THE EXPANSION OF CLUSTERS OF GALAXIES

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1. Introduction

The possibility of the expansion of clusters of galaxies has been considered from two points of view. First, if one assumes that the redshift of galaxies reflects velocity of recession, the question arises: do the members of clusters of galaxies participate in the general expansion indicated by the law of redshifts, with the clusters themselves expanding? In a study of this problem, Neyman and Scott [1] developed a statistical kinematic test of the stability of systems of galaxies. In 1953 the test was applied to the twelve galaxies in the Coma Cluster for which the necessary data (radial velocities and magnitudes) were available. There was no indication of expansion.

More recently, the possible expansion of systems of galaxies was considered from a different point of view. The suggestion, emphasized particularly by Ambartsumian [2], is that some systems of galaxies are disintegrating rapidly. This suggestion is motivated partly by the observation that the average mass of galaxies derived through the application of the virial theorem, valid for stable clusters, sometimes appears to be considerably larger than the estimates obtained by other methods, for example, by the study of double galaxies by Holmberg [3] and by Page [4]. The discrepancy has often been interpreted as indicating the presence of large masses of intergalactic material. However, some investigators consider that, at least in some clusters, the discrepancy is too large to be explained entirely by dark matter, although it may indeed be present; they conclude that these clusters are unstable and the virial theorem is not applicable.

The basic assumption underlying this argument is that the various galaxies for which divergent estimates have been obtained by different methods have average mass of the same order of magnitude. This may well be true. However, a priori it is not evident that, for example, the members of small systems of

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galaxies studied by Page must be of the same mass as the galaxies of the giant clusters. In the studies concerned with the estimation of masses of galaxies and, in particular, with the inconsistency of estimates obtained through the application of the virial theorem [5]–[11], a considerable number of other assumptions are used. These assumptions vary from one study to another and occasionally lead to diametrically opposite conclusions.

In these circumstances, the possibility of approaching the problem of stability of systems of galaxies from an entirely different point of view, with the test criterion published in 1954, totally independent of any assumption regarding masses of galaxies and, indeed, independent of dynamical considerations of any kind, presents some interest. This interest is enhanced by the fact that, in connection with a contemplated joint study of the Coma Cluster, Mayall [12] undertook an extensive observational program yielding the radial velocities and the apparent magnitudes of a number of presumed members of the Coma Cluster. The number of objects for which both these data are available has increased from 12 in 1953 to 50 galaxies. Also, combining the observations of various investigators, currently we have usable data for a number of other clusters and groups. It is true that, with the exception of the Virgo Cluster, in each case the number of available galaxies is much smaller than for the Coma Cluster. However, there is the possibility of combining the data to perform a summary test.

Actually, two summary tests are performed. Both test the hypothesis that all the clusters and groups considered are stable. However, the first test criterion was deduced to be particularly powerful against the alternative hypothesis that all the systems of galaxies studied are expanding *at the same rate.* The second criterion tests the hypothesis of stability against the alternative that *at least some of the systems* studied are unstable: some may be expanding and some contracting not necessarily at the same rate. The mathematical assumptions concerning the properties of systems of galaxies underlying the deduction of the two tests are the same as in the original paper of 1954; they are reformulated in the following section. Admittedly, these assumptions are rather specific and not very likely to be satisfied exactly. Later, we indicate that whenever the number of observations is substantial so that probability limit theorems become applicable, the domain of validity of the tests is markedly extended.

We remark that applications of the virial theorem test also involve assumptions concerning the systems of galaxies—sometimes specific (for example, [11]) and sometimes understood. In addition to the assumptions regarding the systems of galaxies, the two kinematic tests of stability depend on assumptions of a different character, concerned with the collection of data. Naturally, assumptions of this kind also underly the virial theorem test and, indeed, any other test based on radial velocities and applied to individual clusters or groups of galaxies. One such assumption is that the objects treated as members of a particular cluster or group really are members of this system, rather than foreground or background galaxies. Another assumption of the same category is that the decision as to whether a given object does or does not belong to the particular system of galaxies is reached without reference to the object's radial velocity. These two important assumptions underlying the kinematic tests may be combined, as follows: even though the observations relating to a given system of galaxies may be subjected to selection for brightness or for position or for some other characteristic, they represent a fair sample of the system's membership *as far as radial velocities are concerned.*

At this point it may be relevant to notice that while the validity of the kinematic tests is not harmed in any way by selection of objects for their brightness, this probably is not true for the dynamical tests. The reason is that brightness generally is assumed to be correlated with mass and thus selection for brightness must imply selection for mass.

Whether the data relating to any given system satisfy the combined condition above is a subjective question. In order to give the reader an opportunity to make up his own mind, whenever there is a question of some "outliers," which might be suspected of not belonging to the system studied, the tests for stability were applied twice, including and excluding the outliers. In fact, the summary test was applied to groups of galaxies yet again, excluding those groups that, for one reason or another, appear to have been affected by selection for those objects which have similar radial velocities.

Quite apart from the above precautions, we should mention a particular mechanism through which the selection of galaxies may have modified the distribution of radial velocities and invalidated the tests. There is little doubt that, particularly for distant systems, the objects selected for observation are those with bright apparent magnitude, including the dimming due to redshift. In other words, the available observational material does not include galaxies with peculiar velocities away from the observer so large that the resulting redshift decreases their apparent brightness to the point of making them difficult to observe.

It must be obvious that this kind of selection takes place, and it is our intention to study it in another paper. For the present we note that the values of the K-correction to the apparent magnitude published by Humason, Mayall and Sandage [13] indicate that the numerical effect of the selection described is likely to be negligible.

We first note that the estimates of dispersion of radial velocity within clusters are of the general order of 1000 km/sec. Thus the difference between the radial velocities of two cluster members selected at random is a random variable, probably not far from normal, with zero mean and with dispersion of about 1400 km/sec. This difference rarely would be greater than 3000 km/sec and practically never greater than four times the dispersion, that is, 5600 km/sec. The latter figure differs but little from the radial velocity of the Perseus Cluster (5433 km/sec) for which Humason, Mayall and Sandage list the K-correction as 0.08 mag. Even allowing for the nonlinearity of the K-correction, this result indicates that the differences in dimming of cluster members due to differences in their radial velocities must be less than one-tenth of a magnitude and can have very little effect on the decision of the observer to include a given object in his program.

Whatever the case may be, after calling the reader's attention to the theoretical possibility of bias, in the remainder of the present paper we postulate that, whatever may have been the selection of galaxies for brightness, this selection does not affect the conditional distribution of redshift. What we assume, in effect, is that if a particular galaxy has been left off the observational program because of its apparent faintness, then this galaxy would have been left off the program also if it were brighter by one-tenth of one magnitude.

It is unfortunate that the outcome of the tests depends very much on the objects to which they are applied. Thus, for example, the data for the Coma Cluster appear to include four "outliers," which, for various reasons explained in the sequel, the present authors suspect do not really belong to the Coma Cluster. However, the galaxies in question are located in the general area of the Coma Cluster and, in spite of indications to the contrary, may conceivably belong to the cluster. If these outliers are accepted as members of the cluster, the test leaves little doubt that the cluster is expanding. On the other hand, if the outliers are omitted, the same test fails to show expansion and, if anything, indicates that the Coma Cluster is contracting. From the point of view of the hypothesis that contraction or expansion is a relatively simple phenomenon applying to the cluster as a whole, these contradictory conclusions certainly are disconcerting. However, it is notable that qualitatively they agree with the following conclusion of van Albada [14] suggested earlier by Oort [10]: "At a later stage, relative contraction of the central regions (of a cluster) becomes absolute contraction, while the intermediate and outer regions are still expanding." There is, then, the possibility that the suspected "outliers" in the Coma Cluster actually are members of this system but are located in the "outer region" showing a rapid expansion. At the same time, the central region of the cluster may be contracting.

A somewhat similar dependence of the conclusions on the material included in the calculations prevails in the study of groups of galaxies. The two summary tests applied to all groups for which we have data fail to show evidence of instability. On the other hand, if those groups for which there is a suspicion of selection for similarity in redshift are omitted, then the same tests yield a strong indication of expansion.

The above are examples of objective findings presented below. Our subjective opinion is that the hypothesis that at least some systems of galaxies are unstable cannot be dismissed lightly and deserves further study.

Certain remarks by Dr. R. Minkowski made at the Symposium during the discussion of the present paper, combined with some further analysis presented below, lead us to another conclusion. This is that, if the results of the kinematic tests indicating instability really reflect instability of systems of galaxies, rather than an arbitrary selection of data, then the same results are evidence of the presence of absorbing matter within the systems concerned. The study of this question is in progress.

Throughout this paper, the term "system of galaxies" is used as a synonym of the term "cluster" appearing in the theory of the spatial distribution of galaxies [1], [15]. This change in terminology is adopted in deference to the distinction made in some sections of astronomical literature in which the word cluster is used to denote a system of a large number of galaxies, while a system of relatively few members is described as a group.

Although the methods used in this paper are applied solely to systems of galaxies, the theoretical results described in the next sections could as well be adapted to stellar systems.

2. Kinematic test of the stability of several systems of galaxies against the alternative that all these systems expand or contract at the same rate

In this section we recapitulate and slightly reformulate the assumptions underlying the deduction of the locally most powerful test for stability of a system as given in [1] and indicate an easy generalization of this test so as to refer to any number s of systems assumed to have certain similar properties.

The basic hypotheses underlying the test of stability are those adopted in the Neyman-Scott theory of the spatial distribution of galaxies. Consider an orthogonal system of coordinates with its origin at the observer. Consider a system with its center at $\mathbf{u} = (u_1, u_2, u_3)$ and let D be the distance of this center from the observer. Our hypotheses are:

(i) Given $\mathbf{u}$, the coordinates $\mathbf{X} = (X_1, X_2, X_3)$ of any member of the system are random variables with their conditional probability density $f(\eta)$ depending only on the distance η from the center of the system.

In the applications of the theory, whenever we need to deal with specific functions, it is assumed that

$$f(\eta) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^3 e^{-\eta^2/2\sigma^2}. \tag{1}$$

(ii) Given the positions $u_1, u_2, \cdots, u_s$ of the centers of s systems, the positions of the objects belonging to these systems are mutually independent and follow the same distribution.

(iii) The absolute magnitude $\mathcal{M}$ of a cluster member is a random variable independent of all the other random variables considered and following a normal distribution with mean M_0 and variance σ_M^2.

(iv) The present study was conducted from the point of view of the hypothesis that the amount of absorbing matter in space, whether within the systems considered or outside, is negligible. However, in order to explain certain details of the numerical findings, we feel in need of the hypothesis that the systems can include some dark matter. The actual distribution of this matter is likely to be complex. However, it is hoped that the following simple hypothesis may serve as a first approximation.

Consider a system with its center at distance D from the observer. Consider a particular member of this system and denote by ξ its distance from the observer and by φ its angular separation from the center of the system. Let C denote the dimming of an object located exactly at the center of the cluster, that is, when $\xi = D$ and $\varphi = 0$. We assume that, for small φ, the same number C will represent the dimming of all the cluster members for which $\xi = D \cos \varphi$. Generally, for small φ and arbitrary ξ, we assume that the apparent magnitude m (corrected for redshift) is connected with the absolute magnitude $\mathcal{M}$ of the object by the equation

$$m = \mathcal{M} - 5 + 5 \log_{10}\xi + C + \tau(\xi - D \cos \varphi), \tag{2}$$

where τ is a constant to be described as the absorption coefficient. The value of C does not play any role in our calculations. As mentioned, the principal computations were performed assuming $\tau = 0$. However, certain details indicate that τ may be a positive number and some of the computations were repeated.

(v) We now consider the velocities of cluster members with respect to the center of the system to which they belong.

The velocity vector Ω of a cluster member located at $\mathbf{X}$ is the resultant of two vectors Ω_1 and Ω_2, the first of which is random and the second nonrandom. Given the positions of several cluster members, the random components are mutually independent and identically distributed. More specifically, the distribution of the component $\Omega_1 = (\dot{X}_1, \dot{X}_2, \dot{X}_3)$ is spherical normal, with variance σ_1^2 and with $E(\dot{X}_j) = \dot{u}_j$, where $j = 1, 2, 3$ and $\dot{u} = (\dot{u}_1, \dot{u}_2, \dot{u}_3)$ is the recession velocity of the center of the system to which the given object belongs. As to the nonrandom vector Ω_2, it represents the velocity of the Hubble-type expansion (or contraction) of the system. This vector is assumed to be directed away from the center of the system and its magnitude is assumed to be proportional to the distance between the center and the cluster member in question. Thus, for a member with fixed coordinates x_1, x_2, x_3, the components of Ω_2 are

$$\theta(x_1 - u_1), \theta(x_2 - u_2), \theta(x_3 - u_3), \tag{3}$$

where θ is the factor of proportionality, analogous to the Hubble constant. If $\theta = 0$ then the system is stable, while $\theta > 0$ means expansion and $\theta < 0$ means contraction. Figure 1 illustrates the situation.

(vi) It is assumed that the radial velocity of the center of each system is exactly proportional to the distance of this center from the observer, so that the components of this velocity,

$$\dot{u}_j = H u_j, \qquad j = 1, 2, 3, \tag{4}$$

where H is the Hubble constant.

