SOME THEOREMS ON CHARACTERISTIC FUNCTIONS OF PROBABILITY DISTRIBUTIONS

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1. Introduction

Let X be a real valued random variable with probability measure P and distribution function F. It will be convenient to take F as the *intermediate* distribution function defined by

$$F(x) = \frac{1}{2}[P\{X < x\} + P\{X \leqq x\}]. \tag{1.1}$$

In mathematical analysis it is a little more convenient to use this function rather than

$$F_1(x) = P\{X < x\} \quad \text{or} \quad F_2(x) = P\{X \leqq x), \tag{1.2}$$

which arise more naturally in probability theory. With this definition, if the distribution function of X is $F(x)$, then the distribution function of $-X$ is $1 - F(-x)$. The distribution of X is symmetrical about 0 if $F(x) = 1 - F(-x)$. For F_1 and F_2 the corresponding relations are more complicated at points of discontinuity.

The characteristic function of X, or of F, is

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x), \tag{1.3}$$

defined and uniformly continuous for all real t. The function ϕ is uniquely determined by F. Conversely, F is uniquely determined by ϕ. Every property of F must be implicit in ϕ and vice versa. It is often an interesting but difficult problem to determine what property of one function corresponds to a specified property of its transform.

We know that in a general way the behavior of $F(x)$ for large x is related to the behavior of $\phi(t)$ in the neighborhood of $t = 0$. The main object of this paper is to make some precise and rather simple statements about this relation. We are interested in the behavior of $\phi(t)$ in the neighborhood of $t = 0$ because upon this depend all limit theorems on sums of random variables. For example, suppose that $X_1, X_2, \cdots$ is a sequence of independent, identically distributed random variables with distribution function $F(x)$ and characteristic function

$\phi(t)$. The sum $X_1 + X_2 + \cdots + X_n$ has characteristic function $\phi(t)^n$. Suppose, to take the simplest case, that

$$W_n = \frac{X_1 + X_2 + \cdots + X_n}{B_n} \tag{1.4}$$

has a limit distribution as $n \to \infty$, where B_n is a function of n which $\to \infty$ as $n \to \infty$. The characteristic function of W_n is $\phi(t/B_n)^n$, and must $\to$ a characteristic function $\psi(t)$ as $n \to \infty$. For fixed t, $t/B_n \to 0$ as $n \to \infty$, and so the existence and the nature of the limit $\psi(t)$ will depend only on the behavior of $\phi(t)$ in the neighborhood of $t = 0$.

For $x \geqq 0$, put

$$\begin{aligned} H(x) &= 1 - F(x) + F(-x), && \text{the tail sum,} \\ K(x) &= 1 - F(x) - F(-x), && \text{the tail difference.} \end{aligned} \tag{1.5}$$

If the distribution is symmetrical about 0, then $K(x)$ is identically zero. If X is a nonnegative random variable, $F(x) = 0$ when $x < 0$, and $K(x) = H(x)$ for $x > 0$.

We may write

$$\phi(t) = \int_{-\infty}^{0} e^{itx}\, dF(x) + \int_{0}^{\infty} e^{itx}\, d[F(x) - 1]. \tag{1.6}$$

Integrating by parts and putting

$$\phi(t) = U(t) + iV(t), \tag{1.7}$$

where U, V are real for real t, we finally obtain

$$\begin{aligned} \frac{1 - U(t)}{t} &= \int_{0}^{\infty} H(x) \sin tx\, dx, \\ \frac{V(t)}{t} &= \int_{0}^{\infty} K(x) \cos tx\, dx. \end{aligned} \tag{1.8}$$

We have the inversion formulas,

$$\begin{aligned} H(x) &= \frac{2}{\pi} \int_{0}^{\infty} \frac{1 - U(t)}{t} \sin xt\, dt, \\ K(x) &= \frac{2}{\pi} \int_{0}^{\infty} \frac{V(t)}{t} \cos xt\, dt. \end{aligned} \tag{1.9}$$

