THE DISTRIBUTION OF ENERGY PERTURBATIONS FOR HALLEY'S AND SOME OTHER COMETS

DAVID G. KENDALL
MAGDALEN COLLEGE, OXFORD UNIVERSITY

1. Introduction

As a result of conversations with Dr. J. M. Hammersley the writer became interested in the "random walk" which a comet performs along the energy scale when making its successive revolutions about the sun. The subject is of course far from being a new one; in particular, significant papers by H. N. Russell [13], A. J. J. van Woerkom [17], and J. H. Oort [12] must be mentioned. Recently, however, J. M. Hammersley and R. A. Lyttleton [9] have taken up some of the stochastic problems involved in great detail, and following on their work the relevant integral equations have been investigated (Kendall [10]) from a slightly different point of view. In all this analysis a conspicuous role is played by the frequency distribution of energy perturbations suffered by a comet during the transition from one aphelion to the next, and it therefore seemed appropriate to collect and discuss some empirical evidence on the form of this distribution which is available in virtue of earlier computations.

It is important for the theoretical work that one should know the distribution of energy perturbations for strongly bound as well as for loosely bound comets, and it is fortunate that the evidence is not entirely restricted to the loosely bound comets, even though the fragment of evidence relating to the other end of the energy scale concerns just one comet (Halley's) and that not a specially typical one.

2. Energy perturbations for Halley's comet

In 1907–1908 P. H. Cowell and A. C. D. Crommelin [1]–[5] made an extensive series of computations concerning the motion of Halley's comet. These were inspired in the first instance by the forthcoming reappearance in 1910 and by the desire to predict as accurately as possible the epoch of perihelion. Their best known series of calculations, and the most accurate, relates to the motion

Formerly Skynner Student in Astronomy, Balliol College, Oxford.

of the comet between 1759 and 1910, but independently and by a less precise method Cowell and Crommelin followed the comet back to 240 B.C. and were able to identify most of the 29 perihelia with recorded appearances, early Chinese observations often forming the basis for these identifications.

If we write z for the *negative* of the total energy per unit mass ($z = 0$ for a comet at rest at infinity) then z will be a positive constant multiple of $1/a$, where a is the semimajor axis of the instantaneous orbit, and it will be convenient to measure the *energy state* z of the comet by $1/a$ in (astronomical units)$^{-1}$. We shall then have

$$z = \frac{1}{a} = \frac{1}{P^{2/3}}, \tag{2.1}$$

where P is the instantaneous orbital period in years, and Russell's study [13] of the energy perturbations is based on the relations (2.1). He took Cowell and Crommelin's successive perihelion epochs, and differenced them to obtain successive estimates of P; a second differencing then gives ΔP, from which Δz can be inferred. It is clear that Russell did not intend this to be more than a very rough calculation, and in fact it is unsatisfactory in several respects. It is not clear in all cases what Cowell and Crommelin's final estimates of the perihelion epochs were, and these estimates are besides confused at two points by calendar ambiguities, which of course appear in amplified form in the column of second differences. Another objection to the Russell procedure is that his periods P are from perihelion to perihelion, whereas we are interested in the energy perturbation from aphelion to aphelion. An entirely different method of reducing the Cowell and Crommelin data has therefore been employed.

Cowell and Crommelin worked in terms of the mean motion for the instantaneous orbit; this is $n = 2\pi/P$ in radians per year. For this their first-order perturbation formula was

$$\Delta n = \frac{6\pi a^{1/2} m'}{365.256} \int \left\{ \left(\frac{x' - x}{\rho^3} - \frac{x'}{r'^3} \right) \sin u - (1 - e^2)^{1/2} \left(\frac{y' - y}{\rho^3} - \frac{y'}{r'^3} \right) \cos u \right\} du. \tag{2.2}$$

Here Δn is in seconds of arc per day, a is the unperturbed semimajor axis for the comet in a.u., m' is the mass of the perturbing planet in solar units, $(x', y', z'; r')$ are the coordinates and solar distance for the planet in a.u. while $(x, y, z; r)$ refer similarly to the comet; ρ is the distance from the planet to the comet, e is the eccentricity of the unperturbed orbit, and u is the comet's eccentric anomaly in seconds of arc.

