171	.2220
10	.0154	.2343

As an illustration, consider the case $\nu = 40$ and $t = 2$. Here $t^2/\mu = \frac{1}{20}$, so the terms of (6.1) will decrease rapidly. In order to find β as a function of w, we shall compute β for a few equally spaced values of w as a basis for interpolation. The value $w = 0$ is attractive, since here the terms T_{2a} vanish. Since T_{2a} has the same value at w and $-w$, and since T_{2a+1} merely changes sign, it is convenient to use values such as $w = 0$, $w = \pm 1$, $w = \pm 2$. If we carry 12 decimals in the work, terms beyond T_{16} will not be needed. The computations are exhibited in table IV, the necessary values of M_p having been found as indicated in table III. The five values of β are recorded in table V, where the final figure is subject to rounding error.

As a by-product of the series (6.1) and the moment series developed in section 5, one can obtain the Cornish-Fisher development for β. Substitution of (6.2), (5.10) and (5.12) into (6.1) gives

$$(6.4)\qquad \beta = \Phi(w) + \varphi(w)\left[\frac{P_1}{\mu} + \frac{P_2}{\mu^2} + \cdots\right]$$

TABLE IV

ILLUSTRATION OF (5.1) AT $\nu = 40$, $t = 2$, $w = 0, 1, 2$

k	$t^k M_k/k!$	$w = 0$	$w = 1$	$w = 2$
	$\Phi(w)$	.500 000 000 000	.841 344 746 069	.977 249 868 052
2	.024 841 829 119		−6 010 995 390	−2 682 468 728
3	$.0^3$ 209 580 665	−83 610 588		33 946 388
4	$.0^3$ 308 608 225		149 348 312	−33 324 113
5	$.0^5$ 518 651 54	6 207 361	−2 509 970	−1 400 125
6	$.0^5$ 257 823 31		−3 743 142	2 505 623
7	$.0^7$ 641 896	−384 119	248 512	−38 122
8	$.0^7$ 164 243 6		79 484	−76 262
9	$.0^9$ 531 26	22 254	−16 969	7 142
10	$.0^{10}$ 856 35		−580	878
11	$.0^{11}$ 331 7	−1 251	976	−469
12	$.0^{12}$ 382 1		−87	69
13	$.0^{13}$ 167	69	−50	20
14	$.0^{14}$ 150		9	−7
15	$.0^{16}$ 7	−4	2	
16	$.0^{17}$ 5		−1	

where P_k is a polynomial in t and w, of degree $2k - 1$ in w. This, in turn, can be written in the form

$$\Phi^{-1}(\beta) = w + \frac{Q_1}{\mu} + \frac{Q_2}{\mu^2} + \cdots \tag{6.5}$$

where $Q_k = q_{k0} + q_{k1}w + \cdots + q_{k,2k-1}w^{2k-1}$ and the q_{kj} are polynomials in t. In this development, some of the higher terms in t have coefficient 0, and some of the q_{kj} vanish. After some straightforward algebra one finds for the first five Q_k the expressions

$$\begin{aligned}
&2q_{11} = -t^2;\\
&6q_{20} = -t^3, \quad 8q_{21} = t^2(3t^2 + 2), \quad 6q_{22} = t^3;\\
&12q_{30} = t^3(3t^2 - 1), \quad 16q_{31} = -t^2(5t^4 + 6t^2 - 4), \quad 12q_{32} = -t^3(5t^2 - 1);\\
&240q_{40} = -t^3(75t^4 + 18t^2 - 65), \quad 1152q_{41} = t^2(315t^6 + 316t^4 + 108t^2 - 360);\\
&240q_{42} = t^3(175t^4 + 36t^2 - 65), \quad 72q_{43} = t^4(8t^2 - 9), \quad 40q_{44} = -t^5;\\
&480q_{50} = t^3(175t^6 + 165t^4 - 339t^2 + 375);\\
&2304q_{51} = -t^2(567t^8 - 308t^6 + 2068t^4 - 3240t^2 + 3312);\\
&480q_{52} = -t^3(525t^6 + 427t^4 - 583t^2 + 375);\\
&576q_{53} = -t^4(288t^4 - 316t^2 + 144), \quad 80q_{54} = 3t^5(3t^2 - 1).
\end{aligned} \tag{6.6}$$

These formulas throw some light on the problem of interpolation with respect to δ in the noncentral t tables of Resnikoff and Lieberman [9] and Locks *et al.* [3].