For real t, the real part of $\phi(t)$ depends only on H, and the unreal part only on K. A necessary and sufficient condition for $\phi(t)$ to be real for all real t is that $K(x)$ be identically 0, that is, that the distribution be symmetrical about 0. $U(t)$ is itself a characteristic function; the corresponding distribution function is $[F(x) + 1 - F(-x)]/2$. In investigating the behavior of $\phi(t)$ in the neighborhood of $t = 0$, it is advisable to consider $U(t)$ and $V(t)$ separately. For real values of t these are not closely connected because the former depends only on $H(x)$ and the latter only on $K(x)$, and the only connection between $H(x)$ and $K(x)$ is the relation $H(x) \geqq |K(x)|$.

Consider $H(x)$ and $U(t)$. If the distribution has a finite standard deviation σ, then

$$U(t) = 1 - \frac{\sigma^2 t^2}{2} + o(t^2), \qquad t \to 0. \tag{1.10}$$

Hence,

$$1 - U(t) = \frac{1}{2}\sigma^2 t^2 - o(t^2) \sim \frac{1}{2}\sigma^2 t^2, \qquad t \to 0. \tag{1.11}$$

Later we shall have some statements about the term $o(t^2)$, but now let us note that in order to get anything different from

$$1 - U(t) \sim \frac{1}{2}\sigma^2 t^2, \tag{1.12}$$

we must have a distribution of infinite standard deviation.

2. Distributions of infinite standard deviation

$$\begin{aligned} \frac{1 - U(t)}{t} &= \int_0^\infty H(x) \sin tx \, dx; \\ 1 - U(t) &= \int_0^\infty H(u/t) \sin u \, du. \end{aligned} \tag{2.1}$$

The sort of result we get is

$$1 - U(t) \sim cH(1/t), \qquad t \downarrow 0, \tag{2.2}$$

where c is a constant depending on the distribution. Under what conditions can we expect this?

$$\frac{1 - U(t)}{H(1/t)} = \int_0^\infty \frac{H(u/t)}{H(1/t)} \sin u \, du. \tag{2.3}$$

We want the right side to tend to a limit when $t \downarrow 0$. We can expect this only if

$$\frac{H(u/t)}{H(1/t)} \to \text{a limit } h(u) \text{ when } t \downarrow 0, \tag{2.4}$$

and the limit of the integral is the integral of the limit. If all goes well, the limit of the right side of (2.3) will be

$$\int_0^\infty h(u) \sin u \, du = c, \tag{2.5}$$

say, and we shall have

$$1 - U(t) \sim cH(1/t), \qquad t \downarrow 0. \tag{2.6}$$

For $u > 0$, we want to have

$$\frac{H(u/t)}{H(1/t)} \to h(u) \qquad \text{as} \quad t \downarrow 0. \tag{2.7}$$

Changing the notation, we require for every $\lambda > 0$ that

$$\frac{H(\lambda x)}{H(x)} \to h(\lambda) \qquad \text{as} \quad x \to \infty. \tag{2.8}$$

For $\lambda, \mu > 0$, we have

$$\frac{H(\lambda\mu x)}{H(x)} = \frac{H(\lambda\mu x)}{H(\mu x)} \frac{H(\mu x)}{H(x)} \tag{2.9}$$

and so we must have

$$h(\lambda\mu) = h(\lambda)h(\mu). \tag{2.10}$$

It is a classical theorem that for a measurable h the only solution of this functional equation is

$$h(\lambda) = \lambda^k, \tag{2.11}$$

where k is a constant.

Thus, for the present, we are interested in distributions for which the tail sum $H(x)$ has the property that for every $\lambda > 0$,

$$\frac{H(\lambda x)}{H(x)} \to \lambda^k \qquad \text{as} \quad x \to \infty. \tag{2.12}$$

