ON CONFIDENCE INTERVALS AND SETS FOR VARIOUS STATISTICAL MODELS

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1. Introduction

In addition to obtaining point estimates, one of the central problems of statistical inference is the construction of confidence intervals. In most works, as a rule, considerations are limited to independent sampling, which restricts the range of application without justification. The present paper intends to show that the problems of constructing confidence sets and intervals may be solved for diverse models of mathematical statistics. Underlying the methods is the concept of systems of confidence sets (see [1], [2], [3]). The material expounded below is part of the lectures in a course in mathematical statistics read to students in the Mathematics-Mechanics Faculty of Moscow University in the Fall semester of 1965.

2. Construction of confidence intervals

We consider the statistical model $[X, \mathcal{B}_X, \Theta, P_\theta]$ where $X = \{x\}$ is the set of possible results x of the experiment, and $\mathcal{B}_X$ is a σ-algebra of events. A family of stochastic measures P_θ governing the outcome of the experiment is given on $\mathcal{B}_X$. The object Θ is a set of unknown parameters. Let us consider the subset H of points (θ, x) in the direct product $\Theta \times X$. The sets $H_\theta = \{x\colon (\theta, x) \in H\}$ are called θ-sections of H. The sets $H_x = \{\theta\colon (\theta, x) \in H\} \subseteq \Theta$ are called x-sections of H. The subsets $\{H_x\}$ of the set Θ are called confident with a coefficient of confidence not less than (equal to) γ if the set $\{\theta \in H_x\} \in \mathcal{B}_X$ and

$$\inf_{\theta \in \Theta} P_\theta\{\theta \in H_x\} \geq (=)\gamma. \tag{1}$$

THEOREM 1. (See [1].) *If the θ-sections H_θ of the set H are measurable, and if for every $\theta \in \Theta$ they satisfy the condition*

$$\inf_{\theta \in \Theta} P_\theta\{x \in H_\theta\} \geq (=)\gamma, \tag{2}$$

then the x-sections of the set $H \subseteq \Theta \times X$ form a system of confidence sets with a coefficient of confidence not less than (equal to) γ, or briefly, a γ system.

The proof is a direct consequence of the equivalence of the events

$$\{(\theta, x) \in H\} \equiv \{x \in H_\theta\} \equiv \{\theta \in H_x\}. \tag{3}$$

Let us consider the following example. Let us assume that the space X is formed by sets of integers $x = (d_1, \cdots, d_m)$, $d_i = 0, 1, \cdots$, whose coordinates are mutually independent Poisson random variables with unknown parameters forming the vector $\theta = (\lambda_1, \cdots, \lambda_m)$, that is, $M_\theta d_i = \lambda_i$. Here Θ is a positive quadrant of m-dimensional Euclidean space. Let us construct θ-sections of the set H by means of the formula

$$H_\theta = \{x\colon d_1 + \cdots + d_m \geq d_\gamma\}, \tag{4}$$

where d_γ is the greatest integer for which

$$P_\theta\{d_1 + \cdots + d_m \geq d_\gamma\} = \sum_{k=d_\gamma}^{\infty} \frac{(\lambda_1 + \cdots + \lambda_m)^k}{k!} e^{-(\lambda_1 + \cdots + \lambda_m)} \geq \gamma. \tag{5}$$

It is easy to verify (see [2]) that the x-sections of such a set H are defined by the formula

$$H_x = \{(\lambda_1, \cdots, \lambda_m)\colon \lambda_1 + \cdots + \lambda_m \leq \Delta_{1-\gamma}(d_1 + \cdots + d_m), \lambda_i \geq 0\}, \tag{6}$$

where $\Delta_\alpha(k)$ is the solution of the transcendental equation $\sum_{k=0}^{d} (z^k/k!)e^{-z} = \alpha$. By virtue of (4), condition 2 of theorem 1 is satisfied and the sets H_x given by formula (6) form a γ system.

In statistical models to which multidimensional spaces of unknown parameters Θ correspond, the need often arises for a construction of the confidence interval for the function $f(x, \theta)$. It is assumed that $f(x, \theta)$ is $\mathcal{B}_X$ measurable for each $\theta \in \Theta$. The interval $[\underline{f}(x), \bar{f}(x)]$ is called confident for $f(x, \theta)$ with a coefficient of confidence not less than (equal to) γ, if $\underline{f}(x), \bar{f}(x)$ are $\mathcal{B}_X$ measurable, and

$$\inf_{\theta \in \Theta} P_\theta\{\underline{f}(x) \leq f(x, \theta) \leq \bar{f}(x)\} \geq (=)\gamma. \tag{7}$$

Let us designate such stochastic intervals briefly as γ intervals. The need to extend the problem to functions dependent on x as well as on θ arises in a natural way in problems of statistical acceptance testing, say [2].