TABLE V

VALUES OF β FOR $\nu = 40$, $t = 2$

w	β
2	.974 569 020 346
1	.835 477 157 175
0	.499 922 233 722
−1	.164 518 287 827
−2	.025 496 009 322

It is not at first sight obvious how to extract four decimal values from a table in which the consecutive entries are, to take an extreme example, .4123, .0055, and .0000. Because of the vanishing of some of the terms q_{kj}, we see from (6.6) that with an error of order μ^{-4}, $\Phi^{-1}(\beta)$ is a quadratic function of w and hence of δ, whereas it is a quartic function of w with error of order μ^{-6}. Thus, three- or five-point interpolation of $\Phi^{-1}(\beta)$ with respect to δ should give good results in these tables. This is especially convenient in the Locks table since there the entries are equally spaced in δ, permitting the use of Lagrange coefficients.

In the example of table IV, five-point interpolation based on the values at $w = 0, \pm 1, \pm 2$ reproduces the computed values at $w = \pm.5, \pm 1.5, \pm 2.5$ with an error in the eleventh decimal place, while three-point interpolation based on $w = 0, \pm 1$ gives about seven decimals of accuracy. Thus the computation of β by means of (6.1) need be carried out at only a few values of w to provide the entire power curve with high precision, at least when ν is large and t is moderate.

The good results obtained from quadratic interpolation based on the values of β at $w = 0, \pm 1$ suggests a simple approximation for large μ and moderate t. If we fit a quadratic to $\Phi^{-1}(\beta)$ at the points $w = 0, \pm 1$, we obtain from (6.6) the approximation

$$\beta \doteq \Phi(A_0 + A_1 w + A_2 w^2) \tag{6.7}$$

where

$$\begin{aligned} A_0 &= \frac{q_{20}}{\mu^2} + \frac{q_{30}}{\mu^3} + \frac{q_{40}}{\mu^4} + \frac{q_{50}}{\mu^5}, \\ A_1 &= 1 + \frac{q_{11}}{\mu} + \frac{q_{21}}{\mu^2} + \frac{q_{31}}{\mu^3} + \frac{q_{41} + q_{43}}{\mu^4} + \frac{q_{51} + q_{53}}{\mu^5}, \\ A_2 &= \frac{q_{22}}{\mu^2} + \frac{q_{32}}{\mu^3} + \frac{q_{42} + q_{44}}{\mu^4} + \frac{q_{52} + q_{54}}{\mu^5}. \end{aligned} \tag{6.8}$$

In the example of table IV, this approximation appears to have a maximum error of about $.0^7 5$ over the entire range of w. Even if ν is reduced to 10, the approximation is good to the four decimals of the noncentral t tables. Since in those tables it is often necessary to interpolate with respect to all three arguments, the approximation (6.7) is in many cases easier to use than the tables, as well as being more accurate.

REFERENCES

[1] HARALD CRAMÉR, *Mathematical Methods of Statistics*, Princeton, Princeton University Press, 1946.
[2] Harvard Computation Laboratory, *Tables of the Error Function and of Its First Twenty Derivatives*, Cambridge, Harvard University Press, 1952.
[3] M. O. LOCKS, M. J. ALEXANDER, and B. J. BYARS, *New Tables of the Noncentral t Distributions*, Aeronaut. Res. Labs., U.S.A.F., 1963.
[4] KENNETH S. MILLER, *An Introduction to the Calculus of Finite Differences and Difference Equations*, New York, Holt, 1960.
[5] National Bureau of Standards, *Tables of the Exponential Function* e^x, Applied Mathematics Series 14, Washington, Government Printing Office, 1951.
[6] NIELS ERIK NÖRLUND, *Vorlesungen über Differenzenrechnung*, Berlin, Springer, 1924.
[7] E. S. PEARSON and H. O. HARTLEY, *Biometrika Tables for Statisticians*, I, Cambridge, Cambridge University Press, 1954.
[8] J. PETERS, *Zehnstellige Logarithmentafel*, I, Berlin, 1922. (Reprinted in New York, Ungar, 1957.)
[9] G. J. RESNIKOFF and G. J. LIEBERMAN, *Tables of the Non-Central t-Distribution*, Stanford, Stanford University Press, 1957.