SECOND-ORDER HOMOGENEOUS RANDOM FIELDS

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1. Introduction

The central fact in the theory of second-order stationary random processes is the existence of the spectral representations of the process $\xi(t)$ as

$$\xi(t) = \int_{-\infty}^{\infty} e^{it\lambda} Z(d\lambda), \tag{1.1}$$

and of the corresponding covariance function $B(\tau) = E\{\xi(t+\tau)\overline{\xi(t)}\}$ as

$$B(\tau) = \int_{-\infty}^{\infty} e^{i\tau\lambda} F(d\lambda). \tag{1.2}$$

Here $Z(\Lambda)$ is a completely additive random set function (random measure), while $F(\Lambda)$ is the usual nonnegative bounded measure on the λ-axis $(-\infty, \infty)$, connected with $Z(\Lambda)$ by the relation

$$F(\Lambda) = E|Z(\Lambda)|^2. \tag{1.3}$$

We assume here that the time parameter t of the process takes on all real values. For discrete parameter random processes the limits of integration in (1.1) and (1.2) must be replaced by $-\pi$ to $+\pi$.

Analogous spectral representations exist for stationary processes with a multidimensional parameter $\mathbf{t} = (t_1, t_2, \cdots, t_n)$, that is, for homogeneous random fields $\xi(\mathbf{t})$ in an n-dimensional space R_n, and for a more general class of homogeneous fields on an arbitrary locally compact commutative group G [see formulas (2.21) to (2.23) below]. Moreover, in the case of a homogeneous field $\xi(\mathbf{t})$ with $\mathbf{t} \in R_n$ any additional assumptions about its symmetry impose special restrictions on the covariance function $B(\boldsymbol{\tau})$ and on the spectral measures $F(\Lambda)$ and $Z(\Lambda)$. From the point of view of applications the most interesting is the case of a homogeneous and isotropic random field, that is, the homogeneous field $\xi(\mathbf{t})$ which possesses spherical symmetry. The general form of the covariance function $B(\tau)$, with $\tau = |\boldsymbol{\tau}|$, of such a field in R_n is given by the well-known formula of I. J. Schoenberg [1], namely

$$B(\tau) = \int_0^{\infty} \frac{J_{(n-2)/2}(\tau\lambda)}{(\tau\lambda)^{(n-2)/2}}\, dG(\lambda), \tag{1.4}$$

where $J_{(n-2)/2}$ is a Bessel function of order $(n-2)/2$, and $G(\lambda)$ is a bounded nondecreasing function. The homogeneous and isotropic random vector fields

$\xi(t) = \{\xi_1(t), \cdots, \xi_n(t)\}$ are also considered in the statistical theory of turbulence. The corresponding covariance matrix $B_{ij}(\tau) = E\{\xi_i(t + \tau)\xi_j(\tau)\}$ has its own "spectral representation" similar to (1.4) (see [2], [3], and [4]). The notion of the isotropic random current introduced by K. Itô [5] is a generalization of the notion of the isotropic random vector field. For such currents there also exist spectral representations of a special form.

At first glance, the short note of A. M. Obukhov [6], devoted to homogeneous random fields over a sphere S_2 of the three-dimensional space R_3, seems to have little connection with work in the theory of random fields in Euclidean spaces. Obukhov considered expansions of such a field $\xi(\theta, \varphi)$ in a series of spherical harmonics $Y_l^m(\theta, \varphi)$ as follows

$$\xi(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} z_{lm} Y_l^m(\theta, \varphi) \tag{1.5}$$

and showed that the condition for the homogeneity of the field is equivalent to the condition

$$E z_{lm} \overline{z_{e_1 m_1}} = \delta_{ll_1} \delta_{mm_1} f_l. \tag{1.6}$$

It follows from (1.5), (1.6), and the addition theorem of associated Legendre functions that the corresponding covariance function $B(\theta_{12}) = E\xi(\theta_1, \varphi_1)\xi(\theta_2, \varphi_2)$, where θ_{12} is the angular distance between the points (θ_1, φ_1) and (θ_2, φ_2) on the sphere S_2, has a representation

$$B(\theta) = \sum_{l=0}^{\infty} g_l P_l(\cos \theta), \qquad g_l = \frac{2l+1}{2} f_l \geqq 0, \tag{1.7}$$

which was discovered earlier in another connection by Schoenberg [7]. Conversely, for every z_{lm} and f_l satisfying (1.6) and such that the series (1.7) converges, the field (1.5) is homogeneous and the function (1.7) is a covariance function of a homogeneous random field.

The results (1.2), (1.4), and (1.7) or (1.1) and (1.5) seem quite different. However, it is natural to suppose that they are all included as special cases of some general theory, which comprises wide classes of random functions invariant with respect to some transformation group. It is not hard to see that the only mathematical tool that could be useful in the construction of such a general theory must be the theory of group representations. The idea of the present paper is to apply the theory of group representations to find the general form of homogeneous random fields and of their covariance functions over various manifolds having a transitive transformation group.

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2. Homogeneous random fields on groups

The simplest types of spaces admitting a transitive transformation group are the group spaces $G = \{g\}$ consisting of the elements g of some group. In the following we shall consider only topological groups G. There are two distinct families of continuous transformations of the group space G, namely the transformations

$$V_g: g_1 \to gg_1 \tag{2.1}$$

(*left shifts*) and

$$V'_g: g_1 \to g_1g \tag{2.2}$$

(*right shifts*). We define the random field $\xi(g)$ over the group G as a function on G with values in the Hilbert space $\mathfrak{H}$ of complex random variables with finite variance (the scalar product of the elements of $\mathfrak{H}$ is equal to their covariance), which is continuous in the strong topology in $\mathfrak{H}$. The field $\xi(g_1)$ will be called *homogeneous* if its first and second moments remain invariant on the application to g_1 of at least one of the families of transformations V_g or V'_g. Such a concept of homogeneity is obviously a concept in the wide sense, that is, a second-order concept (see section 3 in chapter 2 in [8]). Depending on whether this is true for the family V_g or V'_g, or for both, we shall speak of left homogeneous fields, right homogeneous fields and two-way homogeneous fields. Due to the transitivity of the transformation group under consideration we always have $E\xi(g) = \text{const.}$ for a homogeneous field, so that without loss of generality we can always assume that $E\xi(g) \equiv 0$. In so doing, the only condition for the left homogeneity of the field is the condition

$$E\xi(g_1)\overline{\xi(g_2)} = E\xi(gg_1)\overline{\xi(gg_2)} = B(g_2^{-1}g_1), \tag{2.3}$$

while the right homogeneity condition is

$$E\xi(g_1)\overline{\xi(g_2)} = E\xi(g_1g)\overline{\xi(g_2g)} = B(g_1g_2^{-1}). \tag{2.4}$$

For two-way homogeneous fields conditions (2.3) and (2.4) must be satisfied simultaneously. From this, in particular, it follows that for such fields the function $B(g_2^{-1}g_1)$ of equation (2.3) remains invariant when the elements $V'_g g_1$ and $V'_g g_2$ are substituted for g_1 and g_2, that is, it must be a constant for a class of conjugate elements, so that

$$B(h) = B(ghg^{-1}) \qquad \text{for} \quad h, g \in G. \tag{2.5}$$

We assume at first that the group G is *compact* (in particular it could be simply finite). This assumption simplifies things considerably as it enables us to use the representation theory of compact groups which has reached a high degree of development (see, for instance, [9] and [10]). According to this theory, for every compact group there exists not more than a countable number of nonequivalent, finite dimensional, unitary, irreducible representations, that is, of homomorphisms of the group G into the group of unitary matrices of finite order, namely,

$$g \to T^{(\lambda)}(g) = \|T_{ij}^{(\lambda)}(g)\|, \qquad 1 \leqq i, j \leqq d_\lambda < \infty, \ \lambda = 1, 2, \cdots \tag{2.6}$$

$$\begin{aligned} T^{(\lambda)}(g_1g_2) &= T^{(\lambda)}(g_1)T^{(\lambda)}(g_2), \\ T^{(\lambda)}(g^{-1}) &= [T^{(\lambda)}(g)]^{-1} = [T^{(\lambda)}(g)]^*. \end{aligned} \tag{2.7}$$

The matrix elements $T_{ij}^{(\lambda)}(g)$ of these representations satisfy the following orthogonality relations

$$\int_G T_{ij}^{(\lambda)}(g)\overline{T_{kl}^{(\mu)}(g)}\, dg = \delta_{\lambda\mu}\delta_{ik}\delta_{jl}\frac{1}{d_\lambda} \tag{2.8}$$

where $\delta_{\lambda\mu}$ is the Kronecker symbol, and dg is the unique measure over G invariant with respect to all shifts V_g and V'_g such that $\int_G dg = 1$. The set of all matrix elements $T_{ij}^{(\lambda)}(g)$ for $1 \leqq i, j \leqq d_\lambda$, $\lambda = 1, 2, \cdots$, form a complete orthogonal system in the space $L^2(G)$ of complex functions over G the squares of whose absolute values are summable with respect to dg.

Let now $\xi(g)$ be an arbitrary random field over G. It follows from the fact that the system of functions T_{ij} is orthonormal and complete in $L^2(G)$ that $\xi(g)$ can be represented by a series of these functions convergent in quadratic mean, that is, in the sense of a strong topology in $\mathfrak{H}$, as follows

$$\xi(g) = \sum_\lambda \sum_{i,j=1}^{d_\lambda} z_{ji}^{(\lambda)} T_{ij}^{(\lambda)}(g), \tag{2.9}$$

where

$$z_{ji}^{(\lambda)} = d_\lambda \int_G \xi(g)\overline{T_{ij}^{(\lambda)}(g)}\, dg \in \mathfrak{H}. \tag{2.10}$$

The integral in (2.10) is also understood in the quadratic mean sense. It is obvious that for fields with $E\xi(g) \equiv 0$, also $Ez_{ji}^{(\lambda)} \equiv 0$. If we assume further that the field $\xi(g)$ is homogeneous, then this naturally imposes further restrictions on the coefficients $z_{ji}^{(\lambda)}$. Namely,

THEOREM 1. *The random field* (2.9) *is left homogeneous if and only if the random variables* $z_{ji}^{(\lambda)}$ *satisfy the conditions*

$$Ez_{ji}^{(\lambda)}\overline{z_{lk}^{(\mu)}} = \delta_{\lambda\mu}\delta_{ik}f_{jl}^{(\lambda)}, \tag{2.11}$$

where the matrices

$$f^{(\lambda)} = ||f_{jl}^{(\lambda)}|| \tag{2.12}$$

are positive definite, and such that

$$\sum_\lambda \sum_j f_{jj}^{(\lambda)} = \sum_\lambda \operatorname{Tr} f^{(\lambda)} < \infty. \tag{2.13}$$

The covariance function $B(g)$ *in* (2.3) *can in this case be represented in the form*

$$B(g) = \sum_\lambda \sum_{j,l} f_{jl}^{(\lambda)} T_{lj}^{(\lambda)}(g). \tag{2.14}$$

Conversely, any function $B(g)$ *of the form* (2.14), *where each* $||f_{jl}^{(\lambda)}||$ *is a positive definite matrix satisfying* (2.13) *is a covariance function of some left homogeneous random field on* G.

PROOF. From equations (2.3), (2.10), (2.7), (2.8), and from the fact that $dg_2^{-1}g = dg$, it follows easily that

$$E z_{jt}^{(\lambda)} \overline{z_{lk}^{(\mu)}} = \delta_{\lambda\mu}\delta_{ik}\, d_\lambda \int_G B(g) \overline{T_{lj}^{(\lambda)}(g)}\, dg = \delta_{\lambda\mu}\delta_{ik} f_{jl}^{(\lambda)}. \tag{2.15}$$

This proves the necessity of condition (2.11). Substituting (2.9) into (2.3), we can verify (2.14). The matrices $f^{(\lambda)}$ are obviously all positive definite. Since $T_{ik}^{(\lambda)}(e) = \delta_{ik}$, where e is the unit of the group G, it follows that condition (2.13) is the condition for the convergence of the right sides of (2.9) and (2.14). If now $f^{(\lambda)}$, for $\lambda = 1, 2, \cdots$, are arbitrary, nonnegative definite $(d_\lambda \times d_\lambda)$ matrices, then one can always select $z_{jt}^{(\lambda)} \in \mathfrak{H}$ such that (2.11) is satisfied. Under condition (2.13) the corresponding series on the right side of (2.9) converges and defines a random field $\xi(g)$ for which $E\xi(g_1 g)\overline{\xi(g_1)}$ is given by formula (2.14). Hence, the field $\xi(g)$ is left homogeneous, which completes the proof of theorem 1.

