WEAK LIMITS OF SEQUENCES OF BAYES PROCEDURES IN ESTIMATION THEORY

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1. Introduction

Let $(X, \mathcal{B}, \mu)$ be a totally σ-finite measure space and $\{f(\cdot, \omega), \omega \in \Omega\}$ a family of (generalized) density functions relative to $(X, \mathcal{B}, \mu)$. If $a \in \Omega$ and b is a (randomized) decision procedure for the decision space $\mathfrak{D}$, we borrow Stein's [5] notation and write

$$K(a, b) = \int W(a, t)b(x, dt)f(x, a)\mu(dx), \tag{1.1}$$

where $W(a, t)$ is the measure of loss if $t \in \mathfrak{D}$ is decided and $a \in \Omega$ is the case.

In the sequel we will always suppose that Ω and $\mathfrak{D}$ are locally compact metric spaces and will make suitable measurability assumptions about W, b, f. As is known from the work of Wald [7], under fairly liberal assumptions an admissible procedure b is Bayes in the wide sense. That is, we may find sequences $\{b_n, n \geq 1\}$ and $\{\lambda_n, n \geq 1\}$ such that if $n \geq 1$, b_n is Bayes relative to λ_n, $K(a, b_n) \leq K(a, b)$ for all a in the support of λ_n, and $\lim_{n\to\infty} \int (K(a, b) - K(a, b_n))\lambda_n(da) = 0$. Under convexity assumptions on W one may suppose $b = \text{weak} \lim_{n\to\infty} b_n$, as is explained in the appendix.

If $\mathfrak{D} = (-\infty, \infty)$ and $(\partial W/\partial t)$ is well-defined, then with suitable hypotheses the statement, b_n is Bayes relative to λ_n, is equivalent to the statement, for almost all x, for all t, in the support of $b_n(x, \cdot)$,

$$0 = \int \left(\frac{\partial W}{\partial t}\right)(\omega, t)f(x, \omega)\lambda_n(d\omega). \tag{1.2}$$

If t is vector-valued, (1.2) may be replaced by a system of equations.

Logically, given that b is Bayes in the wide sense relative to $\{b_n, n \geq 1\}$ and $\{\lambda_n, n \geq 1\}$, one would hope to determine a measure $\lambda(\cdot)$ such that for almost all x, for all t in the support of $b(x, \cdot)$,

$$0 = \int \left(\frac{\partial W}{\partial t}\right)(\omega, t)f(x, \omega)\lambda(d\omega). \tag{1.3}$$

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Since not every admissible procedure is Bayes, (1.3) cannot always be solved using probability measures $\lambda(\cdot)$.

Various people have observed that if λ is allowed to be σ-finite on Ω, then many decision procedures of interest are solutions of equations like (1.3). This provided the basis for admissibility proofs in Karlin [3]. And it is strongly suggested that if $b = \text{weak}\lim_{n\to\infty} b_n$, b_n Bayes relative to λ_n, then one might always be able to choose constants $\{k_n, n \geq 1\}$ such that a nonzero σ-finite $\lambda = \text{weak}\lim_{n\to\infty} k_n^{-1}\lambda_n$ was defined for which (1.3) holds. Sacks [4] tried to prove such a result but discovered that it is false.

The arguments used by Sacks required $\{f(\cdot, \omega), \omega \in \Omega\}$ to be an exponential family of density functions. In particular, if $x, y \in X$ and $x \neq y$, then $f(x, \omega)/f(y, \omega)$ is a finite but unbounded function of ω. It was observed by Sacks that if one weakened the restrictions to allow $f(x, \omega) = 0$ for some $(x, \omega) \in X \times \Omega$, then one could give counter examples to show no σ-finite λ could exist.

In this paper we reformulate the problem somewhat and thereby obtain a theory including many examples not covered by Sacks [4]. In particular, one can suppose $f(x, \omega) > 0$ for all $x \in X$, $\omega \in \Omega$ and make the ratios $f(x, \omega)/f(y, \omega)$ very smooth. Yet the result remains false.

In order to consider weak convergence of sequences of measures, one needs to compactify Ω to Ω^*. Given a suitable compactification, one then easily finds examples where mass escapes to the boundary $\Omega^* - \Omega$. The reformulation of the problem and discussion of such details as escape of mass to the boundary constitute section 2.

Section 2 is an exposition designed to give a reformulation of the problem, a compactification of Ω, the "right" renormalization $k_n^{-1}\lambda_n$ of the measures λ_n, $n \geq 1$. Late in section 2 the results are formulated in the theorem of section 2. Lemmas 2.2 and 2.3 are intended as observations useful in various applications of the theorem.

Section 3 gives a necessary and sufficient condition for admissibility in certain estimation problems where strictly convex loss is used. This condition is similar to a condition used by Blyth [1] and Stein [6] to prove admissibility. Suppose b is admissible. To obtain a necessary condition, the main problem is to show that one can pick a compact set E with the following property. Suppose $\{b_n, n \geq 1\}$, $\{\lambda_n, n \geq 1\}$ are any sequences such that $b = \text{weak}\lim_{n\to\infty} b_n$, b_n Bayes relative to λ_n. There exists an integer N such that if $n \geq N$, then $\lambda_n(E) > 0$, and, for every compact set F such that $E \subset F$,

$$\limsup_{n\to\infty} \lambda_n(F)/\lambda_n(E) < \infty. \tag{1.4}$$

In lemma 3.2 we state sufficient conditions for such a result to be true. The hypotheses of lemma 3.2 are satisfied by the examples studied in sections 4 and 5.

Related is the idea of a procedure being admissible outside every compact parameter set. That is, if $E \subset \Omega$, E compact, then the procedure is admissible relative to the parameter space $\Omega - E$. Kiefer and Schwartz (to appear) have studied examples of Bayes tests for the independence of sets of normal variates. These tests have the property that given any compact subset $E \subset \Omega$ there is a probability measure supported on $\Omega - E$, relative to which the given procedure is Bayes.

In estimation problems using strictly convex loss, suppose $E \subset \Omega$, E is an open set, and the procedure δ is not admissible for the parameter space $\Omega - E$. Suppose $f(x, \omega) > 0$ for all $(x, \omega) \in X \times \Omega$, and for each $x \in X$, $f(x, \cdot)$ is a continuous function on Ω. Then there is another procedure δ^* such that $K(\omega, \delta^*) < K(\omega, \delta)$ for all $\omega \in \Omega - E$, and if $F \subset \Omega - E$ is any compact set, $\inf_{\omega \in F} (K(\omega, \delta) - K(\omega, \delta^*)) > 0$ (see lemma 3.1). If δ is admissible, then we may find sequences $\{\delta_n, n \geq 1\}$ and $\{\lambda_n, n \geq 1\}$ such that δ_n is Bayes relative to λ_n, $n \geq 1$, and

$$\lim_{n \to \infty} (\lambda_n(E))^{-1} \int (K(\omega, \delta) - K(\omega, \delta_n))\lambda_n(d\omega) = 0. \tag{1.5}$$

This is a consequence of results of Stein [5]. Let F be a compact set, $E \subset F$; then $F - E$ is a compact set. Since δ^* is not Bayes, we find

$$\begin{aligned}(\lambda_n(E))^{-1} \int (K(\omega, \delta) - K(\omega, \delta_n))\lambda_n(d\omega) &\geq (\lambda_n(E))^{-1} \int_E (K(\omega, \delta) - K(\omega, \delta^*))\lambda_n(d\omega) \\ &+ (\lambda_n(E))^{-1} \int_{\Omega - E} (K(\omega, \delta) - K(\omega, \delta^*))\lambda_n(d\omega).\end{aligned} \tag{1.6}$$

The first integral is bounded, provided risk functions are bounded on compact sets. The integrand of the second integral is strictly positive, and on $F - E$, is bounded away from zero. We find

$$\limsup_{n \to \infty} \lambda_n(F)/\lambda_n(E) < \infty. \tag{1.7}$$

In the sequel we will see that a nonzero σ-finite measure

$$\lambda(\cdot) = \operatorname*{weak\,lim}_{n \to \infty} (\lambda_n(E))^{-1}\lambda_n(\cdot) \tag{1.8}$$

is defined, and from the above it follows that

$$\int_{\Omega - E} (K(\omega, \delta) - K(\omega, \delta^*))\lambda(d\omega) < \infty. \tag{1.9}$$

This shows that the "growth" properties of the measure λ at infinity are limited by the finiteness of these integrals.

The development in sections 2 and 3 depend on the general results of decision theory. The results as formulated in the appendix have been developed by Wald [7], Le Cam (unpublished), and others, and are widely known. But in this case, the literature seems to be lagging badly behind the development of the subject. It was therefore decided to put a few needed results in an appendix, along with proofs.

Details of a number of examples have been worked out when $\Omega = \mathfrak{D} = (-\infty, \infty)$. In section 4 we classify these examples into three groups, and we work out the details for two groups of examples.

Details for the third group of examples are given in section 5. Here it will be seen that in order to obtain the "right" result, the methods of section 2 apply, but a different functional equation and a different type of normalization are required.

Section 6 gives the construction of an example of an admissible procedure b, sequences $\{b_n, n \geq 1\}$ and $\{\lambda_n, n \geq 1\}$, such that $b = \text{weak} \lim_{n\to\infty} b_n$, and for the renormalized sequence of measures some mass does escape to the boundary.

In case $W(\omega, \cdot)$ is a strictly convex function, $\omega \in \Omega$, all Bayes procedures and all admissible procedures are nonrandomized. Further, every admissible procedure will be a weak limit of Bayes procedures. One can then interpret the results of section 4 as saying every admissible procedure solves a nondegenerate functional equation; and the results of section 5 as saying every nonlinear admissible procedure solves a nondegenerate functional equation.

Using such functional equations and additional smoothness assumptions, one can infer things about the continuity and differentiability of admissible estimators. We do not pursue this subject here.

2. Formulation of the problem

It is not the primary purpose of this paper to prove complete class theorems. Consequently, instead of starting by saying "Suppose δ is admissible," we start by saying "Suppose $\delta = \text{weak} \lim_{n\to\infty} \delta_n$." Throughout we will suppose $\{\delta_n, n \geq 1\}$ is a sequence of Bayes decision procedures, δ_n Bayes relative to $\{\lambda_n, n \geq 1\}$, with λ_n supported on Ω. And we shall restrict the study to those δ's which are weak limits of such sequences. The meaning of weak limit, as explained in the appendix, is as a sequence of bilinear forms acting on elements of a Banach space.

Nonetheless, in section 3 we will use the results of this section to obtain a complete class theorem. The result is about estimation of a vector parameter using a strictly convex loss function.

The discussion of this section represents as much as anything an exposition of a method or concept for treating a certain problem. This makes it difficult to formally state results as theorems. Nonetheless, towards the end of the section a theorem is stated. The statement assumes the preceding discussion as understood.

Throughout we suppose that $\mathfrak{D}$ is a locally compact metric space and that $\{\delta_n(\cdot, \cdot), n \geq 1\}$ is a sequence of decision procedures such that for each $x \in X$, $n \geq 1$, $\delta_n(x, \cdot)$ is a regular Borel probability measure on $\mathfrak{D}$.

By saying δ_n is Bayes relative to λ_n, we mean the following. We suppose given k functions $V_1(\cdot, \cdot), \cdots, V_k(\cdot, \cdot)$, and that each one is a continuous real-valued function on $\Omega \times \mathfrak{D}$ in the product topology. Thus, in terms of the introduction, $k = 1$ and $V_1(\omega, t) = (\partial W/\partial t)(\omega, t)$. We suppose for all $x \in X$, $n \geq 1$, that if t is in the support of the measure $\delta_n(x, \cdot)$, then

$$0 = \int V_i(\omega, t)f(x, \omega)\lambda_n(d\omega), \qquad i = 1, \cdots, k. \tag{2.1}$$

Basically we wish to prove that if $\{\delta_n, n \geq 1\}$ are Bayes and if

$$\operatorname*{weak\,lim}_{n\to\infty} \delta_n = \delta, \tag{2.2}$$

then the probability measures λ_n can be renormalized so that the renormalized sequence converges weakly on compact subsets of Ω. This description will prove to be inadequate, and much of the sequel is about compactification of Ω.

In order to compactify Ω, we assume that there is a positive continuous real valued function $V(\cdot)$ defined on Ω, which we call the normalizing function, satisfying the following.

(i) If E is a compact subset of $\mathfrak{D}$, then

$$\sup_{\omega\in\Omega}\sup_{t\in E} |V_i(\omega, t)|/V(\omega) < \infty, \qquad 1 \leq i \leq k. \tag{2.3}$$

(ii) If E is a compact subset of $\mathfrak{D}$, then $V_i(\omega, t)/V(\omega)$ is a uniformly continuous function of $(\omega, t) \in \Omega \times E$.

It may be helpful to consider the example $\Omega = \mathfrak{D} = (-\infty, \infty)$, $W(\omega, t) = (\omega - t)^\alpha$, $k = 1$, $V_1(\omega - t) = (\partial W/\partial t)(\omega - t) = -\alpha(\omega - t)^{\alpha-1}$. A suitable choice of $V(\cdot)$ would be $V(\cdot) = |\omega|^{\alpha-1} + 1$.

