SOME PROBLEMS IN THE THEORY OF COMETS, II

DAVID G. KENDALL
MAGDALEN COLLEGE, OXFORD UNIVERSITY

1. Let $y_1, y_2, \cdots$ be independent random variables having the distribution

$$g(y)\,dy, \qquad -\infty < y < \infty, \tag{1.1}$$

where $g(y)$ is an *even* function of y, and let R_m and S_m be defined for $m \geqq 1$ by

$$\begin{aligned} S_m &= y_1 + y_2 + \cdots + y_m, \\ R_m &= \min(0, S_1, S_2, \cdots, S_m); \end{aligned} \tag{1.2}$$

let k be a constant in the range $0 \leqq k < 1$. This paper is concerned with the function

$$C(z|x) = \sum_{m=1}^{\infty} (1-k)^m P\{x + R_m > 0, x + S_m \leqq z\}, \tag{1.3}$$

where $x > 0$ and $z \geqq 0$, which we shall study by the methods of Frank Spitzer [6], [7], [8]; the results will then be applied to an astronomical problem formulated in the first part [3] of this paper.

From theorem 4.1 of [6] (or from an earlier theorem of Paul Lévy) we know that lim sup $S_m = +\infty$ and that lim inf $S_m = -\infty$, with probability one, so that infinitely many terms of the sequence

$$x + S_1, x + S_2, \cdots \tag{1.4}$$

will be zero or negative. Let the first such nonpositive term and *all* succeeding terms (of either sign) be removed from (1.4). Let a biased coin show heads with probability k and tails with probability $(1 - k)$, and in an infinite sequence of independent throws (independent also of the y) let the first head occur at the Mth throw; we then remove the Mth and *all* subsequent terms from the sequence (1.4) (if they still survive). The quantity $C(z|x)$ defined at (1.3) above will then be the expected number of terms $x + S_m$ in the curtailed sequence which lie in the half-open interval $(0, z]$. It is not clear from this definition that $C(z|x)$ is finite, but this will be proved in due course.

In the astronomical problem $C(z|x)$ is the expected number of complete circuits described round the sun by a comet initially in the positive energy state x,

Formerly Skynner Student in Astronomy, Balliol College, Oxford.

when we count only those circuits during which the energy state lies in the interval $(0, z]$. Here "energy state" means the negative of the total energy per unit mass (and is zero for a comet at rest at infinity). Comets in zero or negative energy states are unbound and escape from the solar system on parabolic or hyperbolic orbits; this is why we reject $x + S_m$ and all subsequent terms if $x + S_m$ is zero or negative. The y represent the perturbations in energy state acquired by passage through the planetary zone, just before and just after perihelion, so that $x + S_m$ is the energy state of a comet which has survived m perturbations. At each perihelion passage we allow the possibility (with small probability k) that the comet will disintegrate; this is the reason for the second curtailment of the sequence (1.4). Finally $C(z|x)$ itself is needed to permit the calculation of the theoretical "z-spectrum" (the law governing the distribution of the sun's family of comets among the various energy states). Further details of all this will be found in the preceding part [3] of this paper.

2. It will be convenient to introduce auxiliary random variables A and B independent of one another and of all those previously mentioned and having the distributions

$$\begin{aligned} &e^{-\alpha A}\alpha\, dA, \qquad && 0 < A < \infty, \\ &e^{-\beta B}\beta\, dB, \qquad && 0 < B < \infty, \end{aligned} \tag{2.1}$$

respectively, so that we can employ the method of "collective marks" devised by the late David van Dantzig. Consider the random event

$$\{A + S_1 > 0, A + S_2 > 0, \cdots, A + S_m > 0, A + S_m \leqq B\}, \tag{2.2}$$

where at any point we are at liberty to weaken or sharpen the sign of inequality because of the absolute continuity of the A-, B-, and y-distributions (we shall frequently do so without further comment). One evaluation of its probability is

$$\int_0^\infty \int_0^\infty P\{x + R_m > 0, x + S_m \leqq z\}\, e^{-\alpha x - \beta z}\alpha\beta\, dx\, dz, \tag{2.3}$$

while another is

$$E\Big[\int_{A > -R_m} e^{-\alpha A - \beta(A + S_m)}\, \alpha\, dA\Big], \tag{2.4}$$

where $E(\cdot)$ denotes integration with respect to the probability measure for the y. On equating the two evaluations we obtain the identity

$$\begin{aligned} \int_0^\infty \int_0^\infty C(z|x) e^{-\alpha x - \beta z}\, dx\, dz \\ = \beta^{-1}(\alpha + \beta)^{-1} \sum_{m=1}^{\infty} (1 - k)^m\, E\{e^{(\alpha+\beta)R_m - \beta S_m}\} \end{aligned} \tag{2.5}$$

for all positive α and β.

We now appeal to a celebrated identity of Frank Spitzer ([6], theorem 6.1; for a simpler proof see J. G. Wendel [9]), which in the present circumstances assures us that

$$(2.6) \qquad 1 + \sum_{m=1}^{\infty} (1-k)^m E\{e^{(\alpha+\beta)R_m - \beta S_m}\} = \exp\left(\sum_{n=1}^{\infty} \frac{(1-k)^n}{2n} E\{e^{-\alpha|S_n|} + e^{-\beta|S_n|}\}\right),$$

when α, β, and k are restricted as before. (In the first instance we obtain (2.6) for $k > 0$. The monotone convergence theorem then shows that it also holds for $k = 0$.) Another result of Spitzer ([7], lemma 4) tells us that the series

$$(2.7) \qquad \sum_{n=1}^{\infty} n^{-1} E\{e^{-\lambda|S_n|}\}$$

is convergent for each positive λ, and so the right side of (2.6) is finite for all positive α and β, even when $k = 0$. On putting (2.5) and (2.6) together we find that

$$(2.8) \qquad 1 + \beta(\alpha+\beta) \int_0^{\infty} \int_0^{\infty} C(z|x) e^{-\alpha x - \beta z}\, dx\, dz = \exp\left(\sum_{n=1}^{\infty} \frac{(1-k)^n}{2n} E\{e^{-\alpha|S_n|} + e^{-\beta|S_n|}\}\right)$$

where the right side is finite.

We shall transform (2.8) into a more convenient form, but first we use it to prove a number of useful qualitative facts about $C(z|x)$. It is clear from (1.3) that $C(z|x)$ increases (in the weak sense) when z increases and x is fixed; it can also be seen that it increases in the weak sense when x increases and $z - x$ is fixed. Now suppose it possible that $C(z'|x') = \infty$; then it would follow that $C(z|x) = \infty$ whenever both $x > x'$ and $z > z' + (x - x')$, and this would contradict the fact, obvious from (2.8), that $C(z|x)$ is finite save at most on a set of planar Lebesgue measure zero. Thus we conclude that *$C(z|x)$ is finite for all positive x and nonnegative z*. From this, and from the fact that the second sign of inequality in (1.3) can be written both sharply and weakly, it at once follows that *$C(\cdot|x)$ is continuous on $z \geqq 0$ for each fixed $x > 0$*. [In particular, $C(0+|x) = C(0|x) = 0$.] The fact that each term of (1.3) is dominated by the corresponding term in $C(Z|X)$ when $x < X$ and $z < x + (Z - X)$ further shows that *$C(z|\cdot)$ is continuous on $x > 0$ for each fixed $z \geqq 0$.*

The continuity of $C(z|x)$ in x and in z and the fact that it is nonnegative show (via Lerch's theorem) that $C(z|x)$ is in principle uniquely determinable from (2.8). Again from (2.8) we see, because of the weakly increasing character of $C(\cdot|x)$, that

$$(2.9) \qquad \int_0^{\infty} e^{-\alpha x} [e^{-\beta Z} C(Z|x)]\, dx \leqq \beta \int_0^{\infty} e^{-\alpha x} \left[\int_Z^{\infty} e^{-\beta z} C(z|x)\, dz\right] dx,$$

and the right member of this inequality tends to zero when $Z \to \infty$. Thus we can integrate by parts and replace the left side of (2.8) by

$$(2.10) \qquad 1 + (\alpha+\beta) \int_0^{\infty} e^{-\alpha x}\, dx \int_0^{\infty} e^{-\beta z} C(dz|x).$$

We can also transform the right side of (2.8) with the aid of the integral formula

$$e^{-\lambda|S|} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{i\lambda uS}}{1+u^2}\,du, \qquad \lambda > 0, \tag{2.11}$$

and on introducing the characteristic function

$$\phi(t) = \int_{-\infty}^{\infty} e^{ity}g(y)\,dy, \qquad -\infty < t < \infty, \tag{2.12}$$

we obtain

THEOREM 1. *The nonnegative quantity $C(z|x)$ is finite for all $x > 0$ and $z \geqq 0$, and depends continuously on x and also on z. It is uniquely determined by the identity*

$$1 + (\alpha + \beta)\int_0^{\infty} e^{-\alpha x}\,dx \int_0^{\infty} e^{-\beta z}C(dz|x) = \exp\,[f_k(\alpha) + f_k(\beta)], \qquad \alpha, \beta > 0, \tag{2.13}$$

where

$$f_k(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left\{\frac{1}{1-(1-k)\phi(\lambda u)}\right\}\frac{du}{1+u^2}, \qquad \lambda > 0. \tag{2.14}$$

The integral at (2.14) *is absolutely convergent.*

The final step in the proof of (2.13) involves the equality (for $\lambda > 0$) of

$$\sum_{n=1}^{\infty}\frac{(1-k)^n}{n}\int_{-\infty}^{\infty}\frac{[\phi(\lambda u)]^n}{1+u^2}\,du \tag{2.15}$$

and

$$\int_{-\infty}^{\infty}\log\left\{\frac{1}{1-(1-k)\phi(\lambda u)}\right\}\frac{du}{1+u^2}; \tag{2.16}$$

there is no difficulty in establishing this when $k > 0$, but when $k = 0$ a special argument is required, as follows. As in Spitzer's proof of the finiteness of (2.7) we can find positive numbers A and τ such that

$$|\phi(t)| \leqq e^{-At^2} \tag{2.17}$$

when $|t| < \tau$, and we shall have

$$\sup_{|t|\geqq\tau}|\phi(t)| = \rho < 1. \tag{2.18}$$

Thus

$$\sum_{n=1}^{\infty}\frac{1}{n}\int_{-\infty}^{\infty}\frac{|\phi(\lambda u)|^n}{1+u^2}\,du \leqq \lambda^{-1}\left(\frac{2\pi}{A}\right)^{1/2}\sum_{n=1}^{\infty} n^{-3/2} + \pi\sum_{n=1}^{\infty}\frac{\rho^n}{n}, \tag{2.19}$$

and the right side is finite. Thus the series of integrals can be evaluated as desired, by absolute convergence, and incidentally we have shown that the integral in (2.14) is absolutely convergent even when $k = 0$.