It is well known that the class of covariance functions of a random field coincides with the class of positive definite functions of the corresponding space (see, for example, chapter 2, theorem 3.1. in [8]). Hence formula (2.14) defines the general form of the functions $B(g)$ on G, such that for any complex numbers $a_1, a_2, \cdots, a_n$,

$$\sum_{i,k} B(g_k^{-1} g_i) a_i \overline{a_k} \geqq 0. \tag{2.16}$$

This last result was obtained comparatively long ago by S. Bochner [11]. We note that all the other assertions of theorem 1 can be obtained from this if we make use of the general theorem of K. Karhunen [12] and H. Cramér [13] on representations of random functions. However, the proof of theorem 1 given above is more direct and elementary.

The situation is quite analogous for right homogeneous fields over G, the only change being that condition (2.11) is replaced by

$$E z_{jt}^{(\lambda)} \overline{z_{lk}^{(\mu)}} = \delta_{\lambda\mu}\delta_{jl} f_{ik}^{(\lambda)}, \tag{2.17}$$

and for two-way homogeneous fields $\xi(g)$ both conditions (2.11) and (2.17) must hold. Therefore we get the following theorem.

THEOREM 2. *The random field* (2.9) *is two-way homogeneous if and only if the random variables* $z_{jt}^{(\lambda)}$ *satisfy the condition*

$$E z_{jt}^{(\lambda)} \overline{z_{lk}^{(\mu)}} = \delta_{\lambda\mu}\delta_{ik}\delta_{jl} f^{(\lambda)}, \tag{2.18}$$

where $f^{(\lambda)}$ *are nonnegative numbers such that*

$$\sum_\lambda d_\lambda f^{(\lambda)} < \infty. \tag{2.19}$$

The covariance function $B(g)$ *of equation* (2.3) *will then be represented in the form*

$$B(g) = \sum_\lambda f^{(\lambda)} \chi^{(\lambda)}(g), \tag{2.20}$$

where $\chi^{(\lambda)}(g) = \operatorname{Tr}[T^{(\lambda)}(g)]$ *are the characters of the group* G.

Formula (2.20) for general positive definite functions on a compact group G invariant with respect to two-way shifts was first pointed out by Bochner [11].

Theorem 2 is applicable, in particular, to the case of a commutative group G, over which there are of course only two-way homogeneous fields. In this case the assumption of compactness of the group G is superfluous; it can be replaced, for example, by the assumption of local compactness or even by some more general assumption (see, for example, [14]). The general form of a positive definite function over such G can be given by the following known generalization of the classical theorem of Bochner

$$B(g) = \int_{\overline{G}} \chi^{(\lambda)}(g)F(d\lambda), \tag{2.21}$$

where $F(d\lambda)$ is a bounded measure over the set $\overline{G}$ of characters $\chi^{(\lambda)}$ of the group G (see A. Weil [10] and D. A. Raĭkov [14]). In view of the representation theorem of Karhunen-Cramér, it follows from this that any homogeneous field $\xi(g)$ over such a group allows a spectral representation of the form

$$\xi(g) = \int_{\overline{G}} \chi^{(\lambda)}(g)Z(d\lambda), \tag{2.22}$$

where $Z(d\lambda)$ is a random measure over $\overline{G}$ such that

$$EZ(\Lambda_1)\overline{Z(\Lambda_2)} = F(\Lambda_1 \cap \Lambda_2) \tag{2.23}$$

(see [15] and [16]). The usual spectral representation (1.1) and (1.2) of stationary processes $\xi(t)$ is obviously a special case of the representation (2.21) to (2.23).

The situation is much more complicated in the case of an arbitrary locally compact group G, not assumed to be commutative. Such a group, as is well known, may have no unitary finite-dimensional representation. But Gelfand and Raĭkov [17] have shown that it will always have a sufficient number of unitary infinite-dimensional representations, that is, the homeomorphic mappings of G into the group of unitary operators in a Hilbert space $\mathfrak{H}$. However, the problem of expansion of an arbitrary function $\xi(g)$ on the group G into matrix elements of such representations has not been solved in all generality even for ordinary, not random, functions. Therefore, one should not expect to carry over the method of proof of theorem 1 to a wide class of locally compact groups. It seems more hopeful to follow the path which leads to the construction of an analogue of Bochner's theorem characterizing all possible positive definite functions on G.

As was shown in [17] (see also [18]), every positive definite function $B(g)$ on a locally compact group G is represented in the form

$$B(g) = [T(g)\xi_0, \xi_0], \tag{2.24}$$

where ξ_0 is a definite vector of a Hilbert space H in which is acting some unitary representation on the group G, while $T(g)$ is the operator of the representation. In this way the description of all possible positive definite functions can be reduced to the description of all possible unitary representations of G. Every such representation can be decomposed into a direct integral of irreducible representations (see, for example, chapter 8 in [18]). However, in the general case, such a decomposition is far from unique and cannot be used to obtain any more or less definite formulas. Or more precisely, from such a decomposition it follows only that every positive definite function $B(g)$ can be represented as $B(g) = \int_S \mathrm{Tr}\,[T^{(\lambda)}(g)A(d\lambda)]$, where S is a space, far from unique, with measure $d\lambda$, while $T^{(\lambda)}(g)$ is an irreducible unitary representation of the group G depending on $\lambda \in S$, and $A(d\lambda)$ is the "operator measure" on S (compare (2.26) below).

Considerably more satisfactory results can be obtained if we impose some restrictions on the group G under consideration. In the following we shall assume that the group G is a separable group of type I, that is, such a separable group that its every unitary representation generates a ring of operators, which is a ring of type I in the sense of F. J. Murray and J. von Neumann (see chapter 7 in [18]). These conditions will obviously be satisfied by all compact groups and all separable locally compact commutative groups. According to the results of Harish-Chandra [19] they will be satisfied also by all connected semisimple Lie groups; apparently they will be satisfied also for the majority of other "sufficiently well behaved" locally compact groups (see, for example, [20] where it is shown that these conditions will be satisfied by all algebraic Lie groups). At the same time the above condition enables us to make use of the results of G. W. Mackey [21] and A. Guichardet [22], according to which every unitary representation of a separable group G of type I can always be represented in the form of a direct sum of multiplicity free representations which decompose into an integral with respect to *irreducible nonequivalent* representations. More precisely, the spaces H_l of these multiplicity free representations $T_l(g)$ can be represented as topological direct integrals of the Hilbert spaces $H^{(\lambda)}$ of the nonequivalent, irreducible unitary representations $T^{(\lambda)}(g)$ of the group G

$$T_l(g) = \int_{\overline{G}} \otimes\, T^{(\lambda)}(g), \qquad H_l = \int_{\overline{G}} H^{(\lambda)}[\sigma_l(d\lambda)]^{1/2} \tag{2.25}$$

(for explanation of the notations see chapter 8 in [18]). Here $\overline{G}$ denotes the "dual object" of the group G, namely, the set of all equivalence classes of irreducible nonequivalent unitary representations of this group, while $\sigma_l(d\lambda)$ denotes a measure in the space $\overline{G}$, providing a "natural Borel structure" [21], that is, the Borel field of measurable sets. The representation $T_l(g)$ naturally depends only on the class of equivalent measures in $\overline{G}$ to which σ_l belongs. If we now substitute into (2.24) the direct sum of representations (2.25) instead of $T(g)$, then we obtain for $B(g)$ a formula of the form $B(g) = \int_{\overline{G}} \sum_{i,j,l} T_{ij}^{(\lambda)}(g)\xi_{i,l}^{(\lambda)}\overline{\xi_{j,l}^{(\lambda)}}\sigma^{(l)}(d\lambda)$, which can be written in the form

$$B(g) = \int_{\overline{G}} \mathrm{Tr}\,[T^{(\lambda)}(g)F(d\lambda)], \tag{2.26}$$

where $F(d\lambda)$ is some completely additive operator-valued set function in $\overline{G}$, whose values are Hermitian nonnegative definite operators in the space $H^{(\lambda)}$ of representation $T^{(\lambda)}(g)$. If among the unitary representations of the group G there are some finite-dimensional representations, then the integrals (2.25) and (2.26) and all other integrals on $\overline{G}$ decompose naturally into the sum of integrals on the subspaces $\overline{G}_n \subset \overline{G}$, with $n = \infty, 1, 2, \cdots$, corresponding to all nonequivalent n-dimensional irreducible unitary representations of the group G. In all summands of such a sum the spaces $H^{(\lambda)}$, having the same dimension, are considered identical.

We have therefore arrived at the following theorem (see also the note at the end of the paper).

THEOREM 3. *Let G be a separable, locally compact group of type I. Then the function $B(g)$ on G is positive definite if and only if it can be represented in the form* (2.26), *where $F(d\lambda)$ is the "operator measure" over $\overline{G}$, whose values are Hermitian nonnegative operators in the Hilbert space $H^{(\lambda)}$ such that*

$$\int_{\overline{G}} \operatorname{Tr}\left[F(d\lambda)\right] = \operatorname{Tr}\left[F(\overline{G})\right] < \infty. \tag{2.27}$$

In fact it can be easily verified independently that every function of the form (2.26) is positive definite. Condition (2.27) obviously guarantees the convergence of the integral (2.26). Formula (2.26) is a natural generalization of the relation (2.14).

Let now $\xi(g)$ be a random left homogeneous field over the group G satisfying all the conditions of theorem 3, and let the function (2.26) be the covariance function of this field. In order to obtain a "spectral representation" of the field $\xi(g)$, we can use methods similar to those in [12] and [13]. Consider the Hilbert space $L^2(F)$ of operator-valued functions $U^{(\lambda)}$, $\lambda \in G$, with values in a ring of bounded operators in $H^{(\lambda)}$. The norm in $L^2(F)$ we define by the equation

$$\|U^{(\lambda)}\|^2 = \int_{\overline{G}} \operatorname{Tr}\left[U^{(\lambda)}F(d\lambda)U^{(\lambda)*}\right] < \infty. \tag{2.28}$$

In this case the correspondence

$$T^{(\lambda)}(g) \leftrightarrow \xi(g) \tag{2.29}$$

will be an isometric mapping of the set of random variables $\{\xi(g), g \in G\}$ into $L^2(F)$, which can be extended to a linear isometric mapping of the whole space $L^2(F)$ into $\mathfrak{H}$ (see [12]). Now let f_1 and f_2 be two arbitrary vectors of the space $H^{(\lambda)}$ and let Λ be a measurable set in $\overline{G}$. Then the triplet (Λ, f_1, f_2) can be made to correspond in the following way to the operator function $U^{(\lambda)}(\Lambda; f_1, f_2) \in L^2(F)$

$$U^{(\lambda)}(\Lambda; f_1, f_2)f = \begin{cases} (f, f_2)f_1 & \text{if } \lambda \in \Lambda, \\ 0 & \text{if } \lambda \notin \Lambda. \end{cases} \tag{2.30}$$

Let $Z(\Lambda; f_1, f_2)$ be the image of $U^{(\lambda)}(\Lambda; f_1, f_2)$ under the isometric mapping (2.29).