For their purposes Cowell and Crommelin had to take the integration at (2.2) through a complete cycle (perihelion—perihelion), but fortunately for us they recorded separately the results for the four quadrants, so that by fitting these together suitably we can effect the integration for an aphelion-aphelion cycle, which is what we want. Their numerical quadratures were not carried out with the same accuracy throughout, and were subject to more extensive approximations for the earlier revolutions. If we are to use their data we must thus accept the approximations employed by Cowell and Crommelin at about 240 B.C. At this stage of their work, however, their procedure was simplified to such an extent that they were able to tabulate the perturbation Δn in mean motion, per quadrant of the comet's orbit, against the *planet's* mean anomaly in degrees, the latter being referred to the epoch at which the comet entered or left the quadrant. (The perturbations were tabulated for Jupiter and Saturn separately, and "the planet" means Jupiter or Saturn, as appropriate; perturbations due to other planets were neglected at this stage.)

In [2], pp. 177–179, we find a table which can be summarized in our table I.

TABLE I

THE TWO OUTER QUADRANTS

u	Jupiter		Saturn	
	90°	270°	90°	270°
0°	+0.3320	+0.0932	+0.0808	−0.0284
θ°	$f(\theta)$	$g(\theta)$	$F(\theta)$	$G(\theta)$

Here u is the comet's eccentric anomaly at the entry into the second and departure from the third quadrant, respectively, and θ is the mean anomaly of the disturbing planet at the corresponding epoch, tabulated by steps of 10 degrees. Similarly in the same issue of the *Monthly Notices* (pp. 458–459) we find table II.

TABLE II

THE TWO INNER QUADRANTS

	Jupiter		Saturn	
	1st quadrant	4th quadrant	1st quadrant	4th quadrant
0°	+0.85	+0.26	+0.19	−0.03
θ°	$h(\theta)$	$k(\theta)$	$H(\theta)$	$K(\theta)$

Here θ is the mean anomaly of the planet at the epoch of the *preceding* perihelion for the 1st quadrant, and at the epoch of the *following* perihelion for the 4th quadrant. In each table the entry is the perturbation in the instantaneous mean motion in seconds of arc per day.

If we take the instantaneous mean motion at perihelion (identified with the mean motion in the undisturbed orbit) to be n_0 seconds of arc per day, then the eccentric anomaly of the undisturbed comet will be 270°, 0° (perihelion), and 90° when the mean anomaly of the disturbing planet is $\alpha - \beta$, α, and $\alpha + \beta$, respectively, where

$$\beta = \beta_1 = \frac{26919.9}{n_0}\left(1 - \frac{2e}{\pi}\right) = \frac{10344.2}{n_0} \text{ degrees, for Jupiter,} \tag{2.3}$$

and

$$\beta = \beta_2 = \frac{10839.4}{n_0}\left(1 - \frac{2e}{\pi}\right) = \frac{4165.1}{n_0} \text{ degrees, for Saturn.} \tag{2.4}$$

(Here $e = 0.9672$; it is the eccentricity of the undisturbed orbit.) We have to be able to read off the values of $f(\alpha \pm \beta)$, and so on, from the Cowell and Crommelin tables, and so it is necessary to choose a value of β which will be a multiple of 10°. If we take $n_0 = 44.975$ then we shall have $\beta_1 = 230°$, while if we take $n_0 = 46.279$ then we shall have $\beta_2 = 90°$. It is of course inconsistent to use different values of n_0 for the calculations associated with the two planets, but the two calculations are quite independent and the two values of n_0 lie within the range of values found by Cowell and Crommelin; we shall therefore accept this very convenient further approximation.

