# ON SOME STATISTICAL PROPERTIES OF DYNAMICAL SYSTEMS

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## 1. Introduction

It is intended to present in this paper a number of problems and a brief summary of some numerical computations on the asymptotic behavior of certain simple dynamical systems.

These problems refer to the behavior of a few mass points with given mutual interactions and concern the ergodic properties of the system. Broadly speaking, the questions pertain to the time rates with which a statistical or thermodynamical equilibrium-like states, might be attained. That the approach to equilibria as postulated in statistical dynamics might be extremely slow as compared to times obtained by phase-space volumes or relaxation estimates was indicated in some computations performed a number of years ago by J. Pasta, E. Fermi, and the author [1]. This problem dealt with the long time range behavior of a vibrating string with nonlinear forces added to the usual linear ones. In reality, the problem concerned a dynamical system of a finite number (for example, 64) of particles and was pursued numerically over hundreds of cycles, each corresponding to a would-be period, that is, times corresponding to a full period of the purely linear part of the problem. The results were somewhat surprising in that no tendency towards equilibrization of energy between all the possible modes was noted. Instead, these results showed a transfer of energy between the first few modes of oscillation of the string. The high modes (say from number 5 on up to the last), even in their totality do not acquire more than a few per cent of the total potential plus kinetic energy. Ultimately, the system came back practically to the initial condition. An account of this work is also given in my book [2].

Imagine, quite generally, a system of particles with different masses, all considered as mass points which attract each other according to a given law, say with inverse square forces. Let us assume, furthermore, that the system is in a quasi equilibrium in the sense that most of the particles will stay within a certain bounded distance from each other for a time long compared to the time it takes the radius vector of each particle to describe a full rotation through $2\pi$. One question is, how long will it take for the velocities of the particles to be distributed approximately in accordance with the equilibrium law of statistical mechanics, that is, to have Maxwellian distribution? For a bounded system—if we assume that there exist walls confining the whole system, or else if there exist, for example, outside magnetic constraints to help confine charged particles—one could rely on the ergodic theorem and the metric transitivity to bring about an approach to such an equilibrium.

It is known that in a bounded phase-space the continuous ergodic (metrically transitive) transformations are everywhere dense in the space of all continuous measure-preserving transformations. What is more, they form, in a topological sense, the "bulk" of the whole space [3]. A corresponding theorem stating an analogous property of a real dynamical system of $n$ bodies has not yet been obtained.

The discussion which follows is occasioned by speculations contained in a Los Alamos Report [4]. This deals with the following situation: assume a system composed of two or more stellar bodies and a vehicle which, as an additional body of mass infinitesimally small compared to the celestial objects, forms a many (for example, three) body system. Assume furthermore that the "rocket" describes a trajectory under the action of gravitational forces due to the two large masses, but also still has a certain amount of reserve energy available for steering, that is to say, changing its course by suitably emitted impulses. This available energy is roughly of the order of the kinetic energy which the vehicle already possesses at time $t = 0$. The problem is whether one can use this reserve energy in such a way as to obtain, by suitable near collisions with one or the other of the celestial bodies, much more kinetic energy than that at time $t = 0$, perhaps more by an order of magnitude. As an illustration, assume that the vehicle is between two members of a double star system, it is describing a trajectory in between the two. The question is whether by planning the orbit and changing it suitably one could acquire in leaving the system many times the velocity initially present. It is clear that in a two-body system such possibilities do not exist. The orbit, unperturbed by further impulses, would be a Keplerian ellipse and obviously no multiple "collisions" are possible. One might expect that in a double star system such possibilities do exist. Obviously, in a triple star system the chances of finding suitable orbits and suitable maneuvering seem greater. In an $n$-body system, say of equal masses, and a rocket whose mass is infinitesimally small by comparison, we will approach the situation of a volume of gas containing both heavy and light atoms, where in equilibrium the velocities of the light particles are greater. From the ergodic theorem, at least applied to a bounded system, it would also follow that the light particle will require very high velocities. The ergodic behavior guarantees that arbitrarily near the given dynamical motion there exists one which will make the rocket approach as close to a small sphere surrounding any of the given heavy mass points as we please, which in particular implies high velocities. The question of whether such motions can be obtained by small changes effected by impulses emitted from the rocket is not answered by the general theorem, but this seems, in view of the prevalence of ergodic motions near given ones, extremely likely. Nothing precise is known, however, about the *times* necessary for obtaining such motions. They might be of super-astronomical lengths. One could say that our question is that of the existence of a Maxwellian Demon in a restricted and, so to say, more modest sense: is it possible, by using "intelligently" a small amount of available energy, to shorten the times for near-equilibrization by large factors?

