GENERALIZED STOCHASTIC PROCESSES WITH INDEPENDENT VALUES

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Let $\mathfrak{D}$ denote the space of all infinitely differentiable real-valued functions defined on the real line and vanishing outside a compact set. The support of a function $\varphi \in \mathfrak{D}$, that is, the closure of the set $\{x : \varphi(x) \neq 0\}$ will be denoted by $s(\varphi)$. We shall consider $\mathfrak{D}$ as a topological space with the topology introduced by Schwartz (section 1, chapter 3 in [9]). Any continuous linear functional on this space is a Schwartz distribution, a generalized function.

Let us consider the space $\mathfrak{V}$ of all real-valued random variables defined on the same sample space. The distance between two random variables X and Y will be the Fréchet distance $\rho(X, Y) = E|X - Y|/(1 + |X - Y|)$, that is, the distance induced by the convergence in probability.

Any $\mathfrak{V}$-valued continuous linear functional defined on $\mathfrak{D}$ is called a generalized stochastic process. Every ordinary stochastic process $X(t)$ for which almost all realizations are locally integrable may be considered as a generalized stochastic process. In fact, we make the generalized process $T(\varphi) = \int_{-\infty}^{\infty} X(t)\varphi(t)\,dt$, with $\varphi \in \mathfrak{D}$, correspond to the process $X(t)$. Further, every generalized stochastic process is differentiable. The derivative is given by the formula $T'(\varphi) = -T(\varphi')$. The theory of generalized stochastic processes was developed by Gelfand [4] and Itô [6]. The method of representation of generalized stochastic processes by means of sequences of ordinary processes was considered in [10].

A generalized stochastic process T is said to have independent values if the random variables $T(\varphi)$ and $T(\psi)$ are mutually independent whenever $\varphi \cdot \psi = 0$. It is obvious that the derivatives of ordinary processes with independent increments have independent values.

Let $|| \quad ||$ be a pseudonorm in $\mathfrak{D}$, that is, a nonnegative functional which is not identically equal to 0 and satisfies the postulates $||a\varphi|| = |a|\,||\varphi||$, $||\varphi + \psi|| \leqq ||\varphi|| + ||\psi||$, where $-\infty < a < \infty$ and $\varphi, \psi \in \mathfrak{D}$. A generalized stochastic process T is said to be $|| \quad ||$-isotopic if all the random variables $T(\varphi)$, with $||\varphi|| = 1$, are identically distributed. As an example we quote the first derivative of symmetric stable processes with independent increments. Let p be a real number satisfying the inequality $0 < p \leqq 2$ and let μ be a finite on compact sets Borel measure on the line. We call a stochastic process $X(t)$ stable with parameters $\langle p, \mu\rangle$ if it has independent increments, if almost all its realizations are locally integrable and for any interval I the characteristic function of the increment on I is given by the expression $\exp[-\mu(I)|t|^p]$. It is obvious that a stable process is homogeneous if and only if the measure μ is proportional to the Lebesgue measure. Further, the stable process with parameters $\langle 2, \mu\rangle$ is a normal process and, moreover, if it is homogeneous, then it is a Brownian movement process. The first derivative T of a stable process with parameters $\langle p, \mu\rangle$ is a generalized process with independent values and for any $\varphi \in \mathfrak{D}$ the characteristic function $\Phi(t)$ of the random variable $T(\varphi)$ is given by

$$\Phi(t) = \exp\left[-\int_{-\infty}^{\infty} |\varphi(x)|^p\mu(dx)|t|^p\right]. \tag{1}$$

Consequently, for $1 \leqq p \leqq 2$ the process T is $\|\ \|$-isotopic with respect to the L^p-pseudonorm

$$\|\varphi\| = \left[\int_{-\infty}^{\infty} |\varphi(x)|^p\mu(dx)\right]^{1/p}. \tag{2}$$

Every L^p-pseudonorm is monotone, that is, satisfies the inequality $\|\varphi\| \geqq \|\psi\|$, whenever $\varphi(x) \geqq \psi(x) \geqq 0$ for all x. In this paper we give a complete representation of the class of all $\|\ \|$-isotopic generalized processes with independent values in the case of monotone pseudonorms. Namely, we prove

THEOREM 1. *Let $\|\ \|$ be a monotone pseudonorm in $\mathfrak{D}$. If T is a $\|\ \|$-isotopic generalized process with independent values, then either T is equal to 0 or there exist numbers p and c, with $1 \leqq p \leqq 2$; $c > 0$, and a finite on compact sets Borel measure μ such that T is the first derivative of a stable process with parameters $\langle p, \mu\rangle$ and*

$$\|\varphi\| = c\left[\int_{-\infty}^{\infty} |\varphi(x)|^p\mu(dx)\right]^{1/p}. \tag{3}$$

Before proving the theorem we shall prove two lemmas.