In addition we assume

(iii) Ω is a locally compact metric space, and

(iv) the topology on Ω is such that for all $x \in X$, $f(x, \cdot)$ is a continuous function of its second variable. Since each $f(\cdot, \omega)$ is a density function, (iv) implies

$$\lim_{\omega\to\omega_0} \int |f(x, \omega) - f(x, \omega_0)|\mu(dx) = 0. \tag{2.4}$$

We find it necessary to assume

(v) for all $x \in X$, $\omega \in \Omega$ that $f(x, \omega) > 0$.

The main reason for this assumption is to ensure that

(vi) for all $x, y \in X$, $f(x, \omega)/f(y, \omega)$ is a bounded continuous function on compact ω sets.

The basic set of equations studied is

$$\begin{aligned} 0 &= \int V_i(\omega, t)f(x, \omega)\lambda_n(d\omega) \\ &= \int [V_i(\omega, t)/V(\omega)][V(\omega)f(x, \omega)/k_n(x)]\lambda_n(d\omega), \qquad 1 \leq i \leq k. \end{aligned} \tag{2.5}$$

We introduce the normalization

$$k_n(x) = \int V(\omega)f(x, \omega)\lambda_n(d\omega), \qquad x \in X, \quad n \geq 1, \tag{2.6}$$

and define a sequence of probability measures on Ω by

$$k_n(x)\nu_n(x, E) = \int_E V(\omega)f(x, \omega)\lambda_n(d\omega), \qquad x \in X, \quad n \geq 1. \tag{2.7}$$

In common examples, to each $t \in \mathfrak{D}$ we may find a compact subset $C_t \subset \Omega$ such that $\inf_{\omega \notin C_t} W(\omega, t)/V(\omega) > 0$. When this is the case, for any Bayes procedure δ, Bayes relative to λ, with finite Bayes risk, it follows by Fubini's theorem that

$$\int V(\omega) f(x, \omega) \mu(d\omega) < \infty \qquad \text{for almost all} \quad x \in X. \tag{2.8}$$

Throughout the remainder of this paper, we suppose that these integrals are finite, this assumption being the analogue of the supposition that a Bayes procedure has finite Bayes risk.

Then (2.5) may be written as

$$0 = \int [V_i(\omega, t)/V(\omega)] \nu_n(x, d\omega), \qquad 1 \leq i \leq k, \quad n \geq 1. \tag{2.9}$$

The problem is now phrased in terms of integration of bounded continuous functions by probability measures. And we wish to establish results about the weak convergence of $\{\nu_n(x, \cdot), n \geq 1\}$. To do this we need to have these act on a compact space. Below we compactify Ω to Ω^*. Then we may compute weak $\lim_{n\to\infty} \nu_n(x, \cdot) = \nu(x, \cdot)$, $x \in X$. The embedding used is such that Ω is a Borel subset of Ω^*. In the general case $\nu(x, \Omega^* - \Omega) > 0$ is possible, and one can construct examples where $\nu(x, \Omega^* - \Omega) = 1$ for all $x \in X$. In the special case considered by Sacks [4], the only possibility is $\nu(x, \Omega^* - \Omega) = 0$ for all x.

We will show below that with suitable restrictions,

(2.10) if t is in the support of $\delta(x, \cdot)$, then there are t_n,
t_n is in the support of $\delta_n(x, \cdot)$, and
$t = \lim_{n\to\infty} t_n$.

Further,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int [V_i(\omega, t_n)/V(\omega)] \nu_n(x, d\omega) \\ &= \int [V_i(\omega, t)/V(\omega)] \nu(x, d\omega), \qquad 1 \leq i \leq k. \end{aligned} \tag{2.11}$$

To obtain the compactification of Ω, we map Ω into a product space as follows. Let $\rho(\cdot, \cdot)$ be a metric on Ω which gives the topology of Ω and which satisfies $\rho(\omega_1, \omega_2) \leq 1$ for all $\omega_1 \in \Omega$, $\omega_2 \in \Omega$. Take countable dense subsets $\{\omega_i, i \geq 1\}$ of Ω and $\{t_i, i \geq 1\}$ of $\mathfrak{D}$. Then for each $\omega \in \Omega$, we associate the value $\phi(\omega)$ given by

$$\phi(\omega) = \{\rho(\omega, \omega_i), V_j(\omega, t_p)/V(\omega), 1 \leq i, 1 \leq p, 1 \leq j \leq k\}. \tag{2.12}$$

This is a one-to-one continuous mapping into a product space (with a countable number of coordinates). Therefore, the set $\phi(\Omega)$ is a Borel subset of the product space (in the product topology), and the closure Ω^* is a compact subset of the product space (see Hausdorff [2]). We may, and do in the sequel, identify Ω with $\phi(\Omega)$. The functions $V_j(\cdot, t_p)/V(\cdot)$, $1 \leq p$, have continuous extensions to Ω^* which are coordinate mappings (projections). The assumption (ii) allows us to approximate each $V_j(\cdot, t)/V(\cdot)$, $1 \leq j \leq k$, $t \in \mathfrak{D}$, uniformly by coordinate mappings, so these functions also have unique continuous extensions. The sequences $\{\lambda_n, n \geq 1\}$ and $\{\nu_n(x, \cdot), n \geq 1, x \in X\}$ are extended to the Borel sets of Ω^* by $\lambda_n(E) = \lambda_n(E \cap \Omega)$, $\nu_n(x, E) = \nu_n(x, E \cap \Omega)$, $n \geq 1$, $x \in X$. To study the question of convergence, the following lemma is basic.

LEMMA 2.1. *Let $X - F \subset X$ be the set of x such that for* every *t in the support of $\delta(x, \cdot)$ there is an integer sequence $\{n_i(x, t), i \geq 1\}$ and a real number sequence $\{t(x, t, n_i(x, t)), i \geq 1\}$ such that $t(x, t, n_i(x, t))$ is in the support of $\delta_{n_i(x,t)}(x, \cdot)$, $\lim_{i\to\infty} n_i(x, t) = \infty$, and $\lim_{i\to\infty} t(x, t, n_i(x, t)) = t$. Then F is a $\mathcal{B}$-measurable set and $\mu(F) = 0$.*

PROOF. Let $\{U_i, i \geq 1\}$ be a countable base for the open sets of $\mathfrak{D}$. Let

$$F(i, m) = \{x | \delta(x, U_i) > 0, \delta_n(x, U_i) = 0, n \geq m\}. \tag{2.13}$$

We will show that $F = \bigcup_{i,m=1}^{\infty} F(i, m)$.

Let $x \in F$. Then for some t in the support of $\delta(x, \cdot)$, t is bounded away from the support of $\delta_n(x, \cdot)$ for n sufficiently large. That is, there is a U_i in the countable base, and an integer m, such that $t \in U_i$, and if $n \geq m$, the support of $\delta_n(x, \cdot)$ is disjoint from U_i. Therefore, if $n \geq m$, $\delta_n(x, U_i) = 0$ and $\delta(x, U_i) > 0$. Thus $x \in F(i, m)$.

Conversely, if $x \in \bigcup_{i,m=1}^{\infty} F(i, m)$, then $x \in F(i, m)$ for some i, m. Since $\delta(x, U_i) > 0$, there is a number $t \in U_i$, t in the support of $\delta(x, \cdot)$. If $n \geq m$, then $\delta_n(x, U_i) = 0$ so the support of $\delta_n(x, \cdot)$ is bounded away from t by U_i. Hence, t cannot be a limit of a subsequence as described, and $x \in F$ follows.

Let U_i be given and $g_i(\cdot)$ a real valued continuous function on $\mathfrak{D}$ such that $g_i(t) = 0$ if $t \notin U_i$, $g_i(t) > 0$ if $t \in U_i$, $1 \leq i$. Then, by the meaning of weak limits (see appendix),

$$\begin{aligned} 0 &= \lim_{n\to\infty} \iint_{F(i,m)} g_i(t)\delta_n(x, dt)\mu(dx) \\ &= \iint_{F(i,m)} g_i(t)\delta(x, dt)\mu(dx). \end{aligned} \tag{2.14}$$

Since $\delta(x, U_i) > 0$ for all $x \in F(i, m)$, it follows that $\mu(F(i, m)) = 0$. Since this holds for $i \geq 1$, $m \geq 1$, it follows that $\mu(F) = 0$. The proof of lemma 2.1 is complete.

THEOREM. *Let $\delta = \text{weak} \lim_{n\to\infty} \delta_n$. Let $\{\lambda_n, n \geq 1\}$ be a sequence of regular Borel probability measures defined on the subsets of Ω. Suppose that if $n \geq 1$, $x \in X$, and if t is in the support of $\delta_n(x, \cdot)$; then*

$$0 = \int V_i(\omega, t)f(x, \omega)\lambda_n(d\omega), \qquad 1 \leq i \leq k. \tag{2.15}$$

Let the compactification Ω^* and the probability measures $\{\nu_n, n \geq 1\}$ be as above. Let $\{n_i, i \geq 1\}$ be an integer sequence such that for almost all $x[\mu]$, $\nu(x, \cdot) =$ weak $\lim_{i\to\infty} \nu_{n_i}(x, \cdot)$. *Then there exists a $\mathcal{B}$-measurable set F such that $\mu(F) = 0$. If $x \notin F$ and t is in the support of $\delta(x, \cdot)$, then*

$$0 = \int [V_i(\omega, t)/V(\omega)]\nu(x, dt), \qquad 1 \leq i \leq k. \tag{2.16}$$

PROOF. We will suppose that the subsequence $\{n_i, i \geq 1\}$ is the entire sequence. This involves no loss of generality. Let F_1 be a $\mathcal{B}$-measurable set such that if $x \notin F_1$ then $\{\nu_n(x, \cdot), n \geq 1\}$ is weakly convergent. Here $\mu(F_1) = 0$ and we write $\nu(x, \cdot) = \text{weak} \lim_{n\to\infty} \nu_n(x, \cdot)$. Choose $x \notin F_1$ and let t be in the support of $\delta(x, \cdot)$. Let F_2 be chosen in accordance with lemma 2.1 relative to the sequence $\{\delta_n, n \geq 1\}$. Then if $F = F_1 \cup F_2$ and $x \notin F$, there are sequences $\{n_i, i \geq 1\}$ and $\{t_{n_i}, i \geq 1\}$ such that $\lim_{i\to\infty} n_i = \infty$, $\lim_{i\to\infty} t_{n_i} = t$ and t_{n_i} is in the support of $\delta_{n_i}(x, \cdot)$, $i \geq 1$. Therefore,

$$0 = \int [V_i(\omega, t_{n_i})/V(\omega)]\nu_{n_i}(x, d\omega), \qquad 1 \leq j \leq k. \tag{2.17}$$

Since by (ii)

$$\lim_{i\to\infty} V_j(\omega, t_{n_i})/V(\omega) = V_j(\omega, t)/V(\omega) \tag{2.18}$$

uniformly in $\omega \in \Omega$, and since weak $\lim_{i\to\infty} \nu_{n_i}(x, \cdot) = \nu(x, \cdot)$, it follows that

$$0 = \int [V_j(\omega, t)/V(\omega)]\nu(x, d\omega), \qquad 1 \leq j \leq k. \tag{2.19}$$

That completes the proof of the theorem.

The discussion so far has not used (v) or (vi). The above theorem is therefore valid quite generally. In practice the hypothesis that there be a single integer sequence $n \geq 1$ on which $\{\nu_n(x, \cdot), n \geq 1\}$ is weakly convergent for almost all x is difficult to verify. In the examples we explore, (v) and (vi) are used to verify this hypothesis. It will also appear in the exposition of examples that one will want the ratios $f(x, \omega)/f(y, \omega)$ as functions on $\Omega \to (0, \infty)$ to have continuous extensions to functions on $\Omega^* \to [0, \infty]$. In order to achieve this, one may have to modify the construction of Ω^* to have the form

$$\phi(\omega) = \{\rho(\omega, \omega_i), V_j(\omega, t_p)/V(\omega), f(x_j, \omega)/f(y_k, \omega), \quad 1 \leq i, 1 \leq p, 1 \leq j, 1 \leq k\} \tag{2.20}$$

taken over suitable countable dense subsets (see (2.12)).

We now prove several lemmas which are useful in verifying the hypotheses of the theorem.