Let us now write n_- and n_+ for the instantaneous mean motions at the preceding and at the following aphelion, respectively, and let us put

$$\begin{aligned} \Delta_-(n) &= n_0 - n_-, \\ \Delta_+(n) &= n_+ - n_0, \end{aligned} \tag{2.5}$$

and

$$\Delta(n) = \Delta_-(n) + \Delta_+(n) = n_+ - n_-; \tag{2.6}$$

it will be convenient to call these the entry, the exit, and the over-all perturbations, respectively. The contributions to the several perturbations from the two planets will then be as shown in table III. The over-all perturbation $\Delta(n)$ can

TABLE III

Perturbation	Jupiter	Saturn
$\Delta_-(n)$	$k(\alpha_1) + g(\alpha_1 - 230)$	$K(\alpha_2) + G(\alpha_2 - 90)$
$\Delta_+(n)$	$h(\alpha_1) + f(\alpha_1 + 230)$	$H(\alpha_2) + F(\alpha_2 + 90)$

be found by summing the entries in the columns; the total over-all perturbation (both planets combined) then follows on adding the perturbations due to Jupiter and Saturn separately.

The perturbation due to a single planet is thus a function defined on the perimeter of a circle (α ranges from 0° to 360°), while the total perturbation is a function defined on a torus (α_1 and α_2 each range from 0° to 360°). Because the mean anomalies of the two planets are essentially time variables and because there is no permanent relation between their positions in their respective orbits, it is clear that in order to find the distribution of values of the perturbation we must regard α_1 and α_2 as independent random variables uniformly distributed from 0° to 360°. In the present circumstances it will be sufficient to confine α_1

and α_2 to the values 0°, 10°, 20°, $\cdots$, 350°, and to treat the 36 possible values of each α-variable as equally likely. We are therefore replacing the true distribution of the perturbation by the distribution of the values assumed at a lattice of 1,296 points on the torus on which it is defined.

