ON THE STATISTICAL LOSS OF LONG-PERIOD COMETS FROM THE SOLAR SYSTEM, I

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1. Summary

Description of the sun's family of comets, number, nature of orbits, is given first. At each return of a long-period comet, planetary action can change the binding energy which is itself small by amounts that must gradually lead to the systematic expulsion of comets from the solar system.

The distribution of perihelion (and aphelion) points of cometary orbits is found to be definitely nonrandom. It also shows strong evidence of association with the galactic plane. The mean position of the perihelion points is in striking agreement with what would be predicted by the accretion hypothesis of cometary formation.

The energy changes already calculated for some of the so-called "hyperbolic" comets are listed. For other comets the possible energy change at each approach to the sun as a result of Jupiter's action (or that of any other great planet) can be calculated by means of a restricted three-body method in which the motion of the sun and Jupiter are regarded as in fixed circles. By good fortune the very quantity that is needed, namely the change of energy, can be found by means of the Jacobi integral, without it being necessary to investigate any other orbital changes. The formulas for the energy change are reduced to the forms that have actually been utilized for the necessary machine integrations.

2. Object of the paper

The object of this paper is to show that the distribution of the orbits of the comets of the solar system must be changing with time, and by a study of the main causes of these changes to obtain an estimate of the decay period associated with a defined group of comets.

3. Introduction

The problem leads first of all to consideration of the dynamical evolution of individual cometary orbits. But as this cannot be followed out in full detail for the necessary lengths of time, the question of general evolution has to be substituted and investigated statistically, as will appear. It is for this reason that what at first sight seems a purely astronomical problem comes within the realm of statistical work. Accordingly, as all statisticians may not be familiar with the general properties of comets, it may be desirable to begin with a brief description of the relevant properties of the "second solar family" as the huge number of comets associated with the sun have been termed.

As a simple consequence of the average annual number of long-period comets actually observed (about four or five) and their orbital periods (which are of order 50,000 years) we are led to the inescapable conclusion that there must be some 200,000 long-period comets bound gravitationally to the sun at the present time.

4. Physical properties of comets

Throughout the investigation of this paper we shall regard the motion of a comet as representable by that of a point mass under ordinary Newtonian forces. The orbits of comets are always computed on this basis, but the accuracy attainable in predicting cometary positions is of a far poorer order than for planetary positions. Accuracy within a minute of arc at best is as much as can be expected for comets, whereas less than one second of arc would be required for satisfactory planetary values.

Almost all comet workers have concluded that a comet consists in essence of a vast swarm of extremely widely spaced tiny particles. The particles may be spread through an immense volume (linear dimensions $\fallingdotseq 10^{10\pm1}$cm), but the total mass must be extremely small by astronomical standards ($\fallingdotseq 10^{18\pm1,2,3}$gm). Only when at great distances from the sun, of the order of a few hundred astronomical units, could such a system hold together by its own very weak self-gravitational attraction, but it is precisely at such great distances that the long-period comets spend by far the greater part of their existence. However, there is no doubt that during each approach to the sun a comet undergoes various kinds of internal disturbance that must contribute to shortening its existence as a comet. For example, most long-period comets exhibit conspicuous tails while describing the part of their orbit near the sun, and the material so released is undoubtedly completely lost from the comet. Moreover, long-period comets may be disrupted into two or more smaller comets if the passage takes them sufficiently close to the sun; a number are known whose orbits actually take them through the solar corona (the so-called "sun-grazers"). The resulting comets depart from the sun in orbits very similar to but not identical with that of the original single comet. Then again, it seems certain that the small faint short-period comets, of which a number approaching 100 are known, have arisen from the deflection (and possibly at the same time disruption) by the mighty Jupiter of former long-period comets whose paths happened to bring them sufficiently close to the planet for enormous perturbations to occur, changing the orbit in a single encounter from long-period, nearly parabolic form to elliptic form of quite short period. Because of their negligible mass and proximity to the sun, the short-period comets themselves undergo rapid dissolution (in times probably less than 10^5 years) into extended meteor streams consisting of the original constituent particles of the comets.

5. Planetary dissipation of comets

The processes described above imply continual dissipation of long-period comets, but in addition to them there is a further more general type of evolutionary effect that would operate on comets even if they were strictly point masses. This is the direct dynamical perturbative action of the great planets in altering the orbits, and in particular for our purposes changing the binding energy of a comet in regard to the sun. It will appear that at a single approach to the sun, the negative energy (per unit mass) of a comet in its orbit may be changed in either direction by an amount comparable with this energy, and so the comet is subject to changes that may bring about its ejection from the solar system altogether. This is the process that it is our purpose to study.

The orbits of long-period comets are so nearly parabolic that it is only when they have been observed over a considerable arc and the positions covering the whole time of its apparition are available that an orbit is computed for an eccentricity not taken as unity. The arc available will usually be part of the orbit within just a few astronomical units of the sun (about 11 a.u. is the greatest distance at which any comet has been observed, 1927 IV, whereas they recede to thousands of astronomical units), and it not infrequently happens that the computed eccentricity comes out to be slightly greater than 1. These are the so-called "hyperbolic comets" of which about 70 have been found of a total of probably more than 500 in all. But when allowance is made for planetary perturbations, for every such comet investigated it is invariably found that the inward path along which the comet traveled when at great distance well outside planetary effects was definitely elliptic. In other words no truly hyperbolic comets exist, so that no comet has entered from outside the solar system as an individual comet.