The variables that are observable for each object are its radial velocity ρ, its apparent magnitude m, which we shall assume to have been corrected for redshift, and its angular distance φ from the center of the system considered. Of these three random variables we shall assume that the data collected for any system are subject to selection for m and, possibly, for φ. On the other hand, we assume that there is no selection for ρ. The test of stability of the system is, essentially, a test of the hypothesis that the observed values of ρ conform to the

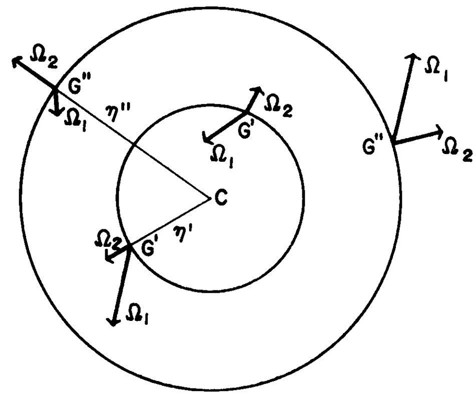

FIGURE 1

Random and nonrandom velocity components.
For illustrative purposes, two cluster members are assumed to be at the same distance η' from the center of the system.
They are denoted G'. Two other cluster members, denoted G'', are assumed to have distance from the center $\eta'' = 2\eta'$. Thus the magnitudes of the corresponding vectors Ω_2 are in the same relation.

conditional distribution of this variable implied by the assumptions (i) to (vi) in combination with the hypothesis of no expansion $\theta = 0$.

Geometrical considerations, combined with the hypotheses regarding the velocity vector Ω, imply

$$\rho = \frac{1}{\xi}\left[\sum_{i=1}^{3} X_i\dot{X}_i + \theta\left(\xi^2 - \sum_{i=1}^{3} u_i X_i\right)\right] = v \cos\psi + HD\cos\varphi + \theta(\xi - D\cos\varphi), \tag{5}$$

where $v = |\Omega_1|$ is the random component of the peculiar velocity and ψ is the angle between Ω_1 and the line of sight. Formula (5) implies that, for fixed $\mathbf{X}$ and $\mathbf{u}$, the conditional distribution of ρ is normal with variance σ_1^2 and with expectation

$$E(\rho|\mathbf{X}, \mathbf{u}) = HD\cos\varphi + \theta(\xi - D\cos\varphi). \tag{6}$$

Consider s systems and assume that the distribution of $\mathcal{M}$, of $X_i - u_i$, and of ρ are identical from one system to the next. Also we assume in this section that the rate of expansion θ is the same for each system. The subscripts i or ij attached to various symbols will indicate the ith system and the jth member in the ith system, respectively. We shall have $i = 1, 2, \cdots, s$ and, for a fixed i, the subscript $j = 1, 2, \cdots, n_i$, where n_i stands for the number of objects in the ith system for which the necessary data are available. Our purpose is to deduce the joint conditional probability density, say p, of all the ρ_{ij}, given the values of all the m_{ij} and φ_{ij}. Because of the postulated independence, p is the product of the conditional densities of the ρ_{ij} given m_{ij} and φ_{ij}.

Using familiar notation and suppressing the subscripts i, j for the time being, we have

$$p_{\rho|m,\varphi} = \frac{p_{m,\varphi,\rho}}{p_{m,\varphi}}. \tag{7}$$

In order to simplify the formulas, rather than work with the variables m, ρ and ξ, it will be convenient to deal with an equivalent system defined by

$$\begin{aligned} \mu &= m + 5 - 5 \log_{10}(D \cos \varphi) - C - M_0, \\ r &= \rho - HD \cos \varphi, \\ t &= \frac{\xi - D \cos \varphi}{\sigma}. \end{aligned} \tag{8}$$

The adopted hypotheses then imply that

$$p_{\varphi,t} = \frac{e^{-D^2\sin^2\varphi/2\sigma^2}}{\sigma^2\sqrt{2\pi}} D^2 \cos^2 \varphi \sin \varphi \, e^{-t^2/2}(1 + qt)^2, \tag{9}$$

where $q = \sigma/D \cos \varphi$ and the region of variation of t extends from $-q^{-1}$ to infinity. Given t and φ, the conditional density of μ is then

$$p_{\mu|\varphi,t} = \frac{1}{\sigma_M\sqrt{2\pi}} \exp\left\{ - \frac{1}{2\sigma_M^2} [\mu - a \log(1 + qt) - \tau\sigma t]^2 \right\}, \tag{10}$$

where the logarithm is taken to the base e and $a = 5 \log_{10}e = 2.1715$. Finally, the conditional probability density of r, given φ and t, is

$$p_{r|\varphi,t} = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-(r-\theta\sigma t)^2/2\sigma_1^2}. \tag{11}$$

Familiar formulas give

$$p_{\mu,\varphi} = \int_{-q^{-1}}^{\infty} p_{\varphi,t}\, p_{\mu|\varphi,t}\, dt, \tag{12}$$

$$p_{\mu,\varphi,r} = \int_{-q^{-1}}^{\infty} p_{\varphi,t}\, p_{\mu|\varphi,t}\, p_{r|\varphi,t}\, dt, \tag{13}$$

and finally

$$p_{r|\mu,\varphi} = \frac{J(r, \mu, \varphi)}{I(\mu, \varphi)}, \tag{14}$$

with

$$\begin{aligned} &J(r, \mu, \varphi) \\ &= \int_{-q^{-1}}^{\infty} (1 + qt)^2 \exp\left\{ -\frac{1}{2}\left[t^2 + \frac{[\mu - a \log(1 + qt) - \tau\sigma t]^2}{\sigma_M^2} - \frac{(r - \theta\sigma t)^2}{\sigma_1^2} \right]\right\} dt, \end{aligned} \tag{15}$$

$$(16) \quad I(\mu, \varphi) = \sigma_1\sqrt{2\pi}\int_{-q^{-1}}^{\infty} (1 + qt)^2 \exp\left\{ -\frac{1}{2}\left[t^2 + \frac{[\mu - a \log(1 + qt) - \tau\sigma t]^2}{\sigma_M^2}\right]\right\} dt.$$

The joint probability density p is then

$$(17) \quad p = \prod_{i=1}^{s} \prod_{j=1}^{n_i} p_{r_{ij}|\mu_{ij},\varphi_{ij}}.$$

Formulas (14) to (16) indicate that, in general, the conditional distribution of r given μ and φ is not normal; this creates the problem of deducing a test of the hypothesis $\theta = 0$ satisfying some optimum condition. However, it will be noted that, if the hypothesis tested is true, then, for fixed μ and φ, the variable r is normally distributed with mean zero and variance σ_1^2. This is an important detail which simplifies the problem considerably. The test to be deduced has the property of being locally most powerful unbiased.

Repeating the reasoning explained in [1], we find that the locally best test of the hypothesis that $\theta = 0$ reduces to the rule of rejecting this hypothesis whenever the absolute value of the criterion

$$(18) \quad T = \frac{\hat{\theta}}{\hat{\sigma}_{\hat{\theta}}}$$

exceeds the tabled value of Student's t corresponding to the chosen level of significance α and to the number of degrees of freedom $\sum n_i - s - 1$. Here $\hat{\theta}$ stands for the least squares estimate of θ,

$$(19) \quad \hat{\theta} = \frac{\sum_i \sum_j (A_{ij} - A_{i\cdot})\, \rho_{ij} \cos \varphi_{ij}}{\sum_i \sum_j (A_{ij} - A_{i\cdot})^2 \cos^2 \varphi_{ij}}$$

with

$$(20) \quad A_{ij} = \frac{\sigma\, E(t_{ij}|\mu_{ij}, \varphi_{ij})}{\cos \varphi_{ij}} = \frac{E(\xi_{ij} - D_i \cos \varphi_{ij}|\mu_{ij}, \varphi_{ij})}{\cos \varphi_{ij}}.$$

As to $A_{i\cdot}$ and a similar symbol $\rho_{i\cdot}$ which we shall need later, they are given by

$$(21) \quad A_{i\cdot} = \frac{\sum_j A_{ij} \cos^2 \varphi_{ij}}{\sum_j \cos^2 \varphi_{ij}}, \qquad \rho_{i\cdot} = \frac{\sum_j \rho_{ij} \cos \varphi_{ij}}{\sum_j \cos^2 \varphi_{ij}}.$$

The quantities A_{ij} can be computed by numerical integration using (9) and (10). These formulas depend on the distance D_i to the cluster. In computations these distances must be replaced by their estimates $\rho_{i\cdot}H^{-1}$. The numerical integration involved is tedious and further below we indicate convenient approximations based on the fact that, ordinarily, the quotients $q_{ij} = \sigma/D_i \cos \varphi_{ij}$ and the angles φ_{ij} are small.

The quantity $\hat{\sigma}_{\hat{\theta}}$ in the denominator of (18) is the estimate of the standard error of $\hat{\theta}$. The variance of $\hat{\theta}$ is given by

$$\sigma_{\hat\theta}^2 = \frac{\sigma_1^2}{\sum_i \sum_j (A_{ij} - A_{i\cdot})^2 \cos^2 \varphi_{ij}}. \tag{22}$$

In order to obtain its estimate to be used in (18), the value of σ_1^2 in (22) is replaced by

$$\hat\sigma_1^2 = \frac{\sum_i \sum_j (\rho_{ij} - \rho_{i\cdot})^2 - \hat\theta \sum_i \sum_j (A_{ij} - A_{i\cdot})\rho_{ij} \cos \varphi_{ij}}{\sum_i n_i - s - 1} \tag{23}$$

which is the pooled estimate of the conditional variance σ_1^2 of ρ.

In the above formulas, the summations for i always extend over all the systems of galaxies included in the test, that is, over $i = 1, 2, \cdots, s$. The summations for j extend over $j = 1, 2, \cdots, n_i$. When we put $s = 1$ the formulas of this section become equivalent to those in the original publication [1]. Whatever the value of s, when the hypothesis tested is true, the T-test will reject it falsely with probability exactly equal to α, irrespective of the values of the D_i and of σ_1, and no matter how the objects were selected for magnitude and position.

It may be useful to point out the relation between the T-test just described and the classical t-test. Operationally, the two coincide. However, the t-test has the property of being the uniformly most powerful test corresponding to the assumption that, whether the hypothesis tested is true or not, the observable random variables are normally distributed with a fixed variance. Because of the nonnormality of the distribution of r(and therefore of ρ), the T-test is not uniformly most powerful (no such test exists) and it has a power function different from that of t. However, when the number of observations is large and θ is small, the power function of T is approximately equal to that of t.

3. Kinematic test of the stability of several systems of galaxies against the alternative that they may be expanding or contracting each at a possibly different rate

In this section we contemplate the possibility that among the systems studied there may be some that are in fact stable, while some others may be expanding each at its own rate, and still others may be contracting. In other words, contrary to what was assumed in the previous section, we now assume that to the ith system of objects there corresponds its own rate of expansion θ_i which need not be equal to the rate θ_j of the jth system. The hypothesis to test is now

$$\theta_1 = \theta_2 = \cdots = \theta_s = 0. \tag{24}$$

All other assumptions used in section 2 are maintained. The analogy of the T-test to the classical t-test suggests the possibility of extending the T-test in the same manner in which the classical t-test is extended for testing the general linear hypothesis. As is well known, this involves the use of the familiar F-distribution.

Considering only one system of objects, say the ith, and using formula (19) we obtain the least squares estimate of θ_i,

$$(25)\qquad \hat{\theta}_i = \frac{\sum_j (A_{ij} - A_{i\cdot})\rho_{ij}\cos\varphi_{ij}}{\sum_j (A_{ij} - A_{i\cdot})^2\cos^2\varphi_{ij}}.$$

The variance of $\hat{\theta}_i$ is obtained from a similar particularization of (22), namely,

$$(26)\qquad \sigma^2_{\hat{\theta}_i} = \frac{\sigma_1^2}{\sum_j (A_{ij} - A_{i\cdot})^2\cos^2\varphi_{ij}}.$$

However, because of the admitted possibility that each system of galaxies may be unstable at a different rate θ_i, the pooled estimate of σ_1^2 is no longer given by (23) but by the formula

$$(27)\qquad \tilde{\sigma}_1^2 = \frac{\sum_i\sum_j (\rho_{ij} - \rho_{i\cdot})^2 - \sum_i \hat{\theta}_i \sum_j (A_{ij} - A_{i\cdot})\rho_{ij}\cos\varphi_{ij}}{\sum_i n_i - 2s},$$

with the number of degrees of freedom $\sum n_i - 2s$.

If the hypothesis tested is true, then each product

$$(28)\qquad \hat{\theta}_i\Big[\sum_j (A_{ij} - A_{i\cdot})^2\cos^2\varphi_{ij}\Big]^{1/2} = \frac{\sum_j (A_{ij} - A_{i\cdot})\rho_{ij}\cos\varphi_{ij}}{\Big[\sum_j (A_{ij} - A_{i\cdot})^2\cos^2\varphi_{ij}\Big]^{1/2}}$$

is a normal variable with expectation zero and variance σ_1^2. It follows that in this case the variable, say,

$$(29)\qquad Z = \frac{1}{s}\sum_i\left(\frac{\hat{\theta}_i}{\tilde{\sigma}_{\hat{\theta}_i}}\right)^2 = \frac{\sum_i \hat{\theta}_i^2 \sum_j (A_{ij} - A_{i\cdot})^2\cos^2\varphi_{ij}}{s\,\tilde{\sigma}_1^2}$$

follows exactly the F-distribution with degrees of freedom $f_1 = s$ and $f_2 = n_i - 2s$. Thus, the test of the hypothesis $\theta_1 = \theta_2 = \cdots = \theta_s = 0$ reduces to the rule of rejecting this hypothesis whenever the computed value of Z exceeds the tabled value of F corresponding to the indicated degrees of freedom and to the chosen level of significance α. When the hypothesis test is true, it will be rejected falsely with probability exactly equal to α, irrespective of the values of the D_i and of σ_1, and no matter how the objects were selected for magnitude and position.

