ON A CLASS OF INFINITESIMAL GENERATORS AND THE INTEGRATION PROBLEM OF EVOLUTION EQUATIONS

KÔSAKU YOSIDA
UNIVERSITY OF TOKYO

1. Introduction

The theory of semigroups of bounded linear operators deals with exponential functions in infinite dimensional function spaces. It has been used, as an operator-theoretical substitute for the Laplace transform method, in the integration problem of temporally homogeneous evolution equations, especially of diffusion equations and wave equations (see Hille and Phillips [5] and Yosida [20], [21]).

The purpose of my paper is to call attention to a class of semigroups which is characterized by either one of the three mutually equivalent conditions to be explained below; one of them reads that the semigroup T_t satisfies

$$\lim_{t \downarrow 0} t \left\| \frac{d}{dt} T_t \right\| < \infty. \tag{1}$$

The semigroups arising from the integration in L_2 of temporally homogeneous diffusion equations belong to this class. And the unique continuation theorem of diffusion equations, inaugurated by Yamabe and Itô [6] may be explained by the time-like analyticity of the corresponding semigroups. The situation has an intimate connection with the theory of analytical vectors published recently by Nelson [14]. There is a procedure to obtain semigroups of our class. Let A be the infinitesimal generator of a contraction semigroup. We can define, following Bochner [3], Feller [4], Phillips [15], and Balakrishnan [1], the fractional powers $-(-A)^{\alpha}$ of A and the semigroups generated by them belong to our class. Balakrishnan gave an interesting application of the operator $-(-A)^{1/2}$ to Hille's reduced Cauchy problem for equations $d^2u/dt^2 + Au = 0$.

Tanabe [19] has recently devised an ingenious method of integration of temporally inhomogeneous evolution equations in Banach spaces: $du/dt = A(t)u$. He assumes, for fixed t, that $A(t)$ is the infinitesimal generator of a semigroup of our class. He further assumes a certain regularity condition with respect to t of $A(t)$ which is the same as that introduced by Kato [8] for the integration of such equations. Under these conditions, Tanabe proved that the solution may be obtained by successive approximation starting with the first approximation $\exp[(t - s)A(s)]$. In this way, he has shown that Levi's classical construction of the fundamental solution $U(t, s)$, with $t > s$, of a diffusion equation may be adapted to the evolution equations in Banach spaces. Komatsu [10] gave an important remark that, if $A(t)$ is analytic in t, then the fundamental solution $U(t, s)$ of Tanabe is also analytic in t and s. In this way, Komatsu proved a unique continuation theorem for temporally inhomogeneous diffusion equations, which was proved in a direct way by Shirota [17].

2. A class of semigroups

Let X be a complex Banach space. A one-parameter family T_t, with $t \geqq 0$, of bounded linear operators in X is said to be a semigroup of type $S(M, \beta)$ if it satisfies the conditions

$$T_tT_s = T_{t+s}, \qquad T_0 = I \text{ (the identity)}, \tag{2}$$

$$s - \lim_{t \to t_0} T_t x = T_{t_0} x, \qquad x \in X;\ t_0 \geqq 0. \tag{3}$$

Such a semigroup satisfies, as was proved by Hille [5]

$$\|T_t\| \leqq M e^{\beta t}, \qquad t \geqq 0, \tag{4}$$

with positive constants M and β. It is well known that the infinitesimal generator A of T_t defined by $A \cdot x = s - \lim_{t \downarrow 0} t^{-1} (T_t - I)x$ generates T_t by various equivalent procedures; one of them states that

$$T_t x = s - \lim_{n \to \infty} \left(I - \frac{t}{n} A\right)^{-n} x. \tag{5}$$

Hence we shall write $T_t = \exp (tA)$.

Let θ be a positive constant $\leqq \pi/2$, and Σ_θ be the sector $|\arg z| < \theta$ in the complex z-plane. If a semigroup $T_t \in S(M, \beta)$ is analytically continuable into Σ_θ in such a way that $T_{t \exp(i\varphi)}$ is, for all φ with $|\varphi| < \theta$, of type $S(M', \beta')$ with a fixed pair (M', β'), then we say that T_t is of type $H(\theta, M', \beta')$.