It is exactly the change of binding energy implied by a change of eccentricity from above unity to below it that we are concerned with, since the same causes on the outward journey after perihelion can increase e above unity (whether it was greater or less at perihelion) and enable the comet to escape completely from the sun. In the early days of such studies, cometary workers were primarily concerned with deciding whether "hyperbolic" comets had come in along truly hyperbolic orbits, with the result that calculations were backward in time and restricted to the inward preperihelion half of the orbit. Table I shows the results of a number of such calculations for the comets shown.

In this table, *negative* numbers correspond to hyperbolic values of $1/a$ and positive to elliptic values. The distances out to which perturbations were cal-

TABLE I

Comet	$1/a$ from Observed Orbit near Perihelion (a in a.u.)	$1/a$ from Earlier Orbit Sufficiently beyond the Planets	$10^4 \times \delta(1/a)$
1853 III	−0.000 819	+0.000 083	9.02
1863 VI	− .000 495	+ .000 017	5.12
1886 I	− .000 694	− .000 007	6.87
1886 II	− .000 477	+ .000 317	7.94
1886 IX	− .000 576	+ .000 063	6.39
1889 I	− .000 692	+ .000 042	7.34
1890 II	− .000 215	+ .000 072	2.87
1897 I	− .000 872	+ .000 040	9.12
1898 VIII	− .000 607	− .000 016	5.91
1902 III	+ .000 081	+ .000 005	0.76
1904 I	− .000 504	+ .000 216	7.20
1905 VI	− .000 142	+ .000 621	7.63
1907 I	− .000 499	+ .000 025	5.24
1908 III	− .000 732	+ .000 158	8.90
1914 V	− .000 146	+ .000 012	1.58
1922 II	− .000 381	+ .000 004	3.85
1925 I	− .000 566	+ .000 054	6.20
1925 VII	− .000 273	+ .000 115	3.88
1932 VI	− .000 595	+ .000 044	6.39
1936 I	− .000 487	+ .000 205	6.92
		Average value of $\delta(1/a) = 5.96 \times 10^{-4}$	

culated range from 16 a.u. for 1932 VI to 52 a.u. for 1922 II. In the two cases in which small hyperbolic values remain (1886 I and 1898 VIII), it is considered that these would have become small and positive had the effects of Uranus and Neptune also been included. The changes of $1/a$ found in the table are all of the same sign because only "hyperbolic" comets were selected for these investigations; changes of either sign can of course result since they arise from dynamical perturbations.

6. The present distribution of cometary orbits

For the very reason that long-period orbits are nearly parabolic, the value of the period derived from observations of its position (which are possible only in a limited part of the path) is subject to tremendous uncertainty. So long do many periods turn out to be, millions of years in some cases, that it is equally possible to represent the observable part of the orbit by a parabola. For this reason, over 250 long-period comets have an assigned eccentricity $e = 1$, and only about 120 have periods small enough (even so, in excess of 200 years) to make its calculation worth while. Most of these periods are measured in thousands of years, but still with great uncertainty, and it is found that different computers will arrive at values of the period differing by amounts comparable with the period itself. The reason for this is in part that only during a limited arc near perihelion is the comet observable, and then with very moderate accuracy, but also because of the ever-present perturbations, as already seen. For this last reason, the period of a comet may change by a factor of ten or more either way from one return to the next, indeed if it chanced to escape from the sun its period becomes infinite.

Accordingly, this inherent uncertainty in knowledge of the precise periods prevents any study of their distribution being possible, and attention instead has been given to the distribution of the orientation of the orbits: that is, the direction of the major axes in relation to the sun as center. This direction must obviously be far less subject to change than is the energy, for if a comet recedes to great distance in a certain direction (as seen from the sun), it must return from more or less that same direction whatever slight change of speed may happen to it while remote from the sun. Only if a comet passes close to a planet can any large change in the direction of its major axis of orbit result.

A number of authors have investigated statistically the distribution of perihelion points, or their antipodal aphelion points, on the celestial sphere. The most recent work has been done by Tyror, who began by marking with very small yellow discs of paper the actual positions of the perihelion points of 448 comets on a large black sphere some 3 feet in diameter representing the celestial sphere. The arrangement of the points could then be immediately inspected by eye and photographed from different aspects to get an over-all picture of the distribution.