LEMMA 2.2. *Suppose for all* $x, y \in X$ *that* $\sup_{\omega\in\Omega} f(y, \omega)/f(x, \omega) < \infty$ *and that* $f(y, \cdot)/f(x, \cdot)$ *has a unique continuous extension to* Ω^*. *Let* $x_0 \in X$, *and let* $\{n_i, i \geq 1\}$ *be a sequence of integers such that* $\lim_{i\to\infty} n_i = \infty$ *and* $\nu(x_0, \cdot) = \text{weak} \lim_{i\to\infty} \nu_{n_i}(x_0, \cdot)$ *exists. Then* $\nu(y, \cdot) = \text{weak} \lim_{i\to\infty} \nu_{n_i}(y, \cdot)$ *exists for all* $y \in X$. *For every Borel set* E *in* Ω^*,

$$\nu(y, E) = \int_E [f(y, \omega)/f(x_0, \omega)]\nu(x_0, d\omega) \Big/ \int [f(y, \omega)/f(x_0, \omega)]\nu(x_0, d\omega). \tag{2.21}$$

If for some $x_0 \in X$ *and for some sequence* $\{n_i, i \geq 1\}$,

$$\nu(x_0, \cdot) = \text{weak} \lim_{i\to\infty} \nu_{n_i}(x_0, \cdot) \text{ and } \nu(x_0, \Omega) = 0, \tag{2.22}$$

then

$$\nu(y, \cdot) = \text{weak} \lim_{i\to\infty} \nu_{n_i}(y, \cdot) \tag{2.23}$$

exists for all $y \in X$ *and* $\nu(y, \Omega) = 0$. *If* t *is in the support of* $\delta(y, \cdot)$, *then*

$$0 = \int [V_j(\omega, t)/V(\omega)]\nu(y, d\omega) \tag{2.24}$$

$$= \int [V_j(\omega, t)f(y, \omega)/V(\omega)f(x_0, \omega)]\nu(x_0, d\omega), \qquad 1 \le j \le k.$$

PROOF. We may consider $f(y, \cdot)/f(x, \cdot)$ as being a continuous function on Ω^*. Then

$$\lim_{i\to\infty} k_{n_i}(y)/k_{n_i}(x_0) = \lim_{i\to\infty} \int [f(y, \omega)/f(x_0, \omega)]\nu_{n_i}(x_0, d\omega) \tag{2.25}$$

$$= \int [f(y, \omega)/f(x_0, \omega)]\nu(x_0, d\omega).$$

If $g(\cdot)$ is a bounded continuous function on Ω^*,

$$\lim_{i\to\infty} \int g(\omega)\nu_{n_i}(y, d\omega) \tag{2.26}$$

$$= \lim_{i\to\infty} [k_{n_i}(x_0)/k_{n_i}(y)] \int [g(\omega)f(y, \omega)/f(x_0, \omega)]\nu_{n_i}(x_0, d\omega)$$

$$= \int [g(\omega)f(y, \omega)/f(x_0, \omega)]\nu(x_0, d\omega) \Big/ \int [f(y, \omega)/f(x_0, \omega)]\nu(x_0, d\omega).$$

A standard approximation argument proves (2.21). The proof of the remainder of the lemma is obvious.

LEMMA 2.3. *Given the hypotheses of lemma* 2.2, *suppose that*

$$\nu(x, \cdot) = \operatorname*{weak\,lim}_{i\to\infty} \nu_{n_i}(x, \cdot) \qquad \text{for all} \quad x \in X. \tag{2.27}$$

Suppose $\nu(x, \Omega) > 0$ *for all* x. *If* $E \subset \Omega$ *is an open set having closure* $\overline{E}$, *and if* $\overline{E} \subset \Omega$ *and* $\nu(x, \overline{E} - E) = 0$ *for all* $x \in X$, *then for each* $x \in X$ *the sequence of measures* $\lambda_{n_i}(\cdot)/k_{n_i}(x)$ *converges weakly when restricted to* $\overline{E}$, *and if* $\lambda(x, \cdot)$ *is the limiting* σ*-finite measure on* Ω,

$$\int_{\overline{E}} V(\omega)f(x, \omega)\lambda(x, d\omega) = \nu(x, \overline{E}). \tag{2.28}$$

PROOF. If $g(\cdot)$ is a continuous real-valued function defined on $\overline{E}$, we let $g^*(\cdot)$ be a continuous extension of $g(\cdot)$ to all of Ω^*, such that $g^*(\cdot)$ has compact support. (Note that by lemma 2.2, the measures $\{\nu(x, \cdot), x \in X\}$ are mutually absolutely continuous with respect to each other.) We have

$$\lim_{i\to\infty} \int_{\overline{E}} g(\omega)\lambda_{n_i}(d\omega)/k_{n_i}(x) = \lim_{i\to\infty} \int_{\overline{E}} [g^*(\omega)/V(\omega)f(x, \omega)]\nu_{n_i}(x, d\omega) \tag{2.29}$$

$$= \int_{\overline{E}} [g^*(\omega)/V(\omega)f(x, \omega)]\nu(x, d\omega)$$

$$= \int_{\overline{E}} [g(\omega)/V(\omega)f(x, \omega)]\nu(x, d\omega).$$

This shows that the limit exists for each such $g(\cdot)$, establishing weak convergence. Choose $g(\omega) = V(\omega)f(x, \omega)$ for $\omega \in \overline{E}$. The conclusion of the lemma follows.

3. A necessary and sufficient condition for admissibility

In this section we will suppose $\Omega = \mathfrak{D}$ is Euclidean n-space and that for each $\omega \in \Omega$, the measure of loss $W(\omega, \cdot)$ is a strictly convex function, for each $t \in \mathfrak{D}$, $W(\cdot, t)$ is a continuous function. We suppose $W(\omega, t) \geq 0$ for all $(\omega, t) \in \Omega \times \mathfrak{D}$. To apply decision theory results we need to assume that if C is a compact subset of Ω, then

$$\lim_{t\to\infty} \inf_{\omega \in C} W(\omega, t) = \infty. \tag{3.1}$$

We consider only procedures having bounded risk on compact Ω sets.

We shall suppose the hypotheses (i)–(vi) and (2.4)–(2.8) of section 2 hold, where the V_j are the partial derivatives of W, and in particular, we will make heavy use of (v), that $f(x, \omega) > 0$ for all $(x, \omega) \in X \times \Omega$. We consider a fixed procedure δ and suppose the following can be proven.

Let $\{\delta_n, n \geq 1\}$ and $\{\lambda_n, n \geq 1\}$ be any sequences such that δ_n is Bayes relative to λ_n, $n \geq 1$, such that $\delta = \text{weak} \lim_{n\to\infty} \delta_n$, and $\lambda_n(\Omega) = 1$, $n \geq 1$. For every such sequence it must follow that there exists an integer sequence $\{n_i, i \geq 1\}$ such that for almost all $x, y[\mu]$,

$$0 < \lim_{i\to\infty} k_{n_i}(x)/k_{n_i}(y) < \infty, \tag{3.2}$$

and

$$\nu(x, \cdot) = \text{weak} \lim_{i\to\infty} \nu_{n_i}(x, \cdot) \quad \text{exists.} \tag{3.3}$$

We suppose $\nu(x, \Omega) > 0$ for almost all $x[\mu]$ necessarily follows.

The examples considered by Sacks [4] satisfy these hypotheses. Several later sections of this paper consider examples where these hypotheses are satisfied. In the sequel we write $K(\omega, \delta^*)$ for the risk of δ^* evaluated at ω.

THEOREM 3.1. *Suppose the hypotheses made above hold. A necessary and sufficient condition that a non-Bayes procedure δ shoula be admissible is that there exist an open set E with compact closure $\overline{E}$ and sequences $\{\delta_n, n \geq 1\}$, $\{\lambda_n, n \geq 1\}$ such that*

$$\delta = \text{weak} \lim_{n\to\infty} \delta_n; \tag{3.4}$$

(3.5) *δ_n is Bayes relative to the probability measure $\lambda_n(\cdot)$, $n \geq 1$. If C_n is the support of $\lambda_n(\cdot)$, then $\overline{E} \subset C_n \subset C_{n+1} \subset \Omega$, $n \geq 1$. The sets C_n are compact, $n \geq 1$;*

(3.6) *for almost all $x \in X$, $\lim_{n\to\infty} \lambda_n(E)/k_n(x)$ exists, finite and positive;*

$$\lim_{n\to\infty} (\lambda_n(\overline{E}))^{-1} \int (K(\omega, \delta) - K(\omega, \delta_n))\lambda_n(d\omega) = 0. \tag{3.7}$$

We will need the following lemma in the proof of sufficiency. We consider only nonrandomized procedures.

LEMMA 3.1. *Suppose δ^* is as good as δ. Let $A_1 = \{x | \delta(x) \neq \delta^*(x)\}$, and suppose $\mu(A_1) > 0$. Let C be a compact parameter set. Then there exists a real number $\gamma > 0$ such that if $\omega \in C$, then*

$$\gamma + K(\omega, (\delta + \delta^*)/2) \leq K(\omega, \delta). \tag{3.8}$$

PROOF. By hypothesis of this section, if $B \in \mathcal{B}$, then $\mu(B) > 0$ if and only if $\int_B f(x, \omega)\mu(dx) > 0$ for all $\omega \in \Omega$. We set

$$A_2(\alpha) = \{x| \text{ if } \omega \in C, \text{ then } \alpha + W(\omega, (\delta(x) + \delta^*(x))/2) \leq (\tfrac{1}{2}) (W(\omega, \delta(x)) + W(\omega, \delta^*(x))\}. \tag{3.9}$$

Since $W(\omega, t)$ is strictly convex in t and continuous in ω, if $\delta(x) \neq \delta^*(x)$, then

$$0 < \inf_{\omega \in C} \{\tfrac{1}{2}(W(\omega, \delta(x)) + W(\omega, \delta^*(x))) - W(\omega, (\delta(x) + \delta^*(x))/2)\}. \tag{3.10}$$

Therefore, $A_1 = \bigcup_{n=1}^{\infty} A_2(1/n)$. If $n \geq 1$, then $A_2(1/n) \subset A_2(1/(n + 1))$, so it follows that

$$\int_{A_1} f(x, \omega)\mu(dx) = \lim_{n\to\infty} \int_{A_2(1/n)} f(x, \omega)\mu(dx). \tag{3.11}$$

By Dini's theorem this limit is uniform in $\omega \in C$, since all functions involved are continuous functions of ω. Further, since $0 < \inf_{\omega\in C} \int_{A_1} f(x, \omega)\mu(dx)$, it follows that we may find an integer n such that if $\omega \in C$, then

$$\int_{A_2(1/n)} f(x, \omega)\mu(dx) \geq 1/n. \tag{3.12}$$

If we take $\gamma = (1/n)^2$, the lemma now follows.

PROOF OF SUFFICIENCY. If δ^* is as good as δ, then by lemma 3.1 we may suppose $K(\omega, \delta^*) + \gamma \leq K(\omega, \delta)$ for all $\omega \in \overline{E}$ where $\gamma > 0$. Then we must have

$$\begin{aligned} \gamma\lambda_n(\overline{E}) &\leq \int_{\overline{E}} (K(\omega, \delta) - K(\omega, \delta^*))\lambda_n(d\omega) \\ &\leq \int (K(\omega, \delta) - K(\omega, \delta_n))\lambda_n(d\omega). \end{aligned} \tag{3.13}$$

Application of (3.7) leads to the contradiction that $\gamma = 0$.

We will need the following lemma in the proof of necessity.

LEMMA 3.2. *There exists an open subset U of Ω having compact closure (in Ω) with the following property. Suppose $\{\delta_n, n \geq 1\}$ is a sequence of decision procedures, δ_n is Bayes relative to λ_n, $n \geq 1$, and $\lambda_n(\Omega) = 1$, $n \geq 1$. Suppose $\delta =$ weak $\lim_{n\to\infty} \delta_n$, and $k_n(x)$, $n \geq 1$, is defined as in (2.6) for the sequence $\{\lambda_n, n \geq 1\}$. Then*

$$0 < \liminf_{n\to\infty} \lambda_n(U)/k_n(x) < \limsup_{n\to\infty} \lambda_n(U)/k_n(x) < \infty. \tag{3.14}$$

(The given open set U is to have this property for all choices of sequences $\{\delta_n, n \geq 1\}$, $\{\lambda_n, n \geq 1\}$.)

PROOF. If the lemma is false, let $\{U_n, n \geq 1\}$ be an increasing sequence of open sets with compact closure, $\Omega = \bigcup_{n=1}^{\infty} U_n$. Corresponding to each U_n there are sequences $\{\delta_{m,n}, m \geq 1\}$ and $\{\lambda_{m,n}, m \geq 1\}$ such that the hypotheses of the lemma are satisfied, yet $\lim_{m\to\infty} \lambda_{m,n}(U_n)/k_{m,n}(x) = 0$. On the basis of the assumptions made in section 2, $\lim_{m\to\infty} \lambda_{m,n}(U_n)/k_{m,n}(x) = \infty$ is impossible.