4. Approximate formula for the symbol A

According to definition (20) and using (9) and (10), we have

$$(30)\qquad A\cos\varphi = \sigma\frac{\int t(1+qt)^2\exp(-\frac{1}{2}\{t^2 + [\mu - a\log(1+qt) - \tau\sigma t]^2/\sigma_M^2\})\,dt}{\int (1+qt)^2\exp(-\frac{1}{2}\{t^2 + [\mu - a\log(1+qt) - \tau\sigma t]^2/\sigma_M^2\})\,dt}$$

where both integrations extend from $-q^{-1}$ to $+\infty$, and the subscripts ij have been suppressed for brevity. Using the value of σ in the range 10^5 to 10^6 as estimated in [16] and the estimates of the distances D_i of the various systems of galaxies involved in the empirical study presented below, we find that, except for the closest group, $q_{ij} = \sigma/D_i \cos \varphi_{ij}$ is always smaller than 0.05. This fact indicates that the integrals in (30) depend mainly on the properties of the integrands in the interval $|t| < 4$. Thus, with the danger of committing only a small error, we may consider these integrals as extending from $-\infty$ to $+\infty$ and also we may use the approximation

$$\log(1 + qt) \doteqdot qt. \tag{31}$$

Adopting these approximations and performing easy transformations, we write

$$A \cos \varphi \doteqdot \sigma \frac{\int t(1 + qt)^2 \exp\left[-\frac{\sigma_M^2 + \lambda^2\sigma^2}{2\sigma_M^2} (t - t_0)^2 \right] dt}{\int (1 + qt)^2 \exp\left[-\frac{\sigma_M^2 + \lambda^2\sigma^2}{2\sigma_M^2} (t - t_0)^2 \right] dt}, \tag{32}$$

where, for the sake of brevity,

$$\lambda = \frac{aq}{\sigma} + \tau, \tag{33}$$

$$t_0 = \frac{\lambda\sigma\mu}{\sigma_M^2 + \lambda^2\sigma^2}, \tag{34}$$

and the two integrals extend from $-\infty$ to $+\infty$. Simple calculations now give

$$A \cos \varphi \doteqdot \sigma \left[t_0 + \frac{2q(1 + qt_0)\sigma_M^2}{(1 + qt_0)^2(\sigma_M^2 + \lambda^2\sigma^2) + q^2\sigma_M^2} \right]. \tag{35}$$

Already, this formula for $A \cos \varphi$ is easy to compute. However, if one decides to neglect the terms in q^2 and also to substitute unity for $\cos \varphi$, which introduces errors of the order of a few percent at most, then

$$A \doteqdot \sigma \left[t_0 + 2q \frac{\sigma_M^2}{\sigma_M^2 + \lambda^2\sigma^2} \right] = \frac{\sigma(\lambda\sigma\mu + 2q\sigma_M^2)}{\sigma_M^2 + \lambda^2\sigma^2}. \tag{36}$$

Numerical computations, involving the evaluation of integrals by quadrature formulas, indicated that, for all but the closest systems, formula (36) provides a very good approximation. For the closest systems the approximation to individual values of the A_{ij} is somewhat less satisfactory. However, since the A_{ij} appear in all statistics as weights, the final effect of the inaccuracies is negligible. For example, in the Ursa Major Cloud, where φ_{ij} extends to 15°, the error in the estimate of θ_i is only five percent.

Returning now to the subscripts ij, formula (36) yields

$$A_{ij} - A_{i\cdot} \doteqdot \frac{\lambda_i\sigma^2}{\sigma_M^2 + \lambda_i^2\sigma^2} (m_{ij} - m_{i\cdot}), \tag{37}$$

with

$$\lambda_i = \tau + \frac{a}{D_i}. \tag{38}$$

Before proceeding further, we recall that the symbol m is used here to denote the apparent magnitude of a galaxy *corrected for redshift*. On the other hand, the apparent magnitudes usually are published without this correction, which is often called the K-correction. The formulas given above could be used directly after each of the apparent magnitudes had been corrected. However, a considerable amount of labor can be saved by noticing that over a very substantial range of redshift the K-corrections are very small and are approximately proportional to ρ. In fact, using the K-corrections published by Humason, Mayall and Sandage [13] for twelve clusters ranging in radial velocities up to 23,365 km/sec, it was found that, with only trivial errors, we can put

$$K(\rho) = \chi\rho \quad \text{with} \quad \chi = 1.55 \times 10^{-5}. \tag{39}$$

Now if, contrary to the notation adopted earlier, m denotes the apparent magnitude *not corrected for redshift,* then expression (37) for $A_{ij} - A_{i\cdot}$ must be replaced by

$$A_{ij} - A_{i\cdot} = \frac{\lambda_i\sigma^2}{\sigma_M^2 + \lambda_i^2\sigma^2}\,[(m_{ij} - m_{i\cdot}) - \chi(\rho_{ij} - \rho_{i\cdot})]. \tag{40}$$

Now denote by $S_i^2(\rho)$ the sample variance of ρ, by $S_i(m, \rho)$ the sample covariance of ρ and the *uncorrected apparent magnitude* and, finally, by $S_i^2(m)$ the sample variance of the *uncorrected apparent magnitude,* all computed from the data on n_i objects within the ith system, so that

$$\begin{aligned} S_i^2(\rho) &= \frac{1}{n_i - 1}\sum_j (\rho_{ij} - \rho_{i\cdot})^2, \\ S_i(m, \rho) &= \frac{1}{n_i - 1}\sum_j (m_{ij} - m_{i\cdot})(\rho_{ij} - \rho_{i\cdot}), \\ S_i^2(m) &= \frac{1}{n_i - 1}\sum_j (m_{ij} - m_{i\cdot})^2. \end{aligned} \tag{41}$$

For purposes of computing, the main elements in the formulas connected with the two tests for stability are easy to express in terms of (41). We have

$$\begin{aligned} \sum_j (A_{ij} - A_{i\cdot})^2 \cos^2\varphi_{ij} &\fallingdotseq \left[\frac{\lambda_i\sigma^2}{\sigma_M^2 + \lambda_i^2\sigma^2}\right]^2 (n_i - 1)\sigma_i^2(m), \\ \sum_j (A_{ij} - A_{i\cdot})\rho_{ij}\cos\varphi_{ij} &\fallingdotseq \frac{\lambda_i\sigma^2}{\sigma_M^2 + \lambda_i^2\sigma^2}(n_i - 1)\sigma_i(m, \rho), \end{aligned} \tag{42}$$

with

$$\begin{aligned} \sigma_i^2(m) &= S_i^2(m) - 2\chi S_i(m, \rho) + \chi^2 S_i^2(\rho), \\ \sigma_i(m, \rho) &= S_i(m, \rho) - \chi S_i^2(\rho) \end{aligned} \tag{43}$$

representing the sample variance of the apparent magnitude corrected for redshift and its sample covariance with redshift, respectively. Now the expressions in (42) can be substituted into the formulas giving the estimates $\hat{\theta}$, $\hat{\theta}_i$, $\hat{\sigma}_1$, $\sigma_{\hat{\theta}}^2$ and so forth used in the T-test and in the Z-test. Using (19) and (42) we have

$$\hat{\theta} \doteq \frac{\sum_i (n_i - 1)\lambda_i\sigma_i(m, \rho)/(\sigma_M^2 + \lambda_i^2\sigma^2)}{\sigma^2 \sum_i (n_i - 1)\lambda_i^2\sigma_i^2(m)/(\sigma_M^2 + \lambda_i^2\sigma^2)^2}. \tag{44}$$

Also,

$$\sigma_{\hat{\theta}}^2 \doteq \frac{\sigma_1^2}{\sigma^4 \sum_i (n_i - 1)\lambda_i^2\sigma_i^2(m)/(\sigma_M^2 + \lambda_i^2\sigma^2)^2}, \tag{45}$$

and so forth.

When considering the formulas above, it is important to be clear how far their values depend on the data involved in the study and, on the other hand, how far they are influenced by constants taken from other sources and possibly subject to doubt. Such constants are χ, σ_M, σ, τ and the Hubble constant H.

The values of the K-correction are deduced using certain hypotheses and approximations, and depend on the type of galaxies involved. However, unless the theory behind them is entirely wrong, the order of magnitude of the figures given by Humason, Mayall and Sandage may be expected to be correct. The values of the expressions (44), (45) and others will not change much if $\chi = 1.55 \times 10^{-5}$ were to be multiplied or divided by a substantial factor. Thus, we do not anticipate any difficulty from the value of χ.

The other constants σ, σ_M, τ and H are much more troublesome. We can write $H = 10^{-4}h$ km/sec/pc where h is a factor between 0.5 and 1.5, depending on the distance scale one wants to adopt. Then, using the assumption in (4), we estimate the distance to the ith cluster by $D_i = H^{-1}\rho_{i\cdot} = 10^4\rho_{i\cdot}/h$ parsecs. The value of the dispersion σ also depends on H, namely, $\sigma = \bar{\sigma}/h$ where $\bar{\sigma}$ is of the order of 7×10^5 parsecs. On the other hand, the dispersion σ_M does not depend on H; the value $\sigma_M = 1.25$ mag used below represents a compromise between several current estimates. As to τ, there is an estimate by Zwicky [9] to the effect that galaxies located behind a large cluster have their apparent magnitudes increased by about 0.7 mag due to absorbing matter within the cluster. Assuming that the radius of a cluster corresponds to about 3.5σ, Zwicky's estimate yields $\sigma\tau = 0.1$ mag.

Taking all these values into account, we notice that

$$\lambda_i^2\sigma^2 = \left(0.1 + \frac{a\bar{\sigma}}{10^4\rho_{i\cdot}}\right)^2 \tag{46}$$

is independent of h and, for not too small $\rho_{i\cdot}$, relatively unimportant compared to σ_M^2. It follows that even substantial errors in τ will not affect the denominators $\sigma_M^2 + \lambda_i^2\sigma^2$ appearing in (44) and (45). On the other hand, the same approximate calculations indicate that if the adopted value of σ_M^2 is an underestimate or an overestimate of the true value by a factor then the estimate $\hat{\theta}$ is multiplied by the same factor. Finally the factor σ^2 appearing in the denominator of (44) indicates that the estimate $\hat{\theta}$ is roughly proportional to h^2.

To summarize, we may say that the estimates $\hat{\theta}$ and $\hat{\theta}_i$ are nearly proportional to σ_M^2/σ^2 and that $\sigma_{\hat{\theta}}^2$ is nearly proportional to the square of the same ratio. This circumstance must be taken into account when examining the numerical estimates obtained.

Should it happen that the outcomes of the T and the Z-tests depend on the assumed values of the above parameters to any comparable extent, this would constitute a considerable weakness in the tests. Luckily, this is not the case. The two tests depend essentially on quotients of the type $\hat{\theta}/\hat{\sigma}_{\hat{\theta}}$ and it so happens that the factors σ_M^2/σ^2 occur in both the numerator and the denominator and cancel. It follows then that, while the estimates of the expansion factors do depend on the uncertain quantities σ_M and σ, the outcomes of the kinematic tests are practically independent of these parameters and depend only on the data. As far as the validity of the tests is concerned, the greatest danger comes from the possible selection on ρ.

The fact that the outcomes of the kinematic tests do not depend on the values of the assumed parameters is not surprising when we consider the intuitive interpretation of the tests. Essentially, they measure the correlation between A_{ij} and ρ_{ij} (using the approximation for A_{ij}, the correlation between m_{ij} and ρ_{ij}) and thus are independent of the value of any scale or location parameter. The heuristic explanation of T_i, for example, is that if there is expansion of the ith cluster then the objects on the front of the cluster (those with negative A_{ij}) will have a component Ω_2 towards the observer and thus will tend to have smaller ρ_{ij} than average, whereas those objects on the back side of the cluster will have positive A_{ij} and will tend to have larger ρ_{ij} than the average. Hence, expansion will tend to produce large positive values of T_i.

One more remark is necessary. An inspection of the formulas (14) to (16) indicates that $\theta\sigma t$ is the conditional expectation of $r = \rho - D \cos \varphi$ whether the hypothesis tested is true or not. This circumstance is the reason why the estimate $\hat{\theta}$ is a good test criterion: if, relative to its standard error, $\hat{\theta}$ is too far from zero then this is a good indication that the true θ differs from zero. On the other hand, it is necessary to realize that the property that $\hat{\sigma}_1^2$ is an unbiased estimate of σ_1^2, the conditional variance of ρ given the position of the galaxy within the cluster, has a "local" character and is subject to the assumption that $\theta = 0$. This is satisfactory unless one adopts the attitude that $\theta \neq 0$. When $\theta \neq 0$ then $\hat{\sigma}_1^2$ becomes the estimate of the conditional variance of ρ given φ and m and includes the variability of ρ due to the variability in the distance ξ with which ρ is correlated and which, for any fixed m, must be considerable. For this reason, the formulas for the standard errors of the estimates $\hat{\theta}$ are not appropriate when it is admitted that $\theta \neq 0$. The problem of estimating the standard errors in this case must be relegated to another study.