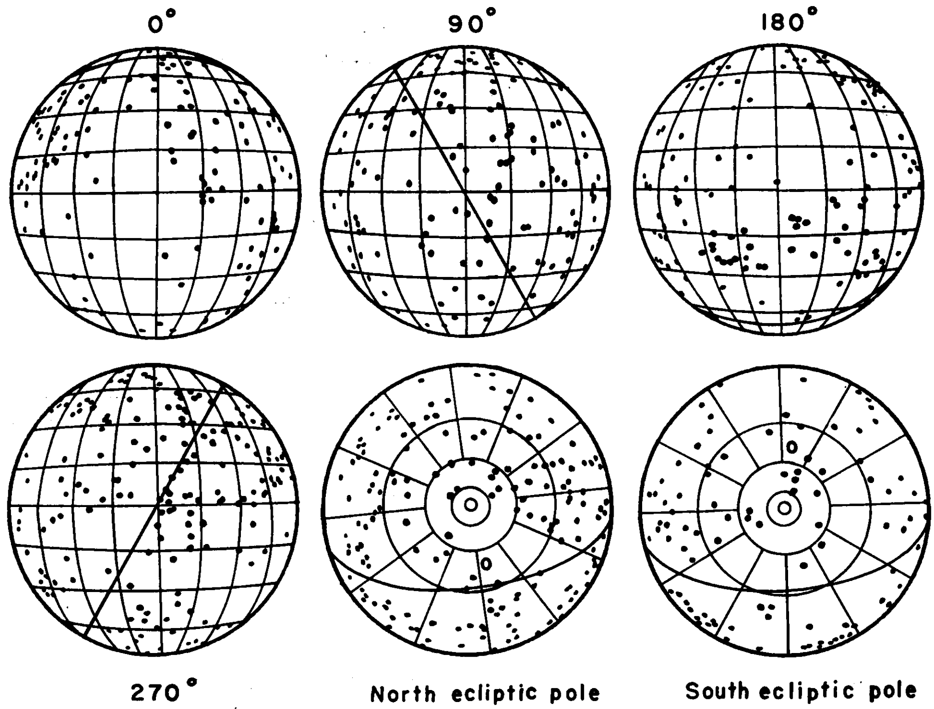

FIGURE 1

Directions of the perihelia of long-period comets.

Figure 1 above makes clear at once that not only are there large areas of the sphere devoid of perihelion points but that there are numerous compact clusters of points. The total number of points (448) more or less uniformly distributed over the sphere would each be separated by about 10° from their nearest neighbors, but the picture shows plenty of instances of clusters of several points all within much smaller distances of each other.

An "equal areas" investigation, in which the whole sphere was divided into 432 (almost) equal areas and the number of points falling in each area were noted, showed a strong preponderance at one end of areas containing many points, and at the other of many areas containing few or no points, a distribution highly typical of grouping. Several other tests applied all indicated a real departure from randomness.

Again, the first-order moments of the positions of the points show that their mean position is significantly displaced from the center of the celestial sphere. With rectangular axes Ox, Oy in the plane of the ecliptic (with Ox in direction $\lambda = 0$) and Oz perpendicular to it, the mean position is

(1) $$\bar{x} = -0.0069, \qquad \bar{y} = -0.0446, \qquad \bar{z} = 0.1316,$$

the radius of the sphere being taken as unit of length. The direction of this point from the sun is toward

(2) $$\lambda = 261°, \qquad \beta = 71°,$$

and this is rather less than 19° away from the apex of the sun's motion, which is at

(3) $$\lambda = 271°, \qquad \beta = 53°.$$

This result is strongly in accord with the accretion hypothesis of the origin of comets. If comets were simply picked up from interstellar space and their (hyperbolic) speeds converted to elliptic ones by some resistance, say, then the mean direction of their perihelion points would be 180° removed from the apex of the sun's motion. But on the accretion hypothesis, interstellar dust converges to the axial line behind the sun to form the comets which then fall toward the sun from great distances at this side. Accordingly their *aphelion* points would in the mean be opposite to the solar motion and the mean of perihelion points in the same direction. The difference of 19° is no more than might be expected for two reasons. First, what is relevant here is the motion of the sun relative to the individual dust clouds it passes through and not to the local group of stars, and there is no reason to suppose these identical, though possibly not very different. Second, perturbations (by passing stars and planets) of the orbits subsequent to formation must cause the perihelion points to be displaced more or less randomly and increasingly as time goes on. The probability of two points on a sphere lying within less than 19° of each other is about 0.025, and so there is only this small order of probability that the coincidence is pure chance.

The second moments of the distribution show that there exists a preferred plane of concentration of the points whose normal has direction cosines (0.7674, 0.0151, −0.6410). That the preferred plane may be closely identified with the galactic plane is shown by comparing this direction with that of the galactic pole, namely, (0.8772, −0.0536, −0.4772). The angle between these two directions is a mere 12°, and the probability of this being pure coincidence is only about 0.01. Accordingly the analysis points very strongly to a correlation between the distribution of the comets and the galactic plane.

In view of the importance of this result, to test it further Tyror selected out those comets of known period lying in the intermediate range of between about 1000 years and 60,000 years which would have major axes in the range of about 200 to 3000 a.u. for the reason that these particular comets might therefore have been least affected by both planetary and stellar perturbations. This group when analyzed by the same method of second moments were found to show even greater preference for the galactic plane.

Similarly a selection of 127 comets believed for various reasons to be the intrinsically brightest comets of the whole set was made for the reason that these probably must be systematically younger objects and so subjected to less perturbation. Here again stronger concentration toward the galactic plane was found than for the whole set of 448.

TABLE II

	No.	$\overline{D^2}$	$1/3 - \overline{D^2}$	s.d.	g	$1 - \cos g$
All comets	448	.276	.057	.014	78°	.022
Intermediate orbits	70	.226	.107	.036	84°.5	.004
Brightest comets	127	.262	.071	.026	81°.5	.012

Table II shows the actual results. The quantity $\overline{D^2}$ is defined by

$$\overline{D^2} = \frac{1}{448} \sum_i (lx_i + my_i + nz_i)^2, \tag{4}$$

where l, m, n are the direction cosines of the normal to the required preferred plane chosen to make $\overline{D^2}$ a minimum. For a perfectly random distribution $\overline{D^2}$ would have the value 1/3. The angle g, tabulated in table II, represents the galactic latitude of the normal to the preferred plane (it would be 90° if this coincided exactly with the galactic plane).