If all the constructions of confidence intervals in mathematical statistics are analyzed, we then shall see that they either explicitly or implicitly follow the plan of the following theorem.

THEOREM 2. *If $\{H_x\}$ is a γ system, and*

$$\underline{f}(x) = \inf_{\theta \in H_x} f(x, \theta) \quad \text{and} \quad \bar{f}(x) = \sup_{\theta \in H_x} f(x, \theta) \tag{8}$$

are $\mathcal{B}_X$ measurable functions, then the interval $[\underline{f}(x), \bar{f}(x)]$ is a γ interval for $f(x, \theta)$.

The proof is a consequence of the relation $\{\theta \in H_x\} \subseteq \{\underline{f}(x) \leq f(x, \theta) \leq \bar{f}(x)\}$ from which we obtain for any $\theta \in \Theta$

$$\gamma \leq P_\theta\{\theta \in H_x\} \leq P_\theta\{\underline{f}(x) \leq f(x, \theta) \leq \bar{f}(x)\}. \tag{9}$$

It is sometimes useful to keep the following in mind.

COROLLARY. *If one is given a priori the supplementary information that* $\theta \in \Theta_0 \subset \Theta$, *then a narrower* γ *interval may be constructed by means of the formulas*

$$(10) \qquad \underline{f}'(x) = \inf_{\theta \in H_x \cap \Theta_0} f(x, \theta) \quad \text{and} \quad \bar{f}'(x) = \sup_{\theta \in H_x \cap \Theta_0} f(x, \theta).$$

The following trivial remark may also turn out to be useful sometimes.

THEOREM 3. *If* $\{H_x^i\}$ *is a* γ_i *system,* $i = 1, 2$, *then the system of sets* $\{H_x = H_x^1 \cap H_x^2\}$ *is a* $(\gamma_1 + \gamma_2 - 1)$ *system.*

It follows from theorems 2 and 3 that in certain cases of statistical models with a space of large dimensionality Θ the problem of seeking the upper bound of a one-sided γ interval $[0, \bar{f}(x)]$ takes the specific form of a concave nonlinear programming problem. More specifically, in some cases it is required to find the upper confidence level for the concave function $f(\theta)$ when the γ system is formed by random convex polyhedra. In such cases the $\max f(\theta)$ is sought at the vertices of the confidence polyhedra. For example, for the γ system described by sets of the form (6) and the function $f(\theta) = \sum_{i=1}^{m} f_i(\lambda_i)$, where the f_i are concave functions, the upper bound of a one-sided γ interval equals

$$(11) \qquad \bar{f}(x) = \max_{i=1,m} \left\{ f_i\left(\Delta_{1-\gamma}\left(\sum_{j=1}^{m} d_i \right) \right) \right\},$$

with $f_i(0) = 0$.

The construction of a γ system is done in a simpler way by using the assignment of stochastic variables dependent on both the outcome of the experiment $x \in X$ and on the parameter $\theta \in \Theta$. Generally, let $g(x, \theta)$ be a vector stochastic variable. In the space G of values of the stochastic variable $g(x, \theta)$ we select for each $\theta \in \Theta$ a subset $G_\theta(\gamma)$ such that

$$(12) \qquad P_\theta\{g(x, \theta) \in G_\theta(\gamma)\} \geq \gamma.$$

It follows from condition (12) that the set $H_\theta = \{x : g(x, \theta) \in G_\theta(\gamma)\}$ may be considered as a θ-section of some set H in the product space $\Theta \times X$. We obtain the following assertion from theorem 1.

THEOREM 4. *The system of sets*

$$(13) \qquad H_x = \{\theta : g(x, \theta) \in G_\theta(\gamma)\},$$

where the $G_\theta(\gamma)$ *satisfy relation* (12), *is a* γ *system.*

The method of constructing γ systems is particularly simple in those cases when $g(x, \theta)$ has a distribution independent of the unknown parameters θ. Here $G_\theta(\gamma) = G(\gamma)$, that is, independent of the unknown parameter θ.

Let us illustrate the method of constructing γ systems by two examples.

