LATTICE METHODS AND SUBMARKOVIAN PROCESSES

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1. Introduction

1.1. *Summary.* This paper is a contribution to the study by lattice methods of stationary submarkovian processes on a denumerable state space. Our restriction to such simple spaces is only for clarity; it is by no means a restriction on the validity of the lattice methods. We have tried to present in this paper a synthesis of results recently obtained in the field of submarkovian processes as well as a certain number of new results.

We thought it preferable to introduce vector lattices only through their cones of positive elements (to be called L-cones); the basic notions concerning them are briefly summed up in this introduction. No representation using Stonian spaces has been introduced in this work.

Let $P = \{P_t, t \geqq 0\}$ be a submarkovian process on the denumerable set A. Then to every stochastic function $\{X_t\}$ defined for $t > 0$, with transition laws given by P, is associated a family $f = \{f_t, t > 0\}$ of positive measures on A of total mass $\leqq 1$ (the instantaneous laws of the X_t) such that

(1.1.1) $$f_s P_t = f_{s+t}, \qquad s, t > 0;$$

and conversely. There exists an initial distribution f_0 such that $f_t = f_0 P_t$ when, but only when, $\{X_t\}$ can also be defined for $t = 0$. Recent advances in the theory have shown the interest of considering solutions of (1.1.1) with no restriction on the total mass of the f_t (which may even be infinite); for these more general solutions of (1.1.1), we prove in section 2.1 the following three basic properties:

(a) the t-functions $f_t(i)$ are continuous on $[0, \infty)$;

(b) the set $F(P)$ of all solutions f is an L-cone;

(c) the Laplace transform maps a "bounded solution" f onto a solution $\hat{f} = \{\hat{f}_x, x > 0\}$ of the equations $\hat{f}_y[I + (y - x)R_x] = \hat{f}_x$, where $x, y > 0$, and conversely.

Since the positivity and the continuity properties of the matrices P_t are the essential conditions under which these statements are true, it is remarked in section 2.2 that similar statements are valid for the positive solutions $g = \{g_t, t > 0\}$ of $P_t g_s = g_{s+t}$ with $s, t > 0$.

Given two processes P (on A) and P' (on A'), families $C = \{C_{s,t}; s, t > 0\}$ of positive matrices on $A \times A'$ that satisfy the relations

$$(1.1.2) \qquad P_uC_{s,t} = C_{u+s,t}, \qquad C_{s,t}P_u = C_{s,t+u}, \qquad s, t, u > 0,$$

will be called forms from section 2.3 onwards. A positive matrix Γ on $A \times A'$ may define a form by $C_{s,t} = P_s\Gamma P'_t$ but the converse is false in general. The relations (1.1.2) are a two-parameter generalization of (1.1.1); therefore, properties analogous to (a) to (c) hold for forms. However, the property analogous to (b) will not be used here.

Properties of type (a) will be seen to be fundamental in proving continuity and differentiability of the transition functions or of other functions related to a process; they also lead to a great many simplifications in several of our proofs. Properties of type (b) give basic structural results. As for the Laplace transform, we have employed it primarily as a tool, always stating the results relative to the process P rather than relative to its resolvent R.

The notion of form assumes its importance after a natural order relation between submarkovian processes has been introduced in section 3.1 under the name of "dominance"; in fact, theorem 3.1.1 can be considered as the central result of this paper. In section 3.2, we show that this theorem immediately implies as a very particular case the so-called generalized Kolmogorov equations (Kolmogorov [16], Austin [1], Chung [5]); this example was also intended as a demonstration of the general fact that the use of solutions of (1.1.1) rather than the use of initial distributions lead to equations (here the generalized Kolmogorov equations) rather than to inequalities (here the classical Kolmogorov inequalities). Given two processes P and P' such that P' dominates P, we study in section 3.3 (see also section 3.5) the mutual relations between $F(P)$ and $F(P')$. Although the development of this paper is purely analytical, the following stochastic interpretation of part of theorem 3.1.1 and theorem 3.3.2 could have been given: if P' dominates P, there corresponds to every initial distribution a probability space and two submarkovian stochastic functions X and X' with the given initial distribution, with respective transition laws P and P' and such that $X'_t = X_t$ as long as X has not "escaped to infinity"; this remains true if instead of an initial distribution two families f and f' of instantaneous distributions are given which are in (u, u') correspondence. Moreover the expression $C_{s,t-s}(i, j)ds$ is the probability that $X'_t = j$ and that X escapes to infinity during the time interval $(s, s + ds)$, conditionally when the initial state is i.