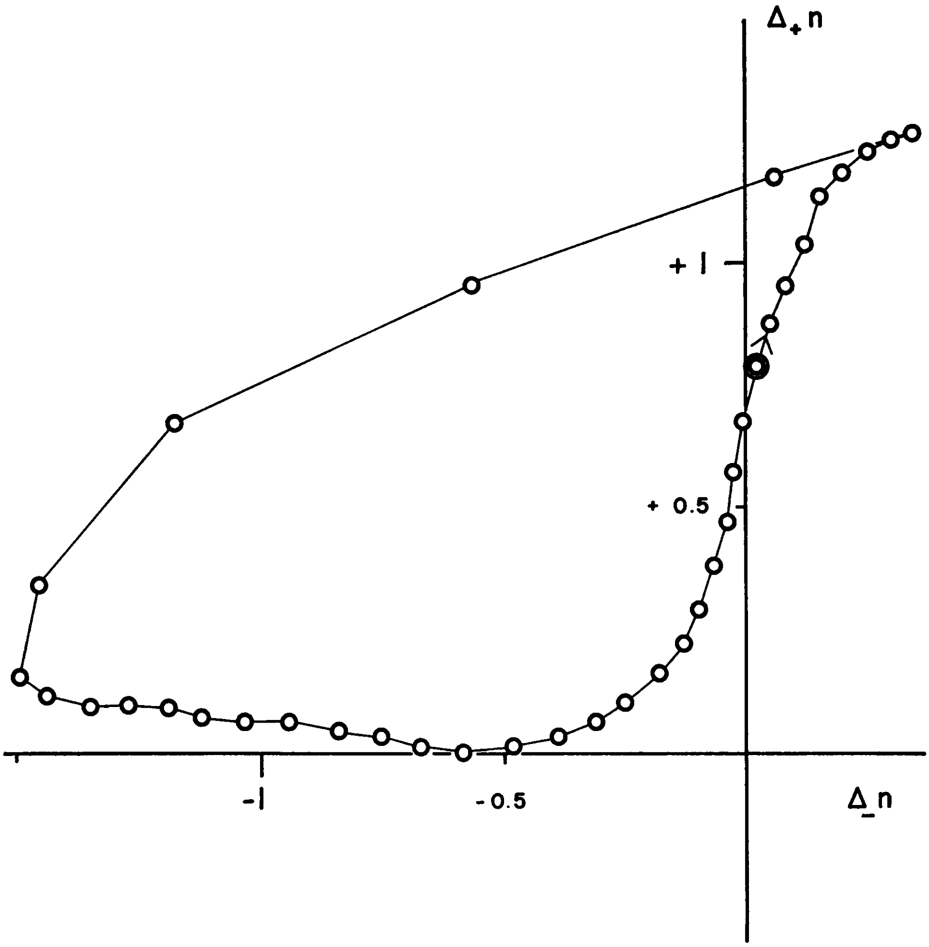

FIGURE 1

Perturbations Δn in the mean motion of Halley's comet when approaching ($-$) and receding from ($+$) perihelion, for each ten degrees in the mean anomaly of Jupiter.

As might be expected, the main perturbation is that due to Jupiter. Figure 1 shows the perturbations $\Delta_-(n)$ and $\Delta_+(n)$ in mean motion (seconds of arc/day) due to the influence of Jupiter alone for the 36 representative values of the mean anomaly α_1 of Jupiter when the comet is at perihelion. The point corresponding

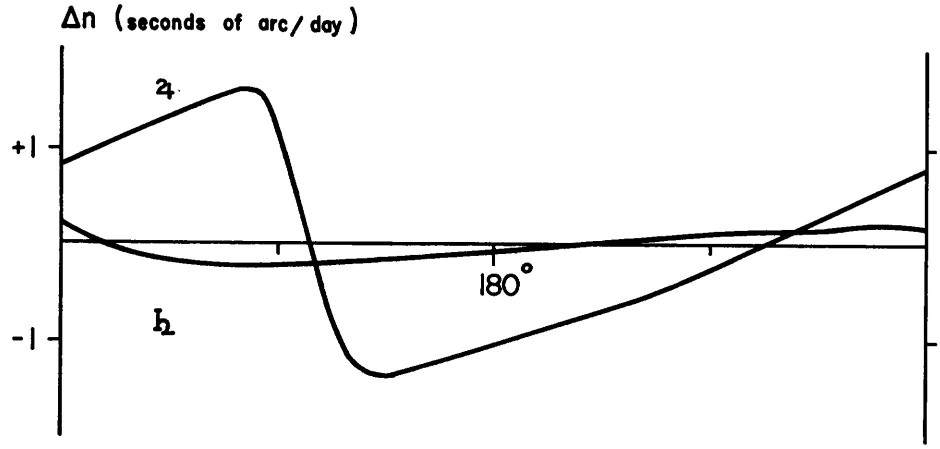

FIGURE 2

Over-all perturbations Δn in the mean motion of Halley's comet, as a function of the mean anomaly of the perturbing planet when the comet is at perihelion.

to $\alpha_1 = 0°$ is specially marked, and α_1 increases in the direction of the arrow.

Figure 2 compares the over-all perturbations $\Delta(n)$ due to Jupiter and Saturn, respectively. Here the horizontal scale measures α_1 for Jupiter and α_2 for Saturn. It must be remembered that in order to find the total over-all perturbation we