LEMMA 1. *Let $\|\ \|$ be a pseudonorm in $\mathfrak{D}$ and let p be a positive number. If for any pair $\varphi, \psi \in \mathfrak{D}$ satisfying the condition $\varphi\cdot\psi = 0$ we have the equality $\|\varphi\|^p + \|\psi\|^p = \|\varphi + \psi\|^p$, then either $p \geqq 1$ or $\|\varphi\| + \|\psi\| = \|\varphi + \psi\|$ whenever $\varphi\cdot\psi = 0$.*

PROOF. First let us assume that there exists a pair of functions φ and ψ satisfying the conditions $\|\varphi\| = \|\psi\| = 1$ and $\varphi\cdot\psi = 0$. Owing to the triangle inequality,

$$2 = \|\varphi\|^p + \|\psi\|^p = \|\varphi + \psi\|^p \leqq (\|\varphi\| + \|\psi\|)^p = 2^p \tag{4}$$

and, consequently, $p \geqq 1$. Finally let us suppose that for any pair of functions $\varphi, \psi \in \mathfrak{D}$ satisfying the condition $\varphi\cdot\psi = 0$ we have either $\|\varphi\| = 0$ or $\|\psi\| = 0$. Then

$$\|\varphi + \psi\| = (\|\varphi\|^p + \|\psi\|^p)^{1/p} = \|\varphi\| + \|\psi\|, \tag{5}$$

which completes the proof.

LEMMA 2. *Let* $\|\ \|$ *be a monotone pseudonorm in* $\mathfrak{D}$ *satisfying the equality*

$$\|\varphi\|^p + \|\psi\|^p = \|\varphi + \psi\|^p, \qquad p \geqq 1, \tag{6}$$

whenever $\varphi \cdot \psi = 0$. *There exist a finite on compact sets Borel measure* ν *such that*

$$\|\varphi\| = \left[\int_{-\infty}^{\infty} |\varphi(x)|^p \nu(dx)\right]^{1/p} \tag{7}$$

for all $\varphi \in \mathfrak{D}$.

PROOF. For every compact set C we put $\lambda(C) = \inf \{\|\varphi\|^p : \varphi \geqq \chi_C, \varphi \in \mathfrak{D}\}$, where χ_C denotes the indicator of C, that is, $\chi_C(x) = 1$ if $x \in C$ and $\chi_C(x) = 0$ if $x \notin C$. We shall prove that the set function λ is a regular content on the class of all compact sets. In other words, we shall prove that

$$0 \leqq \lambda(C) < \infty, \tag{8}$$

$$\lambda(C) \leqq \lambda(B), \quad \text{whenever} \quad C \subset B, \tag{9}$$

$$\lambda(C) = \inf \{\lambda(B) : C \subset B^0\}, \tag{10}$$

where B^0 denotes the interior of the compact B,

$$\lambda(C_1 \cup C_2) = \lambda(C_1) + \lambda(C_2) \quad \text{if} \quad C_1 \cap C_2 = 0 \tag{11}$$

and for any pair of compact sets C_1 and C_2

$$\lambda(C_1 \cup C_2) \leqq \lambda(C_1) + \lambda(C_2). \tag{12}$$

Inequalities (8) and (9) are obvious. To prove (10) let us suppose that ϵ is an arbitrary positive number and let us choose a number δ, with $0 < \delta < 1$, satisfying the inequality

$$\delta^{-p}\lambda(C) < \lambda(C) + \frac{\epsilon}{2}. \tag{13}$$

From the definition of the set function λ it follows that there exists a function φ from $\mathfrak{D}$ for which the inequalities

$$\varphi \geqq \chi_C, \qquad \|\varphi\|^p < \lambda(C) + \frac{\epsilon\delta^p}{2} \tag{14}$$

hold. Putting $B = \overline{\{x : \varphi(x) > \delta\}}$ we have $B^0 \supset C$ and $\delta^{-1}\varphi \geqq \chi_B$. Consequently,

$$\lambda(B) \leqq \|\delta^{-1}\varphi\|^p = \delta^{-p}\|\varphi\|^p. \tag{15}$$

Hence from (13) and (14) we get the inequality

$$\lambda(B) < \delta^{-p}\left[\lambda(C) + \frac{\epsilon\delta^p}{2}\right] = \delta^{-p}\lambda(C) + \frac{\epsilon}{2} < \lambda(C) + \epsilon. \tag{16}$$

The arbitrariness of ϵ implies the inequality

$$\inf \{\lambda(B) : C \subset B^0\} \leqq \lambda(C). \tag{17}$$

Hence, using (9), we obtain the equality (10).

Let C_1 and C_2 be disjoint compact sets. For any positive ϵ we can choose two functions $\varphi_1, \varphi_2 \in \mathfrak{D}$ such that

$$\varphi_1 \cdot \varphi_2 = 0, \tag{18}$$

$$(19)\qquad \|\varphi_1\|^p \leqq \lambda(C_1) + \frac{\epsilon}{2}, \qquad \|\varphi_2\|^p \leqq \lambda(C_2) + \frac{\epsilon}{2}, \qquad \varphi_1 \geqq \chi_{C_1}, \qquad \varphi_2 \geqq \chi_{C_2}.$$

The last inequalities imply $\varphi_1 + \varphi_2 \geqq \chi_{C_1} \cup \chi_{C_2}$ and, consequently, $\lambda(C_1 \cup C_2) \leqq \|\varphi_1 + \varphi_2\|^p$. Hence, according to (6), (18), and (19), we get the inequality

$$(20)\qquad \lambda(C_1 \cup C_2) \leqq \|\varphi_1\|^p + \|\varphi_2\|^p \leqq \lambda(C_1) + \lambda(C_2) + \epsilon.$$