3. We now investigate the limiting behavior of $C(z|x)$ when $x \downarrow 0$; the importance of this (in connection with Oort's theory of the origin of comets) was made clear in [3]. We start with the inequalities

$$\sum_{m=1}^{\infty} (1-k)^m P\{R_m = 0,\, x + S_m \leqq z\} \leqq C(z|x) \leqq \sum_{m=1}^{\infty} (1-k)^m P\{x + R_m > 0,\, S_m < z\}, \tag{3.1}$$

and we note that the terms on the left increase and those on the right decrease when $x \downarrow 0$; also the series on the right is finite because it has the sum $C(z + x|x)$. We can therefore invoke the principle of monotone convergence and in this way we find that $C(z|0+)$ exists and is finite and is given by

$$C(z|0+) = \sum_{m=1}^{\infty} (1-k)^m P\{R_m = 0,\, S_m < z\}. \tag{3.2}$$

Proceeding as before we get

$$\begin{aligned} \int_0^{\infty} C(z|0+)\, e^{-\beta z} \beta\, dz &= \sum_{m=1}^{\infty} (1-k)^m P\{R_m = 0,\, S_m < B\} \\ &= \sum_{m=1}^{\infty} (1-k)^m \int_{R_m = 0} e^{-\beta S_m}\, dP \\ &= \sum_{m=1}^{\infty} (1-k)^m \lim_{\alpha \uparrow \infty} E\{e^{(\alpha+\beta)R_m - \beta S_m}\}. \end{aligned} \tag{3.3}$$

We can take the limit as $\alpha \uparrow \infty$ outside the summation sign, because the terms decrease weakly as α increases and the series is known, from (2.6), to be convergent for each finite positive α; thus we find that

$$1 + \int_0^{\infty} C(z|0+)\, e^{-\beta z} \beta\, dz = \exp\,[f_k(\beta)], \tag{3.4}$$

on using the Spitzer identity and making another appeal to the monotone convergence principle.

The continuity of the nonnegative function $C(\cdot|0+)$ for $z \geqq 0$ can be deduced from (3.2) by imitating the earlier argument, and so the formula just written down determines $C(z|0+)$ uniquely. The integral which occurs in it must converge, and so, because $C(z|0+)$ increases with z, we see that

$$C(z|0+) = o(e^{\epsilon z}) \tag{3.5}$$

as $z \to \infty$, for each positive ϵ. This fact enables us to integrate by parts, and so we have

THEOREM 2. *The limit $C(z|0+)$ exists for each $z \geqq 0$ and determines a finite continuous increasing function of z whose Laplace-Stieltjes transform is given by*

$$1 + \int_0^{\infty} e^{-\beta z} C(dz|0+) = \exp\,[f_k(\beta)], \qquad \beta > 0, \tag{3.6}$$

where $f_k(\beta)$ is given by (2.14).

4. The absolute continuity of the y-distribution implies that we can write

$$P\{R_m = 0,\, S_m < z\} = \int_0^z a_m(t)\, dt, \tag{4.1}$$

where $a_m(\cdot)$ is nonnegative and summable [indeed it is easy to express $a_m(\cdot)$ in terms of the function $g(\cdot)$]; thus we must also have

$$C(z|0+) = \int_0^z c(t)\,dt, \tag{4.2}$$

where

$$c(t) = \sum_{m=1}^{\infty} (1-k)^m a_m(t), \tag{4.3}$$

so that $c(\cdot)$ is nonnegative and summable on bounded Borel sets and $C(z|0+)$ is absolutely continuous and has $c(z)$ as its derivative almost everywhere (in the Lebesgue sense). We shall have

$$\int_0^{\infty} e^{-\beta z} c(z)\,dz = \exp\,[f_k(\beta)] - 1, \qquad \beta > 0. \tag{4.4}$$

We now write

$$c(z|x) = c(|z-x|) + \int_0^{\min(x,z)} c(x-t)c(z-t)\,dt, \qquad x > 0,\ z \geqq 0, \tag{4.5}$$

this function being defined everywhere in the (x, z) plane; its values will be nonnegative but we cannot be sure that they are finite. We shall calculate the double Laplace transform of the function

$$K(z|x) = \int_0^z c(s|x)\,ds. \tag{4.6}$$

A repeated application of Fubini's theorem shows that

$$\int_0^{\infty}\int_0^{\infty} e^{-\alpha x-\beta z} K(z|x)\,dx\,dz = \beta^{-1}(\alpha+\beta)^{-1}\{\gamma(\alpha)+\gamma(\beta)+\gamma(\alpha)\gamma(\beta)\}, \tag{4.7}$$

where $\gamma(\lambda)$ denotes the Laplace transform of $c(\cdot)$, given at (4.4) above. On comparing this with (2.13) we find that

$$\int_0^{\infty}\int_0^{\infty} e^{-\alpha x-\beta z} K(z|x)\,dx\,dz = \int_0^{\infty}\int_0^{\infty} e^{-\alpha x-\beta z} C(z|x)\,dx\,dz, \qquad \alpha, \beta > 0, \tag{4.8}$$

and that $K(z|x)$ (which is plainly nonnegative) is finite for almost all pairs (x, z). Now if we put $C(z)$ for $C(z|0+)$, and substitute for $c(s|x)$ from (4.5) into the expression for $K(z|x)$, we find that

$$K(z|x) = C(x) + C(|z-x|)\,\mathrm{sgn}\,(z-x) + \int_0^{\min(x,z)} C(z-t)c(x-t)\,dt, \tag{4.9}$$

and the last term on the right side can be written as

$$\int_{(x-z)^+}^{x} C(z-x+\tau)c(\tau)\,d\tau, \tag{4.10}$$

so that $K(z|x)$ increases, in the weak sense, (a) when x is constant and z increases, and (b) when $z - x$ is constant and x increases. The argument following equation (2.8) then shows that $K(z|x)$ is finite for all positive x and nonnegative z, so that $c(\cdot|x)$ is summable over bounded Borel sets and consequently $K(\cdot|x)$ is (absolutely) continuous. The last two formulas and the known fact that $C(\cdot) = C(\cdot|0+)$ is continuous yield the further conclusion that $K(z|\cdot)$ is continuous on $x > 0$. Lerch's theorem then tells us that $K(z|x) = C(z|x)$, and we have proved

Theorem 3. *The functions $C(\cdot|x)$ and $C(\cdot) = C(\cdot|0+)$ are absolutely continuous and so can be expressed as the indefinite integrals of their derivatives $c(z|x)$ and $c(z)$. These derivatives are determined up to sets of measure zero by the formulas*

$$c(z|x) = c(|z - x|) + \int_0^{\min(x,z)} c(x - t)c(z - t)\,dt \tag{4.11}$$

and

$$\int_0^\infty e^{-\beta z}c(z)\,dz = \exp\,[f_k(\beta)] - 1, \qquad \beta > 0, \tag{4.12}$$

where $f_k(\beta)$ is the function defined at (2.14).

5. We now return to the astronomical problem (see [3] for the notation and terminology). Formula (2.19) of [3] shows that we can now write

$$R(Z|x) = \frac{1}{2}\,V(x)H(x - Z) + \int_Z^\infty V(z)c(z|x)\,dz \tag{5.1}$$

where $Z \geqq 0$, and by proceeding as in section 4 of [3] we find that *the adjusted z-spectrum* (adjusted so as to incorporate the chance that a comet will have approached perihelion during the period T) *consists of*

(a) *a concentration of ρT "new" comets in the energy state x, and*

(b) *an absolutely continuous component with density $\rho T\, c(z|x)$ at the energy state z, where $0 < z < \infty$.*

On combining this result with theorem 3 we have in principle a completely general solution to the problem of the z-spectrum. In practice there may be some difficulty in inverting the Laplace transform for the function $c(z)$.

6. In order to illustrate the form of the solution we will give the details for the special case

$$g(y)\,dy = \frac{1}{2}\,e^{-|y|/b}\,\frac{dy}{b}, \qquad -\infty < y < \infty, \tag{6.1}$$

already studied in [3], and for another special choice of $g(y)$ which will supplement the earlier result in a useful way.