We take countable dense subsets $\{f_i, i \geq 1\}$ of $L_1(X, \mathcal{B}, \mu)$ and $\{g_i, i \geq 1\}$

of $C(\mathfrak{D}^*)$. Choose $x_0 \in X$. For each $n \geq 1$ we may choose an integer N_n such that if $m \geq N_n$, then for all $1 \leq i, j \leq n$,

$$\left| \int f_i(x)g_j(t)\delta(x, dt)\mu(dx) - \int f_i(x)g_j(t)\delta_{m,n}(x, dt)\mu(dx) \right| \leq 1/n, \tag{3.15}$$

and

$$\lambda_{m,n}(U_n)/k_{m,n}(x_0) \leq 1/n. \tag{3.16}$$

If we interpret $\delta_{m,n}$ as a bilinear form $(\cdot, \cdot)_{m,n}$, then $|(f, g)_{m,n}| \leq \|f\| \, \|g\|$ for all $f \in L_1(X, \mathcal{B}, \mu)$, $g \in C(\mathfrak{D}^*)$. Using this it is easy to show

$$\delta = \operatorname*{weak\,lim}_{n \to \infty} \delta_{N_n, n}. \tag{3.17}$$

If we choose a subsequence $\{n_i, i \geq 1\}$ on which for almost all $x, y \in X$,

$$0 < \lim_{i \to \infty} k_{N_{n_i}, n_i}(x)/k_{N_{n_i}, n_i}(y) < \infty, \tag{3.18}$$

which is possible by hypothesis, and such that for almost all x,

$$\nu(x, \cdot) = \operatorname*{weak\,lim}_{i \to \infty} \nu_{N_{n_i}, n_i}(x, \cdot), \tag{3.19}$$

then for every $g(\cdot)$ continuous on Ω with compact support,

$$\lim_{i \to \infty} \int g(\omega)\nu_{N_{n_i}, n_i}(y, d\omega) = \int g(\omega)\nu(y, d\omega) \tag{3.20}$$

exists for all y. Our construction is such that if $N_n \geq m$, then,

$$\lambda_{N_n, n}(U_m)/k_{N_n, n}(x_0) \leq 1/N_n. \tag{3.21}$$

Consequently, if $g(\omega) = 0$ outside of U_m, it follows that $\int g(\omega)\nu(y, d\omega) = 0$ for almost all $y \in X$. Since we may take m large enough that the support of $g(\cdot)$ is contained in U_m, it follows that $\int g(\omega)\nu(y, d\omega) = 0$ for all $g(\cdot)$ having compact support contained in Ω. Since we suppose Ω is open in Ω^*, $\Omega = \bigcup_{n=1}^{\infty} U_n$, it follows that $\nu(y, \Omega) = 0$ for almost all $y \in X$. This contradicts the basic hypothesis of section 3 that $\nu(y, \Omega) > 0$ for almost all $y \in X$. This contradiction shows that the lemma must be correct.

PROOF OF NECESSITY. In view of the lemma, we may suppose an open set U is given having compact closure such that U has the property stated in lemma 3.2 relative to the admissible procedure δ. We now apply a theorem on admissibility due to Stein [5]. To adapt Stein's notation, if $a \in \Omega$ and b is a decision procedure, we set

$$K(a, b) = \int W(a, t)b(x, dt)f(x, a)\mu(dx). \tag{3.22}$$

With a minor modification of Stein's proofs and results, we may prove the following. Let ξ be a probability measure supported on U. Relative to $\gamma > 0$ and the risk function

$$\int (K(a, b) - K(a, \delta))\xi(da) + \gamma(K(a, b) - K(a, \delta)), \tag{3.23}$$

let b_γ be a minimax procedure. Let

$$(3.24) \qquad -\epsilon_\gamma = \sup_a \left\{ \int (K(a, b_\gamma) - K(a, \delta))\xi(da) + \gamma(K(a, b_\gamma) - K(a, \delta)) \right\}$$

be the minimax risk of b_γ. Then Stein shows $\lim_{\gamma \to 0} \epsilon_\gamma = 0$. The weak compactness condition used by Stein is satisfied in our problem. See theorem 2A of the appendix.

Since b_γ is minimax, for all $a \in \Omega$,

$$(3.25) \qquad \int K(a, \delta)\xi(da) + \gamma K(a, \delta) \geq \int K(a, b_\gamma)\xi(da) + \gamma K(a, b_\gamma).$$

If we divide by γ and let $\gamma \to \infty$, we find

$$(3.26) \qquad \limsup_{\gamma \to \infty} K(a, b_\gamma) \leq K(a, \delta).$$

For each γ we choose a compact parameter set C_γ, satisfying the following. If $\gamma_1 < \gamma_2$, then $C_{\gamma_1} \subset C_{\gamma_2}$, and $\bigcup_{\gamma > 0} C_\gamma = \Omega$. If b_γ^* is minimax for $a \in C_\gamma$ relative to the risk

$$(3.27) \qquad \int (K(a, b) - K(a, \delta))\xi(da) + \gamma(K(a, b) - K(a, \delta)),$$

and Bayes relative to ξ_γ^*, then the minimax risk of b_γ^* is $\geq -2\epsilon_\gamma$. We are using here theorem A4 of the appendix. We find that if $a \in C_\gamma$, then

$$(3.28) \qquad 0 \geq \int (K(a, b_\gamma^*) - K(a, \delta))\xi(da) + \gamma(K(a, b_\gamma^*) - K(a, \delta)),$$

so that as above, if $a \in \Omega$,

$$(3.29) \qquad \limsup_{\gamma \to \infty} K(a, b_\gamma^*) \leq K(a, \delta).$$

If we write the above as follows,

$$(3.30) \qquad 0 \geq (1 + \gamma)\left\{ \int K(a, b_\gamma^*)\xi(da)/(1 + \gamma) + \gamma \int K(a, b_\gamma^*)\xi_\gamma^*(da)/(1 + \gamma) \right.$$
$$\left. - \int K(a, \delta)\xi(da)/(1 + \gamma) - \gamma \int K(a, \delta)\xi_\gamma^*(da)/(1 + \gamma) \right\}$$
$$\geq -2\epsilon_\gamma,$$

then we find b_γ^* to be Bayes relative to the risk $K(a, b)$ and the probability $\xi_1/(1 + \gamma) + \gamma\xi_\gamma^*/(1 + \gamma) = \xi_\gamma$.

Since we assume strictly convex loss, and since we suppose δ is admissible, it follows from (3.29) and theorem A3 of the appendix that

$$(3.31) \qquad \delta = \operatorname{weak} \lim_{\gamma \to \infty} b_\gamma^*.$$

Thus (3.4) and (3.5) hold. Equation (3.7) follows from (3.30). If $\overline{E}$ is the closure of an open subset E of Ω, and if $U \subset E$, then

$$(3.32) \qquad \xi_\gamma(\overline{E}) \geq \xi_\gamma(U) \geq 1/(1 + \gamma).$$

Therefore, by (3.30),

$$\lim_{\gamma\to\infty} (\xi_\gamma(\overline{E}))^{-1} \int (K(a, b_\gamma^*) - K(a, \delta))\xi_\gamma(da) = 0. \tag{3.33}$$

We will complete the proof of the theorem by verifying (3.6). Take $\gamma = n_i$ where $\{n_i, i \geq 1\}$ is an integer sequence on which for almost all x,

$$\nu(x, \cdot) = \text{weak}\lim_{i\to\infty} \nu_{n_i}(x, \cdot). \tag{3.34}$$

We take E to be an open subset of Ω with closure $\overline{E} \subset \Omega$, such that $U \subset \overline{E}$ and such that $\nu(x_0, \overline{E} - E) = 0$. This implies $\nu(x, \overline{E} - E) = 0$ for almost all $x \in X$. Then it follows that for almost all x,

$$\lim_{i\to\infty} \lambda_{n_i}(\overline{E})/k_{n_i}(x) \tag{3.35}$$

exists. We use here (2.26). By lemma 3.2, this limit must be positive. It follows from the construction of section 2 that the limit is finite.

The proof of the theorem is complete.

4. Estimation of a real parameter

We suppose $\Omega = \mathfrak{D} = (-\infty, \infty)$ and examine some of the common examples. We will restrict the discussion to functions $V_1(\cdot, \cdot)$ and $V(\cdot)$ satisfying

$$\lim_{\omega\to-\infty} V_1(\omega, t)/V(\omega) = 1, \qquad \lim_{\omega\to\infty} V_1(\omega, t)/V(\omega) = -1. \tag{4.1}$$

Many typical loss functions $W(\omega, t) = w(\omega - t)$ have this property, where $V_1(\omega, t) = -(dw/dx)_{x=\omega-t}$.

As was suggested in section 2, the analysis depends on the ratios

$$\{f(x, \cdot)/f(y, \cdot), x \in X, y \in X\}. \tag{4.2}$$

We define sets

$$A^y_{\alpha,\beta} = \{x|\lim_{\omega\to-\infty} f(x, \omega)/f(y, \omega) = \alpha, \lim_{\omega\to\infty} f(x, \omega)/f(y, \omega) = \beta\}. \tag{4.3}$$

We consider in this section the following cases.

Case Ia. For all y, $\mu(X - A^y_{1,1}) = 0$.

Case Ib. For all y,

$$X = \bigcup_{\alpha>0,\beta>0} A^y_{\alpha,\beta}, \tag{4.4}$$

and

$$\mu(\{x|\lim_{\omega\to-\infty} f(x, \omega)/f(y, \omega) \neq \lim_{\omega\to+\infty} f(x, \omega)/f(y, \omega)\}) > 0. \tag{4.5}$$

Case II. For all y, $X = A^y_{0,\infty} \cup A^y_{\infty,0}$ and

$$\mu(A^y_{0,\infty}) > 0, \qquad \mu(A^y_{\infty,0}) > 0. \tag{4.6}$$

Case II includes the examples considered by Sacks [4], as well as including many examples of monotone likelihood ratios. The family of Cauchy densities falls in case Ia.

In terms of the construction of section 2, the compactification $\Omega^* = [-\infty, \infty]$.

Consequently, if a limit measure ν puts mass on the boundary, we have only $P_+(x) = \nu(x, \{\infty\})$ and $P_-(x) = \nu(x, \{-\infty\})$ to consider.

Suppose δ is given, $\{\delta_n, n \geq 1\}$ and $\{\lambda_n, n \geq 1\}$ are given, and

$$\delta = \text{weak} \lim_{n \to \infty} \delta_n. \tag{4.7}$$

In case I, we may use lemma 2.2. According to this lemma, we may pick an integer sequence $\{n_i, i \geq 1\}$ such that for all $x \in X$, weak $\lim_{i \to \infty} \nu_{n_i}(x, \cdot) = \nu(x, \cdot)$ exists. Consequently, we may apply the theorem of section 2. If t is in the support of $\delta(x, \cdot)$ (except for $x \in F$, an exceptional set, $\mu(F) = 0$),

$$\begin{aligned} 0 &= \int_{\Omega^*} [V_1(\omega, t)/V(\omega)]\nu(x, d\omega) \\ &= P_-(x) - P_+(x) + \int_\Omega [V_1(\omega, t)/V(\omega)]\nu(x, d\omega). \end{aligned} \tag{4.8}$$

By lemma 2.2,

$$\begin{aligned} P_+(y)/P_-(y) &= \int_{\{\infty\}} [f(y, \omega)/f(x, \omega)]\nu(x, d\omega) \Big/ \int_{\{-\infty\}} [f(y, \omega)/f(x, \omega)]\nu(x, d\omega) \\ &= \frac{\lim_{\omega \to \infty} f(y, \omega)/f(x, \omega)}{\lim_{\omega \to -\infty} f(y, \omega)/f(x, \omega)} (P_+(x)/P_-(x)). \end{aligned} \tag{4.9}$$

Therefore, in case Ia, $P_+(y)/P_-(y) = P_+(x)/P_-(x)$, except for a set of measure zero, whereas in case Ib, $P_+(y)/P_-(y) \neq P_+(x)/P_-(x)$ on a set of positive measure.

If (4.8) holds and $\nu(x, \Omega) = 0$, then we find $P_+(x) = P_-(x) = \frac{1}{2}$. By lemma 2.2, if $\nu(x, \Omega) = 0$ for a single x, then $\nu(x, \Omega) = 0$ for all x, implying $P_+(x) = P_-(x) = \frac{1}{2}$ for all x.

We shall show in section 5 that in case Ia this is indeed possible. We have already seen that in case Ib this is not possible.

LEMMA 4.1. *In case Ib, if* $\delta =$ weak $\lim_{n \to \infty} \delta_n$, δ_n *is Bayes relative to* λ_n, $n \geq 1$, *then there exists an integer sequence* $\{n_i, i \geq 1\}$ *such that*

$$\nu(x, \cdot) = \text{weak} \lim_{i \to \infty} \nu_{n_i}(x, \cdot) \qquad \text{for all } x \text{ and } \nu(x, \Omega) > 0 \text{ for all } x. \tag{4.10}$$

Using lemma 4.1 and lemma 2.3 one can prove at once the following theorem.

THEOREM 4.1. *In case Ib, if* $\delta =$ weak $\lim_{n \to \infty} \delta_n$, δ_n *Bayes relative to* λ_n, $n \geq 1$, *then there exists a nonzero σ-finite regular Borel measure* $\lambda(\cdot)$ *defined on the real line and a real-valued function* $\rho(\cdot)$ *such that for almost all* $x[\mu]$, *if* t *is in the support of* $\delta(x, \cdot)$, *then*

$$0 = \rho(x) + \int V_1(\omega, t) f(x, \omega)\lambda(d\omega). \tag{4.11}$$

In section 6 we give an example of an admissible estimator δ for which (4.10) holds and $\rho(x) \neq 0$.

The main result in case II is as follows.