Our class of semigroups is characterized by any one of the three conditions in the following

THEOREM 1. *Let A be a closed linear operator with domain $D(A)$ dense in X and range in X. Then the following three conditions are mutually equivalent*

$$\exp (tA) \in H(\theta, M, \beta) \quad \textit{for some } \theta, M, \textit{and } \beta, \tag{6}$$

(7) $\exp (tA) \in S(M', \beta')$ *for some* M', β', *and* $\exp (tA)$ *is strongly differentiable in* t *in such a way that* $\overline{\lim}_{t \downarrow 0}\, t \left\| \frac{d}{dt} \exp (tA) \right\| < \infty$,

(8) *there exist positive constants* M'' *and* β'' *such that*

$$\|(\lambda I - A)^{-n}\| \leqq M''(\lambda - \beta'')^{-n} \quad \textit{for} \quad \lambda > \beta'', \quad n = 1, 2, \cdots,$$

$$\overline{\lim_{|\tau| \uparrow \infty}}\, |\tau| \cdot \|[(\sigma + \sqrt{-1}\,\tau)I - A]^{-1}\| < \infty, \quad \textit{for} \quad \sigma > \beta''.$$

The constants M, β, M', β', M'', and β'' are dependent on each other. For the proof of the equivalence of (6) and (7) see Yosida [22], and for that of (6) and (8) see Yosida [22] and Hille and Phillips [5]. We remark that the following facts are used in these proofs. First, (7) implies

$$\frac{d^n}{dt^n} \exp(tA) = A^n \exp(tA) = \left[A \exp\left(\frac{t}{n} A\right)\right]^n. \tag{7'}$$

Secondly, the last condition in (8) implies that $(\lambda I - A)^{-1}$ exists and is analytic in λ for large value of $|\tau| = |\mathrm{Im}\,(\lambda)|$ outside a sector of the left half λ-plane defined by the boundary curve of the form $\lambda(s) = \sigma(s) + i\tau(s)$ such that

$$\lim_{\tau(s)\uparrow\infty} \frac{-\sigma(s)}{\tau(s)} = \tan\epsilon = \lim_{\tau(s)\downarrow-\infty} \frac{\sigma(s)}{\tau(s)}, \qquad \epsilon > 0. \tag{9}$$

Moreover, $\|(\lambda I - A)^{-1}\|$ is of the order $|\tau|^{-1}$ when $|\tau|$ tends to ∞ outside the above sector and lying in the left half λ-plane. Thirdly, the representation theorem of Hille holds for the semigroup satisfying (8)

$$\exp(tA) = \frac{1}{2\pi i} \int_{\lambda(s)} e^{\lambda t}(\lambda I - A)^{-1}\, d\lambda, \qquad t > 0, \tag{10}$$

the integral being taken in the uniform operator topology along the path of integration $\lambda(s) = 2^{-1}\sigma(s) + i\tau(s)$.

3. Unique continuation theorem of the diffusion equations

Consider a diffusion equation

$$\frac{\partial u}{\partial t} = Au, \qquad t > 0, \tag{11}$$

where the differential operator

$$A = a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + b^i(x) \frac{\partial}{\partial x_i} + c(x) \tag{12}$$

is strongly elliptic in a connected region G of an m-dimensional Euclidean space E^m. For the sake of simplicity of the exposition we assume that $G = E^m$. We assume that the real-valued coefficients a, b, and c are C^∞ in E^m and that

(13) *$a^{ij}(x)$ and its first and second partials, $b^i(x)$ and its first partials, and $c(x)$ are, in absolute values, all bounded on E^m by a positive constant γ and δ such that*

$$\gamma \sum_{j=1}^m \xi_j^2 \geqq a^{ij}(x)\xi_i\xi_j \geqq \delta \sum_{j=1}^m \xi_j^2 \qquad \text{on} \qquad E^m \tag{14}$$

for every real vector $(\xi_1, \xi_2, \cdots, \xi_m)$.