We shall now have $\phi(t) = (1 + b^2t^2)^{-1}$, and so

$$f_k(\lambda) = \frac{1}{2\pi}\int_{-\infty}^\infty \log\left\{\frac{1 + b^2\lambda^2u^2}{k + b^2\lambda^2u^2}\right\}\frac{du}{1 + u^2} = \log\frac{1 + b\lambda}{\sqrt{k} + b\lambda}, \tag{6.2}$$

where the last evaluation is an easy consequence of the elementary integral formula

$$\frac{1}{\pi}\int_0^\infty \log\,(1 + c^2u^2)\,\frac{du}{1 + u^2} = \log\,(1 + |c|). \tag{6.3}$$

Thus we must have

$$\int_0^\infty e^{-\beta z}c(z)\,dz = \frac{1-\sqrt{k}}{\sqrt{k}+b\beta}, \tag{6.4}$$

and

$$c(z)\,dz = \frac{1-\sqrt{k}}{b}\exp\left(-\frac{z\sqrt{k}}{b}\right)dz, \tag{6.5}$$

and this confirms formula (4.3) of [3]. The general z-spectrum in the case of double-exponential perturbations can then be found by inserting (6.5) into our formula (4.11) [which incidentally explains and generalizes the curiously symmetrical role played by x and z, partly obscured by the earlier formulas in [3] because we worked then with the integrated z-spectrum $C(z|x)$ instead of with the density $c(z|x)$].

In [3] the theoretical z-spectrum was only calculated when the perturbation distribution had the double-exponential form (6.1), and it is of course very desirable to know whether the general conclusions arrived at in that paper would be much altered by a change in the form of the perturbation distribution. It is therefore of interest that we are now able to calculate the theoretical z-spectrum in another special case; this is the convolution of (6.1) with itself,

$$g(y)\,dy = \frac{1}{4}\left(1+\frac{|y|}{b}\right)e^{-|y|/b}\frac{dy}{b}, \qquad -\infty < y < \infty, \tag{6.6}$$

which differs qualitatively from (6.1) in that $g(\cdot)$ no longer has a "corner" at $y = 0$; in fact (6.6) is appreciably nearer than (6.1) to the Gaussian form of perturbation distribution studied in [2] by J. M. Hammersley and R. A. Lyttleton.

We now have $\phi(t) = (1 + b^2t^2)^{-2}$, and so

$$f_k(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left\{\frac{1+b^2\lambda^2u^2}{c+b^2\lambda^2u^2}\,\frac{1+b^2\lambda^2u^2}{d+b^2\lambda^2u^2}\right\}\frac{du}{1+u^2}, \tag{6.7}$$

where c and d denote $1 \pm (1-k)^{1/2}$. Thus we must have

(6.8)

$$f_k(\lambda) = \log\frac{1+b\lambda}{\sqrt{c}+b\lambda} + \log\frac{1+b\lambda}{\sqrt{d}+b\lambda} = \log\frac{(1+b\lambda)^2}{b^2\lambda^2+(\sqrt{c}+\sqrt{d})b\lambda+\sqrt{k}},$$

and theorem 3 then shows that the continuous part of the z-spectrum is determined by

(6.9) $c(z)\,dz$

$$= \frac{1}{\sqrt{c}-\sqrt{d}}\left\{(1-\sqrt{d})^2\exp\left(-\frac{z\sqrt{d}}{b}\right) - (\sqrt{c}-1)^2\exp\left(-\frac{z\sqrt{c}}{b}\right)\right\}\frac{dz}{b},$$

where $c = 1 + (1-k)^{1/2}$ and $d = 1 - (1-k)^{1/2}$.

When $k = 0$, then d vanishes, and $c(z)$ assumes the form

$$c(z) = \frac{1}{b\sqrt{2}} - \frac{(3\sqrt{2} - 4)}{2b} \exp\left(-\frac{z\sqrt{2}}{b}\right); \tag{6.10}$$

this is to be compared with $c(z) = 1/b$ when the perturbation distribution has the double-exponential form (6.1). Thus, when $k = 0$, we get a "flat" z-spectrum (apart from the concentration at $z = x$) with the perturbation distribution (6.1) *but not with the perturbation distribution* (6.6). This amounts to a formal disproof of H. N. Russell's law [4], according to which the z-spectrum should be flat when $k = 0$, whatever the form of the perturbation distribution.

The use of the parameter b in (6.1) and (6.6) is not very satisfactory, because it is not related to the standard deviation σ of the perturbation distribution in the same way in the two cases; in fact we have $\sigma = b\sqrt{2}$ for (6.1) and $\sigma = 2b$ for (6.6). It is therefore more convenient when $k = 0$, to write for (6.1),

$$c(z) = \frac{\sqrt{2}}{\sigma} \tag{6.11}$$

and for (6.6),

$$c(z) = \frac{\sqrt{2}}{\sigma} - \frac{3\sqrt{2} - 4}{\sigma} \exp\left(-2\sqrt{2}\,\frac{z}{\sigma}\right). \tag{6.12}$$

These formulas show that the effect of switching from (6.1) to (6.6) is to depress the initial value of $c(z)$ to about 83 per cent of its original value. It should be noticed that the two formulas agree to within one per cent as soon as z is as large or larger than σ (about 75 in our customary numerical units).

The function $c(z|x)$ when $k = 0$ now easily follows on substituting from (6.11) and (6.12) into (4.11). We obtain, for the distribution (6.1),

$$c(z|x) = \frac{\sqrt{2}}{\sigma} + \frac{2}{\sigma^2} \min(x, z), \tag{6.13}$$

in agreement with (3.14) of [3], while if the perturbation distribution is (6.6), then

$$\begin{aligned} c(z|x) = {} & \frac{\sqrt{2}}{\sigma} - \frac{3\sqrt{2} - 4}{\sigma} \exp\left(-2\sqrt{2}\,\frac{|z - x|}{\sigma}\right) + \frac{2}{\sigma^2} \min(x, z) \\ & - \frac{3\sqrt{2} - 4}{2\sigma} \left\{\exp\left[2\sqrt{2}\,\frac{\min(x, z)}{\sigma}\right] - 1\right\}\left\{\exp\left[-2\sqrt{2}\,\frac{x}{\sigma}\right] + \exp\left[-2\sqrt{2}\,\frac{z}{\sigma}\right]\right\} \\ & + \frac{17\sqrt{2} - 24}{4\sigma} \left\{\exp\left[4\sqrt{2}\,\frac{\min(x, z)}{\sigma}\right] - 1\right\} \exp\left[-2\sqrt{2}\,\frac{x + z}{\sigma}\right]. \end{aligned} \tag{6.14}$$

Similar formulas can be obtained when $k > 0$, but we shall not write them out here; in practice it may be preferable to compute $c(x)$ from (6.9) and then to substitute in (4.11) and carry out the integration numerically.

Now that $c(z|x)$ is known we can repeat all the numerical calculations contained in [3], but employing the perturbation distribution (6.6) instead of (6.1). This program is carried out in section 12 below.

7. We now return to the general problem, so that we no longer make any particular hypothesis about the form of the perturbation distribution, and we study the function $c(z)$ whose Laplace transform is given by (4.12) and (2.14) and which determines the density $c(z|x)$ of the continuous part of the adjusted z-spectrum via formula (4.11).

We introduced $c(z)$ as the derivative (almost everywhere) of $C(z|0+)$, hence as the density of the continuous part of the adjusted z-spectrum when (as in Oort's theory) the initial energy state of a "new" comet is close to zero. But from (4.11) we now see that there is a second interpretation; *for almost all x, we shall have*

$$c(x) = \lim_{z \downarrow 0} \frac{C(z|x)}{z}. \tag{7.1}$$

To prove this, it is easiest to return to the formula found for $K(z|x)$, which we now know to be identical with $C(z|x)$, just before the statement of theorem 3. This shows that

$$\frac{C(z|x)}{z} = \frac{C(x) - C(x-z)}{z} + \frac{1}{z} \int_0^z C(u)c(x - z + u)\, du \tag{7.2}$$

when $0 < z < x$. Now we know that $C(\cdot)$ is continuous and that $C(0) = 0$, so that $0 \leqq C(u) < \epsilon$ if $0 \leqq u \leqq z$ and z is small enough, and thus the last term on the right side is nonnegative and does not then exceed

$$\frac{\epsilon}{z} \int_{x-z}^{x} c(v)\, dv = \epsilon \left\{ \frac{C(x) - C(x-z)}{z} \right\}. \tag{7.3}$$

But $[C(x) - C(x - z)]/z$ converges to a finite limit $c(x)$ for almost all x, when $z \downarrow 0$, and so, ϵ being arbitrarily small, we obtain (7.1).

The limit at the right side of (7.1) plays an important role in the investigations of Hammersley and Lyttleton [2], and its values for a wide range of values of x have been computed by J. M. Hammersley and K. Wright in the special case when $k = 0$ and the perturbation distribution has the Gaussian form

$$g(y)\, dy = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy, \qquad -\infty < y < \infty. \tag{7.4}$$

Thus an analytical study of the function $c(\cdot)$ has a double value, and may also lead to useful numerical comparisons. Moreover our formula (4.11), when coupled with the Wright-Hammersley computations, will give a complete solution to the problem of the adjusted z-spectrum when $k = 0$ and the perturbations are Gaussian. In the remainder of this paper we shall exploit these various possibilities.