Finally, because of the possibility of the presence of selection effects in the data themselves, as for example from the fact that more comets have been found from northern hemisphere stations, Tyror analyzed those comets whose perihelion points were confined to a fairly broad but equatorial zone. Again there was found nonrandomness within the zone and strong signs of correlation with the galactic plane, as before.

7. Cosmogonical considerations

The theory of the origin of comets will clearly have to conform to any definite evidence that can be obtained as to their ages. The values of $\delta(1/a)$ given in table I make plain that for many comets the change of energy at each approach to the sun is comparable with the binding energy itself, and hence that through successive changes of this kind a comet may eventually achieve genuinely hyperbolic energy with respect to the whole solar system and escape forever into interstellar space. But the following out of the detailed motion of a long-period comet at a sufficient number of returns to decide its ultimate fate, whether it escapes or is drawn into the sun, is an impossible task. The periods involved might be millions of years, and the orbits at their aphelion parts could not be known or found with sufficient accuracy to time the return path in relation to the positions of the (perturbing) great planets. Moreover, from a cosmogonical standpoint, interest attaches not so much to a single comet as to the evolution of all their orbits, and, as we have seen, less than one per cent of all the comets that must be associated with the sun have as yet been observed. The only hope of obtaining any indication of the dynamical age of the comets from the present point of view is therefore by statistical means.

For this, the first requirement is clearly to find the possible energy changes that a comet can undergo according to the position of its orbit. Table I relates to "hyperbolic" comets, and so must tend to have selected just those comets that are already more weakly bound to the sun (so that the slight planetary perturbations can have speeded it up to hyperbolic osculating motion near perihelion), whereas we would wish to know, for comets initially more strongly bound to the sun, what energy changes may be possible for comets approaching from different directions on the celestial sphere.

The main agency in producing such perturbations is the planet Jupiter, and the problem can be simplified to begin with by regarding Jupiter as in given circular motion round the sun undisturbed by the comet itself. The mass of an average comet is probably at most 10^{-12}, that of Jupiter $2 \times 10^{30}g$, with the result that the motion of the comet can be dealt with by means of the restricted three-body problem. Here however the path of the comet is nearly parabolic and extending to very great distance compared with the separation of the sun (S) and the planet (J). The full analysis of the changes of orbit (about S) that a particle of negligible mass would undergo in such circumstances would be beyond the range of perhaps all but the largest of modern computing machines, but it turns out that the very information required for our purposes, namely the change of energy per approach to the sun, can be found to high accuracy without obtaining any of the other changes. Since the planets sensibly affect the motion only when the comet is at the inner part of its path (less than say 50 a.u. from the sun), it is sufficient to regard the undisturbed path as parabolic and calculate the change of energy then resulting. This can be carried out in the restricted three-body case and the integrals involved have been successfully evaluated by use of the Cambridge and Manchester machines. Allowance for the similar effects of Saturn and the other great planets can then be made additively later.

8. The restricted three-body system: coplanar motion

We consider first the case when the undisturbed parabolic cometary orbit is in the same plane as that of the relative circular motion of Jupiter and the sun. It is assumed that S and J move round their common center of mass G in circles of radius a_1 and a_2 respectively with uniform angular speed n. Using the symbols S and J to denote also the masses of the sun and Jupiter, then

$$n^2a^3 = \gamma(S + J), \qquad \text{where} \quad a = a_1 + a_2 \tag{5}$$

by Kepler's relation, γ denoting the constant of gravitation. Let C denote the comet whose mass $(=1)$ is so small as to have no influence on the motion of S and J.

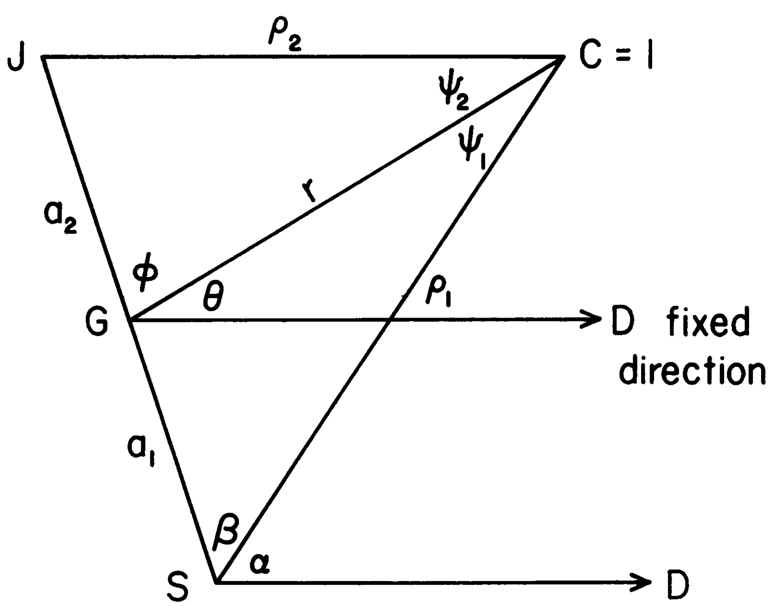

FIGURE 2

Undisturbed parabolic cometary orbit in the same plane as that of the relative circular motion of Jupiter and the sun.