**2.** Dr. Kenneth Ford has studied, with the author [6], a specific version of this problem in the summer of 1959. A rocket, whose mass is negligible, is navigating between two heavy bodies. The question is whether trajectories within such stellar systems can be so arranged that the rocket would finally acquire a velocity which is many times greater than the velocity of the heavy bodies. It is easily seen that the change in its speed after a single collision cannot be greater than twice the speed of the heavy body. The problem has to be also considered with the limitation that the radii of the stellar bodies are finite, which makes it harder to arrange trajectories for repeated collisions which would result in considerable gain of velocity of the rocket. Instead of the speed of the heavy body, the escape velocity from its surface becomes a limit for an additional increase. The model specifically studied, both analytically and numerically, in some detail by K. Ford assumed two heavy bodies of equal mass executing a circular motion about the center of mass. In spite of its very specialized form, this problem is already very complicated, since there is a variety of weird rocket orbits. Ford first finds, in the rotating frame of reference, the properties of a continuously infinite set of periodic solutions and examines solutions slightly perturbed from these looking for net energy changes of the rocket in the laboratory frame. Several orbits are found in which, for example, the rocket arrives with negligible velocity from infinity, is captured in a large orbit of low energy, then eased with judicially applied power into an orbit which loops both stars. At the point of nearest approach to one of these, a downward thrust may be applied giving an orbit in which the kinetic energy increases, and then goes off into infinity with a velocity many times that of the star.

Some of the orbits which are periodic are stable against perturbations. In numerical tries, the greatest final velocity of the rocket was about 3.71 times the velocity of the star.

**3.** A still simpler model to illustrate the general problem will now be considered. A material point (of mass 1) is confined, in one dimension, on a unit interval between two heavy oscillating walls whose mass is infinite. We assume, for example, a harmonic oscillation of the two confining walls. The point is thrown into the interval with the initial velocity, say, equal 1. Assume furthermore that the collisions are always elastic. In succession they will lead to changes in the velocity of our point. In a head-on collision, the point will gain twice the speed of the wall. In a collision which overtakes the wall, the result will be a loss of speed for our point. The maximum velocity of the wall may be assumed to be, say, also equal to 1, and the problem is to study the behavior of the velocity of our point after many collisions. One expects that, after sufficiently long times, the average velocity of the point will become very large, if we look upon the problem as one of statistical mechanics. The tendency towards equipartition of energy would imply this. Mathematically, the problem involves also difficulties of diophantine analysis. The successive collisions take place at times increasingly difficult to compute precisely, and small changes in these lead later on to widely different patterns. Since the collisions which are head-on are slightly more frequent than the unfavorable ones, one might expect a gradual increase in velocity on the average, and the question is to compute or estimate the rate of this increase. One would like such estimates for "almost every" initial position and initial speed of the small particle.

A numerical study of this problem was undertaken with Mark Wells in the Los Alamos Scientific Laboratory. To simplify the computations, the motion of the wall was not assumed to be harmonic, but in a form of a tooth-shaped, that is, broken linear, displacement in time. The problem was studied in two versions, one with velocity of the wall oscillating linearly from 0 to 1, the other with the velocity constant and reversing with a fixed period.

Several thousand successive collisions were computed and great care was taken to examine the influence of the roundoff of errors on the behavior of successive collisions. The results showed a rather surprising behavior. Instead of the expected—perhaps somewhat erratic but, on the average steady increase of speed of particle—enormous fluctuations were observed. With initial speed = 1 of the particle, and speed = 1 of the wall, the velocity of the particle obtained during several thousand collisions sometimes varied between 3 and 4, but the periods of time when the velocity was high were followed by longer periods when it dropped back to 1 or below. It was not possible to conclude from these computations whether the long time average of the energy would increase linearly or with a smaller power of time.

The numerical work was performed for the case where the two walls were moving in phase. (Computationally, it is sufficient, of course, to assume just one heavy wall.) The numerical results, such as they were, would rather indicate a very slow approach to situations envisaged in a statistical mechanics picture, and definitely large fluctuations which seemed to be increasing with time.