With no continuity or measurability assumption, a semigroup $\{P_t, t > 0\}$ of submarkovian matrices on a set A is also shown in theorem 3.3.1 to be continuous, that is, to be a submarkovian process on a subset of A, whenever it is dominated by an auxiliary arbitrary process. Given a submarkovian process P this general result leads us in section 3.4 to an elementary construction of the process P^B which describes the evolution of the submarkovian functions up to the time they reach the "taboo set" B (see Chung [7]); P^B is a process only on a subset of B^c, say $(\bar{B})^c$. We show that $B \to \bar{B}$ is the closure operation in a Hausdorff topology on A, that the process P^B only depends on B through $\bar{B}$ and that the processes P^B all together satisfy a capacity-type inequality (see Choquet

[4]). The process P^B may also be characterized as the largest process on a subset of B which is dominated by the given process P on A. We put special emphasis on the case B finite; in this case it is possible to give a satisfactory description of the process P from the data of P^B; this description is independent of whether the states of B are stable or instantaneous. More details may be found in another paper (Neveu [20]).

The last chapter is devoted to construction of submarkovian processes. The general problem of characterizing and constructing all processes P' which dominate a given process P is considered and nearly solved with the aid of two types of constructions: (1) a perturbation construction which generalizes a well-known construction due to Feller, so as to include escape to infinity and immediate return in the set A of states according to given probability distributions on A; (2) a construction related to absolute dominance, which necessitates the notion of local time (Lévy [17]) in order to be fully interpreted stochastically (for such an interpretation, see Neveu [20]).

Our great debt to Feller's previous works (Feller [10], [11]) probably appears most clearly in section 4; his two papers originate the systematic study of submarkovian processes by lattice methods, to which we bring the present contribution.

1.2. *Submarkovian processes.* Let A be a denumerable nonvoid set. A matrix P on A is said to be submarkovian [Markovian] if it is positive and if $\sum_j P(i, j) \leqq 1$ [$= 1$] for every $i \in A$. In particular, for any subset B of A, the matrix I_B on A is submarkovian if it satisfies the two conditions (1) $I_B(i, j) = 1$ if $i = j \in B$ and (2) $I_B(i, j) = 0$ otherwise. The matrix $I_A \equiv I$ is Markovian.

For any submarkovian matrix P and any positive measure f [function g] on A with finite or infinite values, we denote by fP [or Pg] the positive measure [function] on A defined by the formula $fP(j) = \sum_i f(i)P(i, j)$ [or $Pg(i) = \sum_j P(i, j)g(j)$] with the convention that $c\cdot\infty = \infty\cdot c = \infty$ or 0 as $c > 0$ or $c = 0$. The cone $L^1_+(A)$ [$L^\infty_+(A)$] of positive bounded measures [functions] on A is invariant under any submarkovian matrix P; more precisely, we have $\|fP\|_1 \leqq \|f\|_1$ and $\|Pg\|_\infty \leqq \|g\|_\infty$ if $\|f\|_1$ is defined on $L^1_+(A)$ as $\sum_i f(i) < \infty$ and if $\|g\|_\infty$ is defined on $L^\infty_+(A)$ as $\sup_i g(i) < \infty$. The function equal to one everywhere on A will be denoted by e, so that $Pe \leqq e$ [$Pe = e$] holds for any submarkovian [Markovian] matrix. For any given positive measure f and function g, we put $\langle f, g\rangle = \sum_i f(i)g(i) \leqq \infty$, so that

$$
\begin{aligned}
&\langle f, g\rangle \leqq \|f\|_1\,\|g\|_\infty && \text{if } f \in L^1_+(A);\, g \in L^\infty_+(A);\\
(1.2.1)\qquad &\langle f, e\rangle = \|f\|_1 && \text{if } f \in L^1_+(A);\\
&\langle fP, g\rangle = \langle f, Pg\rangle && \text{for any } f, g.
\end{aligned}
$$

Convergence of functions, measures, and matrices will always refer, unless otherwise stated, to pointwise convergence; free use will be made of the classical monotone convergence theorems and of the Fatou lemma.

The term semigroup of submarkovian [Markovian] matrices will designate a family $\{P_t, t > 0\}$ of submarkovian [Markovian] matrices such that $P_sP_t = P_{s+t}$

for $s, t > 0$, where $t \in R_+$ is a real strictly positive parameter. However, in sections 3.3 and 3.4 semigroups $\{P_t, t \in K\}$ of submarkovian matrices depending on a parameter t in a dense semigroup K of R_+ will be considered. Such a semigroup which is also continuous in the sense that the t-functions $P_t(i, j)$ are continuous on $[0, \infty)$ with limits $I(i, j)$ as $t \downarrow 0$ for all $i, j \in A$, will be called a submarkovian [Markovian] process on A. The condition $\lim_{s\to 0} P_s(i, i) = 1$ for all $i \in A$, is known to be sufficient for a semigroup of submarkovian matrices to be a submarkovian process; the proof of this fact rests on the following general inequality which is valid at least for any $f \in L^1_+(A)$

(1.2.2) $$\|fP_{s+t} - fP_t\|_1 \leqq 2 \sum_i f(i)[1 - P_s(i, i)] \to 0, \qquad s \downarrow 0; t \geqq 0.$$