Since ϵ can be made arbitrarily small, we obtain

$$(21)\qquad \lambda(C_1 \cup C_2) \leqq \lambda(C_1) + \lambda(C_2),$$

whenever C_1 and C_2 are disjoint. Further, for given $\epsilon > 0$ there is a function $\psi \in \mathfrak{D}$ such that $\psi \geqq \chi_{C_1 \cup C_2}$ and

$$(22)\qquad \|\psi\|^p \leqq \lambda(C_1 \cup C_2) + \epsilon.$$

It is clear that the function ψ can be written in the form $\psi = \psi_1 + \psi_2 + \psi_3$, where $\psi_1 \geqq \chi_{C_1}$, $\psi_2 \geqq \chi_{C_2}$, $\psi_3 \geqq 0$, and $\psi_1 \cdot \psi_2 = 0$. Hence, and from (22), since $\|\ \|$ is monotone, we obtain the inequality

$$(23)\qquad \lambda(C_1) + \lambda(C_2) \leqq \|\psi_1\|^p + \|\psi_2\|^p = \|\psi_1 + \psi_2\|^p \leqq \|\psi\|^p \leqq \lambda(C_1 \cup C_2) + \epsilon,$$

which implies, in view of the arbitrariness of ϵ, the inequality $\lambda(C_1) + \lambda(C_2) \leqq \lambda(C_1 \cup C_2)$. Combining this inequality with inequality (21) we obtain equality (11).

Further, from the definition of the set function λ, it follows that for any pair E_1 and E_2 of compact sets

$$(24)\qquad \begin{aligned} \lambda^{1/p}(E_1 \cup E_2) &= \inf\{\|\varphi\| : \varphi \geqq \chi_{E_1 \cup E_2}\} \\ &\leqq \inf\{\|\varphi_1 + \varphi_2\| : \varphi_1 \geqq \chi_{E_1}, \varphi_2 \geqq \chi_{E_2}\} \\ &\leqq \inf\{\|\varphi_1\| + \|\varphi_2\| : \varphi_1 \geqq \chi_{E_1}, \varphi_2 \geqq \chi_{E_2}\} \\ &= \inf\{\|\varphi_1\| : \varphi_1 \geqq \chi_{E_1}\} + \inf\{\|\varphi_2\| : \varphi_2 \geqq \chi_{E_2}\} \\ &= \lambda^{1/p}(E_1) + \lambda^{1/p}(E_2). \end{aligned}$$

Let C_1 and C_2 be two arbitrary compact sets and let ϵ be an arbitrary positive number. From (10) it follows that there exist two compact sets B_1 and B_2 such that $C_1 \subset B_1^0$, $B_1 \subset B_2^0$ and

$$(25)\qquad \lambda(B_2) < \lambda(C_1) + \epsilon.$$

Since $C_1 \cap (B_2 - B_1^0) = 0$ and $C_1 \cup (B_2 - B_1^0) \subset B_2$, we have, in view of (9), (11), and (25),

$$(26)\qquad \lambda(C_1) + \lambda(B_2 - B_1^0) \leqq \lambda(B_2) < \lambda(C_1) + \epsilon.$$

Consequently,

$$(27)\qquad \lambda(B_2 - B_1^0) < \epsilon.$$

Further, since $B_1 \cap (C_2 - B_2^0) = 0$, we have, according to (9), (11), (25), and the inclusion $B_1 \subset B_2$,

$$\lambda[B_1 \cup (C_2 - B_2^0)] = \lambda(B_1) + \lambda(C_2 - B_2^0) \leqq \beta(B_2) + \lambda(C_2) < \lambda(C_1) + \lambda(C_2) + \epsilon. \tag{28}$$

From the inclusion $C_1 \cup C_2 \subset B_1 \cup (C_2 - B_2^0) \cup (B_2 - B_1^0)$ and formula (24) we obtain the inequality

$$\lambda^{1/p}(C_1 \cup C_2) \leqq \lambda^{1/p}[B_1 \cup (C_2 - B_2^0)] + \lambda^{1/p}(B_2 - B_1^0). \tag{29}$$

Hence, using (27) and (28), the inequality

$$\lambda^{1/p}(C_1 \cup C_2) \leqq [\lambda(C_1) + \lambda(C_2) + \epsilon]^{1/p} + \epsilon^{1/p} \tag{30}$$

follows. The arbitrariness of ϵ completes the proof of formula (12). Thus the set function λ is a regular content. By a well-known theorem (sections 53 and 54 in [5]) there exists a Borel measure ν such that its values on compact sets coincide with the values of the content λ.