In this case $Z(\Lambda; f_1, f_2)$ is a random variable depending bilinearly on f_1 and f_2. It can be written as

$$Z(\Lambda; f_1, f_2) = [Z(\Lambda)f_1, f_2], \tag{2.31}$$

where $Z(\Lambda)$ is a *random linear operator* in $H^{(\lambda)}$, depending completely additively on $\Lambda \subset \bar{G}$. Let $e_1, e_2, \cdots, e_n, \cdots$ be an arbitrary orthonormal basis in $H^{(\lambda)}$. It can be easily verified that in $L^2(F)$ the operator function

$$\int_{\bar{G}} \sum_{i,j} [T^{(\mu)}(g)e_i, e_j] U^{(\lambda)}(d\mu; e_j, e_i) \tag{2.32}$$

coincides with $T^{(\lambda)}(g)$. From this, since $\xi(g)$ is the image of $T^{(\lambda)}(g)$ under our isometric mapping, we immediately obtain the formula

$$\xi(g) = \int_{\bar{G}} \mathrm{Tr}\, [T^{(\lambda)}(g)Z(d\lambda)]. \tag{2.33}$$

Moreover, because this mapping is isometric we have

$$E[Z(\Lambda_1)f_1, f_2]\overline{[Z(\Lambda_2)g_1, g_2]} = \int_{\bar{G}} \mathrm{Tr}\, [U^{(\lambda)}(\Lambda_1; f_1, f_2)F(d\lambda)U^{(\lambda)}(\Lambda_2; g_1, g_2)]^*, \tag{2.34}$$

that is,

$$E[Z(\Lambda_1)f_1, f_2]\overline{[Z(\Lambda_2)g_1, g_2]} = (f_1, g_1)[F(\Lambda_1 \cap \Lambda_2)g_2, f_2]. \tag{2.35}$$

Conversely, for every random field $\xi(g)$ of the form (2.33), where $Z(\Lambda)$ is a random linear operator in $H^{(\lambda)}$, satisfying (2.35), it can be seen easily that $E\xi(gg_1)\overline{\xi(g_1)}$ is equal to the right side of (2.26), that is, $\xi(g)$ is left homogeneous. Hence we have

THEOREM 4. *The random field $\xi(g)$ on separable locally compact group G of type I is left homogeneous if and only if it can be represented in the form* (2.33), *where $Z(\Lambda)$ is a random linear operator in $H^{(\lambda)}$, depending completely additively on the set $\Lambda \subset \bar{G}$ and satisfying, for any $f_1, f_2, g_1, g_2 \in H^{(\lambda)}$, condition* (2.35). *In this condition $F(\Lambda)$ is a Hermitian nonnegative operator in $H^{(\lambda)}$, satisfying* (2.27) *and such that the covariance function $B(g)$ of the field $\xi(g)$ is given by formula* (2.26).

Theorem 4 can be regarded as a generalization of theorem 1 to locally compact groups of type I. In the case of right homogeneous fields we must simply replace condition (2.35) of theorem 4 by the condition

$$E[Z(\Lambda_1)f_1, f_2]\overline{[Z(\Lambda_2)g_1, g_2]} = [F(\Lambda_1 \cap \Lambda_2)f_1, g_1](g_2, f_2). \tag{2.36}$$

If we choose in $H^{(\lambda)}$ a definite "coordinate system," that is, an orthonormal basis, then formulas (2.26), (2.27) and (2.33), (2.35) can be rewritten in a form closely related to (2.10) to (2.14), namely

$$B(g) = \int_{\bar{G}} \sum_{i,j} T_{ij}^{(\lambda)}(g)F_{ji}(d\lambda), \qquad \sum_i F_{ii}(\bar{G}) < \infty, \tag{2.26'}$$

$$\xi(g) = \int_{\hat{G}} \sum_{i,j} T_{ij}^{(\lambda)}(g) Z_{ji}(d\lambda), \tag{2.33$'$}$$

where in the case of left homogeneous fields we have

$$EZ_{ij}(\Lambda_1)\overline{Z_{kl}(\Lambda_2)} = \delta_{jl}F_{ik}(\Lambda_1 \cap \Lambda_2), \tag{2.35$'$}$$

while for right homogeneous fields we have

$$EZ_{ij}(\Lambda_1)\overline{Z_{kl}(\Lambda_2)} = \delta_{ik}F_{jl}(\Lambda_1 \cap \Lambda_2). \tag{2.36$'$}$$

For two-way homogeneous fields both conditions (2.35) and (2.36) must hold. From this it follows that the covariance function $B(g)$ is represented in terms of the traces of the operators of the representation $T^{(\lambda)}(g)$. Since the traces of infinite-dimensional unitary operators are not finite, it follows that in general two-way homogeneous random fields may not exist over noncompact noncommutative groups (see, however, section 4.3 below).

In conclusion we give references to some articles containing explicit formulas for the operators of the representations $T^{(\lambda)}(g)$ of certain groups. Starting from these formulas it is evidently possible to enumerate all homogeneous fields on the corresponding groups. For the rotation group in the three-dimensional Euclidean space the values of all the matrix elements $T_{ij}^{(\lambda)}(g)$ are given in [23] as functions of the three Euler angles, which determine the rotation g uniquely; these matrix elements are represented in [23] by trigonometric functions and Jacobi polynomials of Euler angles. The group of motions of the Euclidean plane have a one-parameter family of infinite-dimensional unitary representations. Explicit formulas for the matrix elements of these representations are given in [24], where these matrix elements are expressed in terms of trigonometric and Bessel functions. For the group of motions of the Lobachevsky plane, the matrix elements $T_{ij}^{(\lambda)}(g)$ are given in [25], where they are expressed by hypergeometric functions. In the general case of an arbitrary classical group explicit expressions for the operators $T^{(\lambda)}(g)$ are found in [26], but not in matrix form. The methods for constructing formulas for all matrix elements of the representation of the group of rotations and the group of motions of the n-dimensional Euclidean space and of the n-dimensional Lobachevsky space are given in [27], [28], and [29]. We note that these matrix elements are expressed by some new, not previously studied, transcendental functions.

3. Homogeneous fields on homogeneous spaces

Let $X = \{x\}$ be an arbitrary homogeneous space, that is, a space which admits a transitive transformation group $G = \{g\}$. We denote by $K = \{k\}$ a stationary subgroup of G, that is, a subgroup which leaves invariant a point $x_0 \in X$. The set of transformations $g \in G$ which map x_0 into a fixed point $x_1 \in X$ obviously form a left coset g_1K of the group G modulo K. In this way there is established a one-to-one correspondence between the points $x_1 \in X$ and the left cosets g_1K, so that X can be identified with the set of these cosets as follows:

$X = G/K$. If $x_1 = g_1K$, then $gx_1 = gg_1K$. In what follows we shall always assume that G is a topological group and that K is its closed compact subgroup. In the particular case when K is the unit subgroup, the homogeneous space X becomes the group space $G = \{g\}$.

The topology of the group space G induces naturally a topology in X (see [10]). It turns out that functions in X are continuous if and only if the corresponding functions assuming a constant value over all left cosets of G modulo K are continuous on G. We shall define the random field $\xi(x)$ over X as a continuous mapping of X into $\mathfrak{H}$. The field $\xi(x)$ is called homogeneous if its first and second moments remain unaltered under the transformations $g \in G$, that is, if $E\xi(x)$ is a constant and if $B(x_1, x_2) = E\xi(x_1)\overline{\xi(x_2)}$ satisfies the conditions

$$B(x_1, x_2) = B(gx_1, gx_2), \qquad g \in G. \tag{3.1}$$

In the following we shall always suppose that $E\xi(x) = 0$. It is obvious that the class of homogeneous random fields over X coincides with the class of homogeneous random fields over G, which are constant over all left cosets modulo K.

We begin again with the simplest case when the group G is compact. Here we can make use of the general theory of spherical functions (spherical harmonics) over compact homogeneous spaces, developed by E. Cartan [30] and H. Weyl [31] (see also [10]). Let us consider the complete system of unitary continuous nonequivalent representations (2.6) of the group G and choose in the space of these representations a basis such that these representations decompose into irreducible representations of the subgroup K. In order that a matrix element $T_{ij}^{(\lambda)}(g)$ be a constant over all left cosets of G modulo K we must have

$$T_{ij}^{(\lambda)}(gk) \equiv \sum_m T_{im}^{(\lambda)}(g)T_{mj}^{(\lambda)}(k) = T_{ij}^{(\lambda)}(g), \qquad g \in G;\ k \in K, \tag{3.2}$$

that is,

$$T_{mj}^{(\lambda)}(k) = \delta_{mj}, \qquad m = 1, \cdots, d_\lambda;\ k \in K. \tag{3.3}$$

From this it is seen that the matrix elements $T_{ij}^{(\lambda)}(g)$ which are constant over the left cosets fill out the columns of $T^{(\lambda)}(g)$, corresponding to the identity representation of K. Suppose, for instance, that the representation $T^{(\lambda)}$ of the group G contains r_λ times the identity representation of K. Suppose that in our basis $e_1, e_2, \cdots, e_{d_\lambda}$ these identity representations correspond to the first r_λ basis vectors so that $T^{(\lambda)}(k)e_j = e_j$ for $k \in K$ and for $j = 1, \cdots, r_\lambda$. In this case the functions of x

$$\Phi_{ij}^{(\lambda)}(x) = T_{ij}^{(\lambda)}(g), \qquad i = 1, \cdots, d_\lambda;\ j = 1, \cdots, r_\lambda;\ \lambda = 1, 2, \cdots \tag{3.4}$$

will be called *spherical functions* over X, while the functions

$$\Phi_{ij}^{(\lambda)}(x) = T_{ij}^{(\lambda)}(g), \qquad i = 1, \cdots, r_\lambda;\ j = 1, \cdots, r_\lambda;\ \lambda = 1, 2, \cdots \tag{3.5}$$

will be called *zonal spherical functions*. It is easy to see that the zonal spherical functions assume constant values over all two-sided cosets KgK of the group G modulo K. In other words, for these functions $\Phi_{ij}^{(\lambda)}(x) = \Phi_{ij}^{(\lambda)}(kx)$, $k \in K$. The set of points kx, $k \in K$, can be called naturally a sphere with center at the point

x_0 ($= K$) and passing through the point x. Hence the functions (3.5) are constant on all spheres with center at x_0. Therefore, the zonal function $\Phi_{ij}^{(\lambda)}(x)$ depends only on the invariants of the ordered pair of points x and x_0, which remain unaltered under all transformations $g \in G$, that is, on the *composite distance from x to x_0*,

$$\Phi_{ij}^{(\lambda)}(x) \equiv \Phi_{ij}^{(\lambda)}(x, x_0) = \Phi_{ij}^{(\lambda)}(gx, gx_0), \qquad g \in G; i, j = 1, \cdots, r_\lambda; \lambda = 1, 2, \cdots. \tag{3.6}$$

According to the general theory of spherical functions the functions (3.4) represent a complete orthogonal system in the space $L^2(X)$ of functions $\varphi(x)$ over X such that $|\varphi(x)|^2$ is summable with respect to the measure dx, which is invariant for all transformations $g \in G$. Only the functions (3.4) enter into the expansion of the function $\varphi(g)$ constant over all left cosets G/K in terms of the matrix elements $T_{ij}^{(\lambda)}(g)$ (see section 23 in chapter 5 of [10]).

The application of this theory to homogeneous random fields $\xi(x)$ leads, in view of theorem 1, to

THEOREM 5. *The random field*

$$\xi(x) = \sum_\lambda \sum_{i=1}^{d_\lambda} \sum_{j=1}^{r_\lambda} z_{ji}^{(\lambda)} \Phi_{ij}^{(\lambda)}(x), \qquad z_{ji}^{(\lambda)} = \frac{\int_X \xi(x)\Phi_{ij}^{(\lambda)}(x)\, dx}{\int_X |\Phi_{ij}^{(\lambda)}(x)|^2\, dx} \tag{3.7}$$

over a compact homogeneous space $X = G/K$ is homogeneous if and only if the random variables $z_{ji}^{(\lambda)}$ satisfy the relations

$$E z_{ji}^{(\lambda)} \overline{z_{lk}^{(\mu)}} = \delta_{\lambda\mu}\delta_{ik} f_{jl}^{(\lambda)}. \tag{3.8}$$

The covariance function $B(x_1, x_2)$ of such a field $\xi(x)$ can be represented in the form

$$B(x_1, x_2) = \sum_\lambda \sum_{j,l=1}^{r_\lambda} f_{jl}^{(\lambda)} \Phi_{lj}^{(\lambda)}(x_1, x_2), \tag{3.9}$$

where $\Phi_{lj}(x_1, x_2)$ are the functions (3.6). *Conversely, any function $B(x_1, x_2)$ of the form* (3.9), *where $||f_{jl}^{(\lambda)}||$ are Hermitian, nonnegative definite matrices such that the series* (3.9) *converges, is a covariance function of some homogeneous field over X.*

Formula (3.9) represents a somewhat improved statement of a result formulated in 1941 by Bochner [11]. The theorem of Obukhov [6] mentioned in the introduction is a special case of theorem 5 for homogeneous fields over a two-dimensional sphere S_2. For the more general case of the homogeneous fields over a sphere S_{n-1} in n-dimensional Euclidean space our theorem states that all such fields can be represented as a series of hyperspherical harmonics

$$Y_{l, m_1, \cdots, m_{n-3}, \pm m_{n-2}}(\theta_1, \cdots, \theta_{n-2}, \varphi), \qquad l = 0, 1, 2, \cdots; \quad 0 \leqq m_{n-2} \leqq m_{n-3} \leqq \cdots \leqq m_1 \leqq l, \tag{3.10}$$

(see, for example, volume 2 of [32]), with noncorrelated coefficients

$$z_{l,m_1,\cdots,m_{n-3},\pm m_{n-2}} \tag{3.11}$$

whose variance depends only on l. The zonal spherical functions will be given in this case by Gegenbauer polynomials (ultraspherical polynomials) $C^{(n-2)/2}(\mu)$ in $\mu = \cos\theta$. Therefore (3.9) becomes in this case the well-known formula of Schoenberg [7] for positive definite functions on an $(m-1)$-dimensional sphere

$$B(\theta_{12}) = \sum_{l=0}^{\infty} f_l C_l^{(n-2)/2}(\cos\theta_{12}). \tag{3.12}$$

Here θ_{12} is the angular distance between the points x_1 and x_2 of the sphere S_{m-1}. The fact that the composite distance θ_{12}, which is given in this case by a single number, is here symmetric in the points x_1 and x_2 so that one can simply speak of the distance between two points, and that $r_\lambda \leqq 1$ for every λ has a general explanation, as will be made clear in what follows.