The matrices $R_x = \int_0^\infty dt \exp(-xt)P_t$ defined for $0 < x < \infty$ are called the resolvent matrices of the process $P = \{P_t, t > 0\}$. To the three elementary properties of the measures $dt \exp(-xt)$ on $[0, \infty)$,

(a) $dt \exp(-xt)$ is a probability on $[0, \infty)$;

(b) $\int_\epsilon^\infty dt\, x \exp(-xt) \to 0$ as $x \to \infty$ for every $\epsilon > 0$;

(c) $dt \exp(-xt) - dt \exp(-yt) + (x - y)\, dt \exp(-xt) * \exp(-yt) = 0$, where $*$ denotes the convolution product on $[0, \infty)$, there correspond the three following properties of the resolvent matrices

(a) xR_x is a submarkovian [Markovian] matrix for every x;

(b) $xR_x \to I$ as $x \to \infty$;

(c) $R_x - R_y + (x - y)R_xR_y = 0$ for every x, y. It will be convenient to rewrite (c) in the equivalent forms

(c′) $R_x = [I + (y - x)R_x]R_y = R_y[I + (y - x)R_x], \qquad 0 < x, y < \infty;$

(c″) $$\begin{aligned} I + (z - x)R_x &= [I + (y - x)R_x][I + (z - y)R_y] \\ &= [I + (z - y)R_y][I + (y - x)R_x], \end{aligned} \qquad 0 < x, y, z < \infty.$$

1.3. *Cones and lattices.* Let E be a partially ordered linear space, that is, a linear space on which a partial order is defined by $f_1 \leqq f_2$ if $f_2 - f_1 \in E_+$ where E_+ is a convex cone in E such that $f \in E_+, -f \in E_+ \Leftrightarrow f = 0$. The elements of E_+ are called positive (0 is positive).

DEFINITION 1.3.1. *An L-cone F in a partially ordered linear space E is a subset of E_+ with the two properties*

(a) *$f_1, \cdots, f_n \in F$ and $\sum_1^n c_if_i \in E_+ \Rightarrow \sum_1^n c_if_i \in F$ for any set $c_1, \cdots, c_n$ of n real constants (positive or negative) and for any $n > 0$;*

(b) *every nonvoid subset H of F which is bounded above by an element of F has a least upper bound (l.u.b.) in F, denoted by $\vee H$.*

It readily follows from this definition that (1) F is a convex subcone of E_+ [a property which is not equivalent to (a)] and (2) any finite nonvoid subset $\{f_1, \cdots, f_n\}$ of F, being bounded above in F by $f_1 + \cdots + f_n$, has a l.u.b., also to be denoted by $f_1 \vee \cdots \vee f_n$.

LEMMA 1.3.1. *Every nonvoid subset H of F has a greatest lower bound (g.l.b.) in F, to be denoted by $\wedge H$ or by $f_1 \wedge \cdots \wedge f_n$ when $H = \{f_1, \cdots, f_n\}$.*

PROOF. The general identity $f_1 \wedge f_2 = f_1 + f_2 - f_1 \vee f_2$ implies that g is a lower bound of H in F if and only if $g \leqq h_0 - (h_0 \vee h) + h$ for every $h \in H$ and for a fixed but arbitrary h_0 in H. As a consequence, $h_0 - \vee H(h_0 \vee h - h)$ is the g.l.b. of H in F.

Two elements f_1, f_2 of an L-cone F are called disjoint when $f_1 \wedge f_2 = 0$, or equivalently when $f_1 \vee f_2 = f_1 + f_2$.

DEFINITION 1.3.2. *A convex subcone F_1 of an L-cone F in E is said to be thick in F if it satisfies the condition*

(1) $f \in F, f_1 \in F_1, f \leqq f_1 \Rightarrow f \in F_1$.

If it also satisfies the condition

(2) *every nonvoid subset H of F_1 that is bounded above in F has a l.u.b. in F_1 (that is, $\vee H \in F_1$ whenever it is in F), the convex subcone F_1 of F will be called a positive band of F.*

It is obvious that a convex subcone F_1 of F that is thick in F is itself an L-cone in the partially ordered linear space E.

An important remark. The notion of L-cone is equivalent to that of complete vectorial lattice. Indeed it is easily verified that (1) the cone of positive elements of any linear subspace of E which is a complete vector lattice for the order induced by E, builds an L-cone; (2) the linear subspace of E generated by an L-cone F (to be denoted here by $[F]$) is a complete vector lattice for the order induced by E of which F is the cone of positive elements. Moreover, F_1 is thick in F if and only if $[F_1]$ is thick (Bourbaki [3], chapter II, section 1, no. 6, ex. 4) in $[F]$; similarly F_1 is a positive band in F if and only if $[F_1]$ is a band (*bande* in Bourbaki [3], closed l-ideal in Birkhoff [2]).

We introduced the notion of L-cone because it seemed to us that in the applications to the study of submarkovian processes, only the positive elements of the complete vector lattices to be considered were of interest (see also theorem 1.3.3 below and the related remark). It is also our opinion that the statement of the fundamental Riesz theorem (theorem 1.3.1) is more natural (see its proof in Bourbaki [3]) and more intuitive when it is given for L-cones than when it is given for complete vector lattices.