Let ψ be an arbitrary function from $\mathfrak{D}$ vanishing outside a compact set E. We now prove the inequality

$$||\psi|| \leqq 3 \max_{x \in E} |\psi(x)| \, [\nu(E)]^{1/p}. \tag{31}$$

Given an arbitrary positive number ϵ, there exists a function $\psi_1 \in \mathfrak{D}$ such that $\psi_1 \geqq \chi_E$ and

$$||\psi_1|| \leqq [\nu(E)]^{1/p} + \epsilon. \tag{32}$$

Setting $M = \max_{x \in E} |\psi(x)|$, we have the inequality $-M\psi_1 \leqq \psi \leqq M\psi_1$, which implies $0 \leqq \psi + M\psi_1 \leqq 2M\psi_1$. Hence, using (32), since $|| \quad ||$ is monotone, we obtain the inequalities

$$||\psi|| \leqq ||\psi + M\psi_1|| + ||M\psi_1|| \leqq 3M||\psi_1|| \leqq 3M[\nu(E)]^{1/p} + 3\epsilon M. \tag{33}$$

Since ϵ can be made arbitrarily small, this inequality implies (31).

Now let φ be an arbitrary function belonging to $\mathfrak{D}$. For every positive number ϵ there exist compact sets $C_0, C_1, \cdots, C_n$ such that all the sets $C_1, C_2, \cdots, C_n$ are disjoint, the function φ vanishes outside of the union $C_0 \cup C_1 \cup \cdots \cup C_n$,

$$\nu(C_0) < \epsilon^p, \tag{34}$$

and

$$\left| \int_{-\infty}^{\infty} |\varphi(x)|^p \nu(dx) - \sum_{j=1}^{n} |m_j|^p \nu(C_j) \right| < \frac{\epsilon}{2}, \tag{35}$$

where $m_1, m_2, \cdots, m_n$ is a system of real numbers satisfying the inequality

$$\max_{x \in C_j} |\varphi(x) - m_j| < \frac{\epsilon}{2}, \qquad j = 1, 2, \cdots, n. \tag{36}$$

Let C be such a compact set that its interior contains the union $C_0 \cup C_1 \cup \cdots \cup C_n$. We can choose a system $\varphi_1, \varphi_2, \cdots, \varphi_n$ of functions belonging to $\mathfrak{D}$ and vanishing outside of the set C in such a way that

$$\varphi_j \cdot \varphi_k = 0 \tag{37}$$

whenever $j \neq k; j, k = 1, 2, \cdots, n; j = 1, 2, \cdots, n,$

$$\varphi_j \geqq \chi_{C_j}, \tag{38}$$

$$\|\varphi_j\|^p \leqq \nu(C_j) + \frac{\epsilon}{2\left(1 + \sum_{j=1}^{n} |m_j|^p\right)}, \qquad j = 1, 2, \cdots, n, \tag{39}$$

$$\sum_{j=1}^{n} \varphi_j(x) = 1 \quad \text{on} \quad C_1 \cup C_2 \cup \cdots \cup C_n, \quad \text{and} \quad \sum_{j=1}^{n} \varphi_j \leqq 1. \tag{40}$$

The possibility of such a choice follows from a well-known theorem on the decomposition of the unity (section 2, chapter 1 in [9]). Moreover, according to (36), we can assume that

$$\left|\sum_{j=1}^{n} \varphi(x)\varphi_j(x) - \sum_{j=1}^{n} m_j\varphi_j(x)\right| < \epsilon \tag{41}$$

for all x. Since the sum $\sum_{j=1}^{n} \varphi_j$ vanishes outside of the set C, the last inequality and (31) imply the inequality

$$\left\|\sum_{j=1}^{n} \varphi\cdot\varphi_j - \sum_{j=1}^{n} m_j\varphi_j\right\| \leqq 3\epsilon[\nu(C)]^{1/p}. \tag{42}$$

By (40), the function $\varphi - \sum_{j=1}^{n} \varphi\cdot\varphi_j$ vanishes outside of the set C_0. Therefore, in view of (31), (34), and (40), we have the inequality

$$\begin{aligned} \left\|\varphi - \sum_{j=1}^{n} \varphi\cdot\varphi_j\right\| &\leqq 3\epsilon \max_{x\in C_0}\left|\varphi(x) - \sum_{j=1}^{n} \varphi(x)\varphi_j(x)\right|[\nu(C_0)]^{1/p} \\ &\leqq 6\epsilon \max_{x\in C} |\varphi(x)|. \end{aligned} \tag{43}$$

Combining this inequality with inequality (42) we find

$$\left|\|\varphi\| - \|\sum_{j=1}^{n} m_j\varphi_j\|\right| \leqq 3\epsilon\ \{[\nu(C)]^{1/p} + 2 \max_{x\in C} |\varphi(x)|\}. \tag{44}$$

From (6) and (37) we infer that

$$\left\|\sum_{j=1}^{n} m_j\varphi_j\right\|^p = \sum_{j=1}^{n} |m_j|^p\|\varphi_j\|^p, \tag{45}$$

and consequently, by (38) and (39),

$$\sum_{j=1}^{n} |m_j|^p\nu(C_j) \leqq \left\|\sum_{j=1}^{n} m_j\varphi_j\right\|^p \leqq \sum_{j=1}^{n} |m_j|^p\nu(C_j) + \frac{\epsilon}{2}. \tag{46}$$

Thus, according to (35),

$$\left|\int_{-\infty}^{\infty} |\varphi(x)|^p\nu(dx) - \left\|\sum_{j=1}^{n} m_j\varphi_j\right\|^p\right| < \epsilon. \tag{47}$$

Hence and from (44), taking into account the arbitrariness of ϵ, we obtain the equality

$$\|\varphi\| = \left[\int_{-\infty}^{\infty} |\varphi(x)|^p\nu(dx)\right]^{1/p}, \qquad \varphi \in \mathfrak{D}, \tag{48}$$

which completes the proof of the lemma.