Let $H_1 = H_1(E^m)$ be the space of complex-valued C^∞ functions $f(x) = f(x_1, x_2, \cdots, x_m)$ in E^m for which

$$\|f\|_1 = \left(\int_{E^m} |f(x)|^2\, dx + \sum_{j=1}^m \int_{E^m} |f_{x_j}(x)|^2\, dx\right)^{1/2} < \infty. \tag{15}$$

Then the completion of H_1 by the norm

$$\|f\| = \Big(\int_{E^m} |f(x)|^2\,dx\Big)^{1/2} \tag{16}$$

is the space $L_2(E^m) = L_2$.

LEMMA 1. *Let us consider A as an operator defined on $\{f; f \in H_1, Af \in H_1\} \subseteq L_2$ into L_2. Then the smallest closed extension $\hat{A}$, in L_2, of A is the infinitesimal generator of a semigroup $T_t = \exp(tA)$ of type $S(1, \beta)$.*

For the proof, see Yosida [21] and [23]. See also Phillips [15]. The proof is based upon the Milgram-Lax theorem [7].

LEMMA 2. *A satisfies the condition* (8).

PROOF. (Yosida [23]. Compare Phillips [16].) Let $\sigma > 0$ be sufficiently large. Then we obtain, by partial integration, and by making use of the inequality $|\epsilon\kappa| \leqq 2^{-1}(|\epsilon|^2 + |\kappa|^2)$,

$$\begin{aligned}
&\operatorname{Re}\{[(\sigma + \sqrt{-1}\,\tau)I - A]w, w\} \\
&\quad = \sigma\|w\|^2 + \operatorname{Re}\Big\{\int_{E^m} a^{ij}\frac{\partial w}{\partial x_i}\frac{\partial \bar{w}}{\partial x_j}\,dx + \int_{E^m}\frac{\partial a^{ij}}{\partial x_i}\frac{\partial w}{\partial x_j}\bar{w}\,dx \\
&\qquad - \int_{E^m} b^i\frac{\partial w}{\partial x_i}\bar{w}\,dx - \int_{E^m} cw\bar{w}\,dx\Big\} \\
&\quad \geqq (\sigma - \delta - \eta_0)\|w\|^2 + (\delta - m\eta\nu)\|w\|_1^2,
\end{aligned} \tag{17}$$

where $(\delta - m\eta\nu) > 0$ and $\eta_0 = m\eta(m\nu^{-1} - \nu + m^{-1}) > 0$ for sufficiently small $\nu > 0$. Similarly we have

$$\begin{aligned}
&|\operatorname{Im}\{[(\sigma + \sqrt{-1}\,\tau)I - A]w, w\}| \\
&\quad \geqq |\tau|\cdot\|w\|^2 - m\eta\{\|w\|_1^2 + m\|w\|^2\} = (|\tau| - m^2\eta)\|w\|^2 - m\eta\|w\|_1^2.
\end{aligned} \tag{18}$$

If we assume that there exists $w \in H_1$, with $w \neq 0$, and sufficiently large $|\tau|$ such that $m\eta\|w\|_1^2 \geqq 2^{-1}(|\tau| - m^2\eta)\|w\|^2$, then

$$\operatorname{Re}\{[(\sigma + \sqrt{-1}\,\tau) - A]w, w\} \geqq (\delta - m\eta\nu)\frac{|\tau| - m^2\eta}{2m\eta}\|w\|^2. \tag{19}$$

Therefore, by virtue of the Schwarz inequality,

$$\|[(\sigma + \sqrt{-1}\,\tau) - A]w\|\cdot\|w\| \geqq |\{[(\sigma + \sqrt{-1}\,\tau) - A]w, w\}|, \tag{20}$$

we see that A satisfies the latter condition in (8). The former condition in (8) is clear since we have proved, in lemma 1, that $\hat{A}$ is the infinitesimal generator of a semigroup of type $S(1, \beta)$.

LEMMA 3. *For any $f \in L$, $u(t, x) = \exp(t\hat{A})f(x)$ is C^∞ in $t > 0$ and in $x \in E^m$ and satisfies the Cauchy problem*

$$\begin{aligned}
&\frac{\partial u}{\partial t} = Au, \qquad t > 0, \\
&L_2 - \lim_{t \downarrow 0} u(t, x) = f(x).
\end{aligned} \tag{21}$$

PROOF. If we apply, in the sense of the distribution of Schwartz, the elliptic differential operators $(\partial^2/\partial t^2 + A)$ any number of times to $u(t, x)$, then the result is locally square integrable in the product space $(0 < t < \infty) \times E^m$. Thus, by the Weyl-Schwartz theorem, $u(t, x)$ is equivalent to a function which is C^∞ in $(0 < t < \infty) \times E^m$. Hence the lemma is proved.