THEOREM 4.2. *Suppose in case II that* $\delta = \text{weak} \lim_{n\to\infty} \delta_n$, δ_n *Bayes relative to* λ_n. *Then for all* x, y, $\liminf_{n\to\infty} k_n(x)/k_n(y) > 0$. *There exists an integer sequence* $\{n_i, i \geq 1\}$ *such that for all* $x \in X$,

$$\nu(x, \cdot) = \text{weak} \lim_{i\to\infty} \nu_{n_i}(x, \cdot) \tag{4.12}$$

exists. $\nu(x, \Omega) = 1$ *for all* x. *There exists a nonzero* σ*-finite regular Borel measure* $\lambda(\cdot)$ *such that for almost all* $x[\mu]$, *if* t *is in the support of* $\delta(x, \cdot)$, *then*

$$0 = \int_\Omega V_1(\omega, t) f(x, \omega)\lambda(d\omega). \tag{4.13}$$

The proof of this theorem involves considerable detail. This is broken down into several lemmas.

LEMMA 4.2. *Given the hypotheses of theorem* 4.2, *for all* x, y,

$$\liminf_{n\to\infty} k_n(x)/k_n(y) > 0. \tag{4.14}$$

PROOF. We suppose to the contrary that for some x_0, y_0, and sequence $\{n_i, i \geq 1\}$, that $\lim_{i\to\infty} k_{n_i}(x_0)/k_{n_i}(y_0) = 0$. That is,

$$0 = \lim_{i\to\infty} \int [f(x_0, \omega)/f(y_0, \omega)]\nu_{n_i}(y_0, d\omega). \tag{4.15}$$

Under the conditions of case II we must have either subcase A:

$$\lim_{\omega\to\infty} f(x_0, \omega)/f(y_0, \omega) = 0, \tag{4.16}$$

or subcase B:

$$\lim_{\omega\to-\infty} f(x_0, \omega)/f(y_0, \omega) = 0. \tag{4.17}$$

Subcase A. Let x' be such that $\lim_{\omega\to\infty} f(x', \omega)/f(y_0, \omega) = \infty$. We will show that $\nu_{n_i}(x', \cdot) \to \nu(x', \cdot)$ satisfying $\nu(x', \{\infty\}) = 1$.

In subcase A, $\lim_{\omega\to-\infty} f(x_0, \omega)/f(y_0, \omega) = \infty$. Therefore, for every integer N, $1 = \lim_{i\to\infty} \nu_{n_i}(y_0, [N, \infty])$. We use this in the calculation.

Let g_1, g_2 be nonnegative real-valued continuous functions on Ω^* satisfying the following. For some integer n, if $\omega \geq n$, then $g_2(\omega) = 0$, and if $\omega \leq n$ $g_1(\omega) = 0$. If $\omega \geq n$, then $g_1(\omega) > 0$ and $g_1(\infty) = 1$.

Let $\{m_i, i \geq 1\}$ be a subsequence of $\{n_i, i \geq 1\}$ on which

$$\text{weak} \lim_{i\to\infty} \nu_{m_i}(x', \cdot) = \nu(x', \cdot) \tag{4.18}$$

exists. Then

$$\int g_1(\omega)\nu(x', d\omega) \Big/ \int g_2(\omega)\nu(x', d\omega) \tag{4.19}$$
$$= \lim_{i\to\infty} \frac{[k_{m_i}(x')]^{-1} \int g_1(\omega)V(\omega)f(x', \omega)\lambda_{m_i}(d\omega)}{[k_{m_i}(x')]^{-1} \int g_2(\omega)V(\omega)f(x', \omega)\lambda_{m_i}(d\omega)}$$
$$= \lim_{i\to\infty} \frac{\int [f(x', \omega)/f(y_0, \omega)]g_1(\omega)\nu_{m_i}(y_0, d\omega)}{\int [f(x', \omega)/f(y_0, \omega)]g_2(\omega)\nu_{m_i}(y_0, d\omega)} = \infty.$$

Since $\int g_1(\omega)\nu(x', d\omega) < \infty$, this can happen only if $\int g_2(\omega)\nu(x', d\omega) = 0$. Since the choice of $g_2(\cdot)$ is arbitrary subject to $g_2(\omega) = 0$ if $\omega \geq n$, n arbitrary, $\nu(x', \{\infty\}) = 1$ follows. This argument shows that every limit point of the sequence $\{\nu_{n_i}(x', \cdot)\}$ is the same. Therefore, weak $\lim_{i\to\infty} \nu_{n_i}(x', \cdot) = \nu(x', \cdot)$ where $\nu(x', \{\infty\}) = 1$.

By definition of case II,

$$A_{0,\infty}^{y_0} = \{x| \lim_{\omega\to\infty} f(x, \omega)/f(y_0, \omega) = \infty\} \tag{4.20}$$

has positive μ measure. Suppose $x \in A_{0,\infty}^{y_0}$ and t is in the support of $\delta(x, \cdot)$. By lemma 2.1, for almost all such x we may find a subsequence (depending on x) $\{m_i, i \geq 1\}$ of $\{n_i, i \geq 1\}$ and real numbers t_{m_i} in the support of $\delta_{m_i}(x, \cdot)$, $i \geq 1$, such that $t = \lim_{i\to\infty} t_{m_i}$. Then we obtain

$$\begin{aligned} 0 &= \lim_{i\to\infty} \int [V_1(\omega, t_{m_i})/V(\omega)]\nu_{m_i}(x, d\omega) \\ &= \int [V_1(\omega, t)/V(\omega)]\nu(x, d\omega) = -1. \end{aligned} \tag{4.21}$$

This contradiction shows that subcase A cannot happen.

Subcase B is dual to subcase A, and similar arguments lead to a similar contradiction. That proves lemma 4.2.

LEMMA 4.3. *Suppose for some $x \in X$ and integer sequence $\{n_i, i \geq 1\}$ that* weak $\lim_{i\to\infty} \nu_{n_i}(x, \cdot) = \nu(x, \cdot)$. *Then $\nu(x, \{-\infty\} \cup \{\infty\}) = 0$.*

PROOF. Suppose $\nu(x, \{\infty\}) > 0$. Choose y such that $\lim_{\omega\to\infty} f(y, \omega)/f(x, \omega) = \infty$. Then

$$\lim_{i\to\infty} k_{n_i}(y)/k_{n_i}(x) = \lim_{i\to\infty} \int [f(y, \omega)/f(x, \omega)]\nu_{n_i}(x, d\omega) = \infty. \tag{4.22}$$

Contradiction. The supposition $\nu(x, \{-\infty\}) > 0$ leads to a similar contradiction of lemma 4.2.

LEMMA 4.4. *Choose $x_0 \in X$ and an integer sequence $\{n_i, i \geq 1\}$ such that $\lim_{i\to\infty} n_i = \infty$ and for every continuous function g having compact support,*

$$\lim_{i\to\infty} \int g(\omega)\lambda_{n_i}(d\omega)/k_{n_i}(x_0) \tag{4.23}$$

exists. Then the weak $\lim_{i\to\infty} \lambda_{n_i}(\cdot)/k_{n_i}(x_0) = \lambda(\cdot)$ *is a nonzero σ-finite measure. Further,* weak $\lim_{i\to\infty} \nu_{n_i}(x_0, \cdot) = \nu(x_0, \cdot)$ *exists, and for every integrable real-valued function $g(\cdot)$,*

$$\int g(\omega)\nu(x_0, d\omega) = \int g(\omega)V(\omega)f(x_0, \omega)\lambda(d\omega). \tag{4.24}$$

PROOF. Let $g(\cdot)$ be continuous with compact support. Then if E is the support of $g(\cdot)$, $\inf_{\omega\in E} V(\omega)f(x_0, \omega) > 0$. This implies

$$\sup_{n\geq 1} \int g(\omega)\lambda_n(d\omega)/k_n(x_0) < \infty. \tag{4.25}$$

By considering a countable subset of g's (dense in the continuous functions vanishing at $\pm\infty$) with compact support, we may choose an integer sequence

$\{n_i, i \geq 1\}$ such that for every continuous function with compact support,

$$\lim_{i\to\infty} \int g(\omega)\lambda_{n_i}(d\omega)/k_{n_i}(x_0) \tag{4.26}$$

exists. Use a diagonalization argument. Let $\lambda(\cdot)$ be the limiting σ-finite measure determined by these limits.

If $g(\cdot)$ is a continuous function with compact support, then so is

$$g(\cdot)V(\cdot)f(x_0, \cdot). \tag{4.27}$$

We find

$$\begin{aligned}\lim_{i\to\infty} \int g(\omega)\nu_{n_i}(x_0, d\omega) &= \lim_{i\to\infty} \int g(\omega)V(\omega)f(x_0, \omega)\lambda_{n_i}(d\omega)/k_{n_i}(x_0)\\ &= \int g(\omega)V(\omega)f(x_0, \omega)\lambda(d\omega).\end{aligned} \tag{4.28}$$

If we choose from $\{n_i, i \geq 1\}$ a subsequence $\{m_i, i \geq 1\}$ such that $\nu(x_0, \cdot) =$ weak $\lim_{i\to\infty} \nu_{m_i}(x_0, \cdot)$ exists, then for every compact set $E \subset \Omega$, (4.28) uniquely determines $\nu(x_0, E)$. By lemma 4.3, $\nu(x_0, \{\infty\} \cup \{-\infty\}) = 0$. Therefore, $\nu(x_0, \cdot)$ is uniquely determined by (4.28), which proves that weak $\lim_{i\to\infty} \nu_{n_i}(x_0, \cdot) = \nu(x_0, \cdot)$ exists. By lemma 4.3, (4.28), and a standard approximation argument, (4.24) follows.

LEMMA 4.5. *Let x_0, $\{n_i, i \geq 1\}$ and $\lambda(\cdot)$ be as in lemma 4.4. Then for every $y \in X$,*

$$\nu(y, \cdot) = \text{weak}\lim_{i\to\infty} \nu_{n_i}(y, \cdot) \text{ exists.} \tag{4.29}$$

In addition,

$$\lim_{i\to\infty} k_{n_i}(y)/k_{n_i}(x_0) = \int_\Omega f(y, \omega)V(\omega)\lambda(d\omega), \tag{4.30}$$

and for every ν integrable Ω^* measurable g,

$$\int_{\Omega^*} g(\omega)\nu(y, d\omega) = \int_\Omega g(\omega)f(y, \omega)V(\omega)\lambda(d\omega) \Big/ \int_\Omega f(y, \omega)V(\omega)\lambda(d\omega). \tag{4.31}$$

PROOF. Let $\{m_i, i \geq 1\}$ be a subsequence of $\{n_i, i \geq 1\}$ such that $\nu(\cdot) =$ weak $\lim_{i\to\infty} \nu_{m_i}(y, \cdot)$ exists. If $g(\cdot)$ is a function having compact support, then

(4.32)

$$\begin{aligned}\int g(\omega)\nu(d\omega) &= \lim_{i\to\infty} \int g(\omega)\nu_{m_i}(y, d\omega)\\ &= \lim_{i\to\infty} \left[\int g(\omega)[f(y, \omega)/f(x_0, \omega)]\nu_{m_i}(x_0, d\omega)\right] [k_{m_i}(x_0)/k_{m_i}(y)].\\ &= \left[\int g(\omega)f(y, \omega)V(\omega)\lambda(d\omega)\right] [\lim_{i\to\infty} k_{m_i}(x_0)/k_{m_i}(y)].\end{aligned}$$

By lemma 4.3, $\nu(\Omega) = 1$. Using a standard approximation argument, we obtain

$$\lim_{i\to\infty} k_{m_i}(y)/k_{m_i}(x_0) = \int_\Omega f(y, \omega)V(\omega)\lambda(d\omega). \tag{4.33}$$

Since this holds on every subsequence $\{m_i, i \geq 1\}$ for which $\{\nu_{m_i}, i \geq 1\}$ has a weak limit, (4.30) follows.

Since the value of the limit in (4.32) is independent of the subsequence used, (4.31) follows. For continuous $g(\cdot)$ having compact support, since $\nu(y, \{-\infty\} \cup \{\infty\}) = 0$, these functions are dense in the L_1 space of $\nu(y, \cdot)$, which proves (4.31).

PROOF OF THEOREM 4.2. We may pick an integer sequence $\{n_i, i \geq 1\}$ with $\lim_{i\to\infty} n_i = \infty$ such that for every $x \in X$, weak $\lim_{i\to\infty} \nu_{n_i}(x, \cdot)$ exists. Therefore, the theorem of section 2 applies. For almost all x, if t is in the support of $\delta(x, \cdot)$, then

$$0 = \int [V_1(\omega, t)/V(\omega)]\nu(x, d\omega) \tag{4.34}$$
$$= \int V_1(\omega, t)f(x, \omega)\lambda(d\omega).$$

Thus (4.13) is proven and the proof is complete.

5. Estimation in case Ia

We will suppose that for all x, y in X,

$$\lim_{\omega\to-\infty} f(x, \omega)/f(y, \omega) = \lim_{\omega\to\infty} f(x, \omega)/f(y, \omega) = 1. \tag{5.1}$$

In keeping with section 4, we suppose $\Omega = \mathfrak{D} = (-\infty, \infty)$. The results of this section depend on assuming X is Euclidean p space.