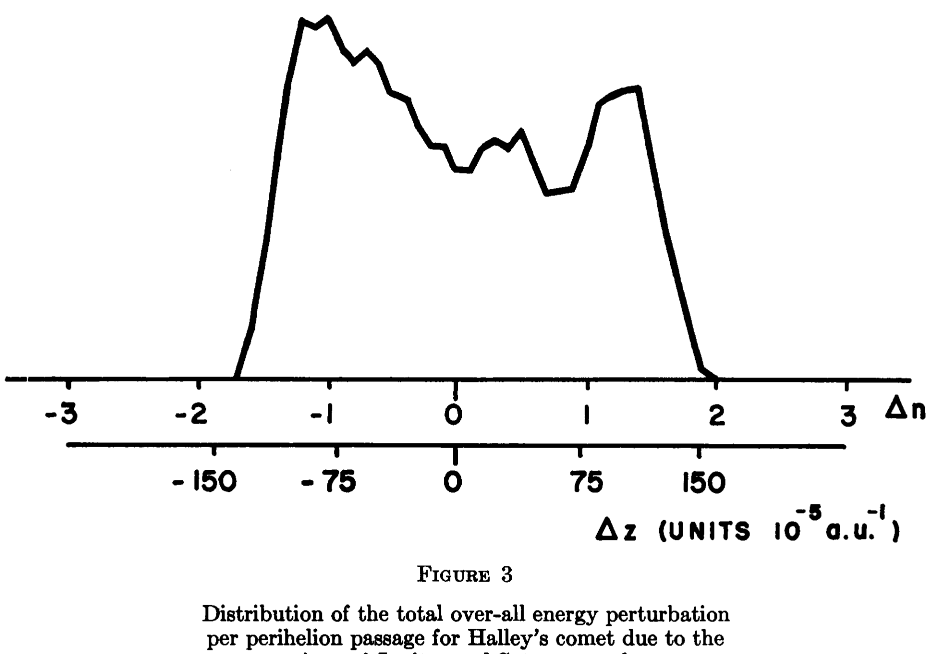

FIGURE 3

Distribution of the total over-all energy perturbation per perihelion passage for Halley's comet due to the actions of Jupiter and Saturn together.

have to combine each ordinate of the Jupiter curve equifrequently with each ordinate of the Saturn curve.

Histograms for the over-all perturbations due to Jupiter and Saturn were constructed separately and then combined by convolution; the result is shown in figure 3, where the two horizontal scales show the over-all total perturbation in n (in seconds of arc per day) and in the energy state z (in a.u.$^{-1}$). (The conversion from n to z was based on the formulas $dn/n = -3\, da/2a = +3\, dz/2z$, with $n_0 = 44.975$.)

The root-mean-square total over-all perturbation in energy state was estimated to be 0.000 765 a.u.$^{-1}$.

We can also calculate the average total entry and exit perturbations, taking account of algebraic sign; we find

$$\text{algebraic mean value of } \Delta_-(z) = -0.000\,432 \text{ a.u.}^{-1}; \qquad (2.7)$$
$$\text{algebraic mean value of } \Delta_+(z) = +0.000\,428 \text{ a.u.}^{-1}.$$

We shall refer to these figures below in connection with the work of E. Sinding. They indicate that, on the average, the orbit of Halley's comet is nearer to the parabolic form at perihelion than at aphelion.

3. Energy perturbations for some near-parabolic comets

In 1910 G. Fayet [6] made approximate determinations of the preorbits (that is, the orbits prior to entry into the planetary zone) of 146 comets and in 1933 [7] he published similar determinations of the postorbits (the orbits after emergence from the planetary zone) of 36 comets, all the comets in each of the two series having very long periods and nearly parabolic orbits. These results of Fayet have often been quoted, but it does not seem to have been noticed that there are 28 comets common to both lists. One of these (1897 I) has to be rejected because (according to Sinding) Fayet used erroneous elements for it, but even so this leaves us with 27 comets for which both the preorbit and postorbit are known, at least approximately. We are therefore able to find approximate values for $\Delta_-(z)$, $\Delta_+(z)$, and $\Delta(z)$, and to examine their distribution.

Fayet worked with the eccentricity e instead of with the semimajor axis a, and because of this a further approximation is needed before we can convert his findings into energy perturbations. We have $q = a(1 - e)$, where q is the perihelion distance, and so

$$\Delta\left(\frac{1}{a}\right) \cong \frac{-\Delta(e)}{q} + (1 - e)\Delta\left(\frac{1}{q}\right). \qquad (3.1)$$

The value of $\Delta(q)$ is not known, but we can neglect the second term on the right side of (3.1) because $|1 - e|$ is small for these near-parabolic comets; we thus have

$$\Delta(z) = \Delta\left(\frac{1}{a}\right) \cong \frac{-\Delta(e)}{q_0}, \qquad (3.2)$$

where q_0 is the perihelion distance actually observed. The resulting distribution for $\Delta(z)$ (the over-all perturbation in energy state from *entry into* to *exit from* the planetary zone) is shown in figure 4. *The root-mean-square over-all perturbation in energy state is* 0.000 791 a.u.$^{-1}$.