Now, by the time-like analyticity, as proved in lemma 2, of the semigroup $T_t = \exp(t\hat{A}) \in H(\theta, M, \beta)$, it is easy to see that the above solution $u(t, x)$ satisfies the *time-like unique continuation theorem*, Yosida [23]: if, for a certain $t_0 > 0$, $u(t_0, x) = 0$ on an open domain G of E^m, then $u(t, x) = 0$ for all $t > 0$ and all $x \in G$. Hence, by applying the space-like *unique continuation theorem* of Mizohata [13], we see that $u(t, x)$ satisfies the *unique continuation theorem*, Itô and Yamabe [6]: if, for a certain $t_0 > 0$, $u(t_0, x) = 0$ on an open domain G of E^m, then $u(t, x) = 0$ for all $t > 0$ and all $x \in E^m$.

REMARK. By virtue of (7′) and (7), we see that $u(t, x) = \exp(tA)f(x)$ is an analytic vector in the sense of Nelson [14], that is, for any fixed $t > 0$,

$$\sum_{n=1}^{\infty} \epsilon^n (n!)^{-1} \| A^n \exp(t_0 A) f \| < \infty \tag{22}$$

for sufficiently small $\epsilon > 0$. Hence, if we assume that the coefficients of the differential operator A are real analytic in x, then $u(t, x)$ is real analytic in x. Therefore, in this case, the unique continuation theorem for $u(t, x)$ may be proved without appealing to Mizohata's result. This observation is due to Komatsu [10], [11].

4. Fundamental solutions of temporally inhomogeneous evolution equations

Consider an equation of evolution

$$\frac{du}{dt} = A(t)u(t), \qquad a \leqq t \leqq b, \tag{23}$$

where $u(t) \in X$ and $A(t)$ is a linear operator in X. Such an equation was investigated by Kato [8] under the assumption that $A(t) \in S(1, 0)$, and recently by Tanabe [19] under the assumption that $A(t) \in H(\theta, M, \beta)$. A family of bounded linear operators $U(t, s)$ in X is called a fundamental solution of (23) if it satisfies the following conditions:

(24) *$U(t, x)$ is defined for $t \geqq s$ and is strongly continuous there,*

$$U(t, t) = I, \tag{25}$$

(26) *for every $x \in D[A(s)]$, $U(t, s)x$ belongs to the domain $D[A(t)]$ and is strongly differentiable in t such that $\frac{d}{dt} U(t, s)x = A(t)U(t, s)x$.*

If a fundamental solution $U(t, s)$ exists and if the uniqueness theorem for the Cauchy problem of (23) holds, then every solution $u(t)$ of (23) is expressed as

$u(t) = U(t, s)u(s)$. The uniqueness theorem was proved by Kato [8] under the sole solution that $A(t) \in S(1, 0)$.

Tanabe proved that (23) has a fundamental solution under the following conditions:

(27) $\exp[\tau A(t)] \in S(\theta, M, \beta)$ *for* $t \in [a, b]$, $\tau > 0$, *and that the constants* θ, M, *and* β *are independent of* t,

(28) *the domain* $D[A(t)] = D$ *is independent of* t,

(29) *there exists a bounded linear operator* A_0 *which maps* X *onto* D *in a one-to-one manner and such that*

$$B(t) = A(t)A_0 \tag{30}$$

is uniformly Lipschitz continuous in t *in the uniform topology of operators and is strongly continuously differentiable in* t.