$SC = \rho_1, \quad JC = \rho_2, \quad GC = r.$

GD is a fixed direction parallel to the axis of the (undisturbed) parabola in which C moves round S.

$DSC = \alpha, \quad CSG = \beta, \quad DGC = \theta.$

$SCG = \psi_1, \quad JCG = \psi_2, \quad JGC = \phi.$

9. The Jacobi integral

If x, y, z are rectangular coordinates of C relative to fixed axes at G, with Gx, Gy in the plane of motion of S and J, then the equations of motion always possess the so-called Jacobi integral, which has the form

$$\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{\gamma S}{\rho_1} - \frac{\gamma J}{\rho_2} - n(x\dot{y} - \dot{x}y) = \text{constant}. \tag{6}$$

The first three terms on the left represent the total energy E (kinetic + potential) of the comet. If E is positive when C is at great distance from S and J, the comet will escape.

If the term in n is written nh, then h is simply the angular momentum of C round the z-axis at G, and the Jacobi integral can be written

$$E - nh = \text{constant}. \tag{7}$$

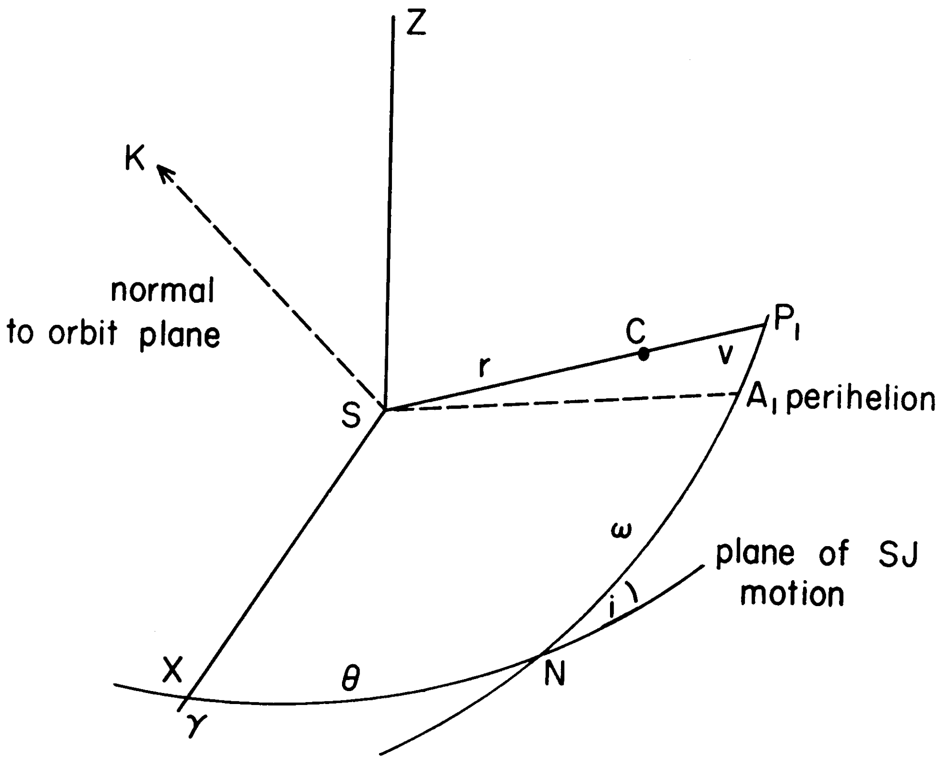

FIGURE 3

Undisturbed parabolic cometary orbit not in the same plane as that of the relative circular motion of Jupiter and the sun.

N = longitude of node = θ, i = inclination of orbit,

NA_1 = distance of perihelion from node,

$A_1P_1 = v$, $SC = r$ (= ρ_1 formerly).

Accordingly, in order to find the total change in E resulting from an approach to the (S, J) system, it is sufficient to calculate dh/dt at a general instant and sum this over the whole time range of the motion of C from its start at very great distance to its return there in the undisturbed parabola.

10. Calculation of the rate of change of *h*

To effect this, the components of gravitational force acting on C resolved perpendicular to GC are

$$
(8) \qquad \begin{aligned} \gamma J \rho_2^{-2} \sin \psi_2 &= \gamma J \rho_2^{-3} a_2 \sin \phi, \quad \text{due to } J, \\ -\gamma S \rho_1^{-2} \sin \psi_1 &= -\gamma S \rho_1^{-3} a_1 \sin \phi, \quad \text{due to } S. \end{aligned}
$$

Hence we have at once

$$
(9) \qquad \frac{dh}{dt} = (\gamma S \rho_2^{-3} a_2 - \gamma S \rho_1^{-3} a_1) r \sin \phi,
$$

and if we write $\mu = \gamma J a_2 = \gamma S a_1$, this becomes

$$
(10) \qquad \frac{dh}{dt} = \mu \left(\frac{1}{\rho_2^3} - \frac{1}{\rho_1^3} \right) r \sin \phi.
$$

11. The undisturbed path of the comet *C*

In accordance with the standard procedure of celestial mechanics, namely that of successive approximation, it will be assumed for the purpose of calculating dh/dt that C pursues a parabolic orbit with S as focus. It is further supposed that C does not pass close to J; in other words, we exclude the case of very large perturbations for which the approximation must fail. This would be the case involved in the conversion of long-period orbits to short-period orbits, but we are not here concerned with this (additional) means whereby comets tend to get lost.