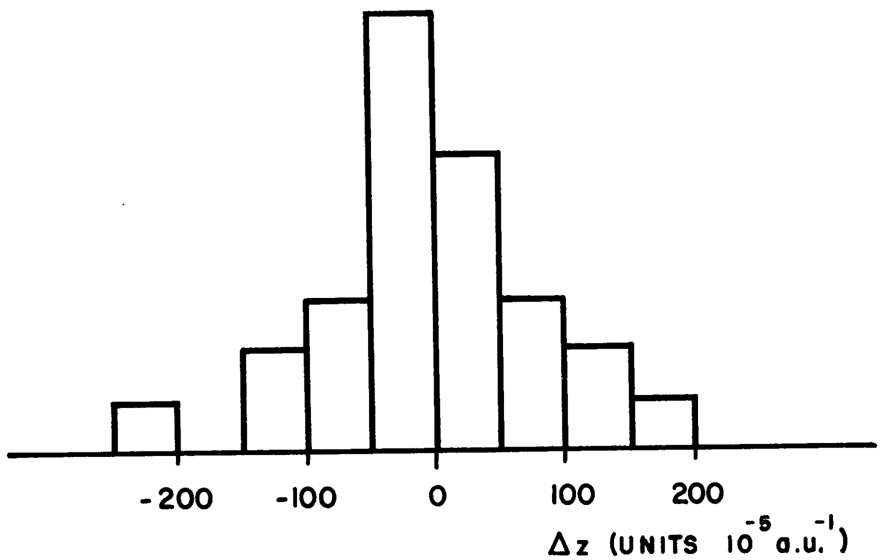

FIGURE 4

Distribution of Δz.

On comparing the preorbit with the perihelion orbit, and the latter with the postorbit, we further find

$$\text{(3.3)} \qquad \begin{aligned} &\text{algebraic mean value of } \Delta_-(z) = -0.000\,492 \text{ a.u.}^{-1}; \\ &\text{algebraic mean value of } \Delta_+(z) = +0.000\,469 \text{ a.u.}^{-1}. \end{aligned}$$

As with Halley's comet, we see that on the average the orbit moves toward the hyperbolic form during the approach to perihelion, and moves back toward the elliptic form during the recession from perihelion.

The correlation coefficient between $\Delta_-(z)$ and $\Delta_+(z)$ is $+0.35$, but this is not significantly different from zero ($t_{25} = +1.85$).

The historical importance of Fayet's work was the demonstration that the hyperbolic form occasionally found for the perihelion orbit could be attributed to a large negative value for $\Delta_-(z)$, the preorbit being parabolic or even elliptic. The approximate character of this investigation made it necessary for Fayet's findings to be checked by exact computation, and this program was initiated by E. Strömgren [16] in 1914. Of 24 comets which have now been examined in this way, 21 proved to have elliptic preorbits, and the three (1886 I, 1898 VII, and 1899 I) with hyperbolic preorbits had a value for z_- not significantly different from zero. Evidence continues to accumulate and largely supports the view that there are no genuinely hyperbolic preorbits. (A few preorbits have also been computed by Galibina [8], using Makover's method. For comet 1914 III she finds a hyperbolic preorbit with $z_{-} = -0.000\,094$. I do not know how much weight should be given to this exception.)

The calculations by Strömgren and others yield much more accurate values for $\Delta_{-}(z)$ than are available from the work of Fayet, but until quite recently there were few correspondingly accurate values for $\Delta_{+}(z)$. In 1955 S. G. Makover [11] presented a new method for computing accurate pre- and postorbits and this, according to I. V. Galibina [8], is sufficiently simple for one to expect in the future that accurate pre- and postorbits will be computed for all adequately observed long-period comets as a matter of course. If this view is correct, data of the kind we are discussing here may be expected to accumulate very rapidly in the near future. Already Galibina has computed 20 postorbits; 12 were hyperbolic and 8 were elliptic. The most extreme example of a hyperbolic postorbit is that of comet 1899 I, for which $1/a_{+} = -0.001\,300$ a.u.$^{-1}$; previously Sinding [15] had found $1/a_{+} = -0.000\,775$ for comet 1898 VII. These are comets which have definitely been lost to the solar system.