Tanabe's construction of the fundamental solution may be written as

$$\begin{aligned}
U(t, s) &= \exp[(t - s)A(s)] + W(t, s),\\
W(t, s) &= \int_s^t \exp[(t - r)A(r)]R(r, s)\, dr,\\
R(t, s) &= \sum_{m=1}^{\infty} R_m(t, s),\\
R_1(t, s) &= \begin{cases} [A(t) - A(s)]\exp[(t - s)A(s)], & t > s,\\ 0, & t = s,\end{cases}\\
R_m(t, s) &= \int_s^t R_1(t, r)R_{m-1}(r, s)\, dr, \qquad m = 2, 3, \cdots.
\end{aligned} \tag{31}$$

He proved that every operator appearing in the above formulas is strongly continuous in t and s, with $s < t$, and that every integral and series converge. The crucial points in his proof are revealed in the following lemmas.

LEMMA 4. $\exp[(t - s)A(s)]$ *is strongly differentiable in* s *and* t *and* $(\partial/\partial t + \partial/\partial s)\exp[(t - s)A(s)]$ *is uniformly bounded in* $a \leqq s < t \leqq b$.

LEMMA 5. *There exist positive constants* K_1, K_2, *and* ρ, $0 < \rho < 1$, *such that, when* $a \leqq s < \tau < t \leqq b$, *we have*

$$\|R(t, s) - R(\tau, s)\| \leqq K_1(t - \tau)(t - s)^{-1} + K_2(t - s)^{\rho}(t - \tau)^{1-\rho}. \tag{32}$$

Tanabe applied his result to the integration of the temporally inhomogeneous diffusion equation

$$\frac{\partial u}{\partial t} = a^{ij}(t, x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b^i(t, x)\frac{\partial u}{\partial x_i} + c(t, x)u + f(t, x) \tag{33}$$

in a bounded domain of E^m and for $a \leqq t \leqq b$.

5. Analyticity of the fundamental solution

Komatsu [10] gave an important remark to Tanabe's result. Let Δ be a convex complex neighborhood of the real segment $[a, b]$. Suppose that $A(t)$ is defined on Δ and satisfies the conditions

(34) *$A(t) \in S(\theta, M, \beta)$ for $t \in \Delta$, where θ, M, and β are independent of t,*

(35) *$D[A(t)] = D$ is independent of $t \in \Delta$,*

(36) *there exists a bounded linear operator A_0 which maps X onto D in a one-to-one manner and such that $B(t) = A(t)A_0$ is analytic in t for $t \in \Delta$.*

Under these conditions, Komatsu [11] proved that the fundamental solution $U(t, s)$ of (25) constructed as in (33) is analytic in t and s if $|\arg(t - s)| < \theta$. His proof is based upon the following

LEMMA 6. *We write $t >_\theta s$ when $|\arg(t - s)| < \theta$. Let $P(t, s)$ and $Q(t, s)$ be bounded linear operators in X defined for $t >_\theta s$ with t and $s \in \Delta$. If they are uniformly bounded and analytic there, then*

$$\int_s^t P(t, r)Q(r, s)\,dr \tag{37}$$

is uniformly bounded and analytic in t and $s \in \Delta$ when $t >_\theta s$.

This result may be applied, as in the case of temporally homogeneous equations discussed in section 3, to the unique continuation theorem of diffusion equations. This is Komatsu's proof of the extension of Shirota [17], [18] of the unique continuation theorem of Itô and Yamabe [6].

6. Fractional powers of infinitesimal generators and the analyticity of the semigroups generated by them

Let $T = \exp(tA)$, with $t > 0$, be a semigroup of type $S(M, 0)$. A fractional power of A

$$-(-A)^\alpha, \qquad 0 < \alpha < 1, \tag{38}$$

was defined by Bochner [3] and Phillips [15] as the infinitesimal generator of the semigroup

$$\hat{T}_t x = \hat{T}_{t,\alpha} x = \int_0^\infty T_\lambda x \, d\gamma_{t,\alpha}(\lambda), \tag{39}$$

where the measure $d\gamma_{t,\alpha}(\lambda) \geqq 0$ is defined through the Laplace integral

$$\exp(-ta^\alpha) = \int_0^\infty \exp(-\lambda a)\, d\gamma_{t,\alpha}(\lambda), \qquad t, a > 0; 0 < \alpha < 1. \tag{40}$$