THEOREM 1.3.1 (*F. Riesz*). *Let H be a nonvoid subset of an L-cone F. Then the set H_1 of all elements f of F such that $f \leqq \sum_1^n h_i$ for at least a finite family $\{h_1, \cdots, h_n\} \subset H$ is the smallest convex subcone of F thick in F which contains H. The set B_1 of all existing l.u.b. of subsets of H_1 is the smallest positive band of F containing H. The set B_2 of all elements of F disjoint from all elements of H is a positive band; the set B_2 is also the set of all elements of F disjoint from every element of B_1.*

Every element f of F is uniquely decomposable into the sum of an element of B_1 and an element of B_2; that is, $F = B_1 + B_2$. The set B_1 is also the set of all elements of F disjoint from every element of B_2.

For the proof of this theorem we refer the reader to Bourbaki [3]. To every positive band B of F there corresponds the so-called complementary positive band $B^\perp$ of all elements of F which are disjoint from every element of B; then, $F = B + B^\perp$ and $(B^\perp)^\perp = B$.

The next theorem will be seen to be of basic importance.

THEOREM 1.3.2. *Let F and F' be two L-cones and let u and u' be two additive increasing mappings respectively defined from F into F' and from F' into F such that*

$$u'u \geqq I \text{ on } F, \qquad uu' \leqq I' \text{ on } F'.$$

Here I and I' are identical mappings of F onto F and of F' onto F' respectively. Then the L-cone F is isomorphic by u and u' to the positive band $\bar{F}'$ of F' of the elements f' such that $f' = u[u'(f')]$; the complementary positive band $(\bar{F}')^\perp$ of $\bar{F}'$ in F' is composed of the elements f' such that $u'(f') = 0$.

PROOF. We first show that $u'u = I$ on F. Indeed, if $f \in F$ let $h = u'[u(f)] - f \in F$. It then follows from $u(h) = uu'[u(f)] - u(f) \leqq 0$ that $u(h) = 0$ and that $h = 0$ since $h \leqq u'[u(h)] = 0$.

The subset $G' = \{g' : u'(g') = 0\}$ of F' is clearly a convex subcone of F' thick in F'; to show that G' is a positive band, let us consider a subset H' of G' for which $\vee H'$ exists in F' and let us prove that $\vee H' = G'$. But for every $h' \in H'$, we have

$$u[u'(\vee H')] = u[u'(\vee H' - h')] \leqq \vee H' - h'. \tag{1.3.1}$$

Hence $u[u'(\vee H')] = 0$ and $u'(\vee H') = u'u[u'(\vee H')] = 0$, that is, $\vee H' \in G'$.

If $\bar{F}'$ denotes the complementary positive band of G' so that $G' = (\bar{F}')^\perp$ also, we finally show that $uu' = I'$ on $\bar{F}'$ so that our theorem will be reduced to lemma 1.3.2 below. If $f' \in \bar{F}'$, let $h' = f' - uu'(f') \in F'$; from $h' \leqq f'$ it follows that $h' \in \bar{F}'$; from $u'(h') = u'(f') - u'u[u'(f')] = 0$, it follows that $h' \in G'$. This is possible only if $h' = 0$, that is, $uu' = I'$ on $\bar{F}'$.

LEMMA 1.3.2. *The two L-cones F and F' are isomorphic if and only if there exist two additive and increasing mappings u and u' respectively defined from F into F' and from F' into F such that $u'u = I$ on F, and $uu' = I'$ on F'.*

PROOF. Being additive and increasing, the mappings u and u' are necessarily linear in the sense that

$$u\left(\sum_1^n c_i f_i\right) = \sum_1^n c_i u(f_i), \qquad f_1, \cdots, f_n \in F; \sum_1^n c_i f_i \in F. \tag{1.3.2}$$

Since the mappings u and u' are clearly onto F, it only remains to show that they commute with the $\vee$ operation. But for every subset H of F which is bounded above, we have $u(\vee H) \geqq \mathsf{V}_H u(h)$, since u is increasing. These two elements of F' are necessarily equal since their u'-images in F are equal by virtue of

$$\vee H = u'[u(\vee H)] \geqq u'[\mathsf{V}_H u(h)] \geqq \mathsf{V}_H u'[u(h)] = \vee H. \tag{1.3.3}$$

We end this section by giving a class of L-cones which one frequently encounters (Feller [10]) in the theory of submarkovian processes.

THEOREM 1.3.3. *Let E be an L-cone and let q be a set of additive increasing and continuous mappings from E into E which commute 2 by 2. Then the convex subcone F of the elements of E which remain invariant under the mappings of q is an L-cone for the order induced by E.*

An increasing mapping Q from E to E is said to be order-continuous if $Q(\vee H) = \vee_H Q(h)$ for every subset H of E which is bounded above and filtering to the right.