We thus have available 24 accurate values for $\Delta_{-}(z)$ computed by Strömgren and his fellow workers, and 20 accurate values for $\Delta_{+}(z)$ computed by Galibina; unfortunately the number of comets for which both $\Delta_{-}(z)$ and $\Delta_{+}(z)$ are available is rather small, and this has given rise to a method for estimating the distribution of $\Delta(z)$ which must now be explained.

We have already remarked on the tendency for $\Delta_{-}(z)$ to be negative and for $\Delta_{+}(z)$ to be positive. A theoretical explanation of this effect has been worked out by E. Sinding [14], who showed that the algebraic mean values of $\Delta_{-}(z)$ and $\Delta_{+}(z)$ should be about $-(+)0.000\,54$. Sinding's calculations are approximate (for example, perturbations other than those due to Jupiter are neglected), but it will be seen that his predictions agree both as to sign and order of magnitude with the values found by Fayet for long-period comets, and even with the values we have found for Halley's comet. In order to determine the distribution of $\Delta(z)$ when only that of $\Delta_{-}(z)$ [or of $\Delta_{+}(z)$] is known, it has been suggested (see, for example, Oort [12]) that we can assume that $\Delta_{-}(z)$ and $\Delta_{+}(z)$ are identically and independently distributed about equal and opposite mean values; if this is so then we can estimate the root-mean-square value of $\Delta(z)$ by calculating the standard deviation of $\Delta_{-}(z)$ [or of $\Delta_{+}(z)$] and then multiplying it by the square root of two. In this way we find

For 24 *long-period comets* (*computations by Strömgren et al.*)

$$\begin{aligned} \text{algebraic mean value of } \Delta_{-}(z) &= -0.000\,573; \\ \text{standard deviation of } \Delta_{-}(z) &= 0.000\,273; \\ \text{estimated r.m.s. value of } \Delta(z) &= 0.000\,386. \end{aligned}$$

For 20 *long-period comets* (*computations by Galibina*)

$$\begin{aligned} \text{algebraic mean value of } \Delta_{+}(z) &= +0.000\,473; \\ \text{standard deviation of } \Delta_{+}(z) &= 0.000\,474; \\ \text{estimated r.m.s. value of } \Delta(z) &= 0.000\,671. \end{aligned}$$

It is suggested that the (significant) difference between the two standard deviations is due to a selection effect; in selecting comets for study in this way priority has been given to comets with hyperbolic perihelion orbits. If there are no genuinely hyperbolic preorbits, this means that we are in effect selecting for large negative values of $\Delta_-(z)$, and this must have reduced the resulting standard deviation for $\Delta_-(z)$ (and made more negative the algebraic mean value). *We can eliminate this selection effect by using the estimate of the root-mean-square over-all perturbation in energy state based on the values of* $\Delta_+(z)$, *namely,* 0.000 671 a.u.$^{-1}$.