8. The function $c(\cdot)$ is determined up to a set of measure zero by the relation

$$(8.1) \qquad 1 + \int_0^\infty e^{-z/\mu} c(z)\, dz = \exp\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \log\left\{\frac{1}{1-(1-k)\phi(\tau)}\right\}\frac{\mu\, d\tau}{1+\mu^2\tau^2}\right), \qquad \mu > 0,$$

and we shall begin by considering the important special case when $k = 0$. By Karamata's theorem (Widder [10], chapter 5, theorem 4.3) we can evaluate the limit of $C(z)/z$ when $z \uparrow \infty$ by dividing each side of (8.1) by μ and then letting μ tend to infinity. Obviously there will be no contribution from the unit on the left side, and in virtue of the elementary formula

$$(8.2) \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} \log(\tau^2)\frac{\mu\, d\tau}{1+\mu^2\tau^2} = -\log \mu,$$

we find that

$$(8.3) \qquad \lim_{z\uparrow\infty}\frac{C(z)}{z} = \lim_{\mu\uparrow\infty}\exp\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left\{\frac{\tau^2}{1-\phi(\tau)}\right\}\frac{\mu\, d\tau}{1+\mu^2\tau^2}\right).$$

If σ is the (assumed finite) standard deviation of the perturbation distribution then the logarithm in the integrand will differ from $\log(2/\sigma^2)$ by less than ϵ whenever $|\tau| < \delta$, if δ is suitably small, and the logarithm in the integrand is dominated by an expression of the form $A + |\log(\tau^2)|$ when $|\tau| \geqq \delta > 0$. The contribution from the term A to the integral is clearly $0(1)$ when $\mu \to \infty$ for fixed $\delta > 0$, and the same is true of the term in $\log(\tau^2)$ because

$$(8.4) \qquad \int_\delta^\infty |\log \tau| \frac{\mu\, d\tau}{1+\mu^2\tau^2} \leqq \int_{\mu\delta}^\infty \frac{|\log v| dv}{1+v^2} + |\log \mu| \int_{\mu\delta}^\infty \frac{dv}{1+v^2},$$

and on putting these results together we obtain

THEOREM 4. *When $k = 0$ and when the perturbation distribution has a finite variance σ^2, then*

$$(8.5) \qquad \lim_{z\uparrow\infty}\frac{C(z)}{z} = \lim_{z\uparrow\infty}\frac{1}{z}\int_0^z c(t)\, dt = \frac{\sqrt{2}}{\sigma}.$$

This appears to be the true result to which Russell's "law" (a flat adjusted z-spectrum when $k = 0$) is an approximation. It does of course lie very close to a calculation by Spitzer in [7].

When the perturbation distribution does not have a finite variance then a simple adaptation of the above argument shows that the limit at (8.5) exists and is zero; that is, $C(z)$ tends to infinity less rapidly than z. In specific cases we can investigate the rate of growth of $C(z)$ precisely. For example, if

$$(8.6) \qquad g(y)\, dy = \frac{1}{\pi}\frac{dy/s}{1+(y/s)^2}, \qquad -\infty < y < \infty,$$

which is sometimes called the Cauchy distribution, then $\phi(\tau) = e^{-s|\tau|}$, and the above type of argument shows that then

$$\lim_{z\uparrow\infty} \frac{C(z)}{\sqrt{z}} = \lim_{z\uparrow\infty} \frac{1}{\sqrt{z}} \int_0^z c(t)\,dt = \frac{2}{\sqrt{\pi s}}; \tag{8.7}$$

that is, $C(z)$ grows asymptotically like $\sqrt{z}$. One would therefore expect that in the Gaussian case $c(z) \to \sigma^{-1}\sqrt{2}$, and that in the Cauchy case $c(z) \to 0$, but such a conclusion could not be justified without a more delicate Tauberian argument than we feel inclined to embark upon here. Let us notice, however, that when the perturbation distribution has one of the forms (6.1), (6.6), then the limit of $c(z)$ as $z \to \infty$ does exist; it is correctly given, as of course it must be, by formula (8.5) of theorem 4.

We turn now to the behavior of $C(z)$ and $c(z)$ when $z \downarrow 0$. A similar Tauberian argument (Widder [10], *loc. cit.*) shows that we can calculate the limit of $C(z)/z$ when $z \downarrow 0$ by dividing each side of (8.1) by μ and then letting $\mu \downarrow 0$. In this case there is no advantage to be gained in letting $k = 0$, so we assume merely that $0 \leqq k < 1$. The logarithmic term inside the integral on the right side tends to zero (because $|\phi(\tau)|$ does so) when $|\tau|$ tends to infinity, and thus the whole expression inside the exponential tends to zero when $\mu \downarrow 0$. This shows that we have to examine the behavior of

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \log \left\{ \frac{1}{1 - (1 - k)\phi(\tau)} \right\} \frac{d\tau}{1 + \mu^2\tau^2} \tag{8.8}$$

when μ tends to zero. Let us call this expression J, and write $J_1(A)$ and $J_2(A)$ for the contributions from the ranges $(0, A)$ and (A, ∞), respectively. Because $g(\cdot)$ and so also $\phi(\cdot)$ is even we can confine attention to the contributions from $(0, \infty)$. Now we have to make a new assumption, namely that the integral

$$\Psi(t) = \frac{1}{2\pi} \int_t^{\infty} \log \left\{ \frac{1}{1 - (1 - k)\phi(\tau)} \right\} d\tau \tag{8.9}$$

is conditionally (perhaps not absolutely!) convergent for each $t > 0$. This will be so, for example, if the density $g(\cdot)$ of the perturbation distribution belongs to the class $L^2(-\infty, \infty)$ and satisfies some smoothness condition at $y = 0$, for example differentiability or locally bounded variation, sufficient to ensure the convergence there of its Fourier series representations, for then ϕ will be conditionally and ϕ^2 absolutely integrable at infinity, and (for large enough t) the integrand of the integral defining $\Psi(t)$ will equal $(1 - k)\phi + \theta\phi^2$, where $|\theta| \leqq 1$. Assuming this to be so, we observe that

$$\begin{aligned} J_2(A) &= \frac{1}{2\pi} \int_A^{\infty} \log \left\{ \frac{1}{1 - (1 - k)\phi(\tau)} \right\} \frac{d\tau}{1 + \mu^2\tau^2} \\ &= \frac{\Psi(A)}{1 + \mu^2 A^2} - 2 \int_A^{\infty} \Psi(\tau) \frac{\mu^2\tau\,d\tau}{(1 + \mu^2\tau^2)^2}, \end{aligned} \tag{8.10}$$

and the last term on the right side is bounded by

$$\sup_{\tau > A} |\Psi(\tau)| \int_0^\infty \frac{2x\,dx}{(1+x^2)^2}, \tag{8.11}$$

which can be made less than ϵ by taking A sufficiently large. Let A be so chosen, and then fixed. The first term on the right side of the last formula will then converge to $\Psi(A)$ when $\mu \downarrow 0$.

We now examine $J_1(A)$. When $k > 0$ it is trivial that this converges (for $\mu \downarrow 0$) to the expression obtained by setting $\mu = 0$, but when $k = 0$ then a further argument is needed. We have merely to note (see for example the "Remark" on page 26 of H. Cramér [1]) that $\phi(\tau)$ is bounded away from unity on any closed set which excludes $\tau = 0$, and that in some neighborhood $(-\delta, \delta)$ of $\tau = 0$ the function ϕ is nonnegative and satisfies the inequality $1 - \phi(\tau) > k\tau^2$, for some k. It then follows that the logarithm in the integrand of J is dominated over $(0, A)$ by some multiple of $(|\log \tau| + 1)$, and thus $J_1(A)$ converges (for $\mu \downarrow 0$) to the value obtained by putting $\mu = 0$, even in the awkward case when k is zero.

On putting together these various results we obtain

THEOREM 5. *The limit formula,*

$$\lim_{z \downarrow 0} \frac{C(z)}{z} = \lim_{z \downarrow 0} \frac{1}{z} \int_0^z c(t)\,dt = \frac{1}{2\pi} \int_{-\infty}^\infty \log\left\{\frac{1}{1-(1-k)\phi(\tau)}\right\} d\tau, \tag{8.12}$$

holds whenever the integral on the right side converges at $\pm\infty$ (*conditional convergence is sufficient*).

The required convergence (even absolute convergence) can easily be verified for all the specific perturbation distributions which we have mentioned so far. Also, for (6.1) and (6.6), the limit $c(0+)$ actually exists, but we are not able to prove this in general.

9. Theorem 4 says nothing about the behavior of $c(\cdot)$ when $z \uparrow \infty$ and $k > 0$; we may expect from the examples which we have studied in detail that $c(z)$ will then converge to zero, and we know that when the perturbation distribution has one of the forms (6.1), (6.6) then the convergence is actually exponentially fast. It is desirable to establish a general result of this kind, and to this we now turn. First let us consider the integral $\int_0^\infty c(z)\,dz$, which will of course be divergent if $k = 0$ and σ is finite. To find this we have only to allow μ to tend to to infinity in (8.1), and we obtain

THEOREM 6. *The expected* (*adjusted*) *total number of bound comets is equal to*

$$\rho T + \rho T \int_0^\infty c(z)\,dz \tag{9.1}$$

when the initial energy state x is nearly equal to zero, and the integral in (9.1) *has the value*

$$\int_0^\infty c(z)\,dz = \frac{1}{\sqrt{k}} - 1, \tag{9.2}$$

so that the expected adjusted total of comets is $\rho T/\sqrt{k}$ and is finite if $k > 0$.