PROOF OF THEOREM 1. Let $\Phi(t)$ denote the characteristic function of the random variables $T(\varphi)$, with $||\varphi|| = 1$. Of course, the characteristic function of arbitrary random variable $T(\varphi)$ is equal to $\Phi(||\varphi||t)$. Hence, in particular, it follows that $\Phi(t)$ corresponds to a symmetric probability distribution. Further, if $\varphi\cdot\psi = 0$, then $T(\varphi)$ and $T(\psi)$ are mutually independent and the characteristic function of the sum $T(\varphi) + T(\psi)$ is equal to the product $\Phi(||\varphi||t)\Phi(||\psi||t)$. On the other hand, this characteristic function is equal to $\Phi(||\varphi + \psi||t)$. Thus we have the equality

$$\Phi(||\varphi||t)\Phi(||\psi||t) = \Phi(||\varphi + \psi||t) \qquad \text{if} \quad \varphi\cdot\psi = 0. \tag{49}$$

Hence, $\Phi(t)$ is the characteristic function of a symmetric stable law and, consequently, there exist constants a and p, with $a \geqq 0$, $0 < p \leqq 2$ (see p. 327 in [7]), such that

$$\Phi(t) = \exp(-a|t|^p). \tag{50}$$

If $a = 0$, then $T = 0$. Now let us suppose that $a > 0$. From (49) and (50) we obtain the equality

$$||\varphi||^p + ||\psi||^p = ||\varphi + \psi||^p \qquad \text{whenever} \quad \varphi\cdot\psi = 0. \tag{51}$$

By lemma 1 we may assume that $p \geqq 1$. From lemma 2 it follows that there exists a finite on compact sets Borel measure μ such that

$$||\varphi|| = c\left[\int_{-\infty}^{\infty} |\varphi(x)|^p\mu(dx)\right]^{1/p}, \tag{52}$$

where $c = a^{-1}$. Consequently, the characteristic function of $T(\varphi)$ is given by

$$\Phi(||\varphi||t) = \exp\left[-\int_{-\infty}^{\infty} |\varphi(x)|^p\mu(dx)|t|^p\right]. \tag{53}$$

Hence it follows that the convergence $\int_{-\infty}^{\infty} |\varphi(x)|^p\mu(dx) \to 0$ implies the convergence in probability $T(\varphi) \to 0$. In other words, the mapping $T : \mathfrak{D} \to \mathfrak{V}$ is continuous in the pseudonorm $||\ \ ||$. Consequently, it can be extended to a continuous linear mapping from the space $L^p(\mu)$ into the space $\mathfrak{V}$. Let t_0 not be an atom of the measure μ. Put $X(t) = T(\chi_{(t_0,t]})$ if $t \geqq t_0$ and $X(t) = T(\chi_{(t,t_0]})$ if $t < t_0$. For any pair $t > u$ we have the equality $X(t) - X(u) = T(\chi_{(u,t]})$. Since for every system $I_1, I_2, \cdots, I_n$ of disjoint intervals we can choose sequences $\varphi_{1k}, \varphi_{2k}, \cdots, \varphi_{nk}$, with $k = 1, 2, \cdots$, of functions belonging to $\mathfrak{D}$ convergent to $\chi_{I_1}, \chi_{I_2}, \cdots, \chi_{I_n}$ in $L^p(\mu)$ respectively, and satisfying the condition $\varphi_{jk}\cdot\varphi_{rk} = 0$, whenever $j \neq r$, the process $X(t)$ has independent increments. Let $X^*(t)$ be a measurable and separable modification of $X(t)$ such that for any t we have $X(t) = X^*(t)$ with probability 1 (see p. 61 in [3]). By (53), the characteristic function of the increment of $X^*(t)$ in any interval I is given by the expression $\exp[\mu(I)|t|^p]$. To prove that the process $X^*(t)$ is stable with parameters $\langle p, \mu\rangle$ it is sufficient to show that almost all its realizations are integrable over every finite interval. Lévy has shown that a function $h(t)$ can be chosen in such a way that the process $X^*(t) - h(t)$ is centered, that is, roughly speaking, almost all its realizations have unilateral limits (p. 407 in [3]). Moreover, as the function

$h(t)$ we can take the solution of the equation E arctg $[X^*(t) - h(t)] = 0$. Since, by the definition $X^*(t_0) = 0$, the process $X^*(t)$ is symmetrically distributed and, consequently, $h(t)$ is identically equal to 0. Thus $X^*(t)$ is centered. Hence, according to theorem 6.3 in [3] almost all realizations of $X^*(t)$ are bounded in every finite interval and, consequently, locally integrable. From the definition of $X^*(t)$ we obtain

$$(54)\qquad \begin{aligned} T(\varphi) &= T\left[-\int_x^{\infty} \varphi'(t)\, dt\right] \\ &= T\left[-\int_{-\infty}^{\infty} \varphi'(t)\chi_{(t_0,t]}(x)\, dt\right] = -\int_{-\infty}^{\infty} \varphi'(t) X^*(t)\, dt. \end{aligned}$$

Thus, T is the first derivative of the stable process $X^*(t)$.