PROOF. There is no restriction in supposing that q is an Abelian semigroup of additive, increasing, and continuous mappings from E into E; then the relation $Q_1 \ll Q_2$ defined as "there exists a $Q_0 \in q$ such that $Q_2 = Q_1 Q_0$" is transitive and q is filtered by its subsets $q_Q = \{Q_1 : Q \ll Q_1\}$, where $Q \in q$.

Let H be a subset of F bounded above by $f \in F$; since f is also an upper bound of H in E, sup H exists as a l.u.b. in E and sup $H \leqq f$. In general, however, sup $H \notin F$. But since the subset of E, that is, $\{Q(\sup H); Q \in q\}$, is filtering to the right and is bounded above in E by f, as is shown by the successive inequalities in E,

$$(1.3.4)\qquad \begin{aligned} &h \leqq \sup H \leqq f; h = Qh \leqq Q(\sup H) \leqq Qf = f, && h \in H,\\ &h \leqq Q_1(\sup H) \leqq Q_2(\sup H) \leqq f, && h \in H; Q_1 \ll Q_2,\end{aligned}$$

we may introduce the following element of E

$$(1.3.5)\qquad \vee H = \sup_q Q(\sup H) = \lim_q Q(\sup H) \leqq f.$$

From the continuity of the mappings Q it follows that

$$(1.3.6)\qquad Q_1(\vee H) = \lim_q Q_1[Q(\sup H)] = \vee H$$

so that $\vee H \in F$. Then since $\vee H \leqq f$ for every upper bound f of H in F, the element $\vee H$ is the l.u.b. of H in F.

REMARK. This theorem becomes false if L-cones are replaced by complete vector lattices in its statement.

2. The basic cones

In sections 2.1 and 2.2 of this chapter a fixed submarkovian process P defined on the denumerable set A will be considered.

2.1. *The cone F.* Let $E^1 = [M(A)]^{R_+}$ be the partially ordered vector space of the mappings $f = \{f_t, t > 0\}$ of the positive real half-line R_+ into the space $M(A)$ of measures on A. The lattice structure of E^1 will be considered only in the proof of theorem 2.1.2 below but never again. We denote by F or $F(P)$ the subset of E^1 of the mappings f such that

$$(2.1.1)\qquad 0 \leqq f_t < \infty \text{ on } A, \qquad t > 0;$$

$$(2.1.2)\qquad f_{s+t} = f_s P_t, \qquad s, t > 0.$$

We denote by F_0 the subset of $M(A)$ of the positive measures f_0 on A such that

$0 \leqq f_0 P_t < \infty$ on A for $t > 0$. Among other consequences, the following theorem will allow us to imbed F_0 in F in a natural way.

THEOREM 2.1.1. *For every $f \in F$, the t-functions $f_t(i)$, where $i \in A$, are continuous on the semiclosed interval $[0, \infty)$; moreover $f_0 = \lim_{t \to 0} f_t$ is an element of F_0 such that $f_0 P_t \leqq f_t$ for $t > 0$.*

PROOF. First, it is clear from the formula

$$f_{t+s}(j) = \sum_i f_t(i) P_s(i, j), \qquad t, s > 0; j \in A, \tag{2.1.3}$$

by letting s vary, that the functions $f_t(i)$ are lower semicontinuous on $(0, \infty)$, that is, that $f_t(i) = \lim_{u \to t} \inf f_u(i)$ for $t > 0$, $i \in A$. The inequality

$$f_{t+s}(i) \geqq f_{t+u}(i) P_{s-u}(i, i), \qquad t \geqq 0, s > 0, -t < u < s; i \in A, \tag{2.1.4}$$

implies, by letting $u \to 0$, that

$$f_{t+s}(i) \geqq \limsup_{u \to t} f_u(i) P_s(i, i), \qquad t \geqq 0, s > 0; i \in A. \tag{2.1.5}$$

By letting $s \downarrow 0$, we see that the right limits $f_t^+(i) = \lim_{u \downarrow t} f_u(i)$ exist for $t \geqq 0$ $i \in A$, and that, at least when $t > 0$, we have

$$f_t^+(i) = \limsup_{u \to t} f_u(i) \geqq f_t(i), \qquad t > 0; i \in A. \tag{2.1.6}$$

On the other hand, it follows from $f_t P_s = f_{t+u} P_{s-u}$ for $t > 0, s > u > 0$, by letting $u \downarrow 0$, that $f_t P_s \geqq f_t^+ P_s (t, s > 0)$. This conclusion is compatible only with $f_t^+ \geqq f_t$ if $f_t^+ = f_t$, since by what has been proved,

$$\begin{aligned} 0 &\leqq [f_t^+(i) - f_t(i)] P_s(i; i) \leqq [(f_t^+ - f_t) P_s](i) \\ &= f_t^+ P_s(i) - f_t P_s(i) \leqq 0, \end{aligned} \tag{2.1.7}$$

and since $P_s(i, i) > 0$. There remains to prove only that $f_t \geqq f_0^+ P_t$ where $t > 0$, since this inequality will imply immediately that $f_0^+ \in F_0$. This is a consequence of $f_t = f_u P_{t-u}$ with $t > u > 0$ by letting $u \downarrow 0$.