We now take up the case of locally compact homogeneous spaces X. We begin by determining the general form of the positive definite function $B(x_1, x_2)$ over X, satisfying condition (3.1). To every such function we make correspond uniquely a positive definite function $B(g)$ over G,

$$B(g) = B(x_1, x_2) \quad \text{if} \quad g = k_1 g_2^{-1} g_1 k_2, \qquad x_1 = g_1 K,\ x_2 = g_2 K. \tag{3.13}$$

Obviously, $B(g) = B(k_1 g k_2)$ for any $k_1, k_2 \in K$, that is, $B(g)$ assumes constant values over all two-sided cosets of G modulo K. Conversely, a positive definite function over G that is constant on all two-sided cosets over K can be put into correspondence by means of equation (3.13) with a positive definite function over X satisfying (3.1). Hence our problem reduces to finding all positive definite functions over G that are constant over two-sided cosets modulo K.

By [17] every positive definite function over G is given by (2.24). In order that it be a constant over all two-sided cosets over K it is necessary and sufficient that the vector ξ_0 satisfy the condition

$$T(k)\xi_0 = \xi_0, \qquad \text{for all} \quad k \in K. \tag{3.14}$$

In particular when the unitary representation $T(g)$ of the group G is irreducible, the function (2.24) with ξ_0 satisfying (3.14) is a zonal spherical function corresponding to this representation. In general the function $\Phi(x)$ is called a zonal spherical function over X, corresponding to an irreducible unitary representation $T^{(\lambda)}(g)$, if it can be represented in the form

$$\Phi(x) = \Phi(x, x_0) = [T^{(\lambda)}(g)\xi, \eta], \qquad \text{where} \quad T^{(\lambda)}(k)\xi = \xi,\quad T^{(\lambda)}(k)\eta = \eta \tag{3.15}$$

for all $k \in K$ and it is called simply a spherical function if

$$\Phi(x) = [T^{(\lambda)}(g)\xi, \eta], \qquad \text{where} \quad T^{(\lambda)}(k)\xi = \xi \tag{3.16}$$

for all $k \in K$. Obviously the zonal spherical function (3.15) depends only on the composite distance from x to x_0, while the function (3.16) depends on the point $x = gK \in X$.

We next suppose that the group G is a separable, locally compact group of type I. In this case, by [21] and [22] we can decompose the representation $T(g)$ in (2.24) into a direct sum of multiplicity-free representations, each of which in turn can be decomposed into a continued direct sum of irreducible nonequivalent representations. If ξ_0 satisfies (3.14) then the projection of this vector into the space of any irreducible representation $T^{(\lambda)}(g)$, belonging to $T(g)$, will be an invariant vector of the corresponding (reducible) representation $T^{(\lambda)}(k)$ of the group K. Arguing as we did when deriving formula (2.26), we find that an arbitrary positive definite function over X satisfying (3.1) will be given by

$$B(x_1, x_2) = \int_{\overline{G}_K} \operatorname{Tr}\,[P_K^{(\lambda)} T^{(\lambda)}(g_2^{-1}g_1) P_K^{(\lambda)} F_K(d\lambda)]. \tag{3.17}$$

Here $\overline{G}_K$ is the subset of those $\lambda \in \overline{G}$ for which in the space $H^{(\lambda)}$ there is at least one vector invariant with respect to all transformations $T^{(\lambda)}(k)$ with $k \in K$, $P_K^{(\lambda)}$ is a projection operator in $H^{(\lambda)}$ onto the maximum invariant with respect to all $T^{(\lambda)}(k)$ subspace $H_K^{(\lambda)}$, and $F_K(d\lambda)$ is a Hermitian nonnegative definite "operator measure" over $\overline{G}_K$ with values in a ring of operators in the space $H_K^{(\lambda)}$ while g_1 and g_2 are arbitrary elements of cosets modulo K, defining the points x_1 and x_2 of X. The operator $P_K^{(\lambda)} T^{(\lambda)}(g) P_K^{(\lambda)}$ in (3.17) is considered as an operator in the subspace $H_K^{(\lambda)}$, and the integral over $\overline{G}_K$ here must be understood as the sum of integrals, taken over the subspaces $\overline{G}_n \subset \overline{G}_K$ for $n = \infty, 1, 2, \cdots$ such that for $\lambda \in \overline{G}_n$ the subspace $H_K^{(\lambda)}$ is n-dimensional.

In this way we come to the following theorem.

THEOREM 6. *Let the group of motions G of the homogeneous space $X = G/K$ be a separable locally compact group pf type I. Then the function $B(x_1, x_2)$ on x is a positive definite function invariant with respect to all motions if and only if it can be represented in the form* (3.17), *where $F_K(d\lambda)$ is a Hermitian nonnegative definite "operator measure" over $\overline{G}_K$, whose values are operators in $H_K^{(\lambda)}$ such that*

$$\operatorname{Tr}\,[F_K(\overline{G}_K)] < \infty. \tag{3.18}$$

If we choose in each of the subspaces $H_K^{(\lambda)}$, corresponding to $\overline{G}_n$, a definite basis, we can rewrite formula (3.17) in the form

$$B(x_1, x_2) = \int_{\overline{G}_K} \sum_{i,j} \Phi_{ij}^{(\lambda)}(x_1, x_2) F_{ji}(d\lambda), \tag{3.19}$$

which generalizes (3.9). Here $\{\Phi_{ij}^{(\lambda)}(x_1, x_2)\}$ is a complete family of linearly independent zonal spherical functions corresponding to an irreducible representation $T^{(\lambda)}(g)$. This family will be finite (n^2 members) for $\lambda \in \overline{G}_n$, where $n = 1, 2, \cdots$, and infinite for $\lambda \in \overline{G}_\infty$.

There exists still another class of homogeneous spaces for which a general formula can be written, even simpler than (3.17), for the arbitrary positive definite function $B(x_1, x_2)$ satisfying (3.1). This is E. Cartan's class of symmetric homogeneous spaces. A homogeneous space $X = G/K$ is called symmetric if the group G of its motions contains an involuntary automorphism $g \to g'$, that is, an isomorphic mapping of G into G such that $(g')' = g$ such that it sets apart the stationary subgroup K in the sense that $g' = g$ if and only if $g \in K$. It is not hard to see that this condition is satisfied, for example, for any homogeneous spaces of constant curvature (see [33] for this and other examples of homogeneous symmetric spaces). For arbitrary symmetric homogeneous spaces the following important theorem was proved by Gelfand [34] (see also section 31.10 in [18]).

THEOREM 6′. *If the space $X = G/K$ is a symmetric homogeneous space, then to every irreducible, unitary representation $T^{(\lambda)}(g)$ of the group G corresponds not more than one zonal spherical function $\Phi^{(\lambda)}(x_1, x_2)$, so that the subspace $H_K^{(\lambda)}$ for any $\lambda \in \overline{G}$ is in this case not more than one-dimensional. The function $B(x_1, x_2)$ over such X is a positive definite function invariant with respect to all motions if and only if it can be represented by the formula*

$$B(x_1, x_2) = \int_{\overline{G}_1} \Phi^{(\lambda)}(x_1, x_2) F(d\lambda), \tag{3.20}$$

where $F(d\lambda)$ is a nonnegative measure over $\overline{G}_1$, which coincides in this case with $\overline{G}_K$, such that the integral on the right of (3.19) *converges.*

Theorem 6′ supplements theorem 6. In the case of a symmetric space X the function $\Phi^{(\lambda)}(x_1, x_2)$ will depend symmetrically on x_1 and x_2, since here motions always exist which interchange the order of these two points. In other words, the zonal spherical functions will depend here only on the composite distance *between the points* x_1 and x_2 (see [33]).

Let us now consider the "spectral representation" of the homogeneous random field $\xi(x)$ itself. We suppose that the covariance function $B(x_1, x_2)$ of this field can be represented in the form (3.17). In particular it could be represented in the form (3.20), which is a special case of (3.17). Our discussion will be similar to the proof of theorem 4. We consider the Hilbert space $L^2(F_K)$ of operator functions $U^{(\lambda)}$ of $\lambda \in \overline{G}_K$ with values in a ring of bounded operators from $H_K^{(\lambda)}$ onto $H^{(\lambda)}$ and with the following norm in $L^2(F_K)$,

$$\|U^{(\lambda)}\|^2 = \int_{\overline{G}_K} \mathrm{Tr}\,[U^{(\lambda)} F_K(d\lambda) U^{(\lambda)*}]. \tag{3.21}$$

Every bounded operator V in $H^{(\lambda)}$ can be considered, if desired, as an operator from $H_K^{(\lambda)}$ onto $H^{(\lambda)}$ by restricting its domain of definition to the subspace $H_K^{(\lambda)}$. To avoid confusion we shall denote such a restricted operator by $VP_K^{(\lambda)}$. In this case the correspondence

$$T^{(\lambda)}(g) P_K^{(\lambda)} \leftrightarrow \xi(x), \qquad x = \{g'K\} \ni g \tag{3.22}$$

will be an isometric mapping of the set $\{\xi(x), x \in X\}$ of the space $\mathfrak{H}$ into $L^2(F_K)$, which can be extended to an isometric mapping of $L^2(F_K)$ into $\mathfrak{H}$. If we now take the function $U^{(\lambda)}(\Lambda; f_1, f_2) \in L^2(F_K)$ corresponding to a pair of vectors $f_1 \in H^{(\lambda)}$, $f_2 \in H_K^{(\lambda)}$ and to a measurable set $\Lambda \in \overline{G}_K$, by means of formula (2.30), then under the mapping (3.22) we have

$$U^{(\lambda)}(\Lambda; f_1, f_2) \leftrightarrow Z(\Lambda; f_1, f_2) = [Z(\Lambda)f_1, f_2], \tag{3.23}$$

where $Z(\Lambda)$ is a random linear operator from $H^{(\lambda)}$ onto $H_K^{(\lambda)}$, depending additively on $\Lambda \in \overline{G}_K$. Furthermore, analogous to the proof of formula (2.33), it may be shown that

$$\xi(x) = \int_{\overline{G}_K} \mathrm{Tr}\,[Z(d\lambda)T^{(\lambda)}(g)P_K^{(\lambda)}], \qquad x = \{g'K\} \ni g \tag{3.24}$$

and that for any $f_1, g_1 \in H^{(\lambda)}$, $f_2, g_2 \in H_K^{(\lambda)}$, we have

$$E[Z(\Lambda_1)f_1, f_2]\overline{[Z(\Lambda_2)g_1, g_2]} = (f_1, g_1)[F_K(\Lambda_1 \cap \Lambda_2)g_2, f_2]. \tag{3.25}$$

From this we can obtain easily

THEOREM 7. *Let the group of motions G of a homogeneous space $X = G/K$ be a separable, locally compact group of type I. Then the random field $\xi(x)$ over X will be homogeneous if and only if it can be represented in the form* (3.24), *where $Z(\Lambda)$ is a random linear operator from $H^{(\lambda)}$ onto $H_K^{(\lambda)}$, depending completely additively on the set $\Lambda \subset \overline{G}_K$ and satisfying condition* (3.25) *for every $f_1, g_1 \in H^{(\lambda)}$, $f_2, g_2 \in H_K^{(\lambda)}$. Here $F_K(\Lambda)$ is a Hermitian nonnegative operator in $H_K^{(\lambda)}$, satisfying* (3.18), *by means of which the covariance function $B(x_1, x_2)$ of the field $\xi(x)$ is expressed by formula* (3.17).