Let the test outcomes be $x = (t_{r_1}^{(1)}, \cdots, t_{r_m}^{(m)})$, where $t_{r_i}^{(i)}$ is the time of the appearance of the r_i-th event in a Poisson process with index i and unknown intensity λ_i. It is assumed that the $t_{r_i}^{(i)}$ are stochastic variables which are mutually independent with respect to i. The sets $\theta = (\lambda_1, \cdots, \lambda_m)$, $\lambda_i \geq 0$ play the part of the unknown parameter θ. It is easy to check that $2\lambda_i t_{r_i}^{(i)}$ has a χ^2 distribution with $2r_i$ degrees of freedom. Correspondingly, the stochastic variable $g(x, \theta) = 2 \cdot \sum_{i=1}^{m} \lambda_i t_{r_i}^{(i)}$ has a χ^2 distribution with $2\sum_{i=1}^{m} r_i$ degrees of freedom.

If we select the interval $[0, \chi^2_\gamma(2 \sum_{i=1}^m r_i)]$ as the set $G_\theta(\gamma)$, where χ^2_γ is the quantile level γ for the χ^2 distribution with $2 \sum_{i=1}^m r_i$ degrees of freedom, then in conformity with (13), the sets

$$H_x = \left\{(\lambda_1, \cdots, \lambda_m): 2 \sum_{i=1}^m \lambda_i t_{r_i}^{(i)} \leq \chi^2_\gamma\left(2 \sum_{i=1}^m r_i\right)\right\} \tag{14}$$

generate a γ system. Thus the confidence sets are formed by points $(\lambda_1, \cdots, \lambda_m)$, cut out of the first quadrant by the hyperplane $2 \sum_{i=1}^m \lambda_i t_{r_i}^{(i)} = \chi^2_\gamma(2 \sum_{i=1}^m r_i)$.

Let us use the second example to illustrate the methods of constructing confidence intervals for one of the components of the unknown parameter, when the other parameters may be considered as nuisances.

Let β_t be a Wiener process with $M\beta_t = \mu \cdot t$ and variance $M(\beta_t - \mu t)^2 = t$. The process β_t is observed up to the cut-off time t^*. It is assumed that $P\{t^* > t\} = e^{-\lambda t}$ and that t^* is independent of the value of β_t. Hence, the trial outcome is a piece of the trajectory of the Wiener process β_s, $0 \leq s \leq t^*$. The unknown parameters are $\theta = (\mu, \lambda)$, that is, the local drift coefficient μ and the cut-off intensity λ. If we start from sufficient statistics [3], then we may limit ourselves to the space of values $X = \{x\}$, where $x = (t, y)$ are the coordinates of the Wiener process $\beta_t = y$ at the cut-off time $t^* = t$. It follows from the conditions of the problem that the probability density is

$$p_\theta(x) = \lambda \cdot e^{-\lambda t} \cdot \frac{1}{\sqrt{2\pi t}} \exp\{(y - \mu t)^2/2t\}. \tag{15}$$

Let us consider the two-dimensional stochastic variable

$$g(x, \theta) = (s_1, s_2); \qquad s_1 = \lambda t,\ s_2 = \sqrt{\lambda}\,(\beta_t - \mu t). \tag{16}$$

It is easy to verify that its distribution is independent of the unknown values of the parameter θ and is given by the density

$$p(s_1, s_2) = \frac{1}{\sqrt{2\pi s_1}} \exp\left\{-\frac{s_2^2}{2s_1} - s_1\right\}. \tag{17}$$

In conformity with theorem 4, we may select any domain $G(\gamma)$ in the plane of the points (s_1, s_2) such that

$$\int_{G(\gamma)} p(s_1, s_2)\, ds_1\, ds_2 = \gamma. \tag{18}$$

In conformity with (13), the confidence sets generating the γ system are

$$H_x = \{(\mu, \lambda): (\lambda t^*, \sqrt{\lambda}(\beta_{t^*} - \mu t^*)) \in G(\gamma)\}. \tag{19}$$

It is possible to formulate and solve the problem of selecting the best domain $G(\gamma)$, which would minimize the chosen "width" index of the confidence interval. In this example we limited ourselves to a rectangular domain of the form $G(\gamma) = \{s_1 \geq a, -b \leq s_2 \leq b\}$ where a and b are connected by means of (18). For such a domain $G(\gamma)$ the confidence domain is

$$H_x = \{(\mu, \lambda): \lambda t^* \geq a, -b \leq \sqrt{\lambda}\,(\beta_{t^*} - \mu t^*) \leq b\}, \tag{20}$$

or finally,

$$H_x = \left\{(\mu, \lambda): \lambda \geq \frac{a}{t^*}, \frac{\beta_{t^*} - b\lambda^{-1/2}}{t^*} \leq \mu \leq \frac{\beta_{t^*} + b\lambda^{-1/2}}{t^*}\right\}. \tag{21}$$