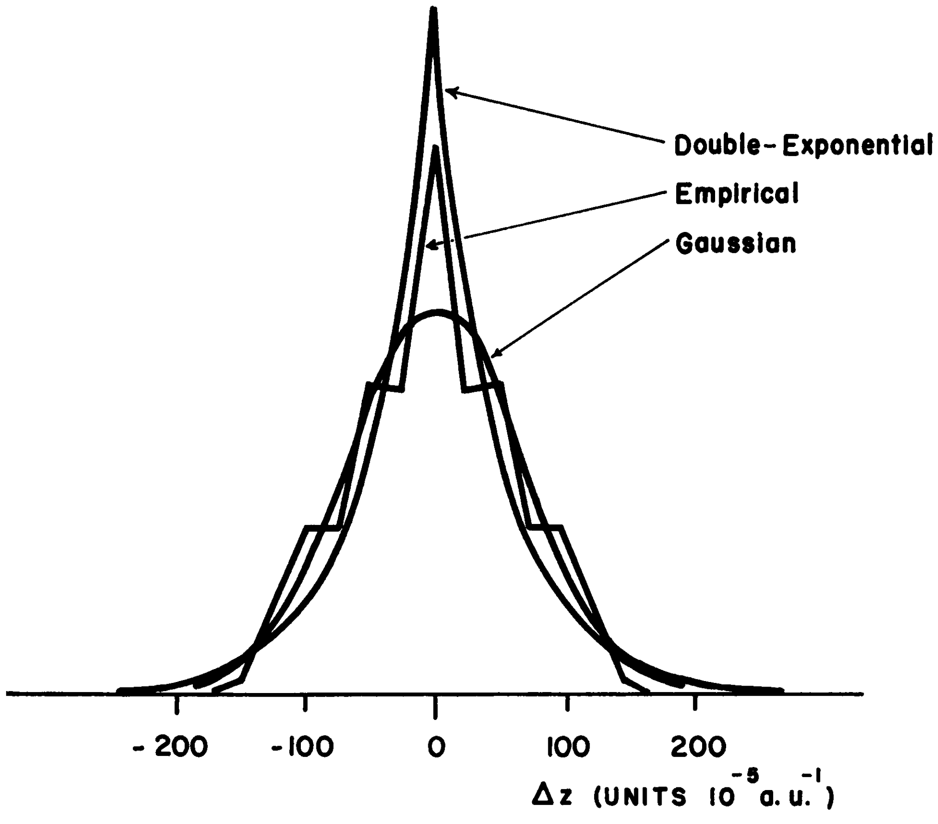

FIGURE 5

Comparison of Gaussian, double-exponential and empirical distribution of Δz.

To obtain an estimate of the distribution of $\Delta(z)$ from Galibina's calculations we have assumed that $\Delta_-(z)$ and $\Delta_+(z)$ are independent, and that $\Delta_-(z)$ has the same distribution as $-\Delta_+(z)$; in this way we obtain the convolved histogram shown in figure 5. The ordinates in figure 5 have been scaled so that the total enclosed area is the same as in figure 4; thus a direct comparison of the two frequency curves is legitimate. Frequency curves for a Gaussian distribution and for the double-exponential distribution used in [10] have been superimposed on the convolved histogram.

Both figure 5 and figure 4 indicate that the assumption of a Gaussian distribution for $\Delta(z)$ (employed by Oort [12] and by Hammersley and Lyttleton [9]) is quite reasonable. The rather peculiar distribution of $\Delta(z)$ for Halley's comet (shown in figure 3) need not be taken too seriously, because here we have examined the distribution of $\Delta(z)$ corresponding to variations in the phase (α_1, α_2) of the solar system, the orbit of the comet itself being held fixed. It is not to be supposed that the same distribution of $\Delta(z)$ would have been obtained if the Cowell and Crommelin calculations had been carried out for another medium-period comet.

The three estimates of the root-mean-square over-all perturbation in energy state,

$$\begin{aligned} \sigma_1 &= 0.000\,76 \text{ (Halley's comet)}, \\ \sigma_2 &= 0.000\,79 \text{ (27 comets of Fayet)}, \\ \sigma_3 &= 0.000\,67 \text{ (20 comets of Galibina)}, \end{aligned} \tag{3.4}$$

are in general accord with one another and with the value

$$\sigma_0 = 0.000\,78 \tag{3.5}$$

found by van Woerkom [17] for a theoretical ensemble of parabolic comets having a perihelion distance of 1 a.u., to which he applied an analysis of the Fayet type. We may therefore confidently adopt a provisional value of $\sigma = 0.000\,75$ a.u.$^{-1}$ until further data make possible a more accurate estimate.

In conclusion I should like to express my gratitude to Professor H. H. Plaskett, F.R.S., who kindly made available to me the resources of the University Observatory, Oxford, while this paper was being written.

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