Since even this deterministic problem shows unexpected difficulties, it seems futile to superimpose on it the original question of whether, by suitably planned additional small impulses, the rate of increase of speed might be greatly accelerated. Such a restricted Maxwellian Demon would have to be in possession of a super-computer and solve, in addition to the diophantine problem, a game theoretic question. From the greatly fluctuating nature of the nonperturbed motion, it is clear that no local recipe of the kind used in problems of the calculus of variations would be suitable for optimization. It may be that, on the contrary, before certain collisions, occasional "sacrifices," in the sense of the term as used in the game of chess, might be necessary for an overall optimum, that is to say, occasionally a few collisions should be planned which might lead to a lower value of speed so as to have favorable collisions later on.

It might be that the restriction to one dimension makes the problem less typical of situations usually dealt with in statistical mechanics. One could imagine an analogous system in two-space dimension, say a ball colliding with pulsating walls on a billiard table. The additional parameter of the angle of impact might introduce more random-like properties, and the asymptotic behavior of the speed of the particle would be perhaps a less fluctuating one. This is by no means certain, however. It is only plausible that a large number of particles or constraints in a dynamical system is required to insure a thermodynamic-like behavior of the system.

**4.** A more realistic problem would assume, still in its simplest version, instead of an infinite, a large but finite mass of particles corresponding to the walls. The asymptotic properties of this system might be different from the ones in the previous discussion.

Here, an idealized mathematical model, still in one-space dimension but able to test some of the schemes above, could be as follows:

Imagine, on an infinite line, masses put in on every point with integer coordinates. These masses are either 1 or 2, and for each integer value of the coordinate we decide by the throw of a coin which one of the two values of the mass to locate there. In addition, we give to each of these points a velocity of $+1$ or $-1$, again deciding independently with probability $1/2$ which one to use. All this is done at time $t = 0$. We can represent the initial state oi such a system by, say, two real numbers, $\xi$ and $\eta$. One may characterize symbolically the distribution of masses by using the symbol 0 on the $n$th binary of $\xi$ if the mass is 1 at the point $x = n$, and symbol 1 if the mass is 2. The other number $\eta$, similarly, will contain all the information about the velocities of the system by using symbol 0 in $\eta_n$ if the velocity at $x = n$ is $-1$ and symbol 1 if the velocity is $+1$. In this fashion it will be possible to talk about "almost every initial distribution" (in the sense of Lebesgue measure). The mass points will start colliding and, assuming collisions to be perfectly elastic, new velocities will appear and new sets of collisions will ensue.

A whole set of problems now arises. Will the distribution of velocities, which initially was random in the sense of Bernouilli, tend to a distribution more resembling that of a gas? In one dimension high values of velocities will not be established, but the question of the rate of approach to an equilibrium-like situation is of interest. Or perhaps the fluctuations will continue indefinitely on a large scale. The proper way to consider limits is obviously to take any point, an interval of length $2N$ around it, compute the functional in question, and examine the limit as $N \to \infty$.

This model can be varied, of course. One could have, instead of giving each point a velocity $\pm 1$, say, a continuous distribution of velocities to start with, and so forth.

A similar problem was considered previously [5]. The point of view adopted there was different. The collisions, on the contrary, were assumed to be totally inelastic. The points were not put on *every* integer valued coordinate, but only with probability 1/2. The question studied there was that of formation of condensations and superclustering. Needless to say, in our problem above there will be initially, with probability 1, arbitrarily large but increasingly rarely-spaced groups of points with velocities in the same direction. Therefore, the spatial distribution of points after some time will be quite nonuniform. Whether the asymptotic density will remain constant is not obvious a priori.

Coming back to our problem of a light particle colliding with heavy ones: the indications given by numerical tries reported above are then that the rate of energy increase is both slow and irregular. If also true for a random distribution of heavy masses in three dimensions, this would have some consequences for models of mechanisms by which cosmic ray particles acquire very high energies. One such model, considered in literature, postulates charged particles colliding with magnetic fields of stars. These stars move at random and would ultimately transfer some of their energy to the elementary particle. Another model is of a particle moving in continuous and varying magnetic fields in interstellar space. This latter is more difficult to schematize as simply as our model above, but could perhaps provide a more efficient way for endowing the particle with very high energy.

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