It is rather curious that here we have a result which is quite independent of the form and scale of the perturbation distribution. A generalization to the case when $x > 0$ will be given later.

From theorem 6 we see that $\sqrt{k}(1 - \sqrt{k})^{-1}c(z)\,dz$ is a probability distribution, and therefore so is

$$(9.3)\qquad (1 - \sqrt{k})^{-1}\, c\left(\frac{w}{\sqrt{k}}\right) dw, \qquad 0 < w < \infty.$$

We introduce the Laplace-Stieltjes transform

$$(9.4)\qquad \pi(\lambda; k) = (1 - \sqrt{k})^{-1} \int_0^\infty e^{-\lambda w} c\left(\frac{w}{\sqrt{k}}\right) dw, \qquad \lambda > 0,$$

of the associated measure and consider its behavior when $k \downarrow 0$. From (8.1) we see that

$$(9.5)\qquad \sqrt{k} + (1 - \sqrt{k})\pi(\lambda; k) = \exp\left(\frac{1}{2\pi} \int_{-\infty}^\infty \log\left\{\frac{1}{1 + k^{-1}(1 - k)[1 - \phi(\lambda t\sqrt{k})]}\right\} \frac{dt}{1 + t^2}\right),$$

and if we assume that the perturbation distribution has a finite variance σ^2 then we can let $k \downarrow 0$ in the integrand because the integrand is dominated by the absolute value of

$$(9.6)\qquad \log\left\{\frac{1}{1 + \lambda^2\sigma^2t^2/2}\right\} \frac{1}{1 + t^2},$$

which is summable (and is in fact the limiting value of the integrand). Thus, recalling the calculation leading to (6.5), we find that

$$(9.7)\qquad \lim_{k \downarrow 0} \pi(\lambda; k) = \left[1 + \frac{\lambda\sigma}{\sqrt{2}}\right]^{-1} = \int_0^\infty e^{-\lambda w} \exp\left(-\frac{w}{\sigma}\sqrt{2}\right) d\left(\frac{w}{\sigma}\sqrt{2}\right)$$

for every real positive λ. The fact that the right side is the Laplace-Stieltjes transform of a probability measure on $(0, \infty)$ is then sufficient, by a not well-known continuity theorem, to ensure that the distribution (9.3) converges in distribution to a negative-exponential distribution with expectation parameter $\sigma/\sqrt{2}$; that is, we have proved

THEOREM 7. *If the perturbation distribution has a finite variance* σ^2, *then*

$$(9.8)\qquad \lim_{k \downarrow 0} \sqrt{k} \int_{u/\sqrt{k}}^\infty c(z)\, dz = \exp\left(-\frac{u}{\sigma}\sqrt{2}\right), \qquad 0 \leqq u < \infty.$$

This result shows that the exponential fall off of the function $c(\cdot)$ when $k \neq 0$, first noticed in [3] in the special case of double-exponential perturbations, is actually characteristic of all perturbation distributions having a finite variance.

10. In theorem 6 we were able to give the expected (adjusted) total number of bound comets in the system when the initial energy state for a comet was just greater than zero. We shall now obtain the generalization of this result for an arbitrary value x for the initial energy state, by appealing to the formula (4.11) of theorem 3 which expresses the function $c(z|x)$ of two independent variables in terms of the function $c(z)$ of one independent variable. Several of our other results concerning $c(z)$ could be extended to $c(z|x)$ in a similar way, but this example should suffice to illustrate the procedure.

We want to calculate $\int_0^\infty c(z|x)\,dz$, and according to (4.11) this is equal to

$$\int_0^x c(x-z)\,dz + \int_0^x dz \int_0^z c(x-t)c(z-t)\,dt + \int_x^\infty c(z-x)\,dz + \int_x^\infty dz \int_0^x c(x-t)c(z-t)\,dt. \tag{10.1}$$

On inverting the orders of integration in the multiple integrals we obtain

$$\int_0^x c(u)\,du + \int_0^x dt \int_t^x c(x-t)c(z-t)\,dz + \int_0^\infty c(u)\,du + \int_0^x dt \int_x^\infty c(x-t)c(z-t)\,dz, \tag{10.2}$$

which is equal to $\alpha + \beta + \alpha\beta$, where $\alpha = \int_0^\infty c(u)\,du$ and $\beta = \int_0^x c(u)\,du$. We therefore have

THEOREM 8. *The expected (adjusted) total number of bound comets (when the initial energy state x is positive) is given by*

$$\frac{\rho T}{\sqrt{k}}\left[1 + \int_0^x c(u)\,du\right], \tag{10.3}$$

and is finite if $k > 0$. When the initial energy state x becomes very large this approaches the finite limit $\rho T/k$.

11. Formula (4.11) gives us the (adjusted) expected z-spectrum explicitly as soon as the function $c(\cdot)$ is known. Now the Laplace transform of $c(\cdot)$ is given by (4.12), but in practice the inversion may present difficulties; we therefore turn to an alternative way of calculating the basic function $c(\cdot)$.

Let us write $b_1(t) = g(t)$ and

$$b_m(t) = \int_0^\infty g(y_1)\,dy_1 \int_{-S_1}^\infty g(y_2)\,dy_2 \int_{-S_2}^\infty g(y_3)\,dy_3 \cdots \int_{-S_{m-2}}^\infty g(y_{m-1})g(t - S_{m-1})\,dy_{m-1}, \tag{11.1}$$

when $m = 2, 3, \cdots$. We notice that

$$b_{m+1}(t) = \int_0^\infty b_m(u)g(t-u)\,du, \tag{11.2}$$

and that

$$P\{R_m = 0, S_m < x\} = \int_0^x b_m(t)\,dt. \tag{11.3}$$

It now follows from (3.2) that

$$\int_0^x c(t)\,dt = \int_0^x \sum_{m=1}^{\infty} (1-k)^m b_m(t)\,dt, \tag{11.4}$$

and so we have

THEOREM 9. *The function $c(\cdot)$ is given for almost all x by*

$$c(x) = \sum_{m=1}^{\infty} (1-k)^m b_m(x), \tag{11.5}$$

where $b_m(\cdot)$ is defined as at (11.1); thus $c(\cdot)$ is the minimal nonnegative solution to the integral equation

$$f(x) = (1-k)g(x) + (1-k)\int_0^{\infty} f(u)g(x-u)\,du, \qquad x > 0, \tag{11.6}$$

If the equation possesses a bounded nonnegative solution $f(\cdot)$ then we must have $c(x) = f(x)$ for almost all x.

The last clause of the theorem (the assertion that if a bounded nonnegative solution $f(x)$ exists, then it must be equal to $c(x)$ save on a set of measure zero) still requires proof; we establish it as follows. Suppose that such an $f(\cdot)$ has been found; then from the minimality of $c(\cdot)$ we know that $c(\cdot)$ must in this case also be bounded, and so $F(x) = f(x) - c(x)$ will be a bounded nonnegative solution of the homogeneous equation

$$F(x) = (1-k)\int_0^{\infty} F(u)g(x-u)\,du. \tag{11.7}$$

When $k > 0$ it follows trivially that $F(x) = 0$, for the right side of (11.7) will then be a proper contraction of the left side. When $k = 0$, let M be an upper bound for $F(\cdot)$; then we shall have

(11.8)

$$\begin{aligned} 0 \leqq F(x) &\leqq M\int_0^{\infty} g(x-u_1)\,du_1 \int_0^{\infty} g(u_1-u_2)\,du_2 \cdots \int_0^{\infty} g(u_{n-1}-u_n)\,du_n \\ &= M\int_0^{\infty} g(u_1-x)\,du_1 \int_0^{\infty} g(u_2-u_1)\,du_2 \cdots \int_0^{\infty} g(u_n-u_{n-1})\,du_n \\ &= M \iiint\limits_E \cdots \int g(y_1)g(y_2)g(y_3)\cdots g(y_n)\,dy_1\,dy_2\,dy_3\cdots dy_n, \end{aligned}$$

where the set E over which the n-fold integration is to be carried out is defined by the inequalities

$$x + y_1 + y_2 + \cdots + y_s > 0, \qquad s = 1, 2, \cdots, n. \tag{11.9}$$

But this is just to say that

$$0 \leqq F(x) \leqq MP\{x + S_s > 0 \quad \text{for} \quad s = 1, 2, \cdots, n\}, \tag{11.10}$$

and we know that the extreme right member of this inequality tends to zero when n tends to infinity; thus $F(x) = 0$, as required.

We do not know the precise circumstances in which $c(\cdot)$ will be (essentially) bounded, but it is clear that boundedness is not the general rule. For example, $c(\cdot)$ must be unbounded if $g(\cdot)$ is unbounded. Thus the final clause of theorem 9 will only occasionally suffice to identify $c(\cdot)$ among the solutions to (11.6), but it is none the less extremely useful, as we shall see. More general statements would be possible if we had more information about the range of solutions to the Wiener-Hopf equation (11.7); the existing studies of this equation (for example, F. Smithies [5]) depend on hypotheses not always satisfied in our problem. A very thorough study ([7], [8]) by Spitzer of the Wiener-Hopf equation when the kernel is a probability density at first appears very suitable for our purposes, but unfortunately it is only concerned with the monotone solutions to (11.7), and this is a serious defect from our point of view because of the remarkable discovery by Wright and Hammersley (see section 12 below) that $c(\cdot)$ itself need not be monotone.