The practical application of the two tests, in their approximate form indicated in (44) and (45) involve the following preliminary steps:

(i) Using the magnitudes, not corrected for redshift, and the radial velocities, compute for each system the sample variances $S_i^2(m)$ and $S_i^2(\rho)$ and the covariance

$S_i(m, \rho)$ as indicated in (41). These statistics are interesting for their own sake.

(ii) Using the estimates $\hat{D}_i = H^{-1}\rho_{i.}$ and an assumed value of τ, compute $\lambda_i = \tau + a/\hat{D}_i$.

(iii) Using an assumed value of χ, compute $\sigma_i^2(m)$ and $\sigma_i(m, \rho)$, the corrected variance and covariance, from (43).

(iv) Assuming values for σ_M and for σ, and using the approximations in (42), proceed to calculate T_i, T and/or Z.

5. Extended validity of the kinematic tests

As noted at the outset, the hypotheses (i) to (vi), used to deduce tests of stability of systems of galaxies, are not likely to be satisfied exactly by actual systems of galaxies. For example, to mention only a few points, it is likely that the distribution of positions of the particular galaxies within a system does not correspond to formula (1). The same applies to the distribution of the random component Ω_1 of the velocity vector of a galaxy. Also the postulated independence of the absolute magnitude $\mathcal{M}$ and of the random velocity component Ω_1 is subject to doubt. The more massive galaxies are likely to be bright and to have somewhat smaller peculiar velocities. Furthermore, $\mathcal{M}$ need not be normally distributed. Finally, the vector Ω_1 probably depends on the position of the galaxy within the system.

The purpose of this section is to indicate that, while the optimality of the tests deduced does depend on the hypotheses enumerated above, the two tests remain approximately valid in much broader conditions. We combine these conditions in the following definition of a "stable system."

With reference to figure 2, let C be the center of a system of galaxies and P an arbitrary point in space. Draw an arbitrary line L through P and select a posi-

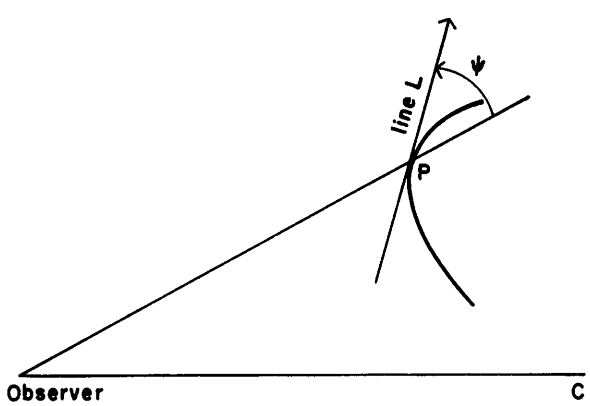

FIGURE 2

Illustration of the definition of a stable system.
The curve passing through P is tangent to L and represents the orbit of a galaxy.

tive direction on the line. Further, let M and v be any possible values of the absolute magnitude and of the peculiar velocity of a galaxy.

DEFINITION. *A system of galaxies will be called stable if, whatever be P, L, M and v, the probability that a galaxy which is at P with its orbit tangent to L and with its absolute magnitude and peculiar velocity equal, respectively, to M and v, will be moving in the positive direction of L equals the probability that it will be moving in the negative direction.*

Now, using formula (5) it will be seen that in a stable system of galaxies, that is, when $\theta = 0$ and the random peculiar velocity satisfies the definition just given, for every given P, L, M, v and for any given value of $|\cos \psi|$, the frequency that $\cos \psi$ is positive will be equal to the frequency that it is negative. Then, it follows from (5) that $E(\rho - HD \cos \varphi|\mu, \varphi)$ is zero so that for a given φ the radial velocity ρ is not correlated with μ. On the other hand, the two test criteria T and Z represent, essentially, the least squares criteria for testing the hypothesis that the regression coefficient of $\rho - HD \cos \varphi$ on m (or, equivalently, on μ) is equal to zero. Thus, if the systems of galaxies considered really are stable in the sense of the above definition and if the number of observations is sufficient for the least squares estimates $\hat{\theta}$ to be approximately normally distributed, then the criteria T and Z will reject the hypothesis tested with relative frequencies approximately equal to the chosen level of significance α.

From the point of view of the frequency with which the two tests indicate instability when the systems of galaxies are in fact stable, we see that the two tests are approximately valid under conditions much broader than those envisaged in their deduction. Unfortunately, this is not true when, in fact, the systems are unstable. Should it happen that, for example, the inner part of a system contracts while the outer part expands then, as is easy to visualize, the statistic θ will be an estimate of the average value of the expansion factors, appropriate to the various distances from the center, and this average may happen to be close to zero. In this way, the two tests may fail to detect instability.

6. Asymptotic power of the two tests for stability

The power of a test of a statistical hypothesis means the probability that this test will reject the hypothesis when, in fact, the hypothesis is false. Usually, there are many different ways in which the hypothesis tested may be false and, naturally, the value of the power will depend on the kind and on the degree of the falsehood of the hypothesis under test. It is obvious also that, in order to calculate the probability representing the power of a test, it is necessary to begin with rather specific conditions for the problem. In this section we return to hypotheses (i) to (vi) adopted in section 2.

The hypothesis of stability of s systems of galaxies is represented by s equations $\theta_1 = \theta_2 = \cdots = \theta_s = 0$ indicating that the postulated expansion coefficient in each of the systems is equal to zero. This hypothesis may be false in many ways including the following:

(a) all the systems expand or contract at exactly the same rate $\theta \neq 0$, so that $\theta_1 = \theta_2 = \cdots = \theta_s = \theta$. In this case the power of the tests considered must be a function of the common rate of instability θ.

(b) The rates of expansion of particular systems need not be equal but at least some of them differ from zero. In this case the power of the tests will depend upon the s arguments $\theta_1, \theta_2, \cdots, \theta_s$.

Case (b) is more general than (a) and therefore in our study we shall consider (b) and then revert to (a) as a particular case.

Reviewing the basic hypotheses (i) to (vi) it will be seen that they are not completely specific and fail to indicate the orders of a number of parameters involved. The parameters D_i and σ_1 represent one category of "nuisance parameters" and the two tests are adjusted so that, when the hypothesis tested is true, it will be rejected with a preassigned probability α, irrespective of the true values of the D_i and σ_1. However, the power of the test is likely to depend upon the actual values of these parameters. Further, the hypotheses (i) to (vi) visualize four more parameters, namely σ, σ_M, τ and H, which are treated as known numbers, determined by some other studies. It was found above that the values of the two criteria are independent of H and are not influenced very much by moderate errors that may be involved in the assumed value of σ, σ_M and τ. However, this need not be the case of the power. In fact, the power is likely to depend on the true values of σ, σ_M, τ and H and also on those assumed values of the parameters σ, σ_M and τ that are used in computing the test criteria. For this reason it is necessary for us to make a clear distinction between the two sets of values. Thus, the original symbols σ, σ_M, and τ will be used to denote the true values of the dispersion of a single coordinate of the position of a galaxy within a system, of the dispersion of the absolute magnitude and of the absorption coefficient. These true values are unknown but this, of course, should not prevent us from discussing them. Along with the true parameter values σ, σ_M and τ, we shall consider their "presumed" values used in the application of the tests. These presumed values will be denoted by $\bar{\sigma}$, $\bar{\sigma}_M$ and $\bar{\tau}$, respectively.

It is known (see, for example, [17]) that the power function of a test based on a criterion of the type T may be approximately calculated from the formula

$$\beta(Q) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\nu(\alpha)-Q}^{+\nu(\alpha)-Q} e^{-x^2/2}\, dx, \tag{47}$$

or

$$\beta_1(Q) = \frac{1}{\sqrt{2\pi}} \int_{\nu(2\alpha)-Q}^{\infty} e^{-x^2/2}\, dx, \tag{48}$$

depending upon whether the test is used against "two-sided" or "one-sided" alternatives, where $\nu(\alpha)$ is adjusted to the adopted level of significance α so that $\beta(0) = \alpha$, and Q depends on the extent and manner in which the hypothesis tested is false. The limitations on formulas (47) and (48) include certain conditions of regularity and the assumptions that the hypothesis tested is only "slightly" false while the number of observations is considerable. It is to this particular assumption that the description "asymptotic" refers.

It is seen that, for the two-sided alternatives, assuming $\theta < 0$ and $\theta > 0$ as possibilities, $\beta(Q)$ is an increasing function of $|Q|$. For the one-sided case $\beta_1(Q)$ is increasing with Q. It follows from (47) and (48) that in order to compute the approximate value of the power of the T-test it is sufficient to compute Q, which must depend on the values of $\theta_1, \theta_2, \cdots, \theta_s$, on the true values of the nuisance parameters D_i and σ_1 and on the true values of the remaining four parameters, σ, σ_M, τ, H and on the "presumed" values $\bar{\sigma}$, $\bar{\sigma}_M$, and $\bar{\tau}$. The process of computing Q consists in making certain substitutions in the formula representing the test criterion T, as follows.

The expression for T in its exact form (18) depends on estimates σ_1 and D_i, the latter entering the formula through the symbols A_{ij}. Our first step in computing Q is to replace these estimates by the true values of the parameters.

Next we notice that the quantities A_{ij} depend also upon "presumed" values of σ, σ_M and τ, that is, on $\bar{\sigma}$, $\bar{\sigma}_M$ and $\bar{\tau}$. These presumed values are left in the expressions for the A and these symbols so calculated will be denoted by $\bar{A}_{ij}$. Correspondingly, if we deal with the approximation (37), we shall write

$$\bar{A}_{ij} - \bar{A}_{i\cdot} \doteq \frac{\bar{\lambda}_i\bar{\sigma}^2}{\bar{\sigma}_M^2 + \bar{\lambda}_i^2\bar{\sigma}^2}(m_{ij} - m_{i\cdot}) \tag{49}$$

where the bars indicate the use of the "presumed" values of the parameters involved. [For the sake of brevity, the symbol m in (49) denotes the apparent magnitude corrected for redshift.]

Our last step in obtaining Q from T consists in replacing each ρ_{ij} by its expectation evaluated with the use of the true parameter values that one wants to consider. Then

$$E(\rho_{ij}|m_{ij}, \varphi_{ij}) = (D_i + \theta_i A_{ij})\cos\varphi_{ij} \tag{50}$$

or, with the approximation (37) and putting $\cos\varphi_{ij} = 1$,

$$E(\rho_{ij}|m_{ij}, \varphi_{ij}) = D_i + \theta_i\left[A_{i\cdot} + \frac{\lambda_i\sigma^2}{\sigma_M^2 + \lambda_i^2\sigma^2}(m_{ij} - m_{i\cdot})\right], \tag{51}$$

where the absence of bars indicates "true" parameter values. As a result of these substitutions and after some easy algebra, we obtain

$$Q = \tilde{\theta}\frac{\sigma^2}{\sigma_1}\frac{\sum_i \bar{w}_i w_i}{(\sum_i \bar{w}_i^2)^{1/2}}, \tag{52}$$

where

$$w_i = \frac{\lambda_i}{\sigma_M^2 + \lambda_i^2\sigma^2}[\sum_j (m_{ij} - m_{i\cdot})^2]^{1/2}, \tag{53}$$

where $\bar{w}_i$ represents the value of w_i corresponding to the presumed values of σ, σ_M and τ, and where $\tilde{\theta}$ is a weighted mean of the θ_i with weights equal to the products $\bar{w}_i w_i$. Note that w_i is simply the approximation to $(1/\sigma)[\sum(A_{ij} - A_{i\cdot})^2]^{1/2}$.

Formula (52) indicates that, if some of the systems of galaxies expand and some contract, the value of Q may well be zero and the power of the T-test may be nil. Otherwise, for a fixed $\tilde{\theta}$, the quantity Q is an increasing function of σ, roughly proportional to the linear dimensions of the system of galaxies and a decreasing function of the dispersion in redshift. Finally, it is important to notice that the greater the variability of the apparent magnitudes of objects within any given system, the greater the power of the test. If for a given system all the n_i objects used in the test had exactly the same apparent magnitude, then the contribution of this system to the power of the test would be exactly zero.

The last factor in (52) is of interest. By the Schwarz inequality, it cannot exceed the limit

$$\frac{\sum_i \bar{w}_i w_i}{(\sum_i \bar{w}_i^2)^{1/2}} \leqq (\sum_i w_i^2)^{1/2}, \tag{54}$$

and this limit is reached when $\bar{w}_i$ is proportional to w_i. In order that this proportionality hold irrespective of the distances D_i, it is necessary and sufficient that the presumed value $\bar{\tau}$ equal the actual absorption coefficient τ and that the dispersions σ and σ_M be underestimated or overestimated by $\bar{\sigma}$ and $\bar{\sigma}_M$ in the same proportion

$$\frac{\bar{\sigma}}{\sigma} = \frac{\bar{\sigma}_M}{\sigma_M}. \tag{55}$$

In all other cases, in particular when the absorption coefficient τ is not guessed correctly, the power of the T-test will be smaller than it could be.

Considering the same formula (52) from a slightly different point of view we notice that, because of the dependence $\lambda_i = \tau + a/D_i$, whatever be the presumed values of the parameters, the power of the T-test increases with an increase in the true absorption coefficient τ, which confirms the remarks of Dr. Minkowski, made at the discussion of this paper.