The fixed direction D is chosen as that of the axis of this unperturbed parabola that C would follow if J were absent, and we take $CSD = \alpha = 0$ at the vertex with $t = 0$ there also. A single inward and outward sweep of C thus corresponds to $t = -\infty$ to $t = +\infty$. The equation of the undisturbed path of C thus may be written in polar form

$$
(11) \qquad \rho_1 = b \sec^2 \frac{1}{2} \alpha = b(1 + \tau^2),
$$

where b = semilatus rectum of the parabola, $\tau = \tan \alpha/2$. For the angular momentum of C in the undisturbed orbit we write

$$
(12) \qquad \rho_1^2 \dot{\alpha} = \text{constant} = c,
$$

say, which in terms of τ becomes

$$
(13) \qquad 2b^2(1 + \tau^2)\, d\tau = c\, dt,
$$

and hence since $t = 0$ when $\tau = 0$ this integrates to the well-known relation

$$ct = 2b^2\left(\tau + \frac{1}{3}\tau^3\right). \tag{14}$$

12. Integrals for the change of energy

We accordingly have

$$\Delta E = \int_{-\infty}^{+\infty} n\frac{dh}{dt}\,dt = \int_{-\infty}^{+\infty} n\mu r \sin\phi(\rho_2^{-3} - \rho_1^{-3})\,dt. \tag{15}$$

Now $\beta = nt + \beta_0 - \alpha$, where $\beta = \beta_0$ at $t = 0$ and $r\sin\phi = \rho_1 \sin\beta$, so that

$$r\sin\phi\,dt = \rho_1 \sin\beta\frac{2b^2}{c}(1+\tau^2)\,d\tau \tag{16}$$

$$= \frac{2b^3}{c}(1+\tau^2)^2 \sin\beta\,d\tau.$$

Since $\sin\alpha = 2\tau/(1+\tau^2)$ and $\cos\alpha = (1-\tau^2)/(1+\tau^2)$, we have

$$\sin\beta = \frac{1-\tau^2}{1+\tau^2}\sin(nt+\beta_1) - \frac{2\tau}{1+\tau^2}\cos(nt+\beta_0). \tag{17}$$

Hence for the second term in ρ_1^{-3} in the integral for ΔE

$$\frac{r\sin\phi}{\rho_1^3}\,dt = \frac{2}{c}\left\{\frac{1-\tau^2}{(1+\tau^2)^2}\sin(nt+\beta_0) - \frac{2\tau}{(1+\tau^2)^2}\cos(nt+\beta_0)\right\}d\tau. \tag{18}$$

If we write $2b^2n/c = \lambda$, a pure number, and write

$$A_1 = \int_0^\infty \frac{1-\tau^2}{(1+\tau^2)^2}\cos\left[\lambda\tau\left(1+\frac{1}{3}\tau^2\right)\right]d\tau, \tag{19}$$

$$B_2 = \int_0^\infty \frac{2\tau}{(1+\tau^2)^2}\sin\left[\lambda\tau\left(1+\frac{1}{3}\tau^2\right)\right]d\tau, \tag{20}$$

then it is readily found that for the second part of ΔE,

$$\Delta E_2 = -n\mu\int_{-\infty}^{+\infty} r\rho_1^{-3}\sin\varphi\,dt = -\frac{4n\mu}{c}(A_1+B_2)\sin\beta_0, \tag{21}$$

wherein the constants are defined by

$$\mu = \frac{\gamma SJa}{S+J}, \qquad n^2a^3 = \gamma(S+J), \qquad c^2 = 2\gamma bS, \qquad \lambda = \frac{2nb^2}{c}. \tag{22}$$

By going over to complex form, the two integrals can be combined, and the required energy change is the real part of the expression

$$\Delta E_2 = -\frac{2\lambda\mu}{b^2}\sin\beta_0\int_0^\infty \frac{1}{(1+i\tau)^2}e^{i\lambda(\tau+\tau^3/3)}\,d\tau. \tag{23}$$

This integral has been studied and evaluated by M. Amouyal on the EDSAC machine at Cambridge.