In case Ia one can give examples of sequences $\{\delta_n, n \geq 1\}$ which converge weakly, δ_n Bayes relative to λ_n, $n \geq 1$, and such that weak $\lim_{n\to\infty} \nu_n(x, \cdot) = \nu(x, \cdot)$ exists for all x, yet $\nu(x, \Omega) = 0$ for all x. We give such an example, where $X = (-\infty, \infty)$.

Let $W(\omega, t) = (\omega - t)^2$, $-\infty < t, \omega < \infty$, and $f(x, \omega) = c/(1 + (x - \omega)^4)$, the constant c being appropriately chosen. Let $\lambda_n(\cdot)$ put mass $\frac{1}{2}$ at n and mass $\frac{1}{2}$ at $-n$. A direct calculation shows that the (nonrandomized) Bayes estimator is

$$\delta_n(x) = \frac{8x^3n + 8xn^4}{2 + 2x^4 + 12x^2n^2 + 2n^4}. \tag{5.2}$$

For given x, $\lim_{n\to\infty} \delta_n(x) = 4x$, which implies weak $\lim_{n\to\infty} \delta_n(x) = 4x$. For given n, $\lim_{x\to\infty} \delta_n(x) = 0$ and $\lim_{x\to-\infty} \delta_n(x) = 0$.

The parameter ω is a location parameter. An easy direct calculation shows that estimators $\delta(x) = \alpha x$ are inadmissible estimators of a location parameter if $\alpha > 1$. In particular, $\delta(x) = 4x$ is not admissible.

The limiting procedure in case Ia depends on the asymptotic behavior of the first and second partial derivatives of $W(\cdot, \cdot)$ on its second variable and upon the asymptotic behavior of the first partial derivatives of $f(\cdot, \cdot)$ with respect to its first variable x. We suppose for $\omega \in X$ that $W(\omega, \cdot)$ is a strictly convex function. In the sequel we write W_2 and W_{22} for the first and second partial derivatives of W on its second variable. Further, if $x = (x_1, \cdots, x_p)$, then we write $f_i(x, \omega)$ for the partial derivative of f with respect to x_i evaluated at (x, ω), $1 \leq i \leq p$.

The "right" normalization in case Ia is not the normalization discussed in section 2. Instead, one wants a normalizing function $V(\cdot)$ satisfying the following. If E is a compact $\mathfrak{D}$ set, then

$$\sup_{t \in E, \omega \in \Omega} |W_{22}(\omega, t)|/V(\omega) < \infty;$$

$$\lim_{|\omega| \to \infty} W_{22}(\omega, t)/V(\omega) = \beta_1 > 0 \quad \text{uniformly in} \quad t \in E; \tag{5.3}$$

$$W_{22}(\omega, t)/V(\omega) \quad \text{is uniformly continuous in} \quad (\omega, t) \in \Omega \times E.$$

We then define

$$k_n'(x) = \int V(\omega) f(x, \omega) \lambda_n(d\omega), \qquad x \in X, \quad n \geq 1, \tag{5.4}$$

and

$$\nu_n'(x, F) = \int_F V(\omega) f(x, \omega) \lambda_n(d\omega)/k_n'(x) \tag{5.5}$$

for Borel subsets of Ω.

THEOREM 5.1. *Given the regularity conditions stated below there exist constants* $\beta_2^{(1)}, \cdots, \beta_2^{(p)}, \beta_3$ *with the following property. Let* $\{n_i, i \geq 1\}$ *be an integer sequence for which* $\nu'(x, \cdot) = \text{weak} \lim_{i \to \infty} \nu_{n_i}'(x, \cdot)$ *exists for all* $x \in X$ (see lemma 2.2). *Then* $\nu'(x, \Omega) = 0$ *for all* x *or* $\nu'(x, \Omega) > 0$ *for all* $x \in X$.

If $\nu'(x, \Omega) = 0$ *for all* $x \in X$, *then* $\delta(x) = (-\beta_3/\beta_1) \sum_{i=1}^p \beta_2^{(i)} x_i$, *where* $x = (x_1, \cdots, x_p)$.

If $\nu'(x, \Omega) > 0$ *for all* $x \in X$, *let* $\alpha = (\alpha_1, \cdots, \alpha_p)$ *be any* p *dimensional row vector of Euclidean p-space. The pair* $\delta(x), \delta(x + \alpha)$ *satisfies the functional equation*

$$\begin{aligned} 0 = &\int W_2(\omega, \delta(x + \alpha))[f(x + \alpha, \omega) - f(x, \omega)]/[V(\omega) f(x, \omega)]\nu'(x, d\omega) \\ &+ \int [(W_2(\omega, \delta(x + \alpha)) - W_2(\omega, \delta(x)))/V(\omega)]\nu'(x, d\omega). \end{aligned} \tag{5.6}$$

We first define the constants $\beta_2^{(i)}$, $1 \leq i \leq p$, β_3, and give the regularity conditions needed. We suppose

$$\lim_{\omega \to -\infty} \frac{\omega f_i(t, \omega)}{f(t, \omega)} = \lim_{\omega \to \infty} \frac{\omega f_i(t, \omega)}{f(t, \omega)} = \beta_2^{(i)}, \qquad 1 \leq i \leq p, \tag{5.7}$$

uniformly in t in compact subsets of X, and

$$\sup_{\omega \in \Omega, t \in K} \left| \frac{\omega f_i(t, \omega)}{f(t, \omega)} \right| < \infty, \qquad 1 \leq i \leq p, \tag{5.8}$$

for every compact subset K of X. We suppose for compact subsets K of $\mathfrak{D}$ that

$$\lim_{|\omega| \to \infty} (\operatorname{sgn} \omega) W_2(\omega, t)/(1 + |\omega| V(\omega)) = \beta_3 > 0, \tag{5.9}$$

uniformly in $t \in K$, and

$$\sup_{\omega \in \Omega, t \in K} |W_2(\omega, t)|/(1 + |\omega| V(\omega)) < \infty. \tag{5.10}$$

We suppose for compact subsets K of $\mathfrak{D}$ that

$$\sup_{\substack{t_1, t_2 \in K \\ \omega \in \Omega}} f(t_1, \omega)/f(t_2, \omega) < \infty \tag{5.11}$$

and

$$\lim_{|\omega| \to \infty} f(t_1, \omega)/f(t_2, \omega) = 1 \qquad \text{uniformly in} \quad t_1, t_2 \in K. \tag{5.12}$$

The theorem is proven from the relation, for all x, α in p-space,

$$\begin{aligned} 0 &= \int W_2(\omega, \delta_n(x + \alpha)) f(x + \alpha, \omega) \lambda_n(d\omega), \\ &= \int W_2(\omega, \delta_n(x + \alpha))[f(x + \alpha, \omega) - f(x, \omega)]\lambda_n(d\omega) \\ &\quad + \int [W_2(\omega, \delta_n(x + \alpha)) - W_2(\omega, \delta_n(x))] f(x, \omega) \lambda_n(d\omega). \end{aligned} \tag{5.13}$$

In case $\nu'(x, \Omega) > 0$ for all x, the final part of the theorem is simply the result of taking weak limits. We note that if weak $\lim_{n\to\infty} \int_{(\cdot)} V(\omega) f(x, \omega)\lambda_n(d\omega)/k_n'(x)$ does not exist for all x, then by lemma 2.2, we may find a subsequence on which this limit does exist for all x. We therefore suppose, without loss of generality, that this is the entire integer sequence. Using lemma 2.1, given x and α, we choose a subsequence on which

$$\lim_{i\to\infty} \delta_{n_i}(x) = \delta(x), \qquad \lim_{i\to\infty} \delta_{n_i}(x + \alpha) = \delta(x + \alpha). \tag{5.14}$$

The argument will not be affected by supposing this is the entire sequence.

On the assumption that $\nu'(x, \Omega) = 0$ for all $x \in X$, using (5.4) and (5.5), the mean value theorem, and (5.3), we find for the third integral in (5.13), after normalization by $(k_n'(x))^{-1}$, and after letting $n \to \infty$, the limiting value $\beta_1(\delta(x + \alpha) - \delta(x))$.

To evaluate the second integral in (5.13), write

$$\begin{aligned} &(k_n'(x))^{-1} W_2(\omega, \delta_n(x + \alpha))[f(x + \alpha, \omega) - f(x, \omega)]\lambda_n(d\omega) \\ &\qquad = (k_n'(x))^{-1} \int \frac{W_2(\omega, \delta_n(x + \alpha))}{1 + |\omega| V(\omega)} (1 + |\omega| V(\omega)) \\ &\qquad\qquad \left(\sum_{i=1}^{p} \alpha_i \frac{f_i(\eta_i(x, \omega), \omega) f(\eta_i(x, \omega), \omega)}{f(\eta_i(x, \omega), \omega) f(x, \omega)} \right) f(x, \omega) \lambda_n(d\omega). \end{aligned} \tag{5.15}$$

The numbers $\eta_1(x, \omega), \cdots, \eta_p(x, \omega)$ are determined by the use of the mean value theorem, and $\eta_i(x, \omega)$ lies in value between x_i and $x_i + \alpha_i$, $1 \le i \le p$. Use of the given regularity conditions and passage to the limit gives for the limiting value $\beta_3 \sum_{i=1}^{p} \alpha_i \beta_2^{(i)}$.

From the two limiting results obtained, the first part of the theorem follows.

In order to get some feeling about sizes, we consider location parameters, and $p = 1$. Suppose

$$f(x - \omega) = c/(1 + |x - \omega|^{-\beta_2}), \qquad \beta_2 < 0. \tag{5.16}$$

Then $(\partial f/\partial x)(x - \omega)/f(x - \omega) \sim \beta_2$ as $\omega \to \infty$. We take $W(\omega, t) = |\omega - t|^{\alpha}$ and $V(\omega) = 1 + |\omega - t|^{\alpha-2}$. Then $\beta_1 = \alpha(\alpha - 1)$ and $\beta_3 = \alpha$.

In order that $\delta(x) = [(-\beta_2)/(\alpha - 1)]x$ have finite risk, we require $1 > \alpha + \beta_2$. It is therefore not possible to attain $-\beta_2\beta_3/\beta_1 = 1$ and have an estimator with finite risk. As previously observed, estimators of a location parameter with $-\beta_2/\alpha - 1 > 1$ are not admissible.

6. An example in which some mass escapes to the boundary

By taking examples in which X, Ω, $\mathfrak{D}$ are compact, and $W(\cdot, \cdot)$, $f(\cdot, \cdot)$ are jointly continuous, it is easy to construct examples, by removing a point from Ω, in which a sequence of probability measures on Ω in the limit puts mass on the boundary, and yet the Bayes procedures converge to an admissible procedure.

We now consider an example for which $W(\cdot, \cdot)$ is unbounded, $\Omega = (0, \infty)$, and in the limit mass is placed at $+\infty$ in the sense of section 2. We suppose X is compact, $\mu(\cdot)$ is a probability measure defined on the Borel subsets of X, $\{f(\cdot, \omega), \omega \in \Omega\}$ is a family of generalized probability measures relative to $\mu(\cdot)$. If $x_0 \in X$, we suppose for all $x \in X$ that $f(x, \omega)/f(x_0, \omega)$ is a bounded uniformly continuous function of $x \in X$, $\omega \in \Omega$, and $\lim_{\omega\to\infty} f(x, \omega)/f(x_0, \omega) = 1$, uniformly in x.

We suppose $W(\cdot, \cdot)$ is a strictly convex function of its second variable and write $W_2(\cdot, \cdot)$ for the partial derivative of $W(\cdot, \cdot)$ on its second variable. We want $W(t, t) = 0$ for all $t \geq 0$, $\lim_{t\to\infty} W_2(\omega, t) > 0$, and $-1 = \lim_{\omega\to\infty} W_2(\omega, t)$, $t \geq 0$, $\omega \geq 0$. For example, $W_2(\omega, t) = \phi(t - \omega)$ for suitable ϕ. We take the normalizing function $V(\omega) = 1$, $0 \leq \omega < \infty$.

In order that mass move to $+\infty$, let $\{\alpha_n, n \geq 1\}$ be a nonnegative real number sequence such that $\lim_{n\to\infty} \alpha_n f(x_0, n) = 1$. In view of our assumption about $f(\cdot, \cdot)$, $\lim_{n\to\infty} \alpha_n f(x, n) = 1$ for all $x \in X$.

Let $\lambda(\cdot)$ be a probability measure on the Borel sets of Ω. We will need to assume λ, W satisfy (6.5) given below. Define $\{\lambda_n(\cdot), n \geq 1\}$ by $\lambda_n(E) = \lambda(E)$ if $n \notin E$, $\lambda_n(E) = \lambda(E) + \alpha_n$ if $n \in E$, $n \geq 1$. Let $\delta_n(\cdot)$ solve the equation

$$(6.1) \qquad 0 = \int W_2(\omega, \delta_n(x))f(x, \omega)\lambda(d\omega) + W_2(n, \delta_n(x))f(x, n)\alpha_n.$$

In the sequel we prove that $\lim_{n\to\infty} \delta_n(x) = \delta(x)$, where $\delta(x)$ solves (6.6), uniformly in $x \in X$ and that $\delta(\cdot)$ is admissible.