We (Kato [9], Yosida [24], and Balakrishnan [1]) can prove that the semigroup T_t is of type $S(\theta, M, \beta)$. To this purpose, invert the Laplace integral (40). Then we see that the measure $d\gamma_{t,\alpha}(\lambda)$ has the density $f_{t,\alpha}(\lambda)$ given by

$$f_{t,\alpha}(\lambda) = (2\pi i)^{-1} \int_{\sigma - i\infty}^{\sigma + i\infty} \exp(z\lambda - z^\alpha t)\, dz \qquad \text{for any} \quad \sigma > 0, \tag{41}$$

so that we have

$$\hat{T}_t x = \hat{T}_{t,\alpha} x = \int_0^\infty f_{t,\alpha}(\lambda) T_\lambda x \, d\lambda. \tag{39'}$$

Take any θ with $\pi/2 \leqq \theta \leqq \pi$. Then we obtain

$$f_{t,\alpha}(\lambda) = \pi^{-1} \int_0^\infty \exp(\lambda r \cos\theta - tr^\alpha \cos\alpha\theta) [\sin(\lambda r \sin\theta - tr^\alpha \sin\alpha\theta + \theta)] \, dr \tag{41'}$$

by deforming the path of integration in (41) to the union of two paths

$$\begin{aligned} &r \exp(-i\theta), && \infty > r > 0, \\ &r \exp(i\theta), && 0 < r < \infty. \end{aligned} \tag{42}$$

Taking $\theta = \theta_\alpha = \pi/(1 + \alpha)$ in (41′) and differentiating (39′) with respect to t, we obtain

$$\frac{d}{dt} \hat{T}_t x = \int_0^\infty T_\lambda x \, d\lambda \left(\int_0^\infty \exp[(\lambda r + tr^\alpha) \cos\theta_\alpha] \{\sin[(\lambda r - tr^\alpha) \sin\theta_\alpha]\} r^\infty \, dr \right). \tag{43}$$

This formal differentiation is justified, since the right side reduces, upon changing the variables of integration, to

$$(t\pi)^{-1} \int_0^\infty T_{\nu t^{1/\alpha}} \, d\nu \left(\int_0^\infty \exp[(s\nu + s^\alpha) \cos\theta_\alpha] \{\sin[(s\nu - s^\alpha) \sin\theta_\alpha]\} s^\alpha \, ds \right) \tag{44}$$

which is, by $\cos\theta_\alpha < 0$ and $\|T_t\| \leqq M$, uniformly convergent in $t \geqq t_0$ for any fixed $t_0 > 0$. We have incidentally proved that $\hat{T}_t$ satisfies (7), and thus $\hat{T}_t$ belongs to the class $S(\theta, M, \beta)$.

From (43) it is easy to deduce the following formulas which were proved earlier by Balakrishnan [1] (see Krasnoselski and Sobolevski [12]),

$$\begin{aligned} -(-A)^\alpha x &= [-\Gamma(-\alpha)]^{-1} \int_0^\infty \lambda^{-\alpha-1}(T_\lambda - I) x \, d\lambda, && x \in D(A), \\ &= \pi^{-1} \sin\alpha\pi \int_0^\infty (\lambda I - A)^{-1} A x \, d\lambda, && x \in D(A). \end{aligned} \tag{45}$$

See Yosida [24] and Kato [8].

7. An application of the fractional power operators to the reduced Cauchy problem

Let A be the infinitesimal generator of a semigroup $T_t = \exp(tA) \in S(M, 0)$. Then for each $u_0 \in D(A)$, we have that $u(t) = \hat{T}_{t,1/2} u_0$ is a solution of

$$\frac{d^2 u}{dt^2} + Au = 0, \qquad t \geqq 0, \tag{46}$$

and satisfies the conditions

$$s - \lim_{t \downarrow 0} u(t) = u_0, \qquad \sup_t \|u(t)\| < \infty, \tag{47}$$

and

$$s - \lim_{t \downarrow 0} \frac{d}{dt} u(t) = s - \lim_{t \downarrow 0} \frac{d}{dt} \hat{T}_{t,1/2} u_0 = B u_0, \tag{48}$$

where $B = -(-A)^{1/2}$. According to Balakrishnan [2], the solution of (46) is uniquely determined by condition (47). Hence, for a solution $u(t)$ of (46) and (47), the initial condition for the first partial derivative $du(0)/dt$ is determined by (48), and other values cannot be prescribed. In this sense, the Cauchy problem for (46) and (47) is reduced. For a general definition of the reduced Cauchy problem introduced by Hille see [5].