The structure of F and F_0 and their relation are described by the following basic result.

THEOREM 2.1.2. *The set F is an L-cone in E^1. The least upper bound $\mathsf{V}_\beta f^\beta$ of the upper-bounded subset $\{f^\beta\}$ of F is given by*

$$(\mathsf{V}_\beta f^\beta)_t = \lim_{s \downarrow 0} \uparrow (\sup_\beta f_s^\beta) P_{t-s}, \qquad t > 0. \tag{2.1.8}$$

Similarly, the upper lower bound of the nonvoid subset $\{f^\beta\}$ of F is given by

$$(\mathsf{V}_\beta f^\beta)_t = \lim_{s \downarrow 0} \downarrow (\inf_\beta f_s^\beta) P_{t-s}, \qquad t > 0. \tag{2.1.9}$$

The set F_0 is an L-cone in $M(A)$ with the lattice operations defined as in $M(A)$.

The set of elements $f = \{f_t, t > 0\}$ of F such that $f_t = f_0 P_t$ where $t > 0$, for $f_0 = \lim_{t \to 0} f_t$ is a positive band of F which is isomorphic to F_0 and will be indentified with F_0; the complementary positive band of F_0 in F is composed of the elements f for which $f_0 = 0$.

The notations "sup" and "inf" are used for and only for the lattice operations in $M(A)$.

PROOF. Clearly, F satisfies the first axiom of the definition of the L-cones; to show that it also satisfies the second axiom, let us consider a nonvoid subset $\{f^\beta\}$ of F with an upper bound in F, say f. The inequalities

$$\begin{aligned}(2.1.10)\qquad &\sup_\beta f^\beta_{t+s} \leqq (\sup_\beta f^\beta_t)P_s \leqq f_tP_s = f_{t+s} && s, t > 0,\\ &(\sup_\beta f^\beta_u)P_{t-u} \leqq (\sup_\beta f^\beta_v)P_{u-v}P_{t-u}\\ &\qquad = (\sup_\beta f^\beta_v)P_{t-v} \leqq f_t, && 0 < \vee < u < t,\end{aligned}$$

show that the limit $\lim_{u\downarrow 0}\uparrow(\sup f^\beta_u)P_{t-u}$ exists for $t > 0$ and is bounded by f_t; by virtue of the continuity of the mappings P_s, these limits define an element of F which is then necessarily the least upper bound $\vee f^\beta$ of the family $\{f^\beta\}$. This proves that F is an L-cone and gives the formula for $\vee f^\beta$; the derivation of the formula for $\wedge f^\beta$ is entirely similar.

The fact that F_0 is an L-cone in $M(A)$ is a direct consequence of the fact that it is a convex subcone of $M(A)$ thick in $M_+(A)$. The second part of theorem 2.1.2 is obtained as an application of theorem 1.3.2 to the L-cones F and F_0 and to the mappings

$$(2.1.11)\qquad u(f) = \lim_{t\to 0} f_t; \qquad u_0(f_0) = \{f_0P_t, t > 0\}.$$

Indeed the existence of u and the inequality $u_0u \leqq I$ on F have been proved in theorem 1.1.1, whereas $uu_0 \geqq I_0$ on F_0, that is, $\lim_{t\to 0} f_0P_t \geqq f_0$ for $f_0 \in F_0$, results from $f_0P_t(i) \geqq f_0(i)P_t(i, i)$.

COROLLARY. *Every element $f \in F$ may be written in one and only one way as $f_t = f_0P_t + f'_t(t > 0)$. Where $f_0 \in F_0$, $f_0 = \lim_{t\downarrow 0} f_t = \lim_{t\to 0} f_0P_t$, and where $\{f'_t, t > 0\} \in F$, $\lim_{t\to 0} f'_t = 0$.*

The following theorem leads to an alternate definition of the cone F as a cone of vectorial measures on $(0, \infty)$ with values in the space $M(A)$; it also gives the clue to theorem 2.1.4 below on the Laplace transform of elements of F. We shall denote by $\mu(t + \cdot)$ the measure on R_+ obtained from a given measure μ on R_+ by a translation of amplitude t along R_+, so that

$$(2.1.12)\qquad \int_{s>0} \mu(t + ds)\,\varphi(s) = \int_{s>t} \mu(ds)\,\varphi(s - t)$$

for any positive measurable real-valued function φ.

THEOREM 2.1.3. *The most general family $\{\mu(\cdot\,; i); i \in A\}$ of positive measures on the open interval $R_+ = (0, \infty)$ such that*

$$(2.1.13)\qquad \mu(t + \cdot\,; j) = \sum_i \mu(\cdot\,; i)P_t(i, j), \qquad j \in A, t > 0,$$

is given by $\mu(ds; i) = f_s(i)\,ds$ where $\{f_s, s > 0\}$ is an arbitrary element of F.