13. The first term in ΔE

To evaluate the first term, $n\mu \int r\rho_2^{-3} \sin\phi \, dt$, it is necessary to begin with to express ρ_2 in terms of ρ_1, a, and β. Thus we have

$$\rho_2^2 = a^2 + \rho_1^2 - 2a\rho_1 \cos\beta,$$

$$(24) \qquad r \sin\phi = \rho_1 \sin\beta = b\{(1 - \tau^2) \sin(nt + \beta_0) - 2\tau \cos(nt + \beta_0)\},$$

$$\rho_1 \cos\beta = b\{(1 - \tau^2) \cos(nt + \beta_0) + 2\tau \sin(nt + \beta_0)\},$$

and it is therefore necessary to evaluate the integral

$$(25) \qquad \frac{2n\mu b}{c} \int \frac{\rho_1^2 \sin\beta \, d\tau}{(a^2 + \rho_1^2 - 2a\rho_1 \cos\beta)^{3/2}}.$$

This can be reduced to neater form by introducing complex numbers, as follows:

$$(26) \qquad \rho_2^2 = a^2 + \rho_1^2 - 2a\rho_1 \cos\beta$$
$$= a^2 + b^2(1 + \tau^2)^2 - 2ab\{(1 - \tau^2) \cos(nt + \beta_0) + 2\tau \sin(nt + \beta_0)\}$$
$$= \{a - b(1 - i\tau)^2 e^{i(nt+\beta_0)}\}\{a - b(1 - i\tau)^2 e^{-i(n\tau+\beta_0)}\},$$

and since these two factors are complex conjugates they have equal magnitudes, and

$$(27) \qquad \rho_2 = |ae^{-i\beta_0} - b(1 - i\tau)^2 e^{int}|.$$

Hence the contribution to the energy change from the Jupiter term in ρ_2^{-3} is

$$(28) \qquad \mu n \int_{-\infty}^{\infty} \frac{b\{(1 - \tau^2) \sin(nt + \beta_0) - 2\tau \cos(nt + \beta_0)\}}{|ae^{-i\beta_0} - b(1 - i\tau)^2 e^{int}|^3} dt$$

$$= \mathcal{I} \mu n b^{-2} \int_{-\infty}^{\infty} \frac{(1 - i\tau)^2 e^{i(nt+\beta_0)}}{\left|\frac{a}{b} e^{-i\beta_0} - (1 - i\tau)^2 e^{int}\right|^3} dt$$

$$= \mathcal{I} \mu n b^{-2} \int_{-\infty}^{\infty} \frac{(1 - i\tau)^2 e^{i(nt+\beta_0)}}{|z - (1 - i\tau)^2 e^{int}|^3} dt,$$

wherein $z = a \exp(-i\beta_0/b)$. If also we write $nt = x = \lambda(\tau + \tau^3/3)$, the integral takes the form

$$(29) \qquad \frac{\mu}{b^2} \mathcal{I} \int_{-\infty}^{+\infty} \frac{(1 - i\tau)^2 e^{i(x+\beta_0)} \, dx}{|z - (1 - i\tau)^2 e^{ix}|^3}.$$

For this second contribution to the energy change, Haselgrove and Kerr have evaluated the integral given by

$$(30) \qquad I_2(\lambda, \beta_0) = (2\lambda^{-2})^{2/3} \mathcal{I} \int_{-\infty}^{+\infty} \frac{(1 - i\tau)^2 e^{i(x+\beta_0)} \, dx}{|z - (1 - i\tau)^2 e^{ix}|^3}$$

$$= -I_2(\lambda, -\beta_0).$$

The increment of energy of the comet due directly to Jupiter is then given by

$$\Delta E_1 = \mu a^{-2} I_2(\lambda, \beta_0). \tag{31}$$

14. Numerical example

We readily find that for the expression given for ΔE_2 at the end of section 11

$$\Delta E_2 = -2^{5/3}\lambda^{-1/3} \frac{\gamma J}{a} \sin \beta_0 \mathcal{R} \int_0^\infty (1 + i\tau)^{-2} e^{i\lambda(\tau + \tau^3/2)}\, d\tau. \tag{32}$$

Since $a \doteqdot 5$ a.u., and the value $b = 1$ would be typical for a long-period comet, while the integral itself has value of order unity, this gives for the magnitude

$$\begin{aligned} \Delta E_2 &= 2^{8/3} \times 10^{-3} \times \frac{1}{5} \sin \beta_0 \text{ (Integral)} && (\text{a.u.})^{-1} \\ &= 1.3 \times 10^{-3}\, (I \sin \beta_0) && (\text{a.u.})^{-1}. \end{aligned} \tag{33}$$

This may be compared with the average value of about $6 \times 10^{-4}(\text{a.u.})^{-1}$ for the *half* sweep of the hyperbolic comets listed in table I.

For the other part,

$$\Delta E_1 = \frac{\gamma J}{a} I_2(\lambda, \beta_0). \tag{34}$$

If, say, $\beta_0 = \pi/10$ and $\lambda = 0.127$, then $I_2 = -0.31$ from Kerr's tables, and if we take $I = 2$, then

$$\Delta E \doteqdot 2.6 \times 10^{-3} \times 0.309 - 0.2 \times 10^{-3} \times 0.31 = 7.4 \times 10^{-4} \ (\text{a.u.})^{-1}. \tag{35}$$

This value would be for a typical long-period comet moving close to the plane of motion of J relative to S and not passing close to Jupiter.

The change of angular momentum h can also be found at once from that of E. From the Jacobi integral $E - nh =$ constant, it easily follows that

$$\frac{\Delta h}{h} = \left(\frac{a^3}{8b}\right)^{1/2} \delta\left(\frac{1}{a}\right) = 4.2 b^{-1/2}\, \delta\left(\frac{1}{a}\right), \tag{36}$$

where a here refers to the semiaxis of the actual cometary orbit. For a comet for which say $b = 1$ a.u. then $h = 1.414$, and if $\delta(1/a) = 5.10^{-4}$ say, then

$$\frac{\Delta h}{h} = 4.2 \times 5.10^{-4} = 2.10^{-3} = 0.002, \tag{37}$$

and the change is small compared with h.