We can however supplement the integral equation (11.6) by an additional condition which enables us in all cases to pick out the correct solution $c(\cdot)$; we shall prove

THEOREM 10. *The function $c(\cdot)$ is equal almost everywhere to the minimal nonnegative solution to the integral equation,*

$$f(x) = (1 - k)g(x) + (1 - k) \int_0^\infty f(u)g(x - u)\, du, \qquad x > 0, \tag{11.11}$$

and this minimal nonnegative solution is uniquely distinguished among the nonnegative solutions to (11.11) *by the facts that*

$$\int_0^\infty c(x)\, dx \int_x^\infty g(y)\, dy = \frac{1}{2} \qquad \text{when} \quad k = 0, \tag{11.12}$$

and

$$\int_0^\infty c(x)\, dx = \frac{1}{\sqrt{k}} - 1 \qquad \text{when} \quad k > 0. \tag{11.13}$$

PROOF. That (11.13) holds when $k = 0$ is part of the content of theorem 6; obviously this property will not hold for any essentially different nonnegative solution $f(x)$ to (11.11), because we must have $f(x) \geqq c(x)$ almost everywhere. Now suppose that $k = 0$, and observe that with probability one we must have either (a) $x + S_1, x + S_2, \cdots, x + S_n$ all positive and $x + S_{n+1}$ negative for some $n = 1, 2, \cdots$, or (b) $x + S_1, x + S_2, \cdots, x + S_n$ all negative and $x + S_{n+1}$ positive for some $n = 1, 2, \cdots$, and also that (a) and (b) each hold with probability 1/2 when $x = 0$. On using (11.1) and (11.5) and the even character of $g(\cdot)$ to express the fact that $P\{(a)\} = 1/2$ when $x = 0$, we obtain (11.12). We must show that this characterizes the solution $c(\cdot)$ among the nonnegative solutions to (11.11). If $f(\cdot)$ is any nonnegative solution to (11.11) and if $f(\cdot)$ like $c(\cdot)$ satisfies (11.12), then $F(x) = f(x) - c(x)$ will be almost everywhere nonnegative and we shall have

$$F(x) = \int_0^\infty F(u)g(x - u)\, du \tag{11.14}$$

for almost all $x > 0$, and

$$\int_0^\infty F(x)\,dx \int_x^\infty g(y)\,dy = 0. \tag{11.15}$$

The desired result $F(x) = 0$ a.e. follows at once if the g-distribution is of infinite extent. If the g-distribution is of finite extent, let Y be the smallest (positive finite) number such that $\int_Y^\infty g(y)\,dy = 0$, so that $F(x) = 0$ almost everywhere in $(0, Y)$ at least. Let X be the largest (positive, but perhaps infinite) extended real number such that $F(x) = 0$ a.e. in $(0, X)$, so that $X \geqq Y$. We shall show that X must be infinite. For otherwise we should have, for $0 < \epsilon < Y$,

$$\begin{aligned} 0 = \int_{X-Y}^{X} F(x)\,dx &= \int_{X-Y}^{X} dx \int_X^\infty F(u)g(x-u)\,du \\ &= \int_X^\infty F(u)\,du \int_{X-Y-u}^{X-u} g(y)\,dy \\ &\geqq \int_X^{X+Y-\epsilon} F(u)\,du \int_{-Y}^{-Y+\epsilon} g(y)\,dy > 0, \end{aligned} \tag{11.16}$$

because the g-distribution is symmetrical. Thus we have a contradiction unless $X = \infty$.

We can use theorem 10 to make clear the relation between the function $c(\cdot)$ studied in this paper and the monotone solution to the Wiener-Hopf equation studied by Spitzer in [7] and [8]. Let us write

$$\Phi(x) = 1 + C(x) = 1 + \int_0^x c(t)\,dt, \qquad x \geqq 0; \tag{11.17}$$

then in virtue of (11.11) and (11.12) we shall have, when $k = 0$,

(11.18)

$$\begin{aligned} \Phi(x) &= 1 + \int_0^x g(t)\,dt + \int_0^x dt \int_0^\infty c(u)g(t-u)\,du \\ &= 1 + \int_0^x g(t)\,dt + \int_0^\infty c(u)\,du \int_{-u}^{x-u} g(y)\,dy \\ &= \frac{1}{2} + \int_0^x g(t)\,dt + \frac{1}{2} + \int_0^\infty c(u)\,du \int_{u-x}^{u} g(y)\,dy \\ &= \int_{-\infty}^x g(t)\,dt + \int_0^\infty c(u)\,du \int_{u-x}^\infty g(y)\,dy \\ &= \int_0^\infty g(x-u)\,du + \int_0^\infty dC(u) \int_{u-x}^\infty g(y)\,dy \\ &= \int_0^\infty g(x-u)\,du + \lim_{U\uparrow\infty} \left[C(U)\int_{U-x}^\infty g(y)\,dy + \int_0^U C(u)g(u-x)\,du\right]. \end{aligned}$$

In all the above expressions all terms are nonnegative, and we see that

$$\int_0^U C(u)g(u-x)\,du \leqq 1 + C(x) < \infty, \tag{11.19}$$

so that, because

$$(11.20)\qquad C(U)\int_{U-x}^{\infty} g(y)\,dy \leqq \int_{U-x}^{\infty} C(x+y)g(y)\,dy = \int_{U}^{\infty} C(u)g(u-x)\,du,$$

the first term within the square brackets in (11.18) tends to zero when $U \to \infty$, and the second term tends to a finite limit. Thus we have

$$(11.21)\qquad \Phi(x) = \int_0^{\infty} g(x-u)\,du + \int_0^{\infty} C(u)g(x-u)\,du,$$

on again making use of the symmetry of the g-distribution. In this way we obtain

THEOREM 11. *Let $k = 0$. Then the function $\Phi(\cdot)$ defined by*

$$(11.22)\qquad \Phi(z) = 1 + C(z|0+) = 1 + \int_0^{z} c(t)\,dt, \qquad z \geqq 0,$$

is absolutely continuous, nondecreasing, and has the value 1 at $z = 0$, and it satisfies the Wiener-Hopf equation

$$(11.23)\qquad \Phi(z) = \int_0^{\infty} \Phi(u)g(z-u)\,du, \qquad z \geqq 0.$$

It therefore coincides with the function $F(\cdot)$ studied by Spitzer in [7] and [8].

12. We shall now illustrate the use of theorems 3, 9, and 10 by calculating the the adjusted z-spectrum when $k = 0$ and when the perturbations follow the Gaussian law (7.4). If we temporarily take the standard deviation σ of the y-distribution as the unit in which x, y, and z are measured, then theorem 9 tells us that

$$(12.1)\qquad c(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) + \frac{1}{\sqrt{2\pi}}\int_0^{\infty}\exp\left[-\frac{(x-u)^2}{2}\right]c(u)\,du,$$

that $c(\cdot)$ is the minimal nonnegative solution to this equation, and that if the equation can be shown to have a nonnegative bounded solution then this must be the desired function $c(\cdot)$.

The equation (12.1) for $c(\cdot)$, which in his notation is called $q(\cdot)$, has also been obtained by Hammersley (see equation (4.9) in [2]), and Wright and he have solved it numerically; they have kindly provided me with a sample of their results, shown in table I.

The (very surprising) ripple in the values of $c(\cdot)$ rapidly decays and ultimately the solution settles down to the limiting value 1.4142. As this is a bounded solution, there can be no question of its not being the right one. That this is the correct solution to (12.1) can also be shown by calculating numerically the integral at (11.12), using the computed values of $c(x)$. We obtain 0.5000. Anv admixture of another solution would have raised this to a value greater than 1/2. Also it is interesting to notice that the initial and final values agree with the predictions of our theorems 4 and 5. Theorem 4 tells us that when $x \to \infty$ then $c(\cdot)$ is $(C, 1)$-limitable to the limit $\sqrt{2}(= 1.41421356)$. Theorem 5 tells us that when $x \to 0$ then $c(\cdot)$ is $(C, 1)$-limitable to the limit

TABLE I

TABLE OF $c(x)$
(by K. W. and J. M. H.)

x/σ	$c(x)$
0.0	1.0422
0.2	1.1433
0.4	1.2275
0.6	1.2937
0.8	1.3424
1.0	1.3759
1.2	1.3969
1.4	1.4087
1.6	1.4144
1.8	1.4164
2.0	1.4166
2.2	1.4159
2.4	1.4152
2.6	1.4147
2.8	1.4143
3.0	1.4142
3.2	1.4141
3.4	1.4141
3.6	1.4142
3.8	1.4142
≧4.0	1.4142

(12.2)
$$\begin{aligned}\lambda &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left\{\frac{1}{1-\exp(-\tau^2/2)}\right\}d\tau\\ &= \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_{-\infty}^{\infty}\exp(-n\tau^2/2)\,d\tau\\ &= \frac{1}{\sqrt{2\pi}}\zeta\left(\frac{3}{2}\right) = 1.04218698.\end{aligned}$$

Here $\zeta(\cdot)$ denotes Riemann's zeta function. Thus the solution computed by Wright and Hammersley takes on the correct values at the two extremes of the range.

If we now substitute the computed values of $c(\cdot)$ into formula (4.11) of theorem 3 we obtain the values in table II for $c(z|x)$ when the distribution of perturbations is Gaussian with standard deviation σ, and $k = 0$.