We shall express this property of H by saying that $H(x)$ is of index k as $x \to \infty$. A function $L(x)$ of index 0 is sometimes called a function of slow growth. It has the property that $L(\lambda x)/L(x) \to 1$ as $x \to \infty$ for $\lambda > 0$. The functions $\log x$, $\log \log x$, $1 + 1/x$ are all of index 0 and so is any constant. Clearly, if $H(x)$ is of index k, then $H(x)/x^k$ is of index 0 and so

$$H(x) = x^k L(x), \tag{2.13}$$

where $L(x)$ is of index 0. Similarly we say that a function $G(x)$ is of index k as $x \downarrow 0$ if for every $\lambda > 0$,

$$\frac{G(\lambda x)}{G(x)} \to \lambda^k \qquad \text{as} \quad x \downarrow 0. \tag{2.14}$$

It is easy to show that if the distribution has infinite standard deviation and if $H(x)$ is of index k as $x \to \infty$, then $-2 \leqq k \leqq 0$. Theorems 1 and 2 are concerned with this case. Theorem 3 gives corresponding results for $K(x)$. The proofs of these three theorems are not given here but will be published elsewhere.

Write

$$S(m) = \begin{cases} \dfrac{\frac{1}{2}\pi}{\Gamma(m) \sin \frac{1}{2} m\pi}, & m > 0, \\ 1, & m = 0, \end{cases}$$

$$C(m) = \frac{\frac{1}{2}\pi}{\Gamma(m) \cos \frac{1}{2} m\pi}, \qquad m > 0. \tag{2.15}$$

$S(m)$ is finite for m not an even positive integer and, for $0 < m < 2$,

$$S(m) = \int_0^\infty \frac{\sin x}{x^m}\, dx. \tag{2.16}$$

$C(m)$ is finite for m not an odd positive integer and, for $0 < m < 1$,

$$C(m) = \int_0^\infty \frac{\cos x}{x^m}\, dx. \tag{2.17}$$

THEOREM 1. *If $H(x)$ is of index $-m$ when $x \to \infty$ and*

(i) $0 < m < 2$, *then* $1 - U(t) \sim S(m)H(1/t)$ *as* $t \downarrow 0$;

(ii) $m = 0$ *and* $H(x + h) \leqq [H(x) + H(x + 2h)]/2$ *when x and h are sufficiently great, then* $1 - U(t) \sim H(1/t)$ *as* $t \downarrow 0$;

(iii) $m = 2$, *then* $1 - U(t) \sim t^2 \int_0^{1/t} xH(x)\, dx$ *as* $t \downarrow 0$.

The condition (ii) is the extension of (i) to the case $m = 0$ with an additional condition. It can be shown by a counterexample that some such additional condition is required for $m = 0$. Statement (i) is an extension of a result of Titchmarsh on Fourier transforms.

There are also comparison theorems of the type

$$\begin{aligned} H_1(x) &= O\{H(x)\}, && x \to \infty, \\ \Rightarrow 1 - U_1(t) &= O\{1 - U(t)\}, && t \downarrow 0, \end{aligned} \tag{2.18}$$

where H satisfies condition (i), (ii) or (iii).

We have the converse

THEOREM 2. *If $1 - U(t)$ is of index m as $t \downarrow 0$ and $0 \leqq m < 2$, then*

$$H(x) \sim \frac{1 - U(1/x)}{S(m)}, \qquad x \to \infty; \tag{2.19}$$

and if $m = 2$, then

$$\int_0^x uH(u)\, du \sim x^2[1 - U(1/x)], \qquad x \to \infty. \tag{2.20}$$

This is more difficult to prove.

For the unreal part of $\phi(t)$, we have

THEOREM 3. *If $K(x)$ is ultimately monotonic and of index $-m$ and*

(i) $0 < m < 1$, *then* $V(t) \sim C(m)K(1/t)$ *as* $t \downarrow 0$;

(ii) $m = 0$, *then*

$$\int_0^t \frac{V(u)}{u}\, du \sim \frac{1}{2}\pi K(1/t) \qquad \textit{as} \quad t \downarrow 0; \tag{2.21}$$

(iii) $m = 1$, *then* $V(t) \sim t \int_0^{1/t} K(x)\, dx$ *as* $t \downarrow 0$.