Analogous results hold for homogeneous random fields over an arbitrary symmetric homogeneous space X. The only difference is that in this case the space $H_K^{(\lambda)}$ is one-dimensional for any $\lambda \in \overline{G}_K = \overline{G}_1$ and therefore $Z(\Lambda)$ is a linear random functional in $H^{(\lambda)}$, satisfying condition

$$EZ(\Lambda_1)f_1\overline{Z(\Lambda_2)g_1} = F(\Lambda_1 \cap \Lambda_2)(f_1, g_1), \tag{3.26}$$

where $F(\Lambda)$ is the nonnegative measure in $\overline{G}_1$ from equation (3.20).

In "coordinate notation" formulas (3.24) and (3.25) will look very much like (3.7) and (3.8). In particular, in the case of a symmetric space X these formulas become

$$\xi(x) = \int_{\overline{G}_1} \sum_n \Phi_n^{(\lambda)}(x)Z_n(d\lambda), \tag{3.27}$$

$$EZ_n(\Lambda_1)\overline{Z_m(\Lambda_2)} = \delta_{nm}F(\Lambda_1 \cap \Lambda_2), \tag{3.28}$$

where $\{\Phi_n^{(\lambda)}(x), n = 0, 1, 2, \cdots\}$ is a complete system of "spherical functions," corresponding to the zonal spherical function $\Phi^{(\lambda)}(x, x_0) \equiv \Phi_0^{(\lambda)}(x)$. Theorem 7, together with theorems 6 and 6′ can be considered as generalizations of theorems 1, 3, 4, and 5 above.

EXAMPLES. It is known that in the n-dimensional Euclidean space R_n, with the usual group of motions G there exists a one-parameter family of zonal spherical functions $\Phi^{(\lambda)}(r)$, depending on the distance r between the points x_1 and x_2,

$$\Phi^{(\lambda)}(r) = \frac{J_{(n-2)/2}(\lambda r)}{(\lambda r)^{(n-2)/2}}, \qquad 0 \leqq \lambda < \infty \tag{3.29}$$

(see, for example, [35] and [28]). In this way, the formula of Schoenberg (1.4) is a special case of the general formula (3.18).

The general spherical functions over the Euclidean plane R_2 were enumerated by M. G. Krein [35] (see also N. Ya. Vilenkin [24]). Substituting these formulas into (3.27) and (3.28) we obtain the following general representation of homogeneous and isotropic random fields over a plane

$$\xi(r, \varphi) = \sum_{n=-\infty}^{\infty} e^{-in\varphi} \int_0^\infty J_n(r\lambda) Z_n(d\lambda), \tag{3.30}$$

where (r, φ) are polar coordinates on a plane and the $Z_n(d\lambda)$ satisfy (3.28). Similarly, for homogeneous and isotropic fields in an n-dimensional Euclidean space R_n we can obtain from the results of [28] (see also volume 2 in [32]) the representation

$$\begin{aligned} \xi(r, \theta_1, \cdots, \theta_{n-2}, \varphi) &= \sum A_{l,m_1,\cdots,m_{n-3},\pm m_{n-2}} Y_{l,m_1,\cdots,m_{n-3},\pm m_{n-2}}(\theta_1, \cdots, \theta_{n-2}, \varphi) \\ &\quad \int \frac{J_{l+(n-2)/2}(r\lambda)}{(r\lambda)^{(n-2)/2}} Z_{l,m_1,\cdots,m_{n-3},\pm m_{n-2}}(d\lambda), \end{aligned} \tag{3.31}$$

where $r, \theta_1, \cdots, \theta_{n-2}, \varphi$ are spherical coordinates in R_n and where the summation on the right side goes over all $l = 0, 1, 2, \cdots, 0 \leqq m_{n-2} \leqq m_{n-3} \leqq \cdots \leqq m_1 \leqq l$, and over both signs of m_{n-2}. In (3.31) the $A_{l,m_1,\cdots,m_{n-3},\pm m_{n-2}}$ are normalizing constants, which can be expressed simply in terms of the Γ-function, and $Y_{l,m_1,\cdots,m_{n-3},\pm m_{n-2}}$ are the corresponding surface harmonics, while $Z_{l,m_1,\cdots,m_{n-3},\pm m_{n-2}}(d\lambda)$ is a countable family of mutually uncorrelated random measures on the line $[0, \infty]$ with equal mathematical expectations of the square of the absolute value.

In case of an n-dimensional Lobachevsky space L_n, the zonal spherical functions have the form

$$\Phi^{(\lambda)}(r) = \frac{P_{\mu(\lambda)}^{-(n-2)/2}(\cosh r)}{(\sinh r)^{(n-2)/2}}, \qquad \mu(\lambda) = -\frac{1}{2} + i\left[\lambda - \left(\frac{n-1}{2}\right)^2\right]^{1/2}, \tag{3.32}$$

where P_μ^ν is a special solution of the well-known Legendre's differential equation (see Krein [35] and Vilenkin [29]; for three-dimensional Lobachevsky space, where $\Phi^{(\lambda)}(r) = [\sin(\lambda - 1)^{1/2} r]/[(\lambda - 1)^{1/2} \sinh r]$, the corresponding result has been obtained previously by Gelfand and Naĭmark [36], [18]). From this we have for the covariance function of an isotropic field L_n the following formula, first established by Krein [35],

$$B(r) = \int_0^\infty \frac{P_{\mu(\lambda)}^{-(n-2)/2}(\cosh r)}{(\sinh r)^{(n-2)/2}} \, d\Phi(\lambda). \tag{3.33}$$

In [35] all the nonzonal spherical functions of the space L_2 are also given. Substituting these into (3.27) we obtain the following form of an isotropic random field over the Lobachevsky plane

$$\xi(r, \varphi) = \sum_{n=-\infty}^{\infty} \gamma_n e^{-in\varphi} \int_0^{\infty} P^n_{-1/2+(1/4-\lambda)^{1/2}}(\cosh r) Z_n(d\lambda),$$

$$\gamma_n = \left\{(-1)^n \frac{\Gamma\left[\left(\frac{1}{4}-\lambda\right)^{1/2} - |n| + \frac{1}{2}\right]}{\Gamma\left[\left(\frac{1}{4}-\lambda\right)^{1/2} + |n| + \frac{1}{2}\right]}\right\}^{1/2}, \tag{3.34}$$

where $Z_n(d\lambda)$ satisfies (3.28). For the n-dimensional Lobachevsky space L_n all the nonzonal spherical functions are given in the note [29]. From this, for an arbitrary isotropic field in L_n, we obtain a representation of the form (3.31) but with the replacement of the function $J_{l+(n-2)/2}(r\lambda)/(r\lambda)^{(n-2)/2}$ by $P^{-(n-2)/2-l}_{\mu(\lambda)}(\cosh r)/(\sinh r)^{(n-2)/2}$ and with the alteration of the values of the constants $A_{l,m_1,\cdots,m_{n-3},\pm m_{n-2}}$.

A number of other examples of complete systems of zonal spherical functions over special homogeneous spaces is given in papers [26], [37], [38]. Some general properties of such functions are studied in [33]. These properties simplify considerably the finding of the functions. The problem of finding arbitrary nonzonal spherical functions is much more difficult. However, for special spaces $X = G/K$ it can also be solved quite effectively in a number of cases.

4. Multidimensional homogeneous fields. Further generalizations

The notion of a homogeneous random field considered in sections 2 and 3 admits a number of further generalizations. We now consider briefly some of them.

4.1. *Multidimensional homogeneous random fields over groups.*

Let $\boldsymbol{\xi}(g) = \{\xi_1(g), \cdots, \xi_N(g)\}$ be an N-dimensional random field over a group G and let $\{U(g)\}$ be some, not necessarily unitary, N-dimensional representation of this group acting in the space A. The field $\boldsymbol{\xi}(g)$ is called *a left homogeneous field of the quantities* $\boldsymbol{\xi}$, *which transform according to the representation* $U(g)$, if for all $g, g_1, g_2 \in G$

$$\begin{aligned} E\xi_i(g_1) &= E[V_g\boldsymbol{\xi}(g_1)]_i, \\ E\xi_i(g_1)\overline{\xi_j(g_2)} &= E[V_g\boldsymbol{\xi}(g_1)]_i\overline{[V_g\boldsymbol{\xi}(g_2)]_j}, \end{aligned} \tag{4.1}$$

where

$$V_g\boldsymbol{\xi}(g_1) = U(g)\boldsymbol{\xi}(g^{-1}g_1). \tag{4.2}$$

In other words, if $\mathbf{M}(g) = E\boldsymbol{\xi}(g)$ is a vector of mean values of the field $\boldsymbol{\xi}(g)$ and if $B(g_1, g_2) = ||E\xi_i(g_1)\overline{\xi_j(g_2)}||$ is its covariance matrix, then

$$\mathbf{M}(g_1) = U(g)\mathbf{M}(g^{-1}g_1) = U(g_1)\mathbf{M}^{(0)}, \tag{4.3}$$

$$B(g_1, g_2) = U(g)B(g^{-1}g_1, g^{-1}g_2)U^*(g) = U(g_2)B^{(0)}(g_2^{-1}g_1)U^*(g_2), \tag{4.4}$$

where $\mathbf{M}^{(0)}$ is some constant vector and $B^{(0)}(g)$ is a matrix depending on one argument. In particular, if the representation $\{U(g)\}$ is a multiple of the identity representation, our N-dimensional field $\boldsymbol{\xi}(g)$ is simply the set of N scalar homogeneous (and homogeneously connected) fields. The field $\boldsymbol{\xi}(g)$, homogeneous with respect to right shifts, is similarly defined. In this case we must only assume that $V_g\boldsymbol{\xi}(g_1) = U(g)\boldsymbol{\xi}(g_1g)$.

Following A. N. Kolmogorov (see Yu. A. Rosanov [39]), the multidimensional field $\boldsymbol{\xi}(g)$ can also be interpreted as a field of linear operators ξ_g from some linear space A onto the space $\mathfrak{H}$ of random variables

$$\xi_g(a) = \sum_k \xi_k(g)a_k, \qquad a = (a_1, a_2, \cdots) \in A. \tag{4.5}$$

In this case the field ξ_g will be called homogeneous with respect to left translations if there exists representation $\{U^+(g)\}$ of the group G in the space A such that for all g, g_1, $g_2 \in G$, we have

$$\begin{aligned} E\xi_{g_1}(a) &= E\xi_{gg_1}[U^+(g)a], \\ E\xi_{g_1}(a_1)\overline{\xi_{g_2}(a_2)} &= E\xi_{gg_1}[U^+(g)a_1]\overline{\xi_{gg_2}[U^+(g)a_2]}. \end{aligned} \tag{4.6}$$

The representation $U^+(g)$ is connected with $U(g)$ by the relation $U^+(g) = [U(g^{-1})]'$, that is, $U^+_{ij}(g) = U_{ji}(g^{-1})$. Such a "noncoordinate" interpretation of multidimensional fields is particularly convenient in the case of infinite-dimensional fields.