First, we show (6.1) is solvable. Since $W_2(\omega, \cdot)$ is strictly increasing, by the monotone convergence theorem,

$$(6.2) \qquad \lim_{t\to 0} \left[\int W_2(\omega, t)f(x, \omega)\lambda(d\omega) + W_2(n, t)f(x, n)\alpha_n\right]$$
$$= \int W_2(\omega, 0)f(x, \omega)\lambda(d\omega) + W_2(n, 0)f(x, n)\alpha_n < 0,$$

whereas

$$\lim_{t\to\infty}\left[\int W_2(\omega, t)f(x, \omega)\lambda(d\omega) + W_2(n, t)f(x, n)\alpha_n\right] \\ = \int W_2(\omega, \infty)f(x, \omega)\lambda(d\omega) + W_2(n, \infty)f(x, n)\alpha_n > 0. \tag{6.3}$$

Therefore, (6.1) is solvable for each $x \in X$, $n \geq 1$.

To prove $\lim_{n\to\infty} \delta_n(x) = \delta(x)$ uniformly in x, suppose $\{x_n, n \geq 1\}$ and $\{t_n, n \geq 1\}$ are sequences such that if $n \geq 1$, then $x_n \in X$, $t_n \geq 0$, $t_n = \delta_n(x_n)$, $\lim_{n\to\infty} x_n = x$ (recall that X is compact) and $\lim_{n\to\infty} t_n = t$ (we allow $t = +\infty$).

Case I. $\text{Limit}_{n\to\infty}\, t_n = \infty$ and $t_n > n$ infinitely often, say on the sequence $\{t_{n_i}, i \geq 1\}$. Since $W_2(n_i, t_{n_i}) > 0$, and since (using the bounded convergence theorem)

$$\lim_{i\to\infty}\int W_2(\omega, t_{n_i})f(x_{n_i}, \omega)\lambda(d\omega) \\ = \lim_{i\to\infty}\int W_2(\omega, t_{n_i})[f(x_{n_i}, \omega)/f(x_0, \omega)]f(x_0, \omega)\lambda(d\omega) \\ = \int W_2(\omega, \infty)f(x, \omega)\lambda(d\omega) > 0, \tag{6.4}$$

equation (6.1) is not solvable for large values of n_i.

Case II. $\text{Limit}_{n\to\infty}\, t_n = \infty$ and $t_n \leq n$ except for a finite number of values of n. Note that $0 \geq W_2(n, t_n) \geq W_2(n, 0)$. Our hypothesis was that $\lim_{n\to\infty} W_2(n, 0) = -1$. Therefore, $\liminf_{n\to\infty} W_2(n, t_n)f(x_n, \omega)\alpha_n \geq -1$. Since $\lim_{n\to\infty} \int W_2(\omega, t_n)f(x_n, \omega)\lambda(d\omega) = \int W_2(\omega, \infty)f(x, \omega)\lambda(d\omega)$, (6.1) will be unsolvable for large values of n if we assume

$$\int W_2(\omega, \infty)f(x, \omega)\lambda(d\omega) > 1, \qquad x \in X. \tag{6.5}$$

Case III ($t < \infty$). Using the bounded convergence theorem

$$0 = \int W_2(\omega, t)f(x_n, \omega)\lambda(d\omega) + \lim_{n\to\infty} W_2(n, t_n)f(x_n, n)\alpha_n \\ = \int W_2(\omega, t)f(x, \omega)\lambda(d\omega) - 1. \tag{6.6}$$

This equation has a unique solution. Therefore $\delta(x) = \lim_{n\to\infty} \delta_n(x_n)$ as was to be shown.

We now prove that the risks converge. That is,

$$0 = \lim_{n\to\infty}\int [W(\omega, \delta(x)) - W(\omega, \delta_n(x))]f(x, \omega)\mu(dx)\lambda(d\omega) \\ + \lim_{n\to\infty} \alpha_n \int [W(n, \delta(x)) - W(n, \delta_n(x))]f(x, n)\mu(d\omega). \tag{6.7}$$

The limit $\delta(x)$ solves (6.6). From this it follows at once that $\sup_x |\delta(x)| < \infty$. Since $\lim_{n\to\infty} \delta_n(x) = \delta(x)$ uniformly in x, there is a $K > 0$ such that $\sup_{x\in X, n\geq 1} |\delta_n(x)| \leq K$. Since $\sup_{\omega,t} |W_2(\omega, t)| < \infty$, it follows that

$$\sup_{\omega,x} |W(\omega, \delta(x)) - W(\omega, \delta_n(x))| < \infty \tag{6.8}$$

and

$$\lim_{n\to\infty} |W(\omega, \delta(x)) - W(\omega, \delta_n(x))| = 0. \tag{6.9}$$

Therefore, using the bounded convergence theorem,

$$0 = \lim_{n\to\infty} \int [W(\omega, \delta(x)) - W(\omega, \delta_n(x))] f(x, \omega)\lambda(d\omega). \tag{6.10}$$

Since $\sup_{x,\omega} f(x, \omega)/f(x_0, \omega) < \infty$, and $\lim_{n\to\infty} \alpha_n f(x_0, n) < \infty$, it follows that $\sup_{n\geq 1, x\in X} \alpha_n f(x, n) < \infty$. Since $\lim_{n\to\infty} |W(n, \delta(x)) - W(n, \delta_n(x))| = 0$ uniformly in x, and since $\mu(X) < \infty$,

$$0 = \lim_{n\to\infty} \alpha_n \int [W(n, \delta(x)) - W(n, \delta_n(x))] f(x, n)\mu(dx) = 0. \tag{6.11}$$

That proves (6.7).

To prove that $\delta(\cdot)$ is admissible, suppose $\delta'(\cdot)$ is as good as $\delta(\cdot)$. Let $K(\omega, \delta)$, $K(\omega, \delta')$, and $K(\omega, \delta_n)$ be the risks of δ, δ', and δ_n evaluated at ω. Since δ_n is a Bayes procedure,

$$\begin{aligned} 0 \leq \int (K(\omega, \delta) - K(\omega, \delta'))\lambda_n(d\omega) &\leq \int (K(\omega, \delta) - K(\omega, \delta_n))\lambda(d\omega) \\ &+ \alpha_n \int (K(\omega, \delta) - K(\omega, \delta_n)). \end{aligned} \tag{6.12}$$

As $n \to \infty$, the right-hand side of (6.12) tends to zero. Since the loss function is strictly convex, using lemma 3.1 of section 3, the admissibility of δ follows.

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APPENDIX. DECISION THEORY

Parts of this paper lean heavily on the interpretation of statistical decision procedures as continuous bilinear forms on certain pairs of Banach spaces. This interpretation is well known, but necessary details do not seem to be available anywhere. We will first discuss bilinear forms abstractly and then discuss statistical procedures.

Let F_1, F_2 be Banach spaces. Given are norms $\|x\|_1$ of $x \in F_1$ and $\|y\|_2$ of $y \in F_2$. A bilinear form $(\cdot, \cdot)$ on $F_1 \times F_2$ is a real-valued function of two variables such that to each $x \in F_1$, $(x, \cdot)$ is a linear functional on F_2, to each $y \in F_2$, $(\cdot, y)$ is a linear functional on F_1.

If a bilinear form is continuous in each variable, then using the uniform boundedness theorem one easily shows there is a constant K satisfying

$$K = \sup_{x\in F_1, y\in F_2} |(x, y)|/\|x\|_1\|y\|_2. \tag{A.1}$$

The constant K is the norm of the bilinear form. Conversely, if a bilinear form satisfies an inequality of the form

$$|(x, y)| \leq K\|x\|_1\|y\|_2 \tag{A.2}$$

for all $x \in F_1$, $y \in F_2$, then the bilinear form is a continuous linear functional in each variable.

The space F_{12} of continuous bilinear forms on $F_1 \times F_2$ is a Banach space under the norm defined above. The weak topology in F_{12} is the weakest topology such that for each pair $(x, y) \in F_1 \times F_2$, the mapping $(\cdot, \cdot) \to (x, y)$ defined on F_{12} to the real line is a continuous mapping. A standard argument of embedding F_{12} in a product space will show the unit ball of F_{12} is compact in the weak topology.

Generally, in most decision theory discussions, the assertion that a set of decision procedures is compact is the assertion that a set of bilinear forms is weakly compact. We develop this idea here.

In the context of this paper, $\mathfrak{D}$ is the set of decisions, and we assume that $\mathfrak{D}$ is a locally compact Hausdorff space and that $\mathfrak{D}^*$ is the one point compactification of $\mathfrak{D}$. We suppose $F_2 = C(\mathfrak{D}^*)$, the set of bounded continuous real-valued functions on $\mathfrak{D}^*$. We take $F_1 = L_1(X, \mathfrak{B}, \mu)$. If $\delta(\cdot, \cdot)$ is a statistical decision procedure, then for every Borel subset A of $\mathfrak{D}^*$, $\delta(\cdot, A)$ is a bounded measurable function, and, if $x \in X$, then $\delta(x, \cdot)$ is a probability measure on the Borel subsets of $\mathfrak{D}^*$. We now state and prove a converse to this.

THEOREM A1. *Suppose $F_1 = L_1(X, \mathfrak{B}, \mu)$ and $(X, \mathfrak{B}, \mu)$ is a totally σ-finite measure space. Suppose $\mathfrak{D}^*$ is a compact metric space. Let $F_2 = C(\mathfrak{D}^*)$. If $(\cdot, \cdot)$ is a continuous bilinear form on $F_1 \times F_2$ of norm K, then there exists $\delta(\cdot, \cdot)$ satisfying the following:*

(i) *to each $x \in X$, $\delta(x, \cdot)$ is a countably additive finite measure on the Borel subsets of $\mathfrak{D}^*$;*

(ii) *to each Borel set $E \subset \mathfrak{D}^*$, $\delta(\cdot, E)$ is a bounded $\mathfrak{B}$-measurable function;*

(iii) *if $f(\cdot) \in L_1(X, \mathfrak{B}, \mu)$ and $g(\cdot) \in C(\mathfrak{D}^*)$, then*

$$(f, g) = \iint f(x)g(t)\delta(x, dt)\mu(dx);$$

(iv) *for all $x \in X$ and Borel subsets E of $\mathfrak{D}^*$, $|\delta(x, E)| \leq K$.*

This theorem is known. An unpublished proof, different from the proof given below, has been given by Le Cam.

PROOF. The space $C(\mathfrak{D}^*)$ is a separable metric space. We take a countable dense subset $\{g_n^*(\cdot), n \geq 1\}$ of $C(\mathfrak{D}^*)$. By discarding some of these functions, we may find a subset $\{g_n(\cdot), n \geq 1\}$ which are linearly independent over the rational numbers, and such that every $g_n^*(\cdot)$ is a linear combination of functions in $\{g_n(\cdot), n \geq 1\}$, $n \geq 1$.

Since $(\cdot, g_n)$ is a continuous linear functional on $L_1(X, \mathfrak{B}, \mu)$, and since $(X, \mathfrak{B}, \mu)$ is a totally σ-finite measure space, we may find a bounded $\mathfrak{B}$-measurable function $\delta(\cdot, g_n)$ satisfying

$$(f, g_n) = \int f(x)\delta(x, g_n)\mu(dx) \tag{A.3}$$

for all $f \in L_1(X, \mathfrak{B}, \mu)$, and

$$\sup_x |\delta(x, g_n)| \leq K\|g_n\|_2. \tag{A.4}$$

Let $C_R(\mathfrak{D}^*)$ be the linear span of $\{g_n, n \geq 1\}$ over the rational numbers. I $g \in C_R(\mathfrak{D}^*)$, we represent $g(\cdot)$ uniquely as a finite sum

$$g(\cdot) = \sum r_i g_i(\cdot), \tag{A.5}$$

and define for all $x \in X$,

$$\delta(x, g) = \sum r_i \delta(x, g_i). \tag{A.6}$$

Then if $f \in L_1(X, \mathcal{B}, \mu)$,

$$\int f(x)\delta(x, g)\mu(dx) = \sum r_i(f, g_i) = (f, g) \leq K\|f\|_1\|g\|_2. \tag{A.7}$$

Since this holds for all $f \in L_1(X, \mathcal{B}, \mu)$,

$$\operatorname*{ess\,sup}_x |\delta(x, g)| \leq K\|g\|_2, \qquad \text{for all} \quad g \in C_R(\mathfrak{D}^*). \tag{A.8}$$

We may then find a set $N \in \mathcal{B}$, $\mu(N) = 0$, such that if $g \in C_R(\mathfrak{D}^*)$, then $\sup_{x \notin N} |\delta(x, g)| \leq K\|g\|_2$. This is possible since $C_R(\mathfrak{D}^*)$ is countable.