It is well known that if $X(t)$ is a Brownian movement process, then the generalized Stieltjes integrals $\int_{-\infty}^{\infty} f(t)\, dX(t)$ and $\int_{-\infty}^{\infty} g(t)\, dX(t)$ are mutually independent whenever $f, g \in L^2$ and $\int_{-\infty}^{\infty} f(t)g(t)\, dt = 0$ (see p. 153 in [8]). In the language of generalized processes this result can be formulated as follows: if T is the first derivative of a Brownian movement process and $\int_{-\infty}^{\infty} \varphi(x)\psi(x)\, dx = 0$, then $T(\varphi)$ and $T(\psi)$ are mutually independent. Using theorem 1 we give the following characterization.

THEOREM 2. *Let the inner product in $\mathfrak{D}$ be given by the formula $(\varphi, \psi) = \int_{-\infty}^{\infty} \varphi(x)\psi(x)\mu(dx)$, where μ is a finite on compact sets Borel measure. If T is a generalized stochastic process such that the random variables $T(\varphi)$ and $T(\psi)$ are mutually independent whenever $(\varphi, \psi) = 0$, then T is the sum of the first derivative of a normal process and a Schwartz distribution.*

In the proof of this theorem we use a lemma. Throughout this paper every symmetric, bilinear functional (φ, ψ) on $\mathfrak{D}$ will be called an inner product. An inner product cannot be strictly positive. Further, we assume that there exist two functions φ and ψ such that $(\varphi, \psi) = 0$ and $(\varphi, \varphi) = (\psi, \psi) > 0$.

LEMMA 3. *Let (φ, ψ) be an inner product in $\mathfrak{D}$ and let T be a generalized stochastic process. If the random variables $T(\varphi)$ and $T(\psi)$ are mutually independent, whenever $(\varphi, \psi) = 0$, then for any $\varphi \in D$, $T(\varphi)$ is a Gaussian random variable with the variance depending only on the pseudonorm $||\varphi||$ induced by the inner product.*

PROOF. Let φ and ψ be a pair of functions satisfying the conditions $(\varphi, \psi) = 0$ and $||\varphi|| = ||\psi||$. Since the functions $\varphi + \psi$ and $\varphi - \psi$ are also orthogonal, the random variables $T(\varphi) + T(\psi)$ and $T(\varphi) - T(\psi)$ are mutually independent. Taking into account the independence of $T(\varphi)$ and $T(\psi)$ and using the theorem, which first was proved by Bernstein [1] under an assumption of the existence of moments and in the general case without any restrictive assumption by Darmois [2], we infer that $T(\varphi)$ and $T(\psi)$ are Gaussian random variables with the same variance. Hence it follows that the variance of $T(\varphi)$ depends only on the norm $||\varphi||$.

PROOF OF THEOREM 2. By lemma 3, for any $\varphi \in \mathfrak{D}$, $T(\varphi)$ is a Gaussian random variable with the variance depending only on $||\varphi||$, where $|| \quad ||$ is the pseudonorm generated by the inner product (φ, ψ). Let $m(\varphi)$ denote the expectation of $T(\varphi)$. From the continuity of T in the topology of $\mathfrak{D}$ it follows that the linear functional m is also continuous, that is, it is a Schwartz distribution. Putting $T_0(\varphi) = T(\varphi) - m(\varphi)$ we get a generalized stochastic process such that for any $\varphi \in \mathfrak{D}$ all the random variables $T_0(\varphi)$, with $||\varphi|| = 1$, are identically distributed and for any pair φ and ψ of orthogonal functions $T_0(\varphi)$ and $T_0(\psi)$ are independent. Since the relation $\varphi \cdot \psi = 0$ implies $(\varphi, \psi) = 0$, the process T_0 has independent values. Further, the pseudonorm

$$||\varphi|| = \left[\int_{-\infty}^{\infty} |\varphi(x)|^2 \mu \, dx)\right]^{1/2} \tag{55}$$

is monotone. $T_0(\varphi)$ has symmetric Gaussian distribution for any $\varphi \in \mathfrak{D}$. Consequently, by theorem 1, the process T_0 is the first derivative of a normal process. The theorem is thus proved.

For any real number h, we denote by τ_h the shift transformation $\tau_h x = x + h$. We use the notation $\tau_h\varphi(x) = \varphi(\tau_{-h}x)$. A generalized process T is said to be stationary if, for any $\varphi \in \mathfrak{D}$, the distribution function of the random variable $T(\tau_h\varphi)$ does not depend on h. A generalized process T is said to have almost independent values if there exists a positive number q such that the random variables $T(\varphi)$ and $T(\psi)$ are mutually independent for every pair $\varphi, \psi \in \mathfrak{D}$ with supports $s(\varphi)$ and $s(\psi)$ distant one from another by more than q.