If $K(x)$ is of index $-m$ and

(iv) $1 < m < 3$, *then* $V(t) - \mu_1^* t \sim C(m)K(1/t)$ *as* $t \downarrow 0$, *where*

$$\mu_1^* = \int_0^\infty K(x)\,dx = \lim_{T\to\infty} \int_{-T}^{T} x\,dF(x). \tag{2.22}$$

There are converse theorems.

3. Application

As an application of these results, consider a distribution which is symmetrical about 0 and for which

$$H(x) \sim \frac{\log x}{x}, \qquad x \to \infty. \tag{3.1}$$

$H(x)$ is of index -1 and $\phi(t) = U(t)$. Thus

$$1 - \phi(t) = 1 - U(t) \sim S(1)H(1/t) = \frac{1}{2}\pi t \log(1/t) \tag{3.2}$$

as $t \downarrow 0$. Hence

$$\phi(t) = 1 - \eta(t)|t| \log(1/|t|), \tag{3.3}$$

where $\eta(t) \to \pi/2$ as $t \to 0$.

If $X_1, X_2, \cdots$ are independent random variables, each with this distribution, and B_n is a positive function of n, then

$$\frac{X_1 + X_2 + \cdots + X_n}{B_n} \tag{3.4}$$

will have characteristic function

$$\phi\left(\frac{t}{B_n}\right)^n = \left\{1 - \eta\left(\frac{t}{B_n}\right) \frac{|t|(\log B_n - \log|t|)}{B_n}\right\}^n. \tag{3.5}$$

This will tend to a limit as $n \to \infty$ if $B_n/\log B_n \sim n$. This will be so if the function $B_n = n \log n$.

Thus the characteristic function of

$$\frac{X_1 + X_2 + \cdots + X_n}{n \log n} \tag{3.6}$$

is asymptotically equal to

$$\left\{1 - \frac{\pi|t|}{2n}\right\}^n, \tag{3.7}$$

which tends to the limit $\exp(-\pi|t|/2)$ as $n \to \infty$, and so (3.6) has a limit distribution which is a Cauchy distribution.

4. Distributions of finite standard deviation

We now consider theorems applicable to distributions of finite standard deviation. The nth moment, if it exists, will be denoted by μ_n and we shall write

$$\mu_n^- = \int_{-\infty}^0 x^n\,dF(x), \qquad \mu_n^+ = \int_0^\infty x^n\,dF(x). \tag{4.1}$$

Define

$$
\begin{aligned}
f_0(x) &= \begin{cases} F(x) & \text{for } x \leqq 0, \\ 0 & \text{for } x > 0; \end{cases} \\
f_n(x) &= \begin{cases} \int_{-\infty}^{x} f_{n-1}(u)\, du & \text{for } x \leqq 0, \\ 0 & \text{for } x > 0, \end{cases} \qquad n = 1, 2, 3, \cdots ; \\
g_0(x) &= \begin{cases} 0 & \text{for } x < 0, \\ 1 - F(x) & \text{for } x \geqq 0; \end{cases} \\
g_n(x) &= \begin{cases} 0 & \text{for } x < 0, \\ \int_{x}^{\infty} g_{n-1}(u)\, du & \text{for } x \geqq 0, \end{cases} \qquad n = 1, 2, 3, \cdots .
\end{aligned} \tag{4.2}
$$

From the relations

$$
\begin{aligned}
\int_{-\infty}^{0} x^{m+1} f_{n-1}(x)\, dx &= -(m+1) \int_{-\infty}^{0} x^m f_n(x)\, dx, \\
\int_{0}^{\infty} x^{m+1} g_{n-1}(x)\, dx &= (m+1) \int_{0}^{\infty} x^m g_n(x)\, dx,
\end{aligned} \tag{4.3}
$$

we can show that

$$
f_n(0) = \frac{|\mu_n^-|}{n!}, \qquad g_n(0) = \frac{\mu_n^+}{n!}. \tag{4.4}
$$

We may write

$$
\begin{aligned}
\phi(t) &= \int_{-\infty}^{0} e^{itx}\, df_0(x) - \int_{0}^{\infty} e^{itx}\, dg_0(x) \\
&= 1 - it \int_{-\infty}^{0} e^{itx} f_0(x)\, dx + it \int_{0}^{\infty} e^{itx} g_0(x)\, dx.
\end{aligned} \tag{4.5}
$$