The same expression for λ_i indicates that, other things being equal, the close-by systems of galaxies contribute to the detecting power of the test much more than the distant systems, with large D_i. Finally, the contribution of a system to the power depends very much on the sums $\sum_j (m_{ij} - m_{i\cdot})^2$, that is, on the number of objects for which there are data and also on the degree to which the apparent magnitudes (corrected for redshift) of these objects diverge from each other. Here again, because of the presence of selection of objects bright enough for convenient observation, the more distant systems are at a disadvantage.

Whatever was just said about the power of the T-test, applied to $s > 1$ different systems of galaxies, could be repeated with reference to the application of the same test to any single system and then transferred to the power of the Z-test applied to s systems. Let Q_i denote the value of Q in (52) referring to just one system of galaxies, namely the ith, so that

$$Q_i = \theta_i \frac{\sigma^2}{\sigma_1} w_i. \tag{56}$$

The asymptotic power of the Z-test is represented by the power of the F-test with degrees of freedom $f_1 = s$ and $f_2 = \sum n_i - 2s$, and with noncentrality parameter

$$\sum_i Q_i^2 = \frac{\sigma^4}{\sigma_1^2} \sum_i \theta_{i\cdot}^2\, w_i^2\,. \tag{57}$$

It will be noticed that, if all the s systems of galaxies expand or contract at the same rate θ, then formula (57) reduces to the square of Q in (52) at its maximum value attainable only when the presumed value $\bar{\tau} = \tau$ and equation (55) is satisfied. Coupled with the fact that the noncentrality parameter (57) does not depend upon the signs of the particular θ_i, this circumstance may suggest that the Z-test has a definite advantage over the T-test which, for its top efficiency, requires the sameness of the signs of the θ_i and a lucky guess as to the parameters σ, σ_M and τ. Unfortunately, the situation is not so simple because for a fixed value of the noncentrality parameter (57) the power of the Z-test declines markedly with an increase in s. This phenomenon is illustrated in table I.

TABLE I

DECREASE IN POWER OF Z-TEST WHEN THE NONCENTRALITY PARAMETER IS KEPT CONSTANT AND THE NUMBER s OF SYSTEMS IS INCREASED

$\sum n_i = 120$, $\alpha = 0.05$, $\sum Q_i^2 = 8.00$

s	1	2	3	4	5	6
β	·80	·71	·65	·59	·55	·51

Table I indicates that, if on a priori grounds the possible instability of several systems of galaxies is expected to be characterized by coefficients θ_i all of the same sign, and if one is confident about the presumed values of the three parameters σ, σ_M and τ, then the T-test may have a definite advantage over the Z-test.

Before proceeding to the empirical part of this study we shall give another formula referring to the case where the "presumed" values $\bar{\sigma}$, $\bar{\sigma}_M$ and $\bar{\tau}$ are not equal to the true parameter values σ, σ_M and τ. This formula represents the expected value of $\hat{\theta}_i$, the estimate of the expansion coefficient of the ith system of galaxies. Using (25), the approximations (39) and denoting by bars the intervening quantities based on presumed values of the parameters, we have

$$\begin{aligned} E(\hat{\theta}_i) &= \frac{\bar{\sigma}_M^2 + \bar{\lambda}_i^2 \bar{\sigma}^2}{\bar{\sigma}^2 \bar{\lambda}_i \sum_j (m_{ij} - m_{i\cdot})^2} \sum_j (m_{ij} - m_{i\cdot}) E(\rho_{ij} | \varphi_{ij}, m_{ij}) \\ &= \frac{\sigma^2}{\bar{\sigma}^2} \frac{\bar{\sigma}_M^2 + \bar{\lambda}_i^2 \bar{\sigma}^2}{\sigma_M^2 + \lambda_i^2 \sigma^2} \frac{\lambda_i}{\bar{\lambda}_i} \theta_{i\cdot} \end{aligned} \tag{58}$$

Previous calculations indicated that the first two factors are practically independent of the distance of the ith cluster. On the other hand, depending upon the presumed value $\hat{\tau}$ of τ, the dependence of the third factor upon the distance D_i may well be very strong. In particular, this will be the case if, as in most of our calculations, the presumed value $\hat{\tau} = 0$. Then

$$\frac{\lambda_i}{\bar{\lambda}_i} = 1 + \frac{\tau}{a} D_i \tag{59}$$

and, if $\tau > 0$, the absolute value of the extimate $|\hat{\theta}_i|$ will tend to overestimate $|\theta_i|$ by a factor which is increasing with the distance D_i. On the assumption that $\theta_i = \theta_2 = \cdots = \theta_s = \theta$,

$$E(\hat{\theta}_i) \doteq k\theta(1 + \frac{\tau}{a} D_i), \tag{60}$$

where k stands for $\sigma^2\bar{\sigma}_M^2/\bar{\sigma}^2\sigma_M^2$.

7. Coma Cluster

As mentioned at the outset, since the first application of the kinematic test in 1953, the number of presumed members of Coma Cluster for which the apparent magnitude and the radial velocities are available increased from 12 to 50. Table II gives the totality of these data including the new observations of radial velocity by Mayall and the new observations of magnitude by G. E. Kron and M. F. Walker. We are very grateful to Drs. Kron and Walker for allowing us to list and to use their unpublished observations.

At the end of table II are listed five galaxies for which the radial velocity has been observed but no observations of magnitude are available. These galaxies cannot be used in the present study.

In table II the first column gives the NGC and IC catalogue numbers, when it was possible to decide with some assurance that the identification was correct. Those galaxies designated "Anon" were in most cases given arbitrary numbers in an observing list of galaxies for which it was judged that spectrograms, suitable for redshift measurements, could be obtained with exposures of $1\frac{1}{2}$ to 3 hours with the Crossley nebular spectrograph. In all cases Kodak IIa0 emulsion was used, and the slit width was 0.1mm, or about 4 seconds of arc. Except when the slit was placed across a close pair, it was oriented east-west in position angle, and was kept fixed on the objects. The positions in the second and third columns resulted from measurement of a 48-inch Palomar schmidt plate of the Coma Cluster, kindly provided by Dr. F. Zwicky. In the reductions the catalogued positions of the brighter objects were used to obtain the scale and orientation factors; for some catalogued galaxies the listed quantities represent corrected, or slightly adjusted values. The fourth column gives the radial distance for a center referred to the midpoint of the line joining the two brightest members, NGC 4874 and 4889. The measurements were made by using dividers on a negative paper print, obtained from a film positive contact-exposed through the glass negative. Although these distances are given to 0$'$.1, the larger values may be systematically in error by greater amounts, since only a linear scale

TABLE II

Observations Available for Galaxies in Coma Cluster

Redshifts are corrected for galactic rotation by 300 cos(l − 55°)cos b

Magnitudes are not corrected for latitude effect or K-correction

Identification	$\alpha_{1950.0}$	$\delta_{1950.0}$	φ Radial Distance	ρ Corrected Redshift: Mayall	ρ Corrected Redshift: Humason	A (parsecs)	Magnitude: m_{pg} or P	Magnitude: Source	$(P - V)$ Color
NGC 4670	12h42m.6	+27°24′	200′.5	+1216		−16960	13.3	[18]	+0.32
NGC 4673	42.9	27 20	197.7	+6997		− 8724	14.2	[18]	0.88
Anon 1	44.2	26 50	192.0	+ 885		+ 3420	15.3:	[18]	0.57:
Anon 2	45.1	27 15	172.4	+7124		− 1637	14.9	[18]	0.92
NGC 4692	45.4	27 30	165.0	+7917		− 7867	14.3	[18]	0.90
Anon 3	45.8	27 08	166.5	+6923		+ 427	15.1	[18]	0.87
Anon 4	47.2	27 10	149.9	+7435		− 2711	14.8	[18]	0.86
NGC 4725	47.6	25 46	194.8		+1108	−49479	10.1	[20]	
NGC 4728	48.0	27 42	128.7	+6528		+ 479	15.1:	[18]	0.78:
NGC 4789	51.9	27 21	91.3	+8375		−11017	14.0	[18]	0.95
NGC 4793	52.4	29 13	89.7	+2544		−20277	13.0	[18]	0.48
NGC 4798	52.5	27 41	73.4		+7679	− 5821	14.5	[18]	0.92
IC 3900	53.2	27 32	70.3		+7177	+ 366	15.1	[18]	0.88
NGC 4819	54.0	27 15	74.2	+6702		− 6685	14.4	[18]	0.86
NGC 4821	54.1	27 14	75.6	+6980		+ 1416	15.2	[18]	0.78
NGC 4827	54.3	27 27	63.0	+7656		− 5819	14.5	[18]	0.90
NGC 4842	54.8	27 46	43.4	+7518		+ 2346	(15.3)	[18]	(0.87)
NGC 4839	55.0	27 46	43.0	+7450		− 8844	14.2	[18]	0.92
NGC 4848	55.7	28 31	28.3	+7224		− 4737	14.6:	[18]	0.50:
NGC 4850	55.8	28 14	19.6		+5996	+ 3603	15.4	[13]	
NGC 4849	55.9	26 40	97.0	+5829		− 5525	(14.5)	[18]	(0.93)
NGC 4853	56.2	27 52	27.8		+7561	− 5808	14.5	[18]	0.66
IC 3946	56.4	28 05	16.6		+6107	+ 2567	15.3	[13]	
IC 3949	56.4	28 06	14.5	+7538		− 714	15.0	[18]	0.65
NGC 4860	56.7	28 23	13.9	+7887	+7870	− 768	15.0	[13]	
vF Anon nr. IC 3960	56.7	28 08	10.9		+6880	+23828	17.4	[20]	
NGC 4865	56.8	28 21	9.7		+4655	− 3315	14.7	[13]	
NGC 4869	56.8	28 11	5.8		+6715	− 4657	(14.6)	[18]	(0.90)

TABLE II (Continued)

Identification	$\alpha_{1950.0}$	$\delta_{1950.0}$	φ Radial Distance	ρ Corrected Redshift: Mayall	ρ Corrected Redshift: Humason	A (parsecs)	Magnitude: m_{pg} or P	Magnitude: Source	$(P - V)$ Color
NGC 4864	56.9	28 14	8.3		+6831	− 3657	(14.7)	[18]	(0.88)
NGC 4867	56.9	28 16	7.7		+4827	+ 6842	15.7	[13]	
NGC 4872	57.2	28 14	3.8		+6923	+ 3456	15.4	[13]	
NGC 4874	57.2	28 14	3.2		+7183	−10841	(14.0)	[18]	(0.93)
NGC 4881	57.6	28 31	16.6		+6703	− 2618	14.8	[13]	
NGC 4886	57.7	28 15	3.3		+6226	+ 1530	15.2	[13]	
NGC 4884 = 4889	57.7	28 15	4.0	+6597	+6428	−14808	(13.6)	[18]	(0.95)
IC 4021	57.8	28 19	6.8		+5804	+ 7706	15.8	[13]	
Anon 2′.7 SW NGC 4895	57.8	28 26	12.8	+6774:		+10608	16.1:	[18]	0.58:
NGC 4895	57.9	28 28	15.3		+8422	− 7981	14.3	[13]	
NGC 4898	57.9	28 13	6.1		+6950	− 3676	14.7	[21]	
NGC 4896	58.2	28 37	24.3		+5836	+ 573	15.1	[13]	
IC 4040	58.3	28 20	11.6	+7527		+ 5397	15.6:	[18]	0.37:
IC 4045	58.4	28 22	14.7		+6542	+ 3516	15.4	[13]	
NGC 4907	58.5	28 26	17.2	+5883		− 3507	14.7	[13]	
NGC 4908	58.5	28 18	14.2		+8853	+ 97	15.1	[13]	
IC 4051	58.6	28 17	14.4		+4947	− 2341	14.8	[13]	
NGC 4911	58.6	28 03	17.0	+8021		−15046	13.6	[13]	
NGC 4921	59.1	28 09	21.8	+5474		−14643	13.6	[13]	
NGC 4926	59.5	27 53	34.3	+7676:		− 9899	14.1	[21]	
Anon 3′.4 NE NGC 4926	12 59.7	27 54	36.0	+7186:		+ 3416	15.4	[18]	0.55
NGC 4952	13 02.7	+29 24	96.9	+5886		− 8587	14.2	[18]	0.92
NGC 4676	12h44m.5	+31°00′	244.1	+6564 (from [22])					
NGC 4858	56.7	28 22	13.8	+9398					
NGC 4922	12 59.1	29 35	83.4	+7372					
NGC 4944	13 01.5	28 28	54.4	+7013					
NGC 4961	13 03.5	28 00	80.4	+2579					

factor was used. The new redshifts in the fifth column were obtained in the same way as those reported in [13], and they may be regarded as of comparable accuracy; a colon (:) indicates either a weak plate or lines of poor visibility for measurement. The new magnitudes in the eighth column, identified by [18] in the ninth, and the new colors in the last column resulted from two-color observations, with a 1P21 photomultiplier, filters that define the P, V system, and an aperture of 44 seconds of arc. This size was large enough to include nearly all the light for the majority of the newly observed galaxies; magnitudes in parentheses represent results from single-night observations. The value of A given in the seventh column were computed from formula (30) or (35).