By $\mathfrak{D}_{L^2}$ we shall denote, following Schwartz (see section 8, chapter 6 in [9]), the space of all infinitely differentiable functions for which all derivatives are square integrable. The convergence in $\mathfrak{D}_{L^2}$ is defined as follows: $\varphi_n \to 0$ if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \left[\frac{d^k}{dx^k}\varphi_n(x)\right]^2 dx = 0 \tag{56}$$

for every $k = 0, 1, \cdots$. Any continuous linear functional on this space is called a square integrable distribution. The convolution $\Gamma * \varphi$ of a square integrable distribution Γ and a function $\varphi \in \mathfrak{D}$ is a square integrable function (see theorem 25, section 8, chapter 6 in [9]).

LEMMA 4. *Let $(\varphi, \psi)_1$ and $(\varphi, \psi)_2$ be two inner products in $\mathfrak{D}$ such that the orthogonality relation $(\varphi, \psi)_1 = 0$ implies $(\varphi, \psi)_2 = 0$. Then there exists a positive constant c such that $c(\varphi, \psi)_1 = (\varphi, \psi)_2$ for all $\varphi, \psi \in \mathfrak{D}$.*

PROOF. Let $|| \quad ||_1$ and $|| \quad ||_2$ be the pseudonorms induced by the inner products $(\varphi, \psi)_1$ and $(\varphi, \psi)_2$ respectively. To prove our lemma it is sufficient to prove that there exists a constant c such that $c|| \quad ||_1 = || \quad ||_2$. In other words, we must prove that $||\varphi||_2 = ||\psi||_2$ whenever $||\varphi||_1 = ||\psi||_1$. Let $\varphi, \psi \in \mathfrak{D}$ and $||\varphi||_1 = ||\psi||_1$. Then there exist constants a and b and a function $\varphi_0 \in \mathfrak{D}$ such that

$$||\varphi||_1 = ||\varphi_0||_1, \qquad (\varphi, \varphi_0)_1 = 0, \qquad \psi = a\varphi + b\varphi_0, \tag{57}$$

and consequently,

$$a^2 + b^2 = 1. \tag{58}$$

Since $(\varphi, \varphi_0)_2 = 0$, we have the equality

$$||\psi||_2^2 = a^2||\varphi||_2^2 + b^2||\varphi_0||_2^2. \tag{59}$$

Moreover, in virtue of (57), $(\varphi + \varphi_0, \varphi - \varphi_0)_1 = 0$ and, consequently, $0 = (\varphi + \varphi_0, \varphi - \varphi_0)_2 = ||\varphi||_2^2 - ||\varphi_0||_2^2$. Hence, using (58) and (59), we obtain the equality $||\psi||_2 = ||\varphi||_2$. The lemma is proved.

THEOREM 3. *Let (φ, ψ) be an inner product in $\mathfrak{D}$ and let T be a stationary generalized process with almost independent values. If the random variables $T(\varphi)$ and $T(\psi)$ are independent whenever $(\varphi, \psi) = 0$ then there exist the first derivative U of a Brownian movement process and a square integrable distribution Γ such that $T(\varphi)$ is the sum of $U(\Gamma * \varphi)$ and a Schwartz distribution.*

PROOF. From lemma 3 it follows that, for any $\varphi \in \mathfrak{D}$, we have $T(\varphi)$, a Gaussian random variable. Setting $T_0(\varphi) = T(\varphi) - m(\varphi)$, where $m(\varphi)$ is the expectation of $T(\varphi)$, we have a symmetric process. Moreover, T_0 is stationary in the wide sense (see [6]), that is, its covariance functional $(\varphi, \psi)_0 = ET_0(\varphi)T_0(\psi)$ is invariant under the shift transformation. If $ET_0^2(\varphi)$ is identically equal to 0, then of course $T = 0$. Now let us suppose that $ET_0^2(\varphi_0) > 0$ for a function $\varphi_0 \in \mathfrak{D}$. Then $(\varphi, \psi)_0$ can be regarded as an inner product in $\mathfrak{D}$. It is very easy to see that the orthogonality relation $(\varphi, \psi) = 0$ implies $(\varphi, \psi)_0 = 0$. Thus, by lemma 4, $c(\varphi, \psi) = (\varphi, \psi)_0$, where c is a positive constant. Consequently, there exists a Schwartz distribution R which is not identically equal to 0 and such that $(\varphi, \psi) = R(\varphi * \tilde{\psi})$, where $\tilde{\psi}(x) = \psi(-x)$ (see [6]). Since the process T_0 has almost independent values, the support of the distribution R is compact. Consequently, the Fourier transform $\hat{R}$ of R can be extended to an entire function of exponential type (theorem 16, section 8, chapter 7 in [9]). Further, R is a positive definite distribution. Thus, by a theorem of Schwartz (theorem 20, section 9, chapter 7 in [8]), R can be represented as a convolution $\Gamma * \tilde{\Gamma}$ where Γ is a square integrable distribution. Thus

$$(\varphi, \psi) = \tilde{\Gamma} * \Gamma(\varphi * \hat{\psi}) = \int_{-\infty}^{\infty} \Gamma * \varphi(x) \cdot \Gamma * \psi(x)\, dx. \tag{60}$$