The general form of the vector of mean values of a homogeneous field of quantities $\boldsymbol{\xi}$ can be defined by formula (4.3). The vector $\mathbf{M}^{(0)}$ in this formula is any constant vector of A. It remains to obtain the general form of the matrix $B(g_1, g_2)$ or, what is the same thing, of the matrix $B^{(0)}(g)$ in (4.4). In order to do this we assume that the group G is a separable, locally compact group of type I. In particular it can be an arbitrary compact group. Let a be an arbitrary constant vector of A and let $a(g) = U^+(g)a$. Then by (4.6) the random field

$$\xi_a(g) = \sum_k \xi_k(g)a_k(g) \tag{4.7}$$

will be a left homogeneous one-dimensional field, depending linearly on the parameter a_k. Applying theorem 4 to this field we obtain

$$\xi_a(g) = \int_{\hat{G}} \operatorname{Tr}[T^{(\lambda)}(g)Z_a(d\lambda)], \tag{4.8}$$

where

$$Z_a(\Lambda) = \sum_k Z_k(\Lambda)a_k, \tag{4.9}$$

and $\mathbf{Z}(\Lambda) = \{Z_1(\Lambda), \cdots, Z_N(\Lambda)\}$ is a vector "random operator measure" on G such that

$$E[Z_m(\Lambda_1)f_1, f_2]\overline{[Z_n(\Lambda_2)g_1, g_2]} = (f_1, g_1)[F_{mn}(\Lambda_1 \cap \Lambda_2)g_2, f_2], \tag{4.10}$$

where $F_{mn}(\Lambda)$ are operators in $H^{(\lambda)}$ such that $\sum F_{mn}(\Lambda)a_m\bar{a}_n$ is a Hermitian nonnegative operator for any complex numbers $a_1, \cdots, a_N$. Or, if $Z_m(\Lambda) = ||Z_{ij,m}(\Lambda)||$, then

$$EZ_{ij,m}(\Lambda_1)\overline{Z_{kl,n}(\Lambda_2)} = \delta_{jl}F_{ik,mn}(\Lambda_1 \cap \Lambda_2), \tag{4.11}$$

that is

$$E[Z_a(\Lambda_1)f_1, f_2]\overline{[Z_b(\Lambda_2)g_1, g_2]} = (f_1, g_1)[F(\Lambda_1 \cap \Lambda_2)(a, g_2), (b, f_2)], \tag{4.12}$$

where $F(\Lambda) = ||F_{ik,mn}(\Lambda)||$ is a Hermitian nonnegative operator in the Kronecker product $A \times H^{(\lambda)}$ of the spaces A and $H^{(\lambda)}$, that is, in the direct sum of spaces $e_1H^{(\lambda)}$, $e_2H^{(\lambda)}$, $\cdots$, isomorphic to $H^{(\lambda)}$ where $\{e_1, e_2, \cdots\}$ is a basis in A. The operator $F(\Lambda)$ has a finite trace and depends completely additively on Λ. The formula (4.8) can also be written in the form

$$\xi(g) = \int_{\hat{G}} U(g) \operatorname{Tr} [T^{(\lambda)}(g)\mathbf{Z}(d\lambda)] \tag{4.13}$$

or in the form

$$\xi_m(g) = \sum_n U_{mn}(g) \int_{\hat{G}} \sum_{i,j} T^{(\lambda)}_{ij}(g)Z_{ji,n}(d\lambda). \tag{4.14}$$

Equation (4.3) is obviously equivalent to the condition

$$E\mathbf{Z}(\Lambda) = \begin{cases} \mathbf{M}^{(0)} & \text{if } \lambda_0 \subset \Lambda, \\ 0 & \text{if } \lambda_0 \not\subset \Lambda, \end{cases} \tag{4.15}$$

where $\{T^{(\lambda_0)}(g)\}$ is the identity representation of the group G, so that $T^{(\lambda_0)}(g) = I$. From (4.8) and (4.12) we obtain the following expression for the covariance matrix $B(g_1, g_2)$

$$\begin{aligned} B(g_1, g_2) &= U(g_1) \int_{\hat{G}} \operatorname{Tr} \{T^{(\lambda)}(g_2^{-1}g_1)F(d\lambda)\} U^*(g_2) \\ &= U(g_2)U(g_2^{-1}g_1) \int_{\hat{G}} \operatorname{Tr} [T^{(\lambda)}(g_2^{-1}g_1)F(d\lambda)] U^*(g_2) \end{aligned} \tag{4.16}$$

or in other words

$$B_{mn}(g_1, g_2) = \sum_{s,t} U_{ms}(g_1)\overline{U_{nt}(g_2)} \int_{\hat{G}} \sum_{i,j} T^{(\lambda)}_{ij}(g_2^{-1}g_1)F_{ji,st}(d\lambda), \tag{4.17}$$

$$B^{(0)}_{mn}(g) = \sum_s U_{ms}(g) \int_{\hat{G}} \sum_{i,j} T^{(\lambda)}_{ij}(g)F_{ji,sn}(d\lambda). \tag{4.18}$$

In order that this relation be compatible with (4.3), the following inequality must hold.

$$F_{mn}(\lambda_0) \geqq M^{(0)}_m \overline{M^{(0)}_n}. \tag{4.19}$$

From this it is easy to derive

THEOREM 8. *Let $\xi(g)$ be a multidimensional random field over a separable, locally compact group G of type I. Then $\xi(g)$ is a left homogeneous field of the quantities ξ, which transform according to the representation $\{U(g)\}$, if and only if it is representable in the form* (4.13), *where $\mathbf{Z}(\Lambda)$ is the vector "random operator* *measure" over $\overline{G}$, satisfying* (4.12), *while* $F(\Lambda)$ *is the "operator measure" over* $\overline{G}$, *whose values are Hermitian nonnegative operators with finite traces in the Kronecker product* $A \times H^{(\lambda)}$ *of the spaces* A *and* $H^{(\lambda)}$. *The mean value* $\mathbf{M} = E\boldsymbol{\xi}(g)$ *of such a field* $\boldsymbol{\xi}(g)$ *is given by formula* (4.3), *where* $\mathbf{M}^{(0)}$ *is determined from* (4.15), *and the covariance matrix* $B(g_1, g_2)$ *is given by* (4.16).

Conversely, any matrix of the form (4.16) *is a covariance matrix of some multidimensional left homogeneous random field whose mean value can assume any value of the form* (4.3), *where* $\mathbf{M}^{(0)}$ *is a constant vector, satisfying* (4.19).

4.2. *Multidimensional homogeneous fields over homogeneous spaces.* The multidimensional field $\boldsymbol{\xi}(x) = \{\xi_1(x), \xi_2(x), \cdots\}$ over $X = G/K$ is called homogeneous if its first and second moments remain unaltered when we apply the transformation $\boldsymbol{\xi}(x) \to U(g)\boldsymbol{\xi}(g^{-1}x)$ to $\boldsymbol{\xi}(x)$, where $\{U(g)\}$ is some representation of the group G. It is clear that if the field $\boldsymbol{\xi}(x)$ is homogeneous, then the vector $\mathbf{M}(x) = E\boldsymbol{\xi}(x)$ and the matrix $B(x_1, x_2) = ||E\xi_i(x_1)\overline{\xi_j(x_2)}||$ will satisfy the relations

$$\mathbf{M}(x) = U(g)\mathbf{M}(g^{-1}x), \tag{4.20}$$

$$B(x_1, x_2) = U(g)B(g^{-1}x_1, g^{-1}x_2)U^*(g). \tag{4.21}$$

This can be expressed as follows: the linear operator from A into $\mathfrak{H}$

$$\xi_x(a) = \sum_k \xi_k(x)a_k \tag{4.22}$$

will in this case have the properties

$$\begin{aligned} E\xi_x(a) &= E\xi_{gx}[U^+(g)a], \\ E\xi_{x_1}(a_1)\overline{\xi_{x_2}(a_2)} &= E\xi_{gx_1}[U^+(g)a_1]\overline{\xi_{gx_2}[U^+(g)a_2]}. \end{aligned} \tag{4.23}$$

From formula (4.20) it follows at once that

$$\mathbf{M}(x) = U(g)\mathbf{M}^{(0)}, \tag{4.24}$$

where $\mathbf{M}^{(0)}$ is an arbitrary vector in A, invariant with respect to all transformations $U(k)$ with $k \in K$

$$U(k)\mathbf{M}^{(0)} = \mathbf{M}^{(0)} \qquad \text{for all } k \in K. \tag{4.25}$$

Much more complicated is the question of the general form of the covariance matrix $B(x_1, x_2)$ and the connected question of the "spectral decomposition" of the field $\boldsymbol{\xi}(x)$ itself. We now consider this question in the case of a finite-dimensional field $\boldsymbol{\xi}(x)$ over a compact homogeneous space X. After this, it will not be hard to see what the corresponding formulas should become in the more general locally compact case.

Let $\boldsymbol{\xi}(x) = \{\xi_1(x), \cdots, \xi_N(x)\}$, where $x \in G/K$ and G is a compact group. The field $\boldsymbol{\xi}(x)$ can be considered as an N-dimensional homogeneous random field $\boldsymbol{\xi}(g)$ over G, constant over all left cosets of G with respect to K. Therefore [see (4.14) and (4.11)]

$$\eta_i(g) = \sum_j \xi_j(g)U_{ji}^+(g) = \sum_j U_{ij}(g^{-1})\xi_j(g) \tag{4.26}$$

for every $i = 1, 2, \cdots, N$ will admit the representation

$$\eta_i(g) = \sum_\lambda \sum_{m,n} z^{(\lambda)}_{nm,i} T^{(\lambda)}_{mn}(g), \tag{4.27}$$

where $\{T^{(\lambda)}(g)\}$ are all possible irreducible and nonequivalent unitary representations of G and

$$E z^{(\lambda)}_{nm,i} \overline{z^{(\mu)}_{sr,j}} = \delta_{\lambda\mu}\delta_{mr} f^{(\lambda)}_{ns,ij}. \tag{4.28}$$

In our case $\xi_k(g) = \xi_k(gk)$, $k \in K$. Therefore for any $k \in K$

$$\begin{aligned} \eta_i(gk) &= \sum_j \xi_j(gk) U^+_{ji}(gk) \\ &\equiv \sum_j \sum_l \xi_j(g) U^+_{il}(g) U^+_{il}(k) = \sum_l \eta_l(g) U^+_{il}(k), \end{aligned} \tag{4.29}$$

that is,

$$\sum_\lambda \sum_{m,n} \sum_r z^{(\lambda)}_{nm,i} T^{(\lambda)}_{mr}(g) T_{rn}(k) = \sum_\lambda \sum_{m,n} \sum_l z^{(\lambda)}_{nm,l}(g) T^{(\lambda)}_{mn}(g) U^+_{il}(k). \tag{4.30}$$

Conversely, it follows from (4.30) that $\xi_j(gk) \equiv \xi_j(g)$. Hence condition (4.30) is both necessary and sufficient in order that the field $\boldsymbol{\xi}(g)$ in (4.26) and (4.27) be a homogeneous, multidimensional field over $X = G/K$.

Because of the orthogonality condition (2.8), it follows from (4.30) that for any λ, m, and n and for all $k \in K$

$$\sum_r z^{(\lambda)}_{nm,i} T^{(\lambda)}_{nr}(k) = \sum_l z^{(\lambda)}_{nm,l} U^+_{il}(k). \tag{4.31}$$

We now choose bases in the space A of the representation $U^+(g)$, and in the spaces $H^{(\lambda)}$ of representations $T^{(\lambda)}(g)$, $\lambda = 1, 2, \cdots$, in such a way that these representations will decompose into irreducible representations of the subgroup K. Moreover, we shall require that the equivalent representations of K, which enter into U^+ and $T^{(\lambda)}$ be written identically. This can always be done by a simple change of the bases. Let the representation $U^+(g)$ of the group G decompose into distinct irreducible representations $\{V^{(i)}(k)\}$, for $i = 1, \cdots, J$. We shall denote the multiplicity of the representation $V^{(i)}(k)$ in $U^+(g)$ by L_i, and its dimension by S_i. In this case it is convenient to replace the index i, with $i = 1, 2, \cdots, N$, of the components of the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ by a compound index i_{ls} where $i = 1, 2, \cdots, J$; $l = 1, 2, \cdots, L_i$; $s = 1, 2, \cdots, S_i$. Then $U^+_{ij}(k)$ becomes

$$U^+_{i_{ls}, j_{mt}}(k) = \delta_{ij}\delta_{lm} V^{(i)}_{st}(k). \tag{4.32}$$

Similarly, if the representation $T^{(\lambda)}(g)$ of the group G decomposes into irreducible representations $V^{(n)}(k)$ with $n = 1, \cdots, N_\lambda$, where the multiplicity of $V^{(n)}(k)$ in $T^{(\lambda)}(g)$ is $U_{n\lambda}$, then instead of the index n with $n = 1, \cdots, d_\lambda$, which enumerates the components of the matrix $T^{(\lambda)}(g)$, we can use the compound index n_{ua} with $n = 1, \cdots, N_\lambda$; $u = 1, \cdots, U_{n\lambda}$; $a = 1, \cdots, S_n$. Then

$$T^{(\lambda)}_{n_{ua}, m_{wb}}(k) = \delta_{nm}\delta_{uw} V^{(n)}_{ab}(k). \tag{4.33}$$

Substituting (4.32) and (4.33) into (4.31) we obtain easily

(4.34) $$z^{(\lambda)}_{n_{ua},m,il_s} = \delta_{nl}\delta_{as} z^{(\lambda,i)}_{m,ul},$$

where

(4.35) $$E z^{(\lambda,i)}_{m,ul} \overline{z^{(\mu,j)}_{n,wk}} = \delta_{\lambda\mu}\delta_{mn} f^{(\lambda,ij)}_{uw,lk}.$$