Let us now assume that we are required to construct a γ interval, for the parameter μ when λ plays the part of the nuisance parameter. To apply theorem 2, let us note that $f(\theta) = \mu$. From (8) and (21) we find

$$\underline{\mu} = \inf_{\theta \in H_x} \mu = \frac{\beta_{t^*} - bt^{*1/2}a^{-1/2}}{t^*}, \qquad \bar{\mu} = \sup_{\theta \in H_x} \mu = \frac{\beta_{t^*} + bt^{*1/2}a^{-1/2}}{t^*}. \tag{22}$$

The selection of the optimum a and b may be carried out by using the method of Lagrange multipliers.

Let us now consider the generalization of an example of Fisher [4], [5] to elementary models of Markov processes with a countable set of states which are observed up to a certain stopping time. The material expounded below is an extension of results of L. N. Bol'shev [6], [7], and [2], which are associated with observations of a Poisson process.

Let us assume that the space of trial outcomes X can be made into a completely ordered set by introducing the relation $x \prec y$ meaning "x to the left of y." The reader might at once imagine the points of the stopping limit of a stochastic process as the generalization of a line. Furthermore, let us assume that for any $z \in X$ the probability $\mathcal{F}(z, \theta) = P_\theta\{x \preceq z\}$ is a nonincreasing (nondecreasing) function of the parameter θ, which we consider to be a real number in this case. Let us use the notation $\mathcal{G}(z, \theta) = P_\theta\{z \succeq x\}$. It is convenient to consider that the family of probabilistic measures P_θ assigned on X is consistent in the sense that for each interval $[a, b]$, $P_\theta\{a \preceq x \preceq b\} > 0$ is continuous with respect to θ, with the possible exception of the critical value of θ only. It is thereby implicitly assumed that those points x which cannot be observed as a result of the experiment are excluded from consideration. For each value of θ in the space X let us prescribe an interval $[\underline{x}(\theta), \bar{x}(\theta)]$, as narrow as possible, for which

$$P_\theta\{\underline{x}(\theta) \preceq x \preceq \bar{x}(\theta)\} \geq (=)\, \gamma. \tag{23}$$

Since $\mathcal{F}(x, \theta)$ is nonincreasing (nondecreasing) in θ, the bounds of $\underline{x}(\theta)$, $\bar{x}(\theta)$ may be chosen as nondecreasing (nonincreasing) functions of θ. It is traditional to select such values of the trial outcomes for which

$$\begin{cases} \mathcal{G}(\underline{x}(\theta), \theta) \geq 1 - \epsilon_1 \geq \sup\limits_{x_1 \succ \underline{x}(\theta)} \mathcal{G}(x_1, \theta) \\ \mathcal{F}(\bar{x}(\theta), \theta) \geq 1 - \epsilon_2 \geq \sup\limits_{x_2 \prec \bar{x}(\theta)} \mathcal{F}(x_2, \theta) \end{cases} \tag{24}$$

as $H_\theta = [\underline{x}(\theta), \bar{x}(\theta)]$. Analogous relationships are written down for nondecreasing $\mathcal{F}(x, \theta)$. If ϵ_1 and ϵ_2 satisfy the condition $\gamma = 1 - (\epsilon_1 + \epsilon_2)$, then (23) follows from (24). Hence, the interval $[\underline{x}(\theta), \bar{x}(\theta)]$ may be considered as a θ-section of the set $H \subset \Theta \times X$; the x-sections generate a γ system. From the fact that the functions $\underline{x}(\theta)$, $\bar{x}(\theta)$ are nondecreasing (nonincreasing) it follows that the

x-sections are the intervals $[\underline{\theta}(x), \bar{\theta}(x)]$. The boundaries of the confidence intervals for the observation of the outcome x^* are found from the relations

$$(25)\qquad \begin{cases} \bar{\theta}(x) = \sup \{\theta\colon \mathcal{G}(x^*, \theta) \geq 1 - \epsilon_1 \geq \sup_{x_1 > x^*} \mathcal{G}(x_1, \theta)\}, \\ \underline{\theta}(x) = \inf \{\theta\colon \mathcal{F}(x^*, \theta) \geq 1 - \epsilon_2 \geq \sup_{x_2 < x^*} \mathcal{F}(x_2, \theta)\}, \end{cases}$$