PROOF. Given a family $\{\mu(\cdot\,; i); i \in A\}$ of positive measures on R_+ satisfying the preceding hypothesis, let us put

$$f_t(j) = \frac{1}{t} \sum_i \int_{\{s<t\}} \mu(ds; i) P_{t-s}(i, j), \qquad t > 0, j \in A. \tag{2.1.14}$$

We first show that $f_t(j) < \infty$. Indeed, from the inequality $P_{t-s}(i, j) P_s(j, j) \leqq P_t(i, j)$, it follows that

$$f_t(j) \inf_{\{s<t\}} P_s(j, j) \leqq \frac{1}{t} \sum_i \int_{\{s<t\}} \mu(ds; i) P_t(i, j) = \int_{\{t<s<2t\}} \mu(ds; j) < \infty. \tag{2.1.15}$$

The theorem will follow from formula

$$\sum_i \int_K \mu(ds; i) P_{t-s}(i, j) = \int_K ds \cdot f_t(j), \qquad t > 0; j \in A, \tag{2.1.16}$$

where K is any Borel subset of the interval $(0, t)$; to prove this formula we only need to remark that the first member defines, for fixed $t > 0$, $j \in A$, a set function of K which is a translation-invariant positive measure on $(0, t)$. This set function is then necessarily equal to the second member of the formula as a consequence of the definition of f. We denote by ds the Lebesgue measure.

The fact that $f_s P_t = f_{s+t}$ with $s, t > 0$, that is, $\{f_t, t > 0\} \in F$, is an immediate consequence of the preceding formula. In order to show that $\mu(dt; j) = f_t(j) dt$, we shall prove that for any $u, v > 0$

$$\int_{u<t\leqq u+v} \mu(dt; j) = \int_{u<t\leqq u+v} f_t(j)\, dt \tag{2.1.17}$$

by taking $K = (t - u, t]$ in the above formula and integrating over $(u, u + v]$. Then

$$\begin{aligned} u \int_{(u,u+v]} f_t(j)\, dt &= \sum_i \int_{(u,u+v]} \int_{(0,u]} \mu[(t - u) + ds; i] P_{u-s}(i, j) \\ &= \sum_i \int_{(0,u]} \int_{(0,v]} dt\, \mu(t + ds; i) P_{u-s}(i, j) \\ &= \sum_i \int_{(0,u]} ds \left[\int_{(s,s+v)} \mu(dt; i) \right] P_{u-s}(i, j) \\ &= \int_{(0,u]} ds \int_{(u,u+v]} \mu(dt; j) = u \int_{(u,u+v]} \mu(dt; j). \end{aligned} \tag{2.1.18}$$

This concludes the proof of theorem 2.1.3.

Let F_b designate the convex subcone of the elements of F such that for every $T > 0$, or equivalently for some $T > 0$,

$$\int_0^T ds\, \langle f_s, e \rangle < \infty. \tag{2.1.19}$$

It is readily seen that F_b is thick in F. The hypothesis $P_t e \leqq e$, where $t \geqq 0$ implies that the real-valued function $\langle f_t, e \rangle$ of t is nondecreasing; this function which in general may take infinite values, is finite for every $f \in F_b$.

THEOREM 2.1.4. *The Laplace transform*

$$\hat{f}_x = \int_0^\infty dt\, e^{-xt} f_t, \qquad 0 < x < \infty, \tag{2.1.20}$$

establishes a one-to-one correspondence between the elements $f = \{f_t, t > 0\}$ *of* F_b *and the families* $\hat{f} = \{\hat{f}_x, x > 0\}$ *of bounded positive measures on* A *such that*

$$\hat{f}_x = \hat{f}_z[I + (z - x)R_x], \qquad 0 < x, z < \infty. \tag{2.1.21}$$

PROOF. Let f be an element of F_b. Then

$$\begin{aligned} \langle \hat{f}_x, e\rangle &= \sum_{n \geqq 0} \int_0^T dt\, e^{-x(t+nT)} \langle f_t, P_{nT} e\rangle \\ &\leqq (1 - e^{-xT})^{-1} \int_0^T dt\, \langle f_t, e\rangle \end{aligned} \tag{2.1.22}$$

shows that $\hat{f}_x$ is a bounded positive measure on A for every $x > 0$. To prove the asserted equation it is sufficient to remark [as in the proof of property (c') of the resolvent] that the convolution product on R_+ of e^{-xt} and e^{-zt} is $(x - z)^{-1}(e^{-zt} - e^{-xt})$, since then

$$\begin{aligned} \hat{f}_z R_x &= \int_0^\infty \int_0^\infty ds\, dt\, e^{-zs} e^{-xt} f_s P_t \\ &= \int_0^\infty \int_0^\infty ds\, dt\, e^{-zs} e^{-xt} f_{s+t} \\ &= (x - z)^{-1} \int_0^\infty ds\, (e^{-zs} - e^{-xs}) f_s \\ &= (x - z)^{-1} (\hat{f}_z - \hat{f}_x). \end{aligned} \tag{2.1.23}$$