By continued integration by parts, we obtain

THEOREM 4. *If μ_n exists and is finite,*

$$
\phi(t) = 1 + \sum_{1}^{n-1} \frac{\mu_r}{r!} (it)^r + \frac{\mu_n^- \phi_{n1}(t) + \mu_n^+ \phi_{n2}(t)}{n!} (it)^n, \tag{4.6}
$$

where $\phi_{n1}(t)$ is the characteristic function of the continuous distribution with density function $n! f_{n-1}(x)/|\mu_n^-|$ and $\phi_{n2}(t)$ is the characteristic function of the continuous distribution with density function $n! g_{n-1}(x)/\mu_n^+$. Both distributions are unimodal with mode at the origin; the former is a purely negative distribution and the latter a purely positive distribution.

If n is even,

$$
\phi_n(t) = \frac{\mu_n^- \phi_{n1}(t) + \mu_n^+ \phi_{n2}(t)}{\mu_n} \tag{4.7}
$$

is the characteristic function of the continuous distribution with density function $n![f_{n-1}(x) + g_{n-1}(x)]/\mu_n$. This distribution is unimodal with mode at the origin.

We may restate this result as follows.

THEOREM 4A. *If* $\mu_{2n} < \infty$,

$$\phi(t) = 1 + \sum_{1}^{2n-1} \frac{\mu_r}{r!} (it)^r + \frac{\mu_{2n}}{(2n)!} (it)^{2n} \phi_{2n}(t), \tag{4.8}$$

where $\phi_{2n}(t)$ *is the characteristic function of the continuous distribution with density function* $(2n)![f_{2n-1}(x) + g_{2n-1}(x)]/\mu_{2n}$.

Put

$$\frac{\mu_{2n}(it)^{2n}}{(2n)!} \phi_{2n}(t) = \frac{\mu_{2n}(it)^{2n}}{(2n)!} - W(t) + iV(t), \tag{4.9}$$

where W, V are real for real t. We then have

$$\phi(t) = 1 + \sum_{1}^{2n} \frac{\mu_r}{r!} (it)^r - W(t) + iV(t). \tag{4.10}$$

By applying theorem 1 to $\phi_{2n}(t)$, we obtain

THEOREM 5. *Suppose* $\mu_{2n} < \infty$ *and that* $H(x)$ *is of index* $-m$ *as* $x \to \infty$.

(i) *If* $2n < m < 2n + 2$, *then* $W(t) \sim S(m)H(1/t)$ *as* $t \downarrow 0$.

(ii) *If* $m = 2n$ *and* $H(x + h) \leqq [H(x) + H(x + 2h)]/2$ *when* x *and* h *are sufficiently great, then*

$$W(t) \sim \frac{(it)^{2n}}{(2n-1)!} \int_{1/t}^{\infty} x^{2n-1} H(x)\, dx \qquad as \quad t \downarrow 0. \tag{4.11}$$

(iii) *If* $m = 2n + 2$, *then*

$$W(t) \sim \frac{-(it)^{2n+2}}{(2n+1)!} \int_{0}^{1/t} x^{2n+1} H(x)\, dx \qquad as \quad t \downarrow 0. \tag{4.12}$$

Corresponding results can be obtained for $V(t)$.

5. The derivatives of a characteristic function

The existence of a finite first moment is a sufficient condition for $\phi(t)$ to have a finite derivative for every real value of t. This condition is not necessary. Necessary and sufficient conditions for $\phi(t)$ to have a finite derivative at $t = 0$ are given in [1]. Theorem 6 gives a sufficient condition for $\phi(t)$ to have a finite derivative at every real value of t except possibly $t = 0$. Theorem 7 gives the corresponding result for a lattice distribution.