when $\mathcal{F}(x, \theta)$ is nonincreasing, and from the relations

$$(26)\qquad \begin{cases} \bar{\theta}(x) = \sup \{\theta\colon \mathcal{F}(x^*, \theta) \geq 1 - \epsilon_2 \geq \sup_{x_2 < x^*} \mathcal{F}(x_2, \theta)\}, \\ \underline{\theta}(x) = \inf \{\theta\colon \mathcal{G}(x^*, \theta) \geq 1 - \epsilon_1 \geq \sup_{x_1 > x^*} \mathcal{G}(x_1, \theta)\}, \end{cases}$$

when the function $\mathcal{F}(x, \theta)$ is nondecreasing in θ. The following theorem is therefore proved.

THEOREM 5. *If the set of trial outcomes can be completely ordered in such a way that the function* $\mathcal{F}(z, \theta) = P_\theta\{x^* \leq z\}$ *is nonincreasing (nondecreasing) in* θ, *then the boundaries of the* γ *intervals are found from formulas* (25), ((26)).

Let us note that in those cases where $\mathcal{F}(x, \theta)$ is a continuous function of x, equations (25) and (26) take a simple form. Here $\bar{\theta} = \bar{\theta}(x^*)$, $\underline{\theta} = \underline{\theta}(x^*)$ are solutions of the equations

$$(25')\qquad \mathcal{F}(x^*, \bar{\theta}) = \epsilon_1, \qquad \mathcal{F}(x^*, \underline{\theta}) = 1 - \epsilon_2,$$

$$(26')\qquad \mathcal{F}(x^*, \underline{\theta}) = \epsilon_1, \qquad \mathcal{F}(x^*, \bar{\theta}) = 1 - \epsilon_2,$$

which is, however, a simple consequence of the fact that under this assumption the stochastic variable $\mathcal{F}(x^*, \theta)$ has a uniform distribution in the interval [0, 1] (see [4]).

As a nontrivial application to life-testing of the theorem proved above, let us consider the following statistical model. A Markov process ξ_s of the pure birth type is observed whose possible values are the integers and $\xi_0 = 0$. Let the intensity of the transitions from the state k to the state $k + 1$ be $\mu_k(\theta)$. Here $\theta > 0$ is an unknown value of the parameter which affects this intensity. A set S of stopping points is given in the plane of (t, k) points, where t is the time coordinate and k is the value of the process ξ_t. As a result of testing, the trajectory of the process ξ_s is observed up to the time of first hitting one of the points of the set S. We make the following assumptions relative to the set of stopping points S and the process ξ_s: (a) for any $\theta > 0$ the trajectory of ξ_s reaches one of the points of S with probability 1; (b) the set S can be completely ordered; hence, for an arbitrary nondecreasing function $k(s)$ taking integer values, any of the boundary points x not lying below the graph $(s, k(s))$ will be "to the left" of any of the boundary points lying below this graph, if the graph $(s, k(s))$ is drawn up to the first hit of the set S; (c) $\mu_k(\theta)$ is a nondecreasing function of θ for every k; $\min_k \mu_k(\theta) \to \infty$, $\theta \to \infty$.

It is useful to employ the following rule in establishing the order relation $x_1 < x_2$ or "x_1 is to the left of x_2" for $x \in S$. If x_1 and x_2 belong to one segment in the (t, k) plane formed by stopping points of the form (s, k), $s' \leq s \leq s''$,

$k = \text{const.}$, then we assert that $x_1 = (t_1, k_1) < x_2 = (t_2, k_2)$, when $k_1 = k_2 = k$, $s' \leq t_1 < t_2 \leq s''$. If the values $t_1 = t_2$ coincide at the points x_1, x_2, we assert $x_1 < x_2$ when $k_1 > k_2$. Furthermore, the order relation is established in such a manner as to satisfy (b). This is possible for a broad class of sets S.

THEOREM 6. *Assuming conditions* (a)–(c), *the confidence* γ *intervals for* θ *are defined by* (26), *where* $\mathfrak{F}(x, \theta)$ *is the probability of not stopping "to the right," and* $\mathcal{G}(x, \theta)$ *is the probability of not stopping "to the left" of the point* $x \in S$.