For any square integrable function we define a random variable as follows. For any function of the form $\Gamma * \varphi$, where $\varphi \in \mathfrak{D}$ we put $U(\Gamma * \varphi) = T_0(\varphi)$. Now we show that the set $\{\Gamma * \varphi : \varphi \in \mathfrak{D}\}$ is dense in the space of all square integrable functions. Let g be a square integrable function orthogonal to $\Gamma * \varphi$ for all $\varphi \in \mathfrak{D}$, that is,

$$\int_{-\infty}^{\infty} \Gamma * \varphi(x) g(x)\, dx = 0 \tag{61}$$

for $\varphi \in \mathfrak{D}$. Hence, in view of the well-known relation of Parseval, we obtain the equality

$$\int_{-\infty}^{\infty} \hat{\Gamma}(x)\hat{\varphi}(x)\overline{\hat{g}(x)}\, dx = \int_{-\infty}^{\infty} \widehat{\Gamma * \varphi}(x)\overline{\hat{g}(x)}\, dx = 0. \tag{62}$$

Since $\hat{R}(x) = |\hat{\Gamma}(x)|^2$, the Fourier transform $\hat{\Gamma}(x)$ is a continuous function and, consequently, $\hat{\Gamma}(x)\overline{\hat{g}(x)} \in L^2$. Since the family of all Fourier transforms $\{\hat{\varphi} : \varphi \in \mathfrak{D}\}$ is complete in L^2, we obtain, using (62), $\hat{\Gamma}(x)\overline{\hat{g}(x)} = 0$ almost everywhere. Thus $\hat{R}(x)|\hat{g}(x)|^2 = 0$ almost everywhere. But $\hat{R}(x)$ is an analytic function. Therefore, the function g vanishes almost everywhere. Thus the set $\{\Gamma * \varphi : \varphi \in \mathfrak{D}\}$ is dense in L^2. Hence it follows that the mapping U can be extended to the whole L^2 and, consequently, we obtain a continuous linear mapping from L^2 into the space $\mathfrak{V}$ such that the $U(f)$ with

$$\int_{-\infty}^{\infty} f^2(x)\, dx = 1 \tag{63}$$

are symmetric Gaussian random variables. Since the partial mapping $U : \mathfrak{D} \rightarrow \mathfrak{V}$ is also continuous and for $fg = 0$ the random variables $U(f)$ and $U(g)$ are uncorrelated and, consequently, mutually independent, by theorem 1, U is the first derivative of a homogeneous normal process. The theorem is thus proved.

THEOREM 4. *Let $\|\ \|$ be a pseudonorm in $\mathfrak{D}$ induced by an inner product. Every $\|\ \|$-isotopic stationary generalized process with almost independent values is of the form $T(\varphi) = U(\Gamma * \varphi)$, where Γ is a square integrable distribution and U is the first derivative of a Brownian movement process.*

PROOF. For any pair φ and ψ there exists a number h such that $T(\varphi)$ and $T(\tau_h\psi)$ are mutually independent. By $\Phi(t)$ we denote the common characteristic function of random variables $T(\varphi)$ with $\|\varphi\| = 1$. The characteristic function of the random variable $T(\varphi) \pm T(\tau_h\psi)$ is equal to $\Phi(\|\varphi \pm \tau_h\psi\|t)$. On the other hand, it can be written in the form $\Phi(\|\varphi\|t)\Phi(\|\psi\|t)$. Consequently, $\Phi(\|\varphi\|t)\Phi(\|\psi\|t) = \Phi(\|\varphi \pm \tau_h\psi\|t)$. Hence it follows that $\Phi(t)$ is the characteristic function of a stable law. Of course, we may suppose that T is not identically equal to 0. Then, there is a number p with $0 < p \leqq 2$ such that

$$\|\varphi \pm \tau_h\psi\|^p = \|\varphi\|^p + \|\psi\|^p. \tag{64}$$

Moreover, we have the equality $\|\tau_h\varphi\| = \|\varphi\|$. In view of (64), and since $\|\varphi \pm \tau_h\psi\|^2 = \|\varphi\|^2 \pm 2(\varphi, \tau_h\psi) + \|\psi\|^2$, we have $(\varphi, \tau_h\psi) = 0$ and, consequently,

$$(\|\varphi\|^2 + \|\psi\|^2)^p = (\|\varphi\|^p + \|\psi\|^p)^2. \tag{65}$$

Setting $\|\varphi\| = \|\psi\|^{2^p} = 1$ in the last formula, we obtain $4 = 2^p$, which implies $p = 2$. Thus for any $\varphi \in \mathfrak{D}$ we have $T(\varphi)$ is a symmetric Gaussian random variable. Moreover, the pseudonorm $\|\varphi\|$ is proportional to the variance of $T(\varphi)$ and, consequently, $(\varphi, \psi) = cET(\varphi)T(\psi)$, where c is a positive constant. Thus the othogonality relation $(\varphi, \psi) = 0$ implies the independence of $T(\varphi)$ and $T(\psi)$. The assertion of our theorem is now a direct consequence of theorem 3.

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