THEOREM 6. *Let the distribution function* $F(x)$ *be absolutely continuous with a density function* $f(x)$. *If* a, b *exist such that* $xf(x)$ *is of bounded variation in* $x \leqq a$ *and also in* $x \geqq b$, *then the characteristic function* $\phi(t)$ *has a finite derivative at every real value of* t *except possibly* $t = 0$. *The condition on* $f(x)$ *will be satisfied in either interval if* $f(x)$ *is monotonic and finite in that interval.*

PROOF.

$$\phi(t) = \int_{-\infty}^{a} e^{itx} f(x)\, dx + \int_{a}^{b} e^{itx} f(x)\, dx + \int_{b}^{\infty} e^{itx} f(x)\, dx. \tag{5.1}$$

If $t \neq 0$,

$$\phi'(t) = i \int_{-\infty}^{a} e^{itx} xf(x)\, dx + i \int_{a}^{b} e^{itx} xf(x)\, dx + i \int_{b}^{\infty} e^{itx} xf(x)\, dx \tag{5.2}$$

because, as shown in the next paragraph, the first and last integral on the right side are both uniformly convergent with respect to t in $|t| > h > 0$. Thus $\phi'(t)$ exists and is finite if $t \neq 0$.

Since $xf(x)$ is of bounded variation in $x > b$, it tends to a finite limit as $x \to \infty$. This limit must be 0 because $\int_b^\infty f(x)\, dx < \infty$.

$$\int_b^c e^{itx} xf(x)\, dx = \frac{e^{itc} cf(c) - e^{itb} bf(b)}{it} - \int_b^c \frac{e^{itx}}{it}\, d(xf(x)). \tag{5.3}$$

If $c > b$, and $|t| > h > 0$, the modulus of this is not greater than

$$\frac{cf(c) + bf(b)}{h} + \frac{1}{h} \int_b^c |d(xf(x))|, \tag{5.4}$$

which $\to 0$ as $b \to \infty$. We have assumed, as we may, that $b > 0$. Thus

$$\int_b^\infty e^{itx} xf(x)\, dx \tag{5.5}$$

is uniformly convergent with respect to t in $|t| > h > 0$. A similar proof applies to the other integral.

If $f(x)$ is monotonic in the interval $x \geqq b$ it must be nonincreasing in that interval. If $x > b$,

$$1 \geqq \int_b^x f(u)\, du = xf(x) - bf(b) - \int_b^x u\, df(u). \tag{5.6}$$

Therefore,

$$1 + bf(b) \geqq -\int_b^\infty u\, df(u). \tag{5.7}$$

Hence,

$$0 \leqq -\int_b^\infty u\, df(u) < \infty. \tag{5.8}$$

Also,

$$xf(x) = bf(b) + \int_b^x f(u)\, du + \int_b^x u\, df(u), \tag{5.9}$$

and therefore $xf(x)$ is of bounded variation in the interval $x \geqq b$.

THEOREM 7. *Let X be a lattice variable which takes only the values $h + n\lambda$, where n runs through integral values, and let*

$$P\{X = h + n\lambda\} = f(n). \tag{5.10}$$

If $nf(n)$ is of bounded variation for $n < a$ and also for $n > b$, then the characteristic function $\phi(t)$ has a finite derivative for all real t except possibly $t = 2n\pi/\lambda$, where n is integral. The condition on $f(n)$ will be satisfied in either range if $f(n)$ is monotonic in that range.

The proof is similar to that for theorem 6.

Finally, we note

THEOREM 8. *If $\mu_{2n} < \infty$, then $\phi(t)$ has a 2nth derivative $\phi^{(2n)}(t)$ and*

$$\frac{\phi^{(2n)}(t)}{\phi^{(2n)}(0)} = \frac{\phi^{(2n)}(t)}{(-1)^n \mu_{2n}} \tag{5.11}$$

is the characteristic function of the distribution with distribution function

$$\frac{\int_{-\infty}^{x} u^{2n}\, dF(u)}{\mu_{2n}}. \tag{5.12}$$

REFERENCE

[1] E. J. G. PITMAN, "On the derivatives of a characteristic function at the origin," *Ann. Math. Statist.*, Vol. 27 (1956), pp. 1156–1160.