Among the new redshifts there are four that seem to be "abnormally" low: NGC 4670, Anon 1, NGC 4725 and 4793. Of these, NGC 4725 is by far the largest and brightest, and in fact is $2\frac{1}{2}$ to 3 magnitudes brighter than the two brightest galaxies used to define the cluster center. It is also more than 3° distant from the center of the cluster. Although the other three galaxies have apparent magnitudes within the range of all the others, their colors of +0.32, +0.57: and +0.48 mag are among the bluest of those measured in this region. Moreover, the bluest, NGC 4670, has a strong emission spectrum and is of irregular and mottled appearance, while both Anon 1 and NGC 4793 also have emission 3727 [OII], together with an early-type continuum in the UV. These characteristics all suggest dwarf subluminous systems. Thus these four galaxies of notably smaller redshift could reasonably be considered as foreground objects. On the other hand, these newer data do not suggest the presence of any background objects: none of those observed have redshifts of 9500 km/sec or larger. This result is qualitatively to be expected, since in a given solid angle the probability of encountering the more numerous nearby dwarf galaxies is greater than that of finding the more infrequent giant systems.

The first panel in figure 3 gives the scatter diagram of 50 points exhibiting the relationships between the radial velocity and the apparent magnitude. It will be seen that 46 objects provide a more or less cohesive group of points in the scatter diagram, while the remaining four represent "outliers," already mentioned at the beginning of the paper. Whether these outliers should be treated as members of the Coma Cluster or not is a subjective matter and, in order to allow the reader to follow his own intuition, the kinematic test for stability of the Coma Cluster has been applied twice, once including all 50 objects and next with the omission of the outliers.

Using data of table II and formulas (41), we compute the sample variance of the apparent magnitude, not corrected for redshift, its covariance with the radial velocity and the sample variance of the latter. Then formulas (43) give the sample covariance and variance of the apparent magnitude corrected for redshift. The results are given in table III.

The sample variances of the radial velocities represent squares of 1745 and 932 km/sec, respectively. It should be noted that they bracket the value of 1000 km/sec estimated by Oort [10] using 23 galaxies in the central region of

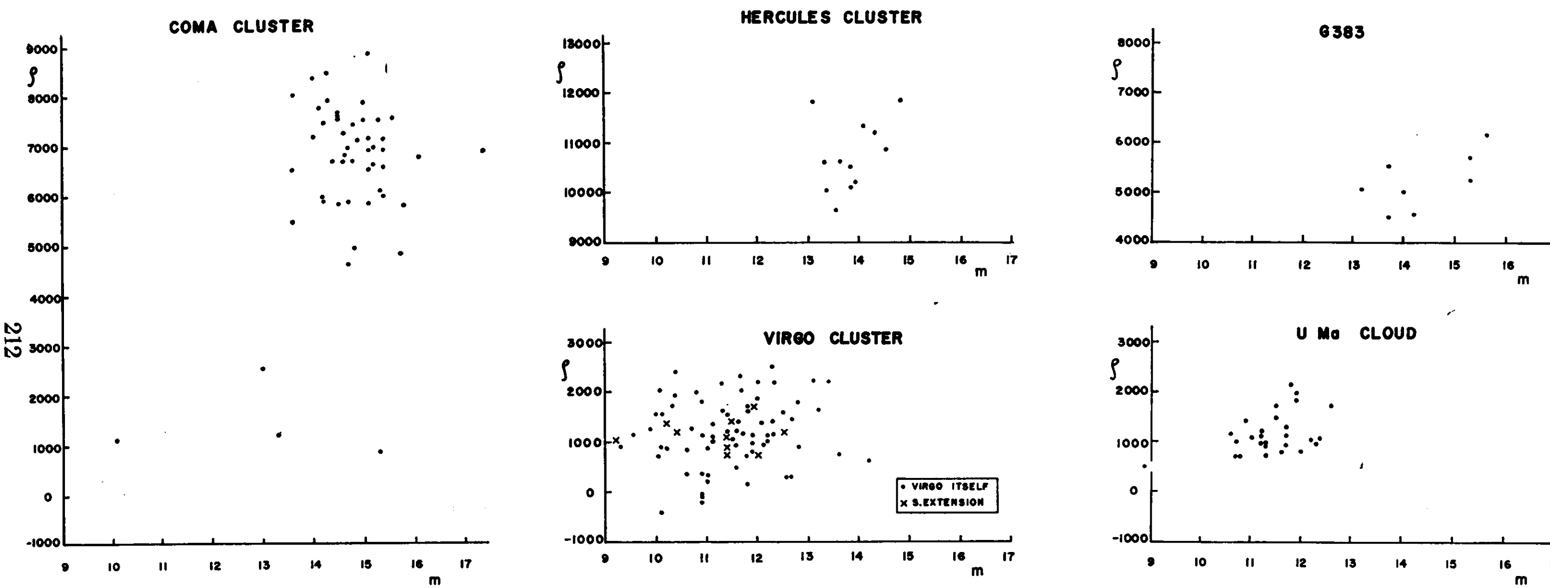

FIGURE 3

Relation between ρ_{ij} and m_{ij} for selected systems.

TABLE III

COMA CLUSTER

VARIANCES AND COVARIANCES OF RADIAL VELOCITY AND APPARENT MAGNITUDE

	n	$\rho.$	$S^2(m)$	$S(m, \rho)$	$S^2(\rho)$	$\sigma(m, \rho)$	$\sigma^2(m)$
With outliers	50	6432	1.000	682.27	3,044,423	636.61	0.980
Without outliers	46	6866	0.481	−102.07	868,448	−115.10	0.484

the cluster. Also, working backwards from the assumption of stability, Oort found the value 583 km/sec.

In order to proceed further, we must now adopt some "presumed" values for the three parameters σ, σ_M, and τ for which no independent estimates are attempted in the present study. The presumed value of σ used below is $\bar{\sigma} = 6.866 \times 10^5/h$. It was obtained by studying Shane's counts of presumed members of the Coma Cluster made in rings about the apparent cluster center. The distribution of these counts represents, approximately, the distribution of sin φ, with $\sigma_\varphi = \sigma/D$. The value of σ_φ found is of the order of 0.01. Using the average radial velocity of presumed members of Coma Cluster (without outliers) given in table III and the estimate of distance $D = 10^4\rho./h$, and solving for σ the above value for $\bar{\sigma}$ is obtained. Subsequent studies of Shane [19] indicate that $\bar{\sigma} = 6.9 \times 10^5/h$ is reasonably consistent with his counts relating to other clusters.

The adopted "presumed" value of σ_M is $\bar{\sigma}_M = 1.25$ mag, which is a compromise of the various current estimates. The "presumed" value of τ used is $\bar{\tau} = 0$.

Using these values further computations yield the results exhibited in table IV.

TABLE IV

KINEMATIC TEST FOR STABILITY OF THE COMA CLUSTER

	Conditional redshift dispersion σ_1 (km/sec)	Expansion coefficient $\hat{\theta}$(km/sec/pc)	S.E. of exp. coefficient $\hat{\sigma}_{\hat{\theta}}$	Test criterion $T = \hat{\theta}/\hat{\sigma}_{\hat{\theta}}$	$P\{T\}$
With outliers	1639	+0.0636 h	0.0232 h	+2.740	0.009
Without outliers	927	−0.0250 h	0.0208 h	−1.201	0.236

The last column of table IV gives the probability that, under conditions of stability $\theta = 0$ and due to chance alone, the absolute value of the criterion T will equal or exceed the value actually obtained for the Coma Cluster, as given in the preceding column. Also, this is the upper bound of the probability of being wrong if we make a rule of rejecting the hypothesis tested in conditions similar to those referring to the Coma Cluster. It is seen that if the four outliers are bona fide members of the Coma Cluster, the risk in asserting that the cluster is unstable is very small. The positive sign of the estimate of θ indicates expansion. However, if the outliers are excluded, which the present authors think appropriate, then the picture changes entirely. First, the upper bound of the probability of error in asserting instability increases to almost one in four. Second, the estimate of θ obtained with outliers omitted is negative and indicates contraction.

In judging these results it must be borne in mind that the limit (the so-called level of significance α) between "small" and "large" probabilities of error is conventional; it is a subjective matter whether to risk an assertion when the probability of error may be as large as 24 per cent. The second relevant consideration is that, if the level of significance is set low, the chance of detecting an effect, when it exists and is reasonably large (that is, the power of the test) is small. In cases where observational data are not very difficult to obtain, the usual values of the level of significance are 0.05 and 0.01. With the high variability of the radial velocity, in order to have a reasonable chance of detecting expansion working with $\alpha = 0.05$, or even $\alpha = 0.10$, the necessary amount of data would be prohibitive (see table X). For these reasons, the present authors are inclined to adopt a much more liberal level of significance and to consider that values $P\{T\} \leqq 0.24$ are a slight indication of instability to be taken into account when considering other evidence. However, one should not be very much surprised if the present indication of contraction of the Coma Cluster is contradicted by further studies. Our final conclusion is that, if the four outliers are members of the Coma Cluster, this cluster must be expanding. If not, then there is slight evidence of contraction.

With a casual approach to the problem of stability or instability of systems of galaxies, one is inclined to treat the phenomenon in its extreme simplicity: a cluster or a group may be stable or, if not, it may either expand as a whole or contract as a whole. However, the phenomenon of instability of systems of galaxies need not be simple. In fact, as suggested by Oort [10] and subsequently reaffirmed by van Albada [14], in certain phases of evolution of a cluster its inner parts may be contracting and its outer parts may be expanding. It is, then, not impossible that the "outliers" are actual members of the Coma Cluster but, being located in its outer regions, are in the process of "evaporating." In this case the quantities $\hat{\theta}$ obtained from the data are estimates of *average rates* of instability, one relating to the cluster including the outer regions, and the other restricted to the inner core. On this particular assumption, the velocities of "evaporation" of cluster members in the outer regions must be substantially higher than might be computed using the estimate given in the table.

With observations of only 50 galaxies available, any subdivision of the data according to the value of φ will give shaky results. Nevertheless, it seemed interesting to carry out the test when the observations were divided into three rings: (i) $\varphi < 0^\circ\!.2$ with 12 galaxies, (ii) $0^\circ\!.2 < \varphi < 0^\circ\!.5$ with 14 galaxies, and (iii) $\varphi > 0^\circ\!.5$ with the remaining 24 galaxies. We find that for the two inner rings there is no evidence of instability. This is not surprising with the small value of n. On the other hand, the results for the outermost ring almost duplicate those for the entire cluster, with strong evidence for expansion or mild for contraction according to whether the outliers are included or not.

The above discussion emphasizes the difficulties which are connected with efforts to study individual systems of galaxies when a substantial effort is made to collect data relating to members of a particular system. Here a subjective judgment is unavoidable and may easily lead to spurious conclusions. The alternative approach is to give up entirely the idea of identifying members of particular systems and to conduct the study on data for all galaxies in a given region of the sky selected on some well defined basis, for example, on the basis of estimated ease in measuring radial velocity. Unfortunately, studies of this kind require a much more complicated theory with more parameters to estimate and, therefore, considerably greater volume of observational data.

The actual values of the two estimates $\hat{\theta}$ given in table IV indicate that the presumed values $\bar{\sigma}$, $\bar{\sigma}_M$ and $\bar{\tau} = 0$ are wrong causing the coefficient of θ in formula (60) to be much greater than unity. In fact, if the actual value of $|\theta|$ were of the order of 0.06, then the "velocity of instability," that is, of expansion or of contraction, of a galaxy at a typical distance from the center of the cluster equal to $\sigma = 6.9 \times 10^5$ pc would have been of the order of 45,000 km/sec, which is fantastic. Although the subject of the present paper is limited to testing the hypothesis that $\theta = 0$ and the problem of estimating the value of θ is not our immediate concern, it is of some interest to consider the effect on the estimate of θ of some plausible errors in the presumed $\bar{\sigma}$, $\bar{\sigma}_M$ and $\bar{\tau}$.

The adopted $\bar{\sigma}$ is very poorly determined and we should not be surprised to find that the actual $\sigma = 2\bar{\sigma}$. Similarly, as indicated in [16], σ_M may well be close to unity. Finally, the studies of Zwicky suggest that τ may be of the order of 10^{-7} mag/pc. Taking $\tau = 2 \times 10^{-7}$ mag/pc we find

$$(61) \qquad \frac{\sigma^2}{\bar{\sigma}^2} \frac{\bar{\sigma}_M^2 + \bar{\lambda}^2\bar{\sigma}^2}{\bar{\sigma}_M^2 + \lambda^2\sigma^2}\left(1 + \tau \frac{D}{a}\right) = 4\,\frac{1.56}{1.10}\left(1 + \frac{13.7}{2.17}\right) = 41.5.$$

The combination of the above errors would then reduce the estimates of θ by a factor of about 40, yielding a value only about 15 times larger than the Hubble constant. At the distance σ from the center of the cluster this would imply the velocity of instability of the order of 1000 km/sec, still very large but not as exorbitant as on the original presumed values of the parameters.

The necessity of revising the originally adopted presumed values of $\bar{\sigma}$, $\bar{\sigma}_M$ and $\bar{\tau} = 0$ is also indicated by the study of the power of the T-test. Table V gives the approximate value of the probability that the T-test applied to the data for the Coma Cluster, without the outliers, will indicate significance at 24 per cent (the observed value of the probability as shown in table IV), computed on several hypotheses regarding the intervening parameters.