From this it follows that

(4.36) $$\eta_{il_s}(g) = \sum_{\lambda} \sum_{m=1}^{d_\lambda} \sum_{u=1}^{U_{i\lambda}} z^{(\lambda,i)}_{m,ul} T^{(\lambda)}_{mi_{u_s}}(g),$$

that is, every component $\eta_j(g)$ of the vector $\boldsymbol{\eta}(g)$ decomposes only along those columns of the matrix $T^{(\lambda)}(g)$ which belong to the same irreducible representation of the group K as the jth column of the matrix $U^+(g)$ and occupies in it the same position at this jth column. By (4.36) the components $\xi_j(x)$ of the field $\boldsymbol{\xi}(x)$ have the form

(4.37) $$\xi_j(x) = \sum z^{(\lambda,i)}_{m,ul} U_{jil_s}(g) T^{(\lambda)}_{mi_{u_s}}(g),$$

where the sum is taken over those indices that are repeated. It is not difficult to check that the functions $U_{jil_s}(g) T^{(\lambda)}_{mi_{u_s}}(g)$ depend in fact only on the cosets of G modulo K, that is, they are functions of $x \in X$, and not of $g \in G$. It follows immediately from (4.37) and (4.35) that the general formula for the correlation matrix $B(x_1, x_2) = ||B_{jk}(x_1, x_2)||$ is

(4.38) $$B_{jk}(x_1, x_2) = \sum f^{(\lambda,il)}_{uv,ab} U_{jia_s}(g_1) T^{(\lambda)}_{l_vi i_{u_s}}(g_2^{-1}g_1) U^*_{l_v lk}(g_2),$$

where the sum is again taken over the indices that are repeated and where $||f^{(\lambda,il)}_{uv,ab}||$ is a nonnegative definite matrix over all its indices, that is, such that for any complex numbers α_{iua}

(4.39) $$\sum_{i,l} \sum_{u,v} \sum_{a,b} f^{(\lambda,il)}_{uv,ab} \alpha_{iua} \overline{\alpha_{lvb}} \geqq 0.$$

Equations (4.37) and (4.35) determine the general "spectral representation" of a multidimensional homogeneous field over a compact homogeneous space X, while (4.38) gives the "spectral representation" of the corresponding covariance function.

In the case of a separable, locally compact group G of type I and its compact subgroup K, the spectral representation of the multidimensional homogeneous field $\boldsymbol{\xi}(x)$ over $X = G/K$ and its covariance matrix $B(x_1, x_2)$ will be given by formulas similar to (4.37), (4.35), (4.38), and (4.39). However, summation over λ must be replaced here by integration of the corresponding functions of the set $d\lambda \subset \overline{G}$ over the whole space $\overline{G}$.

In applications to special manifolds X and to special representations $U(g)$ the general formulas can naturally be considerably simplified. Thus, for example, in the case of fields on a sphere S_2 in three-dimensional Euclidean space R_3 the stationary subgroup $K = O_2$ is the subgroup of rotations around the axis, which is commutative. Therefore all its irreducible representations are one-dimensional representations of the form $V^{(m)}(\varphi) = e^{im\varphi}$, where φ is the angle of rotation. Moreover in this case in all the irreducible representations of the group $G = O_3$, the complete group of rotations in R_3, every representation $V^{(m)}$ of the group O_2 enters not more than once. Thus in this case, instead of the compound indices i_{l_s}, n_{ua}, etc., it is usually possible to get along with simple indices i, n, etc. Making use of the formulas for all matrix elements $T^{(\lambda)}_{mn}$ of representations of the group O_3, given in [23], it is not hard to write down relations (4.37) and (4.38) explicitly for the case, for instance, of vector homogeneous random fields over S_2 or of tensor homogeneous fields of not too high a rank. The same results can be obtained making use of rotationally invariant expansion of vector and tensor (nonrandom) fields of the sphere S_2 over some specially chosen functions as described in [23]. In this connection see also [40], in which the problem of a similar invariant expansion of vector field on the sphere S_n of an $(n + 1)$-dimensional space R_n is considered.

Another important special case is that of homogeneous and isotropic vector fields in n-dimensional Euclidean space $R_n = M_n/O_n$. Here M_n is the group of all motions in R_n, and O_n is the group of n-dimensional rotations. In this case the problem is simplified by the fact that the vector representation U of the group M_n is an irreducible unitary representation of the subgroup O_n, which enters not more than once in every irreducible representation of M_n. Therefore formula (4.36) has in this case the form

$$\eta_s(g) = \int_{\hat{G}} \sum_m T^{(\lambda)}_{ms}(g) z_m(d\lambda), \tag{4.40}$$

where the integral is taken over all irreducible representations of M_n that contain vector representations of the subgroup O_n. Making use of the results of note [28] we can obtain from this a general spectral representation of the covariance matrix $B_{ij}(x_1, x_2)$ of the isotropic vector field, which was found in a different way in [4], as well as the spectral representation of the field $\boldsymbol{\xi}(x)$ itself.

4.3. *Generalized homogeneous random fields.* If the space $X = G/K$, where K can be the identity subgroup, is a finite-dimensional differentiable manifold so that the group G is a Lie group, then together with the usual random field $\xi(x)$ or $\boldsymbol{\xi}(x) = \{\xi_1(x), \cdots, \xi_n(x)\}$ over X we can also consider generalized random fields (random distributions) in the sense of Itô[41] and of Gelfand [42]. According to [41] and [42] a generalized field is a random linear functional $\xi(\varphi)$ or $\boldsymbol{\xi}(\varphi) = \{\xi_1(\varphi), \cdots, \xi_N(\varphi)\}$ defined on the L. Schwartz space D of all infinitely differentiable complex functions $\varphi(x)$ which are zero outside a certain compact. The generalized field $\xi(\varphi)$ is called homogeneous if the mathematical expectations $m(\varphi) = E\xi(\varphi)$ and $B(\varphi_1, \varphi_2) = E\xi(\varphi_1)\overline{\xi(\varphi_2)}$ are such that for every $\varphi \in G$

$$m(\varphi) = m(V_g\varphi), \qquad B(\varphi_1, \varphi_2) = B(V_g\varphi_1, V_g\varphi_2), \tag{4.41}$$

where $V_g\varphi(x) = \varphi(g^{-1}x)$. In the multidimensional case these conditions are replaced by the conditions

$$\mathbf{M}(\varphi) = U(g)\mathbf{M}(V_g\varphi), \qquad B(\varphi_1, \varphi_2) = U(g)B(V_g\varphi_1, V_g\varphi_2)U^*(g), \tag{4.42}$$

where $\mathbf{M}(\varphi)$ is the vector of the mean values of the field $\boldsymbol{\xi}(\varphi)$, and $B(\varphi_1, \varphi_2)$ is its covariance matrix, while $U(g)$ is some representation of the group G. Almost all the results of the present article hold for these generalized homogeneous fields so long as the above formulas for $\xi(x)$, or for $\xi(g)$, are understood in the sense that

$$\xi(\varphi) = \int_X \xi(x)\varphi(x)\,dx \qquad \text{or} \qquad \xi(\varphi) = \int_G \xi(g)\varphi(g)\,dg, \tag{4.43}$$

where dx and dg are the measures invariant with respect to the transformations $g \in G$. The only difference is that with this new approach to the formulas the expressions for $\xi(x)$ and $\xi(g)$ may now be divergent so long as they become convergent after integration with respect to $\varphi(x)\,dx$ or $\varphi(g)\,dg$ where $\varphi \in D$. Therefore, the convergence conditions (2.13), (2.27), (3.18), and so forth, are no longer necessary for generalized fields and must be replaced by less restrictive conditions. The exact form of these weakened conditions will be determined by the asymptotic properties of the corresponding matrix elements $T_{ij}^{(\lambda)}(g)$ and of the spherical functions $\Phi_{ij}^{(\lambda)}(x)$. In the most important concrete examples these asymptotic properties are usually known or can be obtained easily, so that the study of the corresponding homogeneous generalized fields does not present any additional difficulties (see, for example, [5] and [4], devoted to generalized random fields in Euclidean spaces R_n).

Some new problems arise if we consider together with the generalized fields $\xi(\varphi)$ over a functional space D of infinitely differentiable functions, generalized fields over some other functional spaces, for instance over the space D_n of functions, which are differentiable not more than n times and zero outside the compact or over the functional spaces introduced in the book by I. M. Gelfand and G. E. Shilov [43]. It is clear that to a narrower class of functions φ will correspond a wider class of random fields, that is, the random fields with the weakened "convergence conditions" which replace (2.27) and (3.18). The class of ordinary, not generalized, homogeneous random fields is from this point of view an intersection of a whole family of wider and wider classes of generalized random fields, which correspond to narrower and narrower classes of functions φ. The class of ordinary random fields will be followed in this family by the class of homogeneous random measures, which are random functions $\xi(S)$ of the set $S \subset X$ such that $E\xi(S) = E\xi(gS)$, $E\xi(S_1)\overline{\xi(S_2)} = E\xi(gS_1)\overline{\xi(g_1S_2)}$ for any $g \in G$. We note, to avoid misunderstanding, that the expression "random measure" has a different meaning here than earlier in the paper, where it was applied only to random functions of sets, assuming noncorrelated values over nonintersecting sets. The class of random measures can be regarded as the class of generalized random fields, defined on the class of all possible continuous functions φ. Therefore, as distinct from the following classes of generalized fields, it can be defined over any topological homogeneous space, that is, the group G in this case need not necessarily be a Lie group.

In the same sense as for the generalized homogeneous fields of Itô and Gelfand over the space D, the basic results of this paper hold for all the other classes of generalized fields. However, the problem about the exact form of the restrictions imposed on the coefficients $f_{ij}^{(\lambda)}$ or on the "operator measure" $F(d\lambda)$ must be solved separately for each of these classes.

A typical example of results of this kind may be found in the recent article of V. N. Tutubalin [44], where generalized homogeneous random fields on the sphere S_2 are studied.

In this article it was shown that Obukhov's formulas (1.5) and (1.6) hold for the homogeneous random distributions of Itô and Gelfand and for homogeneous random measures on S_2 if we understand (1.5) in the sense (4.43). But in the case of homogeneous random measures the convergence condition

$$\sum_{l=0}^{\infty} (2l + 1)f_l P_l(1) < \infty \tag{4.44}$$

must be replaced by the condition of uniform boundedness of all coefficients f_l, while in the case of homogeneous random distributions condition (4.44) must be replaced by the condition that the coefficients f_l do not increase faster than some finite degree of l for $l \to \infty$.

4.4. *Fields with random homogeneous increments.* In the theory of random processes, together with the study of stationary processes, more general processes with stationary increments are also studied (see, for example, [45] and [8]). Similar generalizations can be proposed in connection with the notion of a homogeneous random field over an arbitrary homogeneous space X. Namely, we shall call the field $\xi(x)$ a field with random homogeneous increments if all the differences $\xi(x_1) - \xi(x_2) = \xi(x_1, x_2)$ will represent a homogeneous, with respect to the transformations $g(x_1, x_2) = (gx_1, gx_2)$, random field over the space $X \times X$.

One should not suppose that the theory of fields with homogeneous increments can be reduced to the theory of homogeneous fields over some other homogeneous space; this is not the case, since in the space $X \times X$, the group $G = \{g\}$ is no longer transitive. Therefore, in the general case, the determination of the "spectral representation" for fields with homogeneous increments requires some new considerations.