The absorption probabilities $\mathfrak{F}(x, \theta)$, $\mathcal{G}(x, \theta)$ may be calculated by using conventional Markov process techniques (see details in [8]). When the points of S have the form $(t, k(t))$, where $k(t)$ is a nonincreasing function of t,

$$\mathfrak{F}(x, \theta) = \sum_{l=0}^{k} p_l(t, \theta), \qquad \mathcal{G}(x, \theta) = \sum_{l \geq k} p_l(t, \theta). \tag{27}$$

Here $x = (t, k)$ and $p_l(t, \theta)$ is the probability that the value of the process is $\xi_t = l$ at time t for free motion without stopping points. For example, for the Poisson process $\mu_k(\theta) = \theta$, $p_l(t, \theta) = ((\theta_t)^l)/l!)e^{-\theta t}$, from which one of the L. N. Bol'shev results [7] easily follows.

The proof of theorem 6 may be obtained by using a stochastic transformation of the time along the trajectory of the process ξ_s. If $\mu_k(\theta'') > \mu_k(\theta')$ for $\theta'' > \theta'$, then the transition to the value θ'' corresponds to a decrease of the sojourn time in the state k, which also increases the probability of stopping the transformed trajectory at the points $y \in S$ "to the left" of x.

In conclusion, let us make several remarks on the construction of the shortest system of confidence intervals. For the case of one-parameter exponential families, the shortest confidence intervals are connected in a definite manner with the most powerful tests of hypotheses (see [3]). If series of statistical models $[X_t, \mathfrak{B}_{X_t}, \Theta, P_{\theta,t}]$ dependent on the "time" t of data accumulation are considered, it is recommended to start from the effective estimates $g(x)$ in constructing confidence intervals for $f(\theta)$. The γ interval is constructed by means of (13). Unfortunately, such a procedure is difficult to carry out in practice for statistical models with spaces of large dimensionality, because of the complexity of the algorithms giving $\underline{f}(x)$, $\bar{f}(x)$ by means of (8). It is necessary to compromise and to construct simpler γ systems which take into account in some way the specific properties of the function $f(\theta)$, and at the same time use comparatively simple algorithms to achieve (8).

For $f(\theta) = \sum_{i=1}^{m} f_i(\lambda_i)$, where the $f_i(\lambda_i)$ are concave functions satisfying conditions leading to the systems (6), better results than those of (11) are obtainable for small values of d_i by using the following system of sets:

(28)
$$H_x = \left\{(\lambda_1, \cdots, \lambda_m): \sum_{i=1}^{m} \lambda_i \leq \Delta_{1-\gamma_1}\left(\sum_{i=1}^{m} d_i\right), 0 \leq \lambda_i \leq \Delta_{1-\gamma_2}(d_i), i = 1, \cdots, m\right\}.$$

In conformity with theorem 3 the sets (28) generate $(\gamma_1 + \gamma_2^m - 1)$ systems.

The H_x are convex polyhedra. The upper confidence bound is $\max f(\theta)$ taken over the vertices of the polyhedron H_x. It is easy to show that the coordinates of those vertices of the polyhedron H_x at which $\max f(\theta)$ is achieved are the following. The set $S \cup i_0$, $S \subseteq (1, \cdots, m)$, for which

$$\sum_{i \in S} \Delta_{1-\gamma_2}(d_i) \leq \Delta_{1-\gamma_1}\left(\sum_{i=1}^{m} d_i\right) < \sum_{i \in S} \Delta_{1-\gamma_2}(d_i) + \max_{j \notin S} \Delta_{1-\gamma_2}(d_j) \tag{29}$$

corresponds to each vertex. The values of the vertex coordinates λ_i, $i \in S$ are assumed to equal $\Delta_{1-\gamma_2}(d_i)$. The remaining are $\lambda_i = 0$ with the exception of $\lambda_{i_0} = x$, $i_0 \notin S$, which is the solution of the equation

$$\sum_{i \in S} \Delta_{1-\gamma_2}(d_i) + x = \Delta_{1-\gamma_1}\left(\sum_{i=1}^{m} d_i\right), \; x < \Delta_{1-\gamma_2}(d_{i_0}). \tag{30}$$

In practice it is impossible to sort out all such vertices since they are many and to find the max $f(\theta)$. However, it is possible to mention a completely realizable algorithm whose complexity depends not so much on m as on the number of different values of d_i.

The algorithmic character of the problems on confidence intervals for complicated spaces Θ is apparently typical.

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