It is seen that, if the presumed values $\bar{\sigma}_M$ and $\bar{\sigma}$ were entirely correct and the expansion coefficient θ coincided with the Hubble constant, then, even with

TABLE V

ASYMPTOTIC POWER OF T-TEST APPLIED TO DATA ON COMA CLUSTER, WITHOUT OUTLIERS

In all cases it is assumed $\sigma_M = \bar{\sigma}_M$

Hypothetical $\lvert\theta\rvert$ and σ	Hypothetical τ $\tau = 0$	$\tau = 10^{-7}$	$\tau = 2 \times 10^{-7}$
$\lvert\theta\rvert = H$, $\sigma = \bar{\sigma}$	0.24	0.24	0.24
$\lvert\theta\rvert = 10\,H$, $\sigma = 2\bar{\sigma}$	0.25	0.42	0.68

$\tau = 2 \times 10^{-7}$, the chance of obtaining the observed T would be as small as with $\theta = 0$. The assumption that $|\theta|$ equals ten times the Hubble constant, combined with the assumption that the actual value of σ is twice the "presumed" and with the assumption $\tau = 0$, does not help much. It is only when we assume that $\tau = 10^{-7}$ combined with $|\theta| = 10\,H$ and $\sigma = 2\bar{\sigma}$ that we reach nonnegligible chance that the test will detect instability. Since, in fact, indications of instability have been detected, our conclusion is that, unless the data available are dominated by effects of selection distorting the distribution of the radial velocity, the figures in table IV are indicative not only of instability but also of the presence of a substantial amount of absorbing matter within the Coma Cluster.

8. Joint study of seven clusters and fifteen groups of galaxies

In the present section we give the results of the application of the kinematic test for stability to all clusters and all groups of galaxies for which we could secure unambiguous data on apparent magnitudes and radial velocity for at least three member objects. Most of the data come from the fundamental paper by Humason, Mayall and Sandage [13]. Here we believe to have exhausted the information given in their table II "Redshift from Bright Nebulae in Clusters" for which we could find apparent magnitudes in table A1. We had somewhat less luck with the data of table III "Redshifts from Faint Nebulae in Clusters," where quite a few clusters had to be omitted. One example is the Hydra Cluster for which three radial velocities are given. However, one of these radial velocities is from a blend of spectra of two different galaxies and it was judged appropriate to discard it. Also in a number of cases, we found it difficult to match the radial velocities with apparent magnitudes. As a result of all such difficulties, out of the 18 clusters treated in [13] we could use only six. The radial velocities for the seventh cluster "around NGC 6166" were communicated to us by Dr. R. Minkowski and the apparent magnitudes by Dr. G. O. Abell, in both cases in advance of publication, and it is a pleasure to record here our indebtedness. Also, Dr. Abell was kind enough to supply us with unpublished apparent magnitudes of several objects in the Hercules and in the Corona Borealis Clusters. Finally, we wish to thank Dr. E. M. Burbidge for her kind help in matching the apparent magnitudes and the radial velocities for several objects in the Hercules Cluster.

Most of the data for groups of galaxies stem from table XI "Data for Representative Groups of Nebulae" [13], a most convenient table giving side by side the radial velocities and the apparent magnitudes for each galaxy. These data were supplemented by those for Messier 81 and Messier 101 as published by Holmberg [23]. The five panels of figure 3 give the scatter diagrams of radial velocities and apparent magnitudes for Coma, Virgo, Ursa Major Cloud, Hercules, and G383. It will be noticed that the Ursa Major Cloud has one "outlier." While the radial velocity of the particular object is comparable to that of other presumed members of this group, its apparent magnitude is much brighter.

The basic statistics are given separately for clusters and for groups in tables VI and VII, respectively. In each case, the systems of galaxies are arranged in

TABLE VI

Basic Statistics for Seven Clusters of Galaxies

Magnitudes corrected for latitude effect but not for K-correction

Cluster	n	ρ. km/sec	m.	$S(m)$	$S(m, \rho)$	$S(\rho)$
Virgo whole	80	1197	11.4	1.00	92.4	641
Virgo without S.E.	70	1207	11.5	1.01	102.9	676
Perseus	4	5516	13.2	0.79	109.8	520
Coma whole	50	6432	14.7	1.00	682.3	1745
Coma without outliers	46	6866	14.8	0.69	−102.1	932
Around NGC 6166	19	9028	16.3	0.42	93.7	864
Hercules	15	10776	15.2	0.52	40.2	652
Pegasus II	3	12821	15.0	0.40	−103.0	662
Cor. Bor.	8	21651	17.4	0.32	− 84.2	1294

the order of increasing mean corrected radial velocities. The first column gives the number n of objects for which the necessary data are available and here we must express our regret that for many systems this number is pitifully small. The next two columns give the mean radial velocity and the mean apparent magnitude exhibiting the familiar redshift-magnitude relation. The next column gives the sample dispersion of the apparent magnitude not corrected for redshift. This column is interesting because of the marked negative correlation between redshift and $S(m)$ which clearly indicates the effect of selection of objects bright enough for convenient measurement of radial velocities. As mentioned earlier, this selection does not invalidate the kinematic tests for stability. However, as was pointed out before, the smaller the variability of the apparent magnitude of objects of a given system, the smaller is the contribution of this system to the power of the tests. The next column, of sample covariances of the radial velocity with the apparent magnitude (not corrected for redshift), shows a regrettable

TABLE VII

BASIC STATISTICS FOR FIFTEEN GROUPS OF GALAXIES
Magnitudes corrected for latitude effect but not for K-correction

Group	n	ρ. km/sec	m.	$S(m)$	$S(m, \rho)$	$S(\rho)$
M81	7	172	10.0	1.39	− 14.0	132
M101	6	478	10.5	1.59	− 29.5	102
G1023	5	513	10.5	0.87	− 51.1	257
Leo Group	18	788	11.1	0.98	36.9	264
Ursa Major Cloud	27	1181	11.4	0.74	146.4	418
U.M. without outliers	26	1208	11.5	0.56	81.1	403
G3190	3	1222	12.2	0.58	− 40.8	70
G1068	3	1604	10.8	1.05	472.1	453
Fornax	3	1631	11.0	0.10	− 20.2	299
G5846	8	1808	12.2	0.87	69.2	384
G7619	6	3836	13.3	0.98	102.8	285
G6928	3	4643	13.8	1.02	−153.7	308
G507	3	4665	13.1	0.74	−265.7	416
G383	8	5264	14.4	0.90	334.9	570
Stephan's Quintet	4	6735	14.2	0.68	− 50.3	470
G68	5	6785	14.9	0.66	117.5	458

lack of regularity including the signs. On the other hand, some regularity is noticeable in table VII in the last column of dispersion in redshift for groups of galaxies. Because of the very unequal and occasionally very small number of observations on which these dispersions are based, in order to see the tendency more clearly, it is advisable to combine several groups, say at intervals of 500 km/sec of their radial velocities, and to compute a "pooled" measure of dispersion for each combination. If one does this, one obtains an unmistakable picture of a systematic increase of dispersion roughly up to the mean radial velocity of 1000 km/sec. Since the validity of the tests introduced here is most sensitive to distortions of the conditional distribution of the radial velocity, the tendency indicated deserves attention, particularly because of the following comments by Humason taken from [13] page 101: "Among the nebulae designated 'anonymous' are several originally observed as possible members of clusters, but later found to be field nebulae with redshifts not in agreement with those obtained from cluster members." In connection with relatively large values of sample dispersion of magnitudes and relatively small dispersions of radial velocities indicated for the four presumably nearest groups, one is led to suspect that the assignment of objects to these groups may have been affected by considerations of similarity of redshift, so that the distribution of this characteristic in these groups became distorted. If this actually happened, then the application of the test would reflect not so much the actual state of affairs within the groups concerned as the arbitrary distortions made in the process of collecting the data. The five groups for which there is the suspicion of such distortions, because of the excessively small dispersion of redshift, are M81, M101, G1023, Leo Group and G3190.

Because of these considerations, the summary treatment of groups of galaxies has been performed twice, once taking into account all the fifteen groups for which the data are available, and next for the ten groups for which no indication of selection for redshift is noticeable.

A glance at the last columns of tables VI and VII is sufficient to indicate the desirability of treating the clusters and the groups of galaxies separately; there is little doubt that the variability of the radial velocity in the systems of these two broad classes is not the same.

Out of all the clusters and groups, the number of observations available justifies individual treatment only for Coma, given above, and for the Virgo Cluster. Table VIII, analogous to table IV, gives the results, separately for the

TABLE VIII

KINEMATIC TEST FOR STABILITY OF THE VIRGO CLUSTER

	Conditional redshift dispersion σ_1(km/sec)	Expansion coefficient $\hat{\theta}$(km/sec/pc)	S.E. of exp. coefficient $\hat{\sigma}_\theta$	Test criterion $T = \hat{\theta}/\sigma_{\hat{\theta}}$	$P\{T\}$
Whole cluster	639	0.00156 h	0.00131 h	1.193	0.237
Without S.E.	674	0.00174 h	0.00148 h	1.172	0.246

whole cluster and separately for its main part, excluding the Southern Extension.

The sign of the expansion coefficient is positive for both the cluster as a whole and for its part after the omission of the Southern Extension. Thus, if there is instability, we would expect expansion. If we admit a priori that the cluster may be expanding or may be contracting, the risk of asserting instability of the Virgo Cluster is just under one in four. On the other hand, if a priori we exclude the possibility of contraction, then the risk is halved. The indication of instability of the Virgo Cluster equals in intensity that of the Coma Cluster without outliers. However, in one case the indicated instability is expansion and in the other contraction. In both cases the conclusion depends on the hypothesis that the collection of data did not distort the conditional distribution of radial velocity.

Remark: out of the 80 objects available for the Virgo Cluster ten objects are within the "Southern Extension" of the cluster. It seemed advisable to perform the analysis twice, first treating all the 80 objects as one cluster and then restricting the calculations to those 70 objects not in the Southern Extension. However, the inclusion or the exclusion of the Southern Extension does not make much difference. This is seen in tables VI and VIII and the agreement persists in further calculations. For this reason, in describing further results, only the figures relating to the Virgo Cluster as a whole are given.

The joint treatment of all clusters together and then of all groups together is based on the presumption that there is a certain amount of similarity in the systems combined into a single class. The particular aspect of similarity is the equality of the residual variance of the radial velocity. Taking the available data, so to speak at their face value, this presumption is far from confirmed. In fact, for example, the data for the Coma and for the Virgo Clusters taken as a whole, or with the indicated omissions, indicate that members of the Coma Cluster have a greater dispersion of radial velocity than those of the Virgo Cluster. The motivation for the joint treatment of all systems belonging to each of the two classes is the suspicion that each of the systems must have been subjected to some selection and that probably the intensity of this selection varied from case to case. In these circumstances it appears interesting to see the results, so to speak, averaged over the whole class of systems of galaxies. In addition, the smallness of the number of observations for particular groups makes any analysis other than by a summary test a rather hopeless undertaking.

Table IX gives pooled estimates of the conditional standard deviation σ_1 of

TABLE IX

POOLED ESTIMATES OF CONDITIONAL DISPERSION OF RADIAL VELOCITIES

Category of Systems of Galaxies	Rate of Instability Assumed							
	Common to all systems				Possibly Different in each			
Outliers	included		excluded		included		excluded	
	σ_1	d.f.	σ_1	d.f.	σ_1	d.f.	σ_1	d.f.
7 clusters	1114.5	171	788.6	167	1078.7	165	791.4	161
15 groups	366.2	93	361.2	92	346.4	79	348.6	78
10 groups	—	—	403.5	58	—	—	401.7	49

the radial velocity calculated separately for clusters and groups, with and without outliers and under the two different assumptions, one that the rate of expansion in all systems of a given category is the same and the other that each system of galaxies may expand or contract at its own rate.

It is seen that, while the omission of the four outlier galaxies in the Coma Cluster considerably influences the pooled estimates of the conditional dispersion in all clusters, the removal of the single outlier galaxy in the Ursa Major Cloud produces hardly any difference in the pooled estimate of σ_1. On the other hand, of course, the elimination of five groups suspected of being affected by selection for similarity in radial velocities, marked with unusually small sample dispersions in redshift, substantially increases the pooled estimates of conditional dispersion in ρ.

The computations also were performed on all groups except the three with very small values of $S(\rho)$, the dispersion in radial velocity. The results are always intermediate between those for all 15 groups and for 10 groups. They are not shown in table IX nor in any later table to save space.

Tables X and XI give the details for the application of the summary T- and Z-tests, separately for clusters and for groups. The conclusions suggested by table X are as follows:

(i) If we take it for granted that the phenomenon of instability is the same for all clusters and is measured by a common value of the expansion coefficient θ, then there is a slight indication of expansion if outliers are included, but none when they are excluded.

(ii) If we admit the possibility that particular clusters may be unstable, each with a different value of θ, then, with outliers included, there is strong evidence that at least some of them (and this would be Coma) are unstable. In this case, the chance of error in asserting instability of at least some of the clusters is less than one in a hundred. If the outliers are excluded, no such evidence exists; the chance of error in asserting instability may be as large as three in four.

(iii) Naturally, lack of clear-cut evidence of instability, does not amount to evidence of stability. The power of the test, as shown in the last columns, is low when $\theta = H$ even with $\alpha = 0.10$. In this connection our attention is attracted by a substantial correlation between the mean radial velocity $\rho_.$ and the absolute value of the estimate $\hat{\theta}$ of the expansion factor. With reference to formula (60) this correlation suggests that some clusters may be expanding, $\theta_i > 0$, and some others contracting, $\theta_i < 0$, but all contain absorbing matter explaining the correlation observed.

Asterisks in table XI mark the five groups of galaxies for which table VII indicated the suspicion that objects assigned to them were those with similar radial velocities, so that the variability of ρ was artificially decreased. It is for this reason that the computations were performed more than once, first for all groups, next for the ten groups not marked with the asterisks, and then for the twelve groups for which the variance in ρ was not too extreme (results not shown).