The basic numerical characteristics of a field with homogeneous increments are the first and second moments of the difference $\xi(x_1, x_2)$, namely,

$$m(x_1, x_2) = E\xi(x_1, x_2), \qquad B(x_1, x_2; x_3, x_4) = E\xi(x_1, x_2)\overline{\xi(x_3, x_4)}. \tag{4.45}$$

The function $m(x_1, x_2)$ is completely characterized by the functional relation $m(x_1, x_2) + m(x_2, x_3) = m(x_1, x_3)$ and by the condition $m(gx_1, gx_2) = m(x_1, x_2)$. From this it is usually not difficult to determine the general form of the function. As to the second moments, $B(x_1, x_2; x_3, x_4)$, they can be expressed easily through their values $B(x_1, x_2; x_3, x_2)$ with $x_4 = x_2$. If the field $\xi(x)$ is a real field, then from the algebraic identity

$$(a - b)(c - d) = \frac{1}{2}[(a - d)^2 + (b - c)^2 - (a - c)^2 - (b - d)^2] \tag{4.46}$$

we can limit our investigation to the functions

$$B(x_1, x_2) = E[\xi(x_1) - \xi(x_2)]^2, \tag{4.47}$$

depending on two variables only. It has been shown by Schoenberg [46] that the function $B(x_1, x_2)$ is completely characterized by the following property (related to the property of being positive definite): *for any n, $x_1, \cdots, x_n \in X$ and any real numbers $a_1, \cdots, a_n$, such that $\sum_i a_i = 0$, we must have*

$$\sum_{i,k=1}^{n} B(x_i, x_k)a_i a_k \leqq 0. \tag{4.48}$$

Therefore the description of all (real) fields with homogeneous increments is in a known sense equivalent to the description of all functions $B(x_1, x_2)$ satisfying (3.1) and (4.48).

In the case of a compact space X it is not difficult to show that the class of all functions $B(x_1, x_2)$ coincides with the class of functions of the form $B_1(x, x) - B_1(x_1, x_2)$, where x is an arbitrary point of X and $B_1(x_1, x_2)$ is a positive definite function on X satisfying (3.1) (see Bochner [11]). From this it follows that in the compact case the class of fields with homogeneous increments coincides with the class of homogeneous fields.

The last assertion does not hold for more general locally compact spaces X: here the class of fields with homogeneous increments can be considerably wider than the class of homogeneous fields. This in particular is the case for the Euclidean space $X = R_n$ having as the group G the group of translations or a general group of motions (see, in this connection, [4]). Fields with homogeneous increments in the space R_n, one-dimensional (scalar) or many-dimensional (vector), play an important role in the statistical theory of turbulence. It is interesting to note that the definition of the fields with homogeneous increments was first formulated by Kolmogorov [47] in connection with his work in turbulence theory. In a still more special case $X = R_1$, that is, for random processes $\xi(t)$, a very general class of processes with homogeneous increments of an arbitrary order was also studied [41], [48], [49]. The theory of such processes could also be extended to the case of fields in some other homogeneous spaces, different from R_1.

In the case where the group G is a Lie group it is convenient to consider from the very beginning the generalized random fields with homogeneous increments. It is not hard to see that such fields can be defined as the fields $\xi(\varphi)$, satisfying (4.41) or (4.42), but defined on the subspace D_1 of functions $\varphi \in D$ such that

$$\int_X \varphi(x)\, dx = 0 \tag{4.49}$$

(see [4]). This definition allows further generalizations, connected with the consideration of homogeneous random linear functionals over some other linear subspaces of the space D; in the particular case of fields over the line R_1 it is possible to construct in this way the theory of fields with homogeneous increments of the nth order.

Note added in proof. Another proof of theorem 3 can be found in the recent articles of M. A. Naĭmark [50], [51]. The general formula (2.26) of our paper is written in these articles in a different but equivalent form.

REFERENCES

[1] I. J. SCHOENBERG, "Metric spaces and completely monotone functions," *Ann. of Math.*, Vol. 39 (1938), pp. 811–841.

[2] A. M. YAGLOM, "Homogeneous and isotropic turbulence in a viscous compressible fluid," *Izv. Akad. Nauk SSSR, Ser. Geograf. Geofiz.*, Vol. 12 (1948), pp. 501–522. (Reprinted as "Homogene und isotrope Turbulenz in einer zähen kompressiblen Flüssigkeit," *Statistische Theorie der Turbulenz*, Berlin, Akademie-Verlag, 1958, pp. 43–69.)

[3] J. E. MOYAL, "The spectra of turbulence in a compressible fluid; eddy turbulence and random noise," *Proc. Cambridge Philos. Soc.*, Vol. 48 (1952), pp. 329–344.

[4] A. M. YAGLOM, "Some classes of random fields in n-dimensional space related to stationary random processes," *Teor. Veroyatnost. i Primenen.*, Vol. 2 (1957), pp. 292–338.

[5] K. ITÔ, "Isotropic random current," *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1956, Vol. 2, pp. 125–132.

[6] A. M. OBUKHOV, "Statistically homogeneous random fields on a sphere," *Uspehi Mat. Nauk*, Vol. 2, No. 2 (1947), pp. 196–198.

[7] I. J. SCHOENBERG, "Positive definite functions on spheres," *Duke Math. J.*, Vol. 9 (1942), pp. 96–108.

[8] J. L. DOOB, *Stochastic Processes*, New York, Wiley, 1953.

[9] L. PONTRJAGIN, *Topological Groups*, Princeton, Princeton University Press, 1939.

[10] A. WEIL, *L'integration dans les Groupes Topologiques et ses Applications*, Paris, Hermann, 1940.

[11] S. BOCHNER, "Hilbert distances and positive definite functions," *Ann. of Math.*, Vol. 42 (1941), pp. 647–656.

[12] K. KARHUNEN, "Ueber lineare Methoden in der Wahrscheinlichkeitsrechnung," *Ann. Acad. Sci. Fenn., Ser. A. I.*, No. 37 (1947), 79 pp.

[13] H. CRAMÉR, "A contribution to the theory of stochastic processes," *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1951, pp. 329–339.

[14] D. A. RAĬKOV, "Harmonic analysis on commutative groups with Haar measure and the theory of characters," *Trudy Mat. Inst. Steklov.*, No. 14 (1945), 86 pp.

[15] J. KAMPÉ DE FERIET, "Analyse harmonique des fonctions aléatoires stationnaires d'ordre 2 sur un groupe abelien localement compact," *C. R. Acad. Sci. Paris*, Vol. 227 (1948), pp. 868–870.

[16] A. BLANC-LAPIERRE and R. FORTET, *Théorie des Fonctions Aléatoires*, Paris, Masson, 1953.

[17] I. M. GELFAND and D. A. RAĬKOV, "Irreducible unitary representations of locally bicompact groups," *Mat. Sb.*, Vol. 13 (1943), pp. 301–316.

[18] M. A. NAĬMARK, *Normed Rings*, Groningen, P. Noordhoff N.V., 1959.

[19] HARISH-CHANDRA, "Representations of a semisimple Lie group on a Banach space," *Trans. Amer. Math. Soc.*, Vol. 75 (1953), pp. 185–243.

[20] J. DIXMIER, "Sur les représentations unitaires des groupes de Lie algébriques," *Ann. Inst. Fourier Grenoble*, Vol. 7 (1957), pp. 315–328.

[21] G. W. MACKEY, "Borel structure in groups and their duals," *Trans. Amer. Math. Soc.*, Vol. 85 (1957), pp. 134–165.

[22] A. GUICHARDET, "Sur une problème posé par G. W. Mackey," *C. R. Acad. Sci. Paris*, Vol. 250 (1960), pp. 962–963.

[23] I. M. GELFAND, R. A. MINLOS, and Z. YA. ŠAPIRO, *Representations of a Group of Rotations and the Lorentz Group*, Moscow, 1958.

[24] N. YA. VILENKIN, "Bessel functions and the representations of the group of Euclidean motions," *Uspehi Mat. Nauk*, Vol. 11, No. 3 (1956), pp. 69–112.

[25] V. BARGMAN, "Irreducible unitary representations of the Lorentz group," *Ann. of Math.*, Vol. 48 (1947), pp. 568–640.

[26] I. M. GELFAND and M. A. NAĬMARK, "Unitary representations of classical groups," *Trudy Mat. Inst. Steklov.*, No. 36 (1950), 288 pp.

[27] N. YA. VILENKIN, E. L. AKIM, and A. A. LEVIN, "Matrix elements of irreducible unitary representations of the group of Euclidean motions of the three-dimensional space and its properties," *Dokl. Akad. Nauk SSSR*, Vol. 112 (1957), pp. 987–989.

[28] N. YA. VILENKIN, "Matrix elements of irreducible unitary representations of the group of real orthogonal matrices and the group of motions of an $(n - 1)$-dimensional Euclidean space," *Dokl. Akad. Nauk SSSR*, Vol. 113 (1957), pp. 16–19.

[29] ———, "Matrix elements of irreducible unitary representations of a group of Lobachevsky space motions and the generalized Fock-Mehler transformations," *Dokl. Akad. Nauk SSSR*, Vol. 118 (1958), pp. 219–222.

[30] E. CARTAN, "Sur la détermination d'un système orthogonal complet dans un espace de Riemann symmétrique clos," *Rend. Circ. Mat. Palermo*, Vol. 53 (1929), pp. 217–252.

[31] H. WEYL, "Harmonics on homogeneous manifolds," *Ann. of Math.*, Vol. 35 (1943), pp. 486–494.

[32] A. ERDÉLYI, *Higher Transcendental Functions*, Vols. I to III, New York, McGraw-Hill, 1953–1955.

[33] F. A. BEREZIN and I. M. GELFAND, "Some remarks on the theory of spherical functions on symmetric Riemann manifolds," *Trudy Moscov. Mat. Obšč.*, Vol. 5 (1956), pp. 311–351.

[34] I. M. GELFAND, "Spherical functions on symmetric Riemann spaces," *Dokl. Akad. Nauk SSSR*, Vol. 70 (1950), pp. 5–8.

[35] M. G. KREIN, "Hermitian-positive kernels on homogeneous spaces," I and II, *Ukrain. Mat. Ž.*, Vol. 1, No. 4 (1949), pp. 64–98; Vol. 2, No. 1 (1950), pp. 10–59.

[36] I. M. GELFAND and M. A. NAĬMARK, "Unitary representations of the Lorentz group," *Izv. Akad. Nauk SSSR, Ser. Mat.*, Vol. 11 (1947), pp. 411–504.

[37] F. A. BEREZIN and F. I. KARPELEVICH, "Zonal spherical functions and Laplace operators on some symmetric spaces," *Dokl. Akad. Nauk SSSR*, Vol. 118 (1958), pp. 9–12.

[38] A. N. BOGAEVSKY, "Calculation of zonal spherical functions," *Dokl. Akad. Nauk SSSR*, Vol. 129 (1959), pp. 484–487.

[39] YU. A. ROSANOV, "Spectral theory of multidimensional discrete parameter stationary processes," *Uspehi Mat. Nauk*, Vol. 13, No. 2 (1958), pp. 93–142.

[40] A. A. KIRILLOV, "Expansions of the group of rotations of the n-dimensional Euclidean space by spherical vector fields," *Dokl. Akad. Nauk SSSR*, Vol. 116 (1957), pp. 538–541.

[41] K. ITÔ, "Stationary random distributions," *Mem. College Sci. Univ. Kyoto, Ser. A*, Vol. 28 (1953), pp. 209–223.

[42] I. M. GELFAND, "Generalized random processes," *Dokl. Akad. Nauk SSSR*, Vol. 100 (1955), pp. 853–856.

[43] I. M. GELFAND and G. A. SHILOV, *Spaces of Basic and Generalized Functions* (*Generalized Functions*, Part 2), Moscow, 1958.

[44] V. N. TUTUBALIN, "Invariant random measures on a sphere," *Teor. Veroyatnost. i Primenen.*, Vol. 6, No. 4 (1961), pp. 125–130.

[45] A. N. KOLMOGOROV, "Curves in Hilbert space, invaiiant with respect to a one-parameter group of motions," *Dokl. Akad. Nauk SSSR*, Vol. 26 (1940), pp. 6–9.

[46] I. J. SCHOENBERG, "Metric spaces and positive definite functions," *Trans. Amer. Math. Soc.*, Vol. 44 (1938), pp. 522–536.

[47] A. N. KOLMOGOROV, "Local structure of turbulence in a non-compressible fluid with very large Reynolds numbers," *Dokl. Akad. Nauk SSSR*, Vol. 30 (1941), pp. 299–303.

[48] A. M. YAGLOM, "Correlation theory of processes with random stationary *n*th increments," *Mat. Sb.*, Vol. 37 (1955), pp. 141–196.

[49] M. S. PINSKER, "Theory of curves in Hilbert space with stationary *n*th increments," *Izv. Akad. Nauk SSSR, Ser. Mat.*, Vol. 19 (1955), pp. 319–345.

[50] M. A. NAĬMARK, "On factor representations of the locally compact group," *Dokl. Akad. Nauk SSSR*, Vol. 134 (1960), pp. 275–277.

[51] ———, "On the decomposition of the unitary representation of the locally compact group into the factor representations," *Sibirsk. Mat. Ž.*, Vol. 2 (1961), pp. 89–99.