We follow Balakrishnan's proof. Put $\hat{T}_{t,1/2} = T_{1/2}(t)$. Let $v(t)$ be a twice strongly and continuously differentiable solution of (46) satisfying the condition

$$s - \lim_{t \downarrow 0} v(t) = u_0 \in D(A), \qquad \sup_t \|v(t)\| < \infty. \tag{47'}$$

Put $w(t) = T_{1/2}(1/n)v(t)$. If we can prove, for all positive integer n, that $dw(0)/dt = Bw(0)$, then we obtain $dv(0)/dt = Bv(0)$ by letting $n \to \infty$.

To prove this fact we first observe that $w(t)$ is a solution of (46), and by (47′), $\|dw(t)/dt\|$ is of exponential growth at $t \to \infty$. Thus we see, by putting $w_0 = w(0)$ and $w_1 = dw(0)/dt$, that

$$\frac{dw(t)}{dt} + Bw(t) = T_{1/2}(t)Bw_0 + T_{1/2}(t)w_1, \tag{49}$$

because both sides satisfy the Cauchy problem

$$\frac{du}{dt} = Bu, \qquad u(0) = Bw_0 + w_1 \tag{50}$$

and are of exponential growth at $t \to \infty$. Hence we have

$$\frac{d}{dt}[T_{1/2}(t)w(t)] = T_{1/2}(2t)w_1 + T_{1/2}(2t)Bw_0, \tag{51}$$

so that

$$T_{1/2}(t)w(t) = T_{1/2}(2t)w_0 + \frac{1}{2}\int_0^{2t} T_{1/2}(s)(w_1 - Bw_0)\, ds. \tag{52}$$

Hence

$$T_{1/2}(t)Bw(t) = T_{1/2}(2t)Bw_0 + \frac{1}{2}[T_{1/2}(2t) - I](w_1 - Bw_0). \tag{53}$$

Since $T_{1/2}(t)$ satisfies (7), $Bw(t) = BT_{1/2}(1/n)v(t)$ is bounded in t by the assumption that $v(t)$ is bounded in t. On the other hand, because of the time-like analyticity of $T_{1/2}(t)$, zero does not belong to the point spectrum of $T_{1/2}(t)$ for any

$t > 0$. We write $T_{1/2}(-t)$ for the inverse of $T_{1/2}(t)$. Hence we see, by applying $T_{1/2}(-t)$ to (53),

$$\sup_t \|T_{1/2}(-t)(w_1 - Bw_0)\| < \infty. \tag{54}$$

From this we can prove that $z = (w_1 - Bw_0) = 0$. To this purpose we put

$$F(\lambda) = \int_0^\infty e^{\lambda t} T_{1/2}(-t) z \, dt, \qquad \text{Re}(\lambda) < \infty. \tag{55}$$

Then $\lambda F(\lambda)$ is bounded when $\text{Re}(\lambda) < 0$, $|\text{Im}(\lambda)|/|\text{Re}(\lambda)| < c$ for any positive constant c. On the other hand, it is easily verified that

$$-F(\lambda) = (\lambda I - B)^{-1} \tag{56}$$

for λ with $\text{Re}(\lambda) < 0$ and in the resolvent set of B. Hence $-\lambda F(\lambda)$ is the analytical extension of $\lambda(\lambda I - B)^{-1}z$ into the left half λ-plane. Moreover, as was indicated in section 2, $\lambda(\lambda I - B)^{-1}$ is bounded in a sector of the form $-\pi/2 - \epsilon \leqq \arg \lambda \leqq \pi/2 + \epsilon$ for $\epsilon > 0$. Hence, by Liouville's theorem, $-\lambda F(\lambda)$ must reduce to a constant vector. Thus, by $s - \lim_{\lambda \downarrow 0} \lambda(\lambda I - B)^{-1}z = z$, we obtain $\lambda(\lambda I - B)^{-1}z = z$. Hence $Bz = 0$ and so $T_{1/2}(t)z = z$. Therefore, by (52),

$$T_{1/2}(t)w(t) = T_{1/2}(2t)w_0 + t(w_1 - Bw_0), \tag{57}$$

and hence the boundedness of $w(t)$ implies that $w_1 = Bw_0$.