Table II gives the density in the continuous part of the (adjusted) expected z-spectrum when σ (the root-mean-square perturbation) is used as the unit of measurement for z and x, for four initial values ($x/\sigma = 0, 1, 2,$ and 3) of the energy state. It should be noted that to get the whole of the (adjusted) expected z-spectrum we have to combine this density with a point concentration of amount

TABLE II

z/σ	Values of $c(z\|x)$			
	$x/\sigma = 0$	$x/\sigma = 1$	$x/\sigma = 2$	$x/\sigma = 3$
0.0	1.042	1.376	1.417	1.414
0.2	1.143	1.640	1.726	1.723
0.4	1.227	1.904	2.060	2.059
0.6	1.294	2.159	2.410	2.416
0.8	1.342	2.393	2.769	2.790
1.0	1.376	2.598	3.126	3.176
1.2	1.397	2.770	3.472	3.568
1.4	1.409	2.914	3.797	3.963
1.6	1.414	3.008	4.094	4.357
1.8	1.416	3.079	4.355	4.743
2.0	1.417	3.126	4.576	5.116
2.2	1.416	3.153	4.756	5.471
2.4	1.415	3.168	4.896	5.800
2.6	1.415	3.174	4.999	6.097
2.8	1.414	3.176	5.070	6.358
3.0	1.414	3.176	5.116	6.579
3.2	1.414	3.175	5.143	6.758
3.4	1.414	3.173	5.157	6.898

unity at $z = x$, and then multiply the spectrum throughout by the scale factor ρT.

In figure 1 we show for these four values of the initial energy state (corresponding to $x = 0$, 75, 150, and 225 in our previous units, when $\sigma = 75$) the complete (adjusted) expected z–spectrum when $k = 0$ and when the distribution of perturbations is (i) double-exponential, (ii) the convolution of a double-exponential with itself, and (iii) Gaussian. The point concentration at $z = x$ has been spread out over an interval of width $\sigma/5$ (= 15 units, when $\sigma = 75$). It will be noticed that the form of the perturbation distribution has only a slight effect.

The function $c(\cdot)$ has not been computed for the case of Gaussian perturbations when $k > 0$, so that we are not (yet) in a position to draw a set of graphs analogous to those in figure 1 for positive k. However, we can test sensitivity to form of distribution when $k > 0$ by comparing the z-spectra for (i) a double-exponential distribution and (ii) the convolution of the double-exponential with itself. This is done in figure 2 for the largest value of k previously considered ($k = 0.04$), when $x/\sigma = 0$ and $1 (x = 0$ and $75)$.

Once again the adjusted z-spectra corresponding to the two perturbation distributions are very similar. On comparing these graphs with figure 3, which presents in a more convenient form the *empirical* z-spectrum for the range $0 \leqq z \leqq 200$ (that is, $0 \leqq z/\sigma \leqq 2.67$) previously shown in figure 3 of the preceding part of this paper, it will be seen that

(a) The observations are not compatible with any value of the initial energy

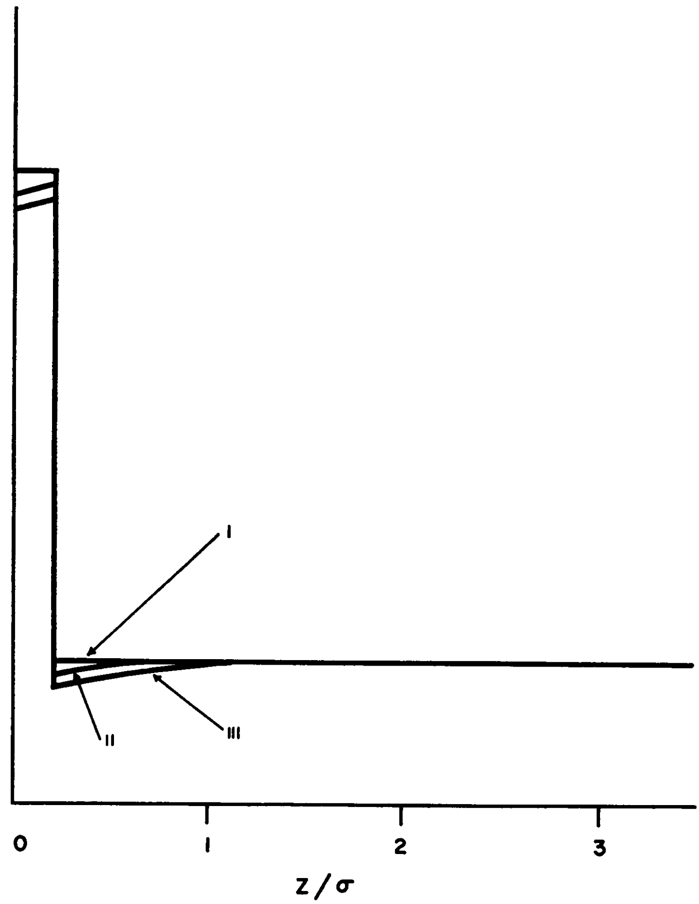

FIGURE 1(a)

Theoretical z-spectra ($x/\sigma = 0$ and $k = 0$) for various perturbation distributions [(i) double-exponential, (ii) double-exponential convolved, (iii) Gaussian].

state x much in excess of (say) 20 units [= 0.00020 (astronomical units)$^{-1}$]. The studies by Hammersley and Lyttleton in [2] offer no support for the suggestion that the system has not had time to enter into statistical equilibrium, so that we seem driven to the conclusion that Lyttleton's mechanism for the origin of comets must be associated with very small values (of the order of 0.3 km/sec) of his parameter v.

(b) Even when we take $x = 0$ (as in Oort's theory, or in Lyttleton's theory when v is negligibly small), the peak near $z = 0$ in the empirical z-spectrum is relatively much more pronounced (in relation to the continuous background) than in the theoretical z-spectrum. This is the effect noted by Oort and attrib-

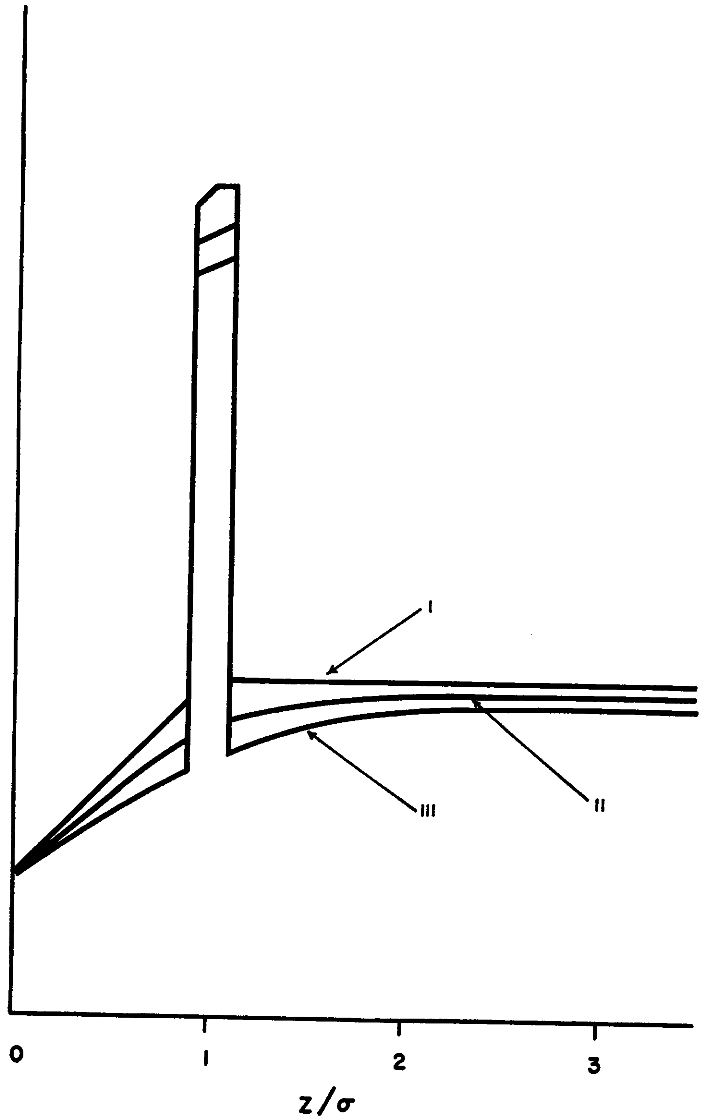

FIGURE 1(b)

Theoretical z-spectra ($x/\sigma = 1$ and $k = 0$)
for various perturbation distributions
[(i) double-exponential, (ii) double-exponential
convolved, (iii) Gaussian].

uted by him to a greater intrinsic luminosity for "new" (as compared with "old") comets.