If $x \notin N$, then $\delta(x, \cdot)$ is a continuous linear functional defined on $C_R(\mathfrak{D}^*)$. It has a unique continuous extension to $C(\mathfrak{D}^*)$. So, if $x \notin N$, $g \in C(\mathfrak{D}^*)$, we write $\delta(x, g)$ for the extension and have $|\delta(x, g)| \leq K\|g\|_2$.

Now if $g \in C(\mathfrak{D}^*)$, we may find a subsequence $\{g_{n_i}, i \geq 1\}$ in $C_R(\mathfrak{D}^*)$ such that $\lim_{i\to\infty} \sup_{t\in\mathfrak{D}^*} |g_{n_i}(t) - g(t)| = 0$. Then if $x \notin N$, $\delta(x, g) = \lim_{i\to\infty} \delta(x, g_{n_i})$. Therefore, $\delta(\cdot, g)$ is a $\mathcal{B}$-measurable function.

By the Riesz representation theorem, the continuous linear functional $\delta(x, \cdot)$ is representable by a countably additive measure. For Borel sets E of $\mathfrak{D}^*$, we write $\delta(x, E)$ for the value of the measure. Then if $x \notin N$, for all Borel subsets E of $\mathfrak{D}^*$, $|\delta(x, E)| \leq K$.

We now show that for each Borel set E, $\delta(\cdot, E)$ is a $\mathcal{B}$-measurable function. Let $\mathcal{C}$ be the set of all Borel subsets of $\mathfrak{D}^*$ for which this is true. The set $\mathcal{C}$ is clearly a monotone class, and by considering monotone sequences of continuous functions, one easily shows $\mathcal{C}$ to contain all compact sets. Therefore, $\mathcal{C}$ contains all Borel sets. (This type of argument works even if the measures are signed measures.)

One may show at once, using the bounded convergence theorem, that in extending δ from $C_R(\mathfrak{D}^*)$ to $C(\mathfrak{D}^*)$, we have for all $f \in L_1(X, \mathcal{B}, \mu)$, $g \in C(\mathfrak{D}^*)$, $(f, g) = \int f(x)\delta(x, g)\mu(dx)$.

From the form of the Riesz representation,

$$(f, g) = \iint f(x)g(t)\delta(x, dt)\mu(dx). \tag{A.9}$$

The integral is absolutely convergent. That completes the proof.

A statistical decision procedure is a bilinear form satisfying (v) if $f \in L_1(X, \mathcal{B}, \mu)$, $f(x) \geq 0$ for all $x \in X$, $g \in C(\mathfrak{D}^*)$, $g(t) \geq 0$ for all $t \in \mathfrak{D}^*$, then $(f, g) \geq 0$; (vi) if $f \in L_1(X, \mathcal{B}, \mu)$, then $(f, 1) = \int f(x)\mu(dx)$. It is easily checked that the set of bilinear forms satisfying these conditions is a weakly closed subset of the unit ball of F_{12}.

COROLLARY. *Let* $(X, \mathcal{B}, \mu)$ *be totally* σ*-finite, let* $L_1(X, \mathcal{B}, \mu)$ *be a separable*

Banach space, and suppose $\mathfrak{D}^*$ is a compact metric space. Then the set of statistical decision procedures is sequentially compact.

In the usual statistical problem a set Ω of density functions is given, if $f \in \Omega$ then $f \in L_1(X, \mathcal{B}, \mu)$. The set of decisions $\mathfrak{D}$ need not be compact, but we suppose $\mathfrak{D}$ has a compactification $\mathfrak{D}^*$ containing $\mathfrak{D}$ as a Borel set. For each $f \in \Omega$, $t \in \mathfrak{D}$ we suppose a measure of loss $W(f, t) \geq 0$ is given. We assume $W(f, \cdot)$ has an extension to $\mathfrak{D}^*$ such that for each $f \in \Omega$, the extended function is lower semicontinuous.

THEOREM A2. *Suppose $\{\delta_n, n \geq 1\}$ is a sequence of statistical decision procedures. There exists a subsequence $\{\delta_{n_i}, i \geq 1\}$ and a procedure δ such that for all $f \in \Omega$,*

$$\int W(f, t)\delta(x, dt)f(x)\mu(dx) \leq \liminf_{i\to\infty} \int W(f, t)\delta_{n_i}(x, dt)f(x)\mu(dx). \tag{A.10}$$

It may be that $\delta(x, \mathfrak{D}^* - \mathfrak{D}) > 0$ for some x.

PROOF. Extend δ_n to $\mathfrak{D}^*$ by $\delta_n(x, \mathfrak{D}^* - \mathfrak{D}) = 0$ for all $x \in X$, $n \geq 1$. Choose a subsequence such that $\delta = \text{weak} \lim_{i\to\infty} \delta_{n_i}$. Let $W_N(f, \cdot)$ be an increasing sequence of continuous functions on $\mathfrak{D}^*$ satisfying $\lim_{N\to\infty} W_N(f, t) = W(f, t)$ for all $f \in \Omega$, $t \in \mathfrak{D}^*$. Then

$$\begin{aligned} \liminf_{i\to\infty} \int W(f, t)\delta_{n_i}(x, dt)f(x)\mu(dx) &\geq \lim_{t\to\infty} \int W_N(f, t)\delta_{n_i}(x, dt)f(x)\mu(dx) \\ &= \int W_N(f, t)\delta(x, dt)f(x)\mu(dx). \end{aligned} \tag{A.11}$$

Let $N \to \infty$ and apply the monotone convergence theorem. The result follows.

In some applications $\mathfrak{D}$ is a finite dimensional space, and for each $f \in \Omega$, $W(f, \cdot)$ is a strictly convex function. If $\lim_{t\to\infty} W(f, t) = \infty$, then we obtain the following result.

THEOREM A3. *Let $\{\delta_n, n \geq 1\}$ be a sequence of decision procedures; let δ be an admissible procedure, and*

$$\limsup_{n\to\infty} \int W(f, t)\delta_n(x, dt)f(x)\mu(dx) \leq \int W(f, t)\delta(x, dt)f(x)\mu(dx) \tag{A.12}$$

for all $f \in \Omega$. Let δ^* be a weak limit point of $\{\delta_n, n \geq 1\}$. Then for all x, $\delta^*(x, \mathfrak{D}^* - \mathfrak{D}) = 0$, and if $A = \{x | \delta(x, \cdot) \neq \delta^*(x, \cdot)\}$, then $\int_A f(x)\mu(dx) = 0$ for all $f \in \Omega$.

PROOF. Let $\delta^* = \text{weak} \lim_{i\to\infty} \delta_{n_i}$ for some subsequence. By theorem A2,

$$\int W(f, t)\delta^*(x, dt)f(x)\mu(dx) \leq \int W(f, t)\delta(x, t)f(x)\mu(dx). \tag{A.13}$$

Since δ is admissible, δ must be nonrandomized; therefore, we write

$$\int W(f, t)\delta(x, t)f(x)\mu(dx) = \int W(f, \delta(x))f(x)\mu(dx). \tag{A.14}$$

Further, we set $\delta^*(x) = \int t\delta^*(x, dt)$ and obtain by Jensen's inequality that

$$\int W(f, \delta^*(x))f(x)\mu(dx) \leq \int W(f, \delta(x))f(x)\mu(dx). \tag{A.15}$$

Since $\delta(\cdot)$ is admissible, if $f \in \Omega$, this must be equality. Let

$$A_1 = \{x|\delta^*(x, \{\delta^*(x)\}) \neq 1\}. \tag{A.16}$$

Since W is a strictly convex function, $\int_{A_1} f(x)\mu(dx) = 0$ for all $f \in \Omega$. Again, since W is a strictly convex function, if $A = \{x|\delta(x) \neq \delta^*(x)\}$, then

$$\int_A f(x)\mu(dx) = 0 \tag{A.17}$$

for all $f \in \Omega$. That completes the proof.

We consider here the minimax theorem in the context needed for section 3. We suppose $W(\ ,\cdot)$ is bounded on compact $\Omega \times \mathfrak{D}$ subsets, $W(\omega, \cdot)$ is lower semicontinuous for each $\omega \in \Omega$, and $W(\cdot, t)$ is continuous for each $t \in \mathfrak{D}$. We assume that $W(\omega, t) \geq 0$ for all $(\omega, t) \in \Omega \times \mathfrak{D}$, and if $C \subset \Omega$ is a compact set, then

$$\lim_{t\to\infty} \inf_{\omega\in C} W(\omega, t) = \infty. \tag{A.18}$$

We suppose that we are given a family $\{f(\cdot, \omega), \omega \in \Omega\}$ of density functions relative to the σ-finite measure space $(X, \mathfrak{B}, \mu)$, and to each $x \in X$, $f(x, \cdot)$ is a continuous function on Ω.

THEOREM A4. *Let $r(\cdot)$ be a nonnegative lower semicontinuous function on Ω, and C a compact parameter set. Assume $\sup_{\omega\in C} r(\omega) < \infty$. Relative to the measure of loss $W(\omega, t) - r(\omega)$ there exists a minimax procedure δ (ω is restricted to C) which is Bayes relative to λ supported on C and*

$$\text{minimax risk} = \iint (W(\omega, t) - r(\omega))\delta(x, dt)f(x, \omega)\mu(dx)\lambda(d\omega). \tag{A.19}$$

PROOF. If δ is an admissible procedure for $\omega \in C$, then the values of δ lie in a compact subset of $\mathfrak{D}$. Indeed, take $a_0 \in \mathfrak{D}$. By hypothesis we can find a compact subset E of $\mathfrak{D}$ such that $a_0 \in E$, and if $a \notin E$, then $\sup_{\omega\in C} W(\omega, a_0) < \inf_{\omega\in C} W(\omega, a)$. Therefore, one always does better to decide a_0 than a.

Consequently, we may suppose there is a constant K such that for all $\omega \in C$, all $x \in X$, and all t in the support of $\delta(x, \cdot)$, $W(\omega, t) \leq K$. It follows that

$$K(\omega, \delta) = \iint W(\omega, t)\delta(x, dt)f(x, \omega)\mu(dx) \tag{A.20}$$

is continuous in ω and that $K(\cdot, \delta) - \delta(\cdot)$ is upper semicontinuous.

We let R_1 be the set of real-valued upper semicontinuous functions on C such that if $g \in R_1$, then for some δ, $g(\omega) = K(\omega, \delta) - r(\omega)$ for all $\omega \in C$. Then R_1 is a convex set of functions. We let R_2 be the set of real-valued continuous functions such that if $g_2 \in R_2$, there is a $g_1 \in R_1$ such that for all $\omega \in C$, $g_1(\omega) \leq g_2(\omega)$. Then R_2 is a convex set of continuous functions, and each $g_1 \in R_1$ is the limit of a monotone decreasing sequence of functions in R_2.

We apply the now classical construction. Let $R(\epsilon)$ be the set of all real-valued continuous functions on C such that if $g \in R(\epsilon)$, then $\sup_{\omega\in C} g(\omega) < \epsilon$. Then for each ϵ, $R(\epsilon)$ is a convex subset of the continuous functions on C, and $R(\epsilon)$ has an interior point in the sup topology. Further, if $\sup_{\omega\in C} r(\omega) < \infty$, there exist ϵ such that $R(\epsilon)$ and R_2 are disjoint. We consider the functions in R_2 as restricted to C. Since $R(\epsilon_0) = \bigcup_{\epsilon<\epsilon_0} R(\epsilon)$, there is a largest ϵ such that $R(\epsilon)$ and R_2 are disjoint.

The convex sets $R(\epsilon)$ and R_2 may be separated by a hyperplane. By the Riesz representation theorem there is a finite countably additive measure ξ on the Borel subsets of C and a number α such that if $g \in R(\epsilon)$, then $\int g(\omega)\xi(d\omega) \leq \alpha$; if $g \in R_2$, then $\int g(\omega)\xi(d\omega) \geq \alpha$. For integer $N \geq 1$, if $C' \subset C$ is a compact subset of C, we may approximate $-N\chi_{C'}$ by monotone limits of functions in $|\epsilon| + R(\epsilon)$. This implies $-N\xi(C') \leq \alpha + |\epsilon|\xi(\Omega)$. Let $N \to \infty$ and obtain $\xi(C') \geq 0$ for every compact subset of C. Since $\xi \neq 0$, we may suppose ξ is normalized to be a probability measure.

Since functions in R_1 may be approximated from above by functions in R_2, it follows that the hyperplane determined by ξ, α separates R_1 and $R(\epsilon)$. Further if $\beta > 0$ and $(\epsilon + \beta)1$ is the constant function of value $\epsilon + \beta$, then R_1 contains a function g satisfying $g(\omega) \leq \epsilon + \beta$ for $\omega \in C$.

Since R_1 has the weak compactness property of theorem A2, R_1 contains the risk function of a minimax procedure satisfying

$$\sup_{\omega \in C} g(\omega) \leq \epsilon; \qquad \int g(\omega)\xi(d\omega) = \epsilon. \tag{A.21}$$

Since every procedure δ which is Bayes with respect to ξ gives rise to a risk function in R_1, and since every such procedure must therefore have Bayes risk $\geq \epsilon$ relative to ξ, it follows that g is Bayes relative to ξ and the class of *all* procedures δ.

That completes the proof.

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