REFERENCES

[1] A. V. BALAKRISHNAN, "An operational calculus for infinitesimal generators of semigroups," *Trans. Amer. Math. Soc.*, Vol. 91 (1959), pp. 330–353.

[2] ———, "Fractional powers of closed operators and the semigroups generated by them," to be published.

[3] S. BOCHNER, "Diffusion equations and stochastic processes," *Proc. Nat. Acad. Sci. U.S.A.*, Vol. 35 (1949), pp. 369–370.

[4] W. FELLER, "On a generalization of M. Riesz' potentials and the semigroups generated by them," *Comm. Sém. Math. Univ. Lund*, Vol. 21 Suppl. (1952), pp. 72–81.

[5] E. HILLE and R. S. PHILLIPS, "Functional analysis and semigroups," *Amer. Math. Soc. Colloq. Publ.*, Providence, 1957.

[6] S. ITÔ and H. YAMABE, "A unique continuation theorem for solutions of a parabolic equation," *J. Math. Soc. Japan*, Vol. 10 (1958), pp. 314–321.

[7] P. D. LAX and A. MILGRAM, "Parabolic equations," *Contributions to the Theory of Partial Differential Equations*, Annals of Mathematics Studies, No. 33, Princeton, Princeton University Press, 1954, pp. 167–190.

[8] T. KATO, "Integration of the equation of evolution in a Banach space," *J. Math. Soc. Japan*, Vol. 5 (1953), pp. 208–234.

[9] ———, "Note on fractional powers of linear operators," *Proc. Japan Acad.*, Vol. 36 (1960), pp. 94–96.

[10] H. KOMATSU, "A characterization of real analytic functions," *Proc. Japan Acad.*, Vol. 36 (1960), pp. 90–93.

[11] ———, "Abstract analyticity in time and unique continuation property of solutions of a parabolic equation," to be published.

[12] M. A. KRASNOSELSKI and P. E. SOBOLEVSKI, "Fractional power operators defined on Banach spaces," *Dokl. Akad. Nauk SSSR*, Vol. 129 (1959), pp. 499–502. (In Russian.)

[13] S. Mizohata, "Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques," *Mem. Coll. Sci. Univ. Kyoto, Ser. A., Math.*, Vol. 31 (1958), pp. 219–239.

[14] E. Nelson, "Analytic vectors," *Ann. of Math.*, Vol. 70 (1959), pp. 572–615.

[15] R. S. Phillips, "On the generation of semigroups of linear operators," *Pacific J. Math.*, Vol. 2 (1952), pp. 343–389.

[16] ———, "On the integration of the diffusion equation with boundary conditions," to be published.

[17] T. Shirota, "A unique continuation theorem of a parabolic differential equation," *Proc. Japan Acad.*, Vol. 35 (1959), pp. 455–460.

[18] ———, "A remark on my paper 'A unique continuation theorem of a parabolic differential equation'," *Proc. Japan Acad.*, Vol. 36 (1960), pp. 133–135.

[19] H. Tanabe, "A class of the equations of evolution in a Banach space," *Osaka Math. J.*, Vol. 11 (1959), pp. 121–145.

[20] K. Yosida, "Semigroup theory and the integration problem of diffusion equations," *Proc. Internat. Congress Math.*, Vol. 1 (1954), pp. 405–420.

[21] ———, "An operator-theoretical integration of wave equations," *J. Math. Soc. Japan*, Vol. 8 (1956), pp. 79–82.

[22] ———, "On the differentiability of semigroups of linear operators," *Proc. Japan Acad.*, Vol. 34 (1958), pp. 337–340.

[23] ———, "An abstract analyticity in time for solutions of diffusion equations," *Proc. Japan Acad.*, Vol. 35 (1959), pp. 109–113.

[24] ———, "Fractional powers of infinitesimal generators and the analyticity of the semigroups generated by them," *Proc. Japan Acad.*, Vol. 36 (1960), pp. 86–89.