(c) There is a strong suggestion that the disintegration probability $k = 0.04$ used in the computation of figure 2 is too small, because the spectral density

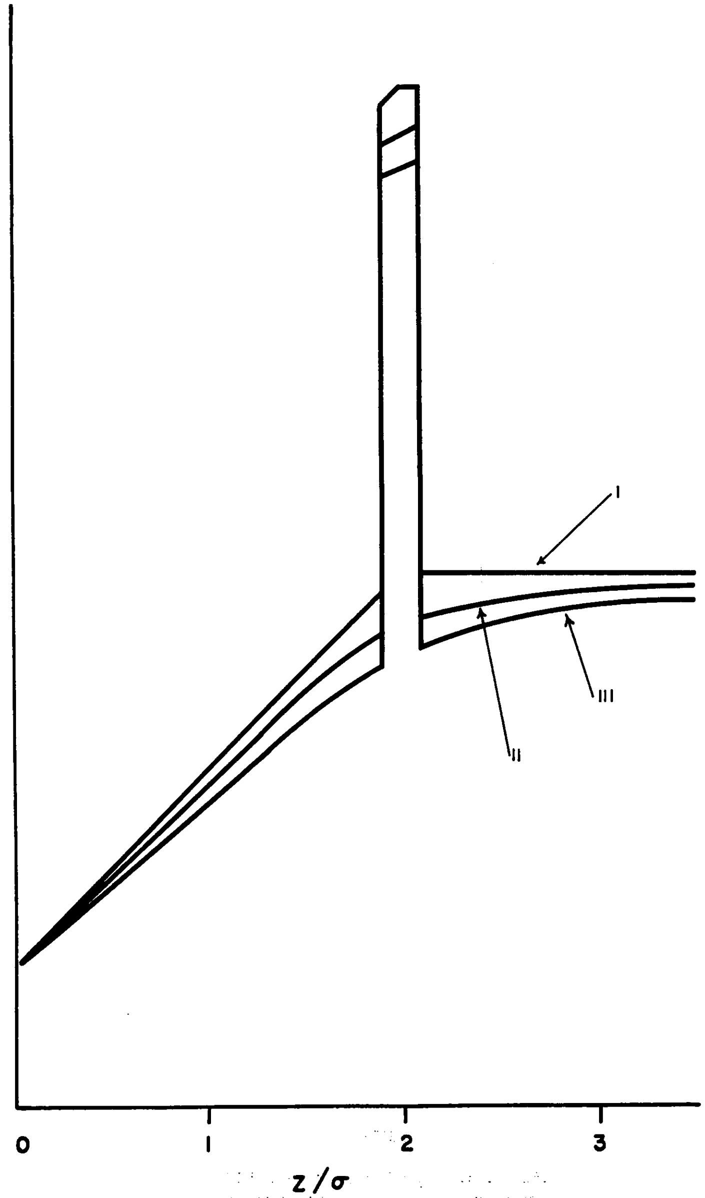

FIGURE 1(c)

Theoretical z-spectra ($x/\sigma = 2$ and $k = 0$) for various perturbation distributions [(i) double-exponential, (ii) double-exponential convolved, (iii) Gaussian].

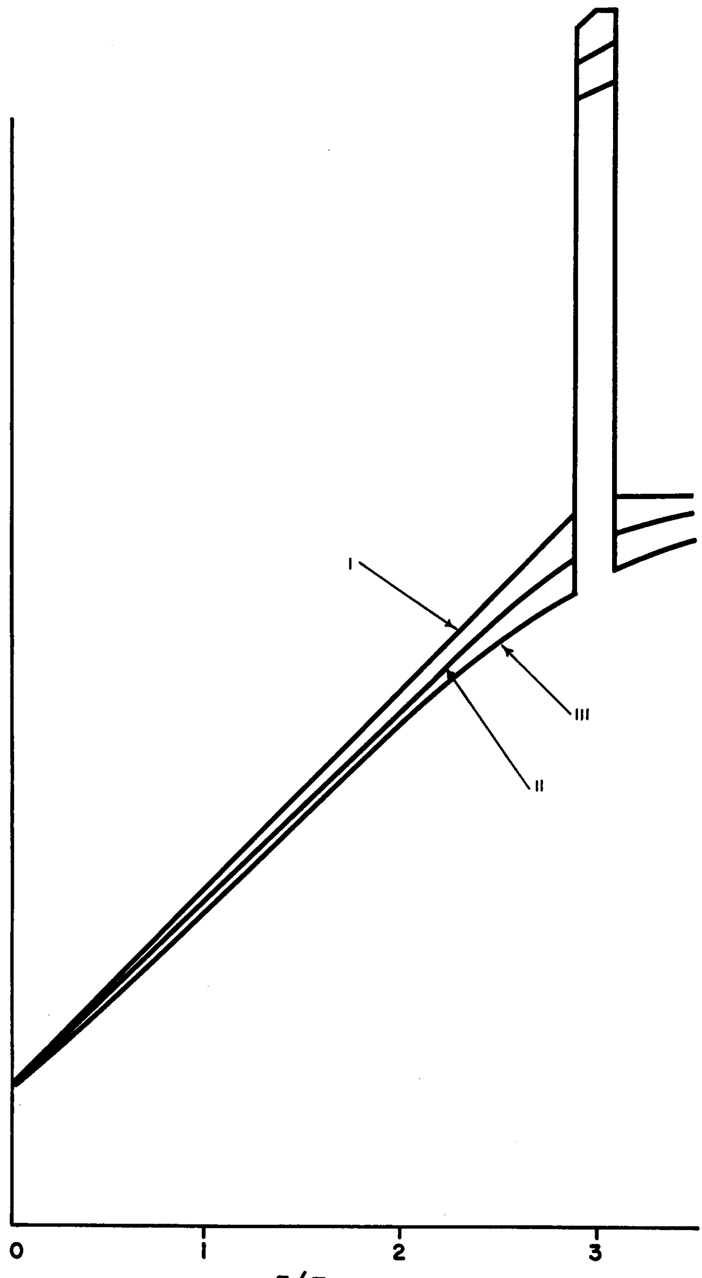

FIGURE 1(d)

Theoretical z-spectra ($x/\sigma = 3$ and $k = 0$)
for various perturbation distributions
[(i) double-exponential, (ii) double-exponential
convolved, (iii) Gaussian].

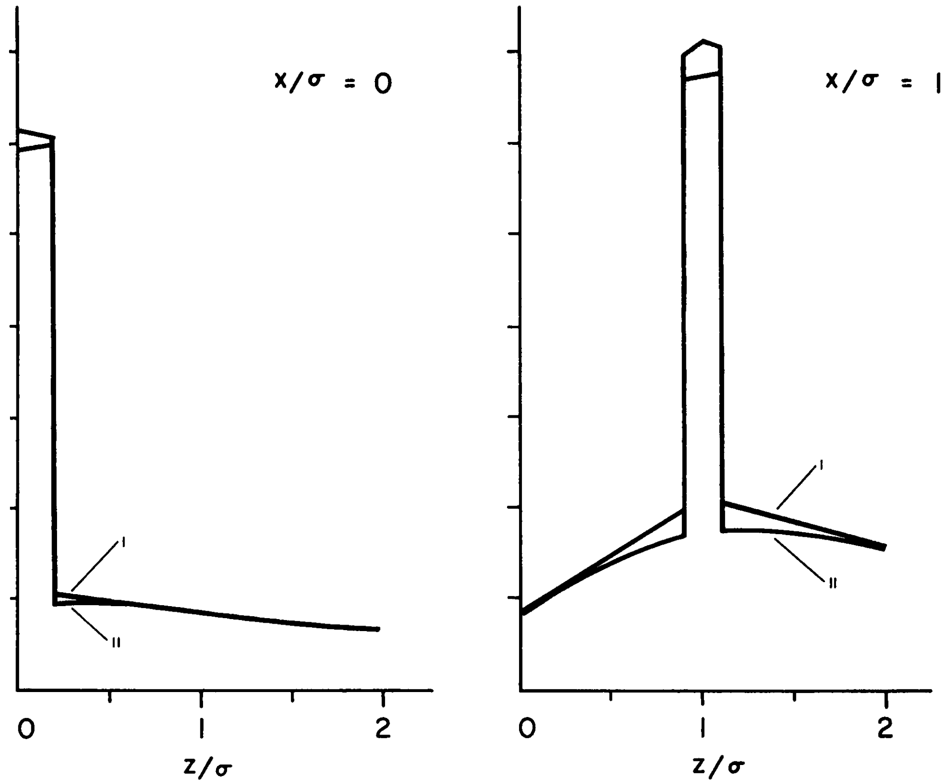

FIGURE 2

Theoretical z-spectra ($x/\sigma = 0$ and 1 with $k = 0.04$) for various perturbation distributions [(i) double-exponential, (ii) double-exponential convolved].

decays more rapidly in figure 3 than in figure 2 (with $x = 0$). It will be worth while looking into this in more detail when our information about the empirical z-spectrum has been extended to greater values of z (the data on which figure 3 is based may be subject to ascertainment errors beyond $z = 150 = 2\sigma$).

I should like in conclusion to thank Dr. J. M. Hammersley for allowing me to make use of the computed values of the function $c(\cdot)$ (Gaussian perturbations), and to express my gratitude to Professor H. H. Plaskett for encouraging me to carry out this work in the University Observatory, Oxford.

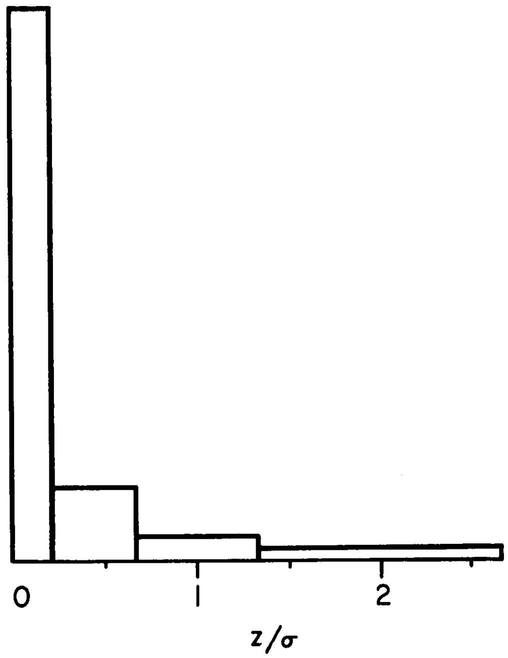

FIGURE 3

The observed z-spectrum.

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