Among the group there is just one, the Ursa Major Cloud, with any "outlier" member. The figures in table XI refer to calculations made with the omission of the outlier.

The conclusions suggested by table XI may be summarized as follows.

(i) As far as the Z-test is concerned, its results are very much independent of whether we include all 15 groups or only 10. In both cases, the probability of error in asserting instability may be as high as one in six. However, it will be remembered that with a substantial number of groups included and with a very small number of observations per group, the power of the Z-test is low.

(ii) If we adopt the a priori attitude that for all the groups treated the instability parameter θ must have the same value, then the conclusions suggested by the T-test differ sharply on whether it is applied to all the 15 groups or only to those without the asterisk. In the first case, there is no evidence of instability.

TABLE X

KINEMATIC SUMMARY TESTS FOR STABILITY OF SEVEN CLUSTERS OF GALAXIES

Cluster	ρ km/sec	$\hat{\theta}/h$ km/sec/pc	Estimate of S.E. of $\hat{\theta}$ based on		T_i based on		$P\{T_i\}$ based on		Power ($\alpha = 0.10$) without outliers			
									Against $\theta = H$		Against $\theta = 10\ H$	
			all objects	without outliers	all objects	without outliers	all objects	without outliers	$\tau = 0$	$\tau = 2 \times 10^{-7}$	$\tau = 0$	$\tau = 2 \times 10^{-7}$
Virgo	1197	0.00156	0.00221	0.00162	0.71	0.96	0.48	0.34	0.10	0.10	0.17	0.35
Perseus	5516	0.01450	0.06694	0.04911	0.22	0.30	0.83	0.77	0.10	0.10	0.10	0.10
Coma whole	6432	0.06346	0.01528	—	4.16	—	0.000	—	—	—	—	—
Coma without outliers	6866	−0.02501	—	0.01777	—	−1.41	—	0.16	0.10	0.10	0.10	0.13
Around NGC 6166	9028	0.06610	0.08467	0.06212	0.78	1.06	0.44	0.29	0.10	0.10	0.10	0.10
Hercules	10776	0.02035	0.09105	0.06680	0.22	0.30	0.82	0.76	0.10	0.10	0.10	0.10
Pegasus II	12821	−0.04481	0.21559	0.15817	−0.21	−0.28	0.84	0.78	0.10	0.10	0.10	0.10
Cor. Bor.	21651	−0.33978	0.41170	0.30204	−0.83	−1.12	0.41	0.26	0.10	0.10	0.10	0.10
All clusters *T*-test	with outliers	0.00289	0.00226	—	1.28	—	0.20	—	0.10	0.10	0.13	0.25
	without	0.00140	—	0.00161	—	0.87	—	0.39	0.10	0.10	0.17	0.37
All clusters *Z*-test	with outliers	—	—	—	2.75	—	0.010	—	—	—	—	—
	without	—	—	—	—	0.62	—	0.74	0.10	0.10	0.12	0.25

TABLE XI

KINEMATIC SUMMARY TESTS FOR STABILITY OF GROUPS OF GALAXIES
Asterisks indicate groups suspected of being affected by excessive selection

Group	ρ km/sec	$\hat{\theta}/h$ km/sec/pc	Estimate of S.E. of $\hat{\theta}$ based on 15 groups	Estimate of S.E. of $\hat{\theta}$ based on 10 groups	T_i based on 15 groups	T_i based on 10 groups	$P\{T_i\}$ based on 15 groups	$P\{T_i\}$ based on 10 groups	Power (α = 0.10) for 15 groups: Against $\theta = H$, $\tau=0$	Against $\theta = H$, $\tau=2\times10^{-7}$	Against $\theta = 10\,H$, $\tau=0$	Against $\theta = 10\,H$, $\tau=2\times10^{-7}$
*M81	172	−0.00002	0.00027	—	−0.07	—	0.94	—	0.12	0.13	0.98	0.99
*M101	478	−0.00009	0.00071	—	−0.12	—	0.90	—	0.10	0.11	0.40	0.60
*G1023	513	−0.00053	0.00156	—	−0.34	—	0.73	—	0.10	0.10	0.17	0.23
*Leo Group	788	0.00045	0.00104	—	0.43	—	0.67	—	0.10	0.10	0.25	0.48
Ursa Major	1208	0.00466	0.00231	0.00266	2.02	1.75	0.05	0.09	0.10	0.10	0.13	0.23
*G3190	1222	−0.00223	0.00786	—	−0.28	—	0.78	—	0.10	0.10	0.10	0.11
G1068	1604	0.01065	0.00581	0.00670	1.83	1.59	0.07	0.12	0.10	0.10	0.11	0.13
Fornax	1631	−0.05046	0.05946	0.06853	−0.85	−0.74	0.40	0.47	0.10	0.10	0.10	0.10
G5846	1808	0.00243	0.00417	0.00481	0.58	0.51	0.56	0.62	0.10	0.10	0.11	0.16
G7619	3836	0.00620	0.00932	0.01074	0.67	0.58	0.51	0.57	0.10	0.10	0.10	0.14
G6928	4643	−0.01054	0.01710	0.01971	−0.62	−0.53	0.54	0.60	0.10	0.10	0.10	0.12
G507	4665	−0.03464	0.02363	0.02723	−1.47	−1.27	0.15	0.21	0.10	0.10	0.10	0.11
G383	5264	0.03311	0.01183	0.01364	2.80	2.43	0.01	0.02	0.10	0.10	0.10	0.14
Stephan's Quintet	6735	−0.01173	0.03015	0.03475	−0.39	−0.34	0.70	0.74	0.10	0.10	0.10	0.11
G68	6785	−0.02731	0.02742	0.03160	1.00	0.86	0.32	0.39	0.10	0.10	0.10	0.11
Summary T	(15)	(10)			0.29	2.44	0.77	0.02	0.13	0.14	0.99	1.00
	0.00007	0.00514	0.00025	0.00211								
Summary Z					1.39	1.54	0.17	0.15	0.11	0.11	1.00	1.00

In the second, the probability of error in asserting instability is less than one in fifty. On the other hand, there is an obvious argument against giving credence to this result. By omitting from the data those items about which the investigator has an intuitive suspicion, one is very much at the mercy of subjective judgment.

(iii) The absolute values of the estimates $\hat{\theta}_i$ exhibited in table XI show the same kind of correlation with the mean radial velocity $\rho_.$ as in table X. This is an interesting phenomenon, suggesting both instability and the presence of absorbing matter within the systems of galaxies.

9. Summary and conclusions

(i) As understood in this paper, the stability of a system of galaxies, a cluster or a group, means, roughly speaking, that for any given distance from the center C of the system and for any given peculiar velocity v, the galaxies approach C with this velocity as frequently as they move away from it. This definition implies that for members of a stable system there shall be no correlation between the redshift and the apparent magnitude, corrected for redshift.

(ii) Two locally and asymptotically most powerful tests for stability of several systems of galaxies have been introduced, both subject to certain specific assumptions regarding the structure of the systems of galaxies and regarding the distribution of peculiar velocities. Also, both tests are optimal with regard to the alternative hypotheses that, if a system of galaxies is unstable, then in addition to random peculiar velocities conforming with the definition of stability, the members of the system have a nonrandom Hubble-type component, either away from the center or towards it, whose magnitude is proportional to the distance between the center of the system and the particular galaxy. The coefficient of proportionality θ is analogous to the Hubble constant.

The first test criterion T is optimal on the assumption that, if the systems studied are unstable, then they are characterized by the same value of θ. The second test criterion Z has optimal properties with regard to the assumption that the systems of galaxies studied may have different coefficients of instability θ. Because the two tests deal only with velocities and apparent magnitudes of the galaxies and are independent from dynamical considerations, they are termed kinematical tests.

(iii) The basic assumption conditioning the validity of the kinematic tests is that, while the objects for which the data are available may have been purposely selected for their brightness or position or some other characteristic, this selection did not affect the conditional distribution of the radial velocity.

(iv) The application of the two tests to empirical data regarding seven clusters and fifteen groups indicated several cases where the hypothesis of stability can be rejected with little risk of error. Unfortunately, conclusions of this kind depend heavily on the basic assumption (iii) and, more specifically, on the objects that one decides to consider members of a given system and on the systems that one is inclined to consider as satisfying the hypothesis (iii).

(v) An outstanding example of this kind is the Coma Cluster. If one accepts that all the 50 objects presumed to be members of the Coma Cluster are really its members, then there is little doubt that the cluster is expanding. On the other hand, if one excludes the "outliers" there appears to be some slight evidence of contraction. It is possible that the "outliers" are located in the outer regions of the cluster, which expands, while the inner part of the cluster contracts.

Similarly, treating jointly all the 15 groups of galaxies one sees no evidence of instability. On the other hand, if one removes from the analysis five groups for which there is some evidence of selection for similarity of redshift, there is a strong indication of rapid expansion.

(vi) The basic numerical work was conducted on the a priori assumption that the systems of galaxies are essentially free of absorbing matter. A study of the asymptotic power of the tests indicated that in the absence of absorption the tests would have little chance of detecting instability even if this were characterized by large values of θ. Since some indications of instability were obtained, this suggests not only that these systems are unstable but also that they contain absorbing matter. The suggestion that absorbing matter is present within cluster and groups of galaxies is reinforced by the positive correlation between the absolute values of the estimates of θ and the mean radial velocities of particular systems. This correlation is exhibited in figure 4. In the first panel of this figure each point corresponds to the actually computed $\rho_.$ and $|\hat{\theta}|$ for each of the seven clusters. In the second panel the exhibited crosses are "normal points" corresponding to groups classified in intervals of 1000 km/sec of their radial velocities. Here each ordinate represents the weighted mean of the particular values of $|\theta_i|$. The presence of a positive correlation is unmistakable, which leads us to the final conclusion.

(vii) With reference to equation (60) it is seen that, if either θ or τ were equal to zero, then there would be no correlation between $\hat{\theta}$ and $\rho_.$. Since τ cannot be negative, positive correlation is possible only when $\theta > 0$ and negative correlation only when $\theta < 0$. The values of θ_i obtained, some positive and some negative, suggest that some of the systems are expanding and some contracting so that θ might be considered as a random variable with some fixed mean, probably close to zero, and a fixed variance. On this assumption, there would be observed changes in signs of the estimates $\hat{\theta}_i$, an increase in the conditional dispersion, proportional to $(1 + \tau\ D/a)$, and a positive correlation between $|\theta_i|$ and $\rho_.$, everything exactly as in the empirical results and as reflected in figure 4. This particular hypothesis, of positive τ and a random θ, appears to be worthy of closer consideration. In studying it, the existence or nonexistence of a correlation between the color index of the galaxies and their radial velocities is likely to be decisive. For expanding systems this correlation should be positive and for contracting negative.

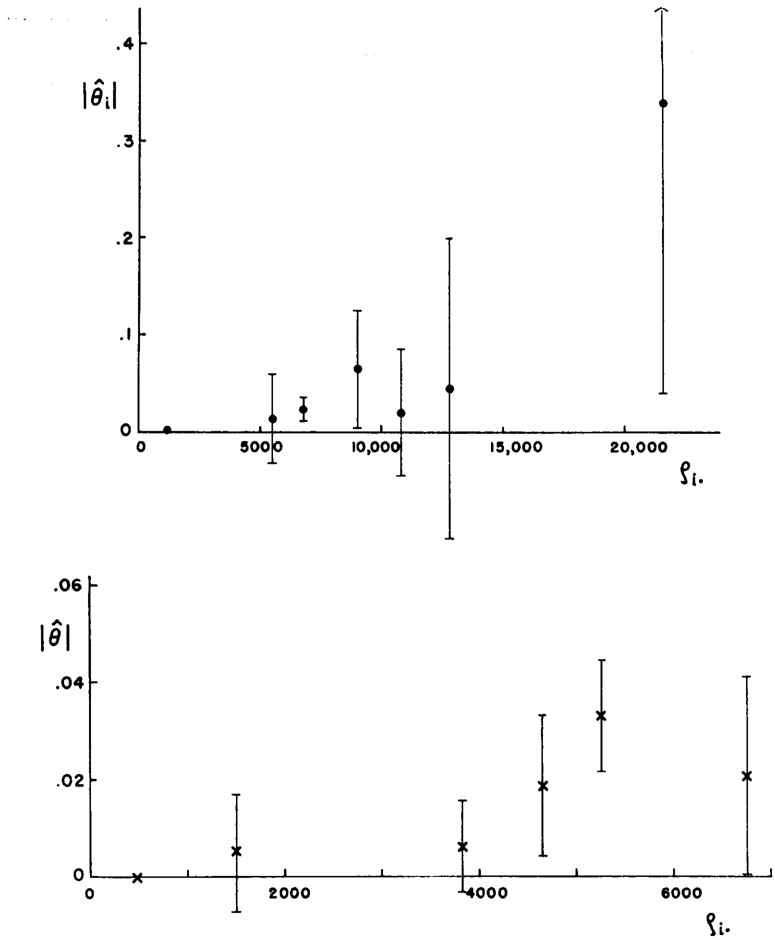

FIGURE 4

Correlation between the estimated absolute value of the expansion coefficient θ and the mean radial velocity of the system.
The vertical bar is of length $2\sigma_{\hat{\theta}}$.
The upper panel refers to clusters, the lower panel refers to groups (normal points).

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