But for a sun-grazer, for which say $b = 10^{-2}$ a.u. or less, with the same value of $\delta(1/a)$ we find $h = 0.1414$, and so $\Delta h/h = 2.10^{-2} = 0.02$, and this means a change of a few per cent at a single approach. Thus comets passing close to the sun must undergo rapid evolution in angular momentum in both directions, increase and decrease, as a result of planetary action. And according to the accretion hypothesis of their formation, most if not all comets must start their careers as sun-grazers.

But the angular momentum, and hence b, is also highly susceptible to the perturbations of passing stars, since these will operate when the comet is near aphelion where it not only spends most of its time but is there moving quite slowly relative to the sun, with the result that the slightest alteration in speed will have large effects on the subsequent perihelion part of the orbit.

15. Three-dimensional case

When the plane of the undisturbed cometary parabola is not coincident with that of the motion of S and J, the Jacobi integral again exists but now has the form $E - nh_3 =$ constant, where h_3 is simply the component of h about Gz, the axes of x and y being in the plane of motion of S and J.

If now with G as origin and $G\Upsilon$ as the axis Gx, the coordinates of C are (X, Y, Z), of J, $(X_J, Y_J, 0)$, and of S, $(X\hat{S}, Y\hat{S}, 0)$, then the third component of the moment about G of the force on C is

$$h_3 = J\rho_2^{-3}(XY_J - YX_J) + S\rho_1^{-3}(XY_S - YX_S). \tag{38}$$

If now r, α, β are the spherical polar coordinates of C relative to parallel axes at S such that the pole is in direction Sz, and β measured from $S\Upsilon$, then

$$\begin{aligned} X + a_1 \cos nt &= r \sin\alpha \cos\beta, \\ Y + a_1 \sin nt &= r \sin\alpha \sin\beta, \qquad Z = r\cos\alpha, \\ (X_J,\ Y_J) &= a_2(\cos nt, \sin nt), \\ (X_S,\ Y_S) &= -a_1(\cos nt, \sin nt), \end{aligned} \tag{39}$$

so that

$$\begin{aligned} XY_J - YX_J &= ra_2 \sin\alpha \sin(nt - \beta), \\ XY_S - YX_S &= -ra_1 \sin\alpha \sin(nt - \beta). \end{aligned} \tag{40}$$

Hence, since $a_1 : a_2 : a = J : S : J + S$, the required value of h_3 is

$$h_3 = \frac{SJa}{S + J}(\rho_2^{-3} - \rho_1^{-3}) r \sin\alpha \sin(nt - \beta). \tag{41}$$

As before, suppose now that C is describing a parabola about S but no longer in the plane of the SJ motion. Then the direction cosines of SP_1 give at once

$$\begin{aligned} \sin\alpha\cos\beta &= \cos\theta\cos(\omega + v) - \sin\theta\sin(\omega + v)\cos i, \\ \sin\alpha\sin\beta &= \sin\theta\cos(\omega + v) + \cos\theta\sin(\omega + v)\cos i, \\ \cos\alpha &= \sin(\omega + v)\sin i, \end{aligned} \tag{42}$$

whence

$$\sin\alpha\sin(nt - \beta) = \cos(\omega + v)\sin(nt - \theta) - \sin(\omega + v)\cos(nt - \theta)\cos i, \tag{43}$$

and so

$$(44)\qquad \Delta E = n\Delta h_3 = \frac{naJS}{S+J}\int_{-\infty}^{\infty} (\rho_2^{-3} - \rho_1^{-3})r\,[\cos(\omega+v)\sin(nt-\theta) - \sin(\omega+v)\cos(nt-\theta)\cos i]\,dt,$$

wherein

(45)

$$\rho_2^2 = a^2 + r^2 - 2ar\,\{\cos(\omega+v)\cos(nt-\theta) + \sin(\omega+v)\sin(nt-\theta)\cos i\}.$$

In this, as before,

$$(46)\qquad \tan\frac{1}{2}v = \tau \quad\text{and}\quad r = b(1+\tau^2),$$

with the time given by

$$(47)\qquad t = \left(\frac{2b^3}{S}\right)^{1/2}\left(\tau + \frac{1}{3}\tau^3\right).$$

16. Change of total angular momentum

When $i \neq 0$, h_3 is only the third component of h, whereas the change of h itself is also of interest in the problem, as being closely associated with the evolution of the perihelion distance. To find this, the direction cosines of the normal SK to the orbital plane are

$$(48)\qquad (\sin\theta\sin i,\quad -\cos\theta\sin i,\quad \cos i),$$

and after a little reduction, the moment about SK of the forces on C gives

$$(49)\qquad \Delta h = \frac{aJS}{S+J}\int_{-\infty}^{\infty}\left(\frac{r}{\rho_2^3} - \frac{1}{r^2}\right)[-\sin(\omega+v)\cos(nt-\theta) + \cos(\omega+v)\sin(nt-\theta)\cos i]\,dt.$$

For $i = 0$ this of course reduces to Δh_3, but curiously the cos i factor is now associated with the other term in brackets from that in Δh_3.

The integrals involved in the three-dimensional case have also been evaluated by Kerr and Haselgrove.