To prove the converse part of theorem 2.1.4, consider a family $\{\hat{f}_x, x > 0\}$ of bounded positive measures on A with the above property. Then as a consequence of Bernstein's theorem on completely monotonic functions and the formula

$$\left(-\frac{d}{dx}\right)^n \hat{f}_x = \hat{f}_x R_x^n \geqq 0, \qquad x > 0;\ n \geqq 0, \tag{2.1.24}$$

there exists a family $\{\mu(\cdot\,; i);\, i \in A\}$ of positive measures on R_+ such that

$$\hat{f}_x(i) = \int_0^\infty e^{-xs} \mu(ds; i), \qquad x > 0;\ i \in A. \tag{2.1.25}$$

To finish the proof of theorem 2.1.4 it is now sufficient in virtue of theorem 2.1.3 to show that this family of measures is such that on R_+

$$\mu(t + \cdot\,; j) = \sum_i \mu(\cdot\,; i) P_t(i, j), \qquad t > 0,\ j \in A. \tag{2.1.26}$$

But this relation is obviously equivalent to

$$\int_0^\infty e^{-xs} \mu(t + ds; j) = \hat{f}_x P_t(j), \qquad x > 0,\ t > 0;\ j \in A. \tag{2.1.27}$$

Since, for every fixed $x > 0$, $j \in A$, the two sides of this equation are right-continuous in t, their equality is also equivalent to

$$\int_0^\infty dt\, e^{-zt} \int_0^\infty c^{-xs}\mu(t + ds; j) = \hat{f}_x R_z(j), \qquad x, z > 0; j \in A. \tag{2.1.28}$$

However, the first member of this equation is also equal to

$$\int_0^\infty (e^{-zt} * e^{-xt})\mu(dt; j) = (x - z)^{-1}(\hat{f}_z - \hat{f}_x), \tag{2.1.29}$$

so that this equation and the preceding equivalent ones follow from the hypothesis.

THEOREM 2.1.5. *In the partially ordered vector space* $[L^1(A)]^{R_+}$, *the subset of the elements* $\hat{f} = \{\hat{f}_x, x > 0\}$ *such that*

(a) $\hat{f}_x \in L^1_+(A)$, $\qquad x > 0$,
(b) $\hat{f}_x = \hat{f}_z[I + (z - x)R_x]$, $\qquad x, z > 0$,

forms an L-cone. The least upper bound of an upper-bounded family $\{f^\beta, \beta \in B\}$ *in this cone is given by*

$$(\mathsf{V}_\beta \beta \hat{f}^\beta)_x = \lim_{y \uparrow \infty} \uparrow (\sup_\beta \hat{f}^\beta_y)[I + (y - x)R_x]. \tag{2.1.30}$$

This L-cone is isomorphic to the L-cone F_b *by the Laplace transform and will be designated by* $\hat{F}_b$.

PROOF. The proof that the elements $\hat{f}$ form an L-cone is entirely similar to the proof of theorem 2.1.2; it essentially rests on the property (c') of the resolvent which implies that

$$\sup_\beta \hat{f}^\beta_x \leqq (\sup_\beta \hat{f}^\beta_y)[I + (y - x)R_x] \leqq f_x, \qquad 0 < x < y, \tag{2.1.31}$$

if f is an upper bound of the family $\{\hat{f}^\beta\}$. The details will be left to the reader. The isomorphism of F_b and $\hat{F}_b$ is a direct consequence of theorem 2.1.4 and of lemma 1.3.2.

Let F_e designate the convex subcone of F, also contained in F_b, of the elements f such that

$$\sup_{t>0} ||f_t||_1 \equiv \lim_{t \downarrow 0} \uparrow \langle f_t, e\rangle < \infty. \tag{2.1.32}$$

It is easily seen that F_e is thick in F. Its image $\hat{F}_e$ in $\hat{F}_b$ by the Laplace transform is the subset of $\hat{F}_b$ of the $\hat{f}$ such that

$$\sup_{x>0} ||x\hat{f}_x||_1 \equiv \lim_{x \uparrow 0} \uparrow \langle x\hat{f}_x, e\rangle < \infty. \tag{2.1.33}$$

In fact the two preceding expressions are equal.

To end this section, we remark that the intersection of the two cones $F_e(P)$ and $F_0(P)$ [to be designated by $F_{e0}(P)$] is isomorphic to $L^1_+(A)$ since

$$\langle f_0, e\rangle = \lim_{t \downarrow 0} \uparrow \langle f_0 P_t, e\rangle \tag{2.1.34}$$

for any $f_0 \in F_0(P)$.

2.2. *The cone G.* Let $E^2 = [N(A)]^{R_+}$ be the partially ordered vector space of the mappings $g = \{g_t, t > 0\}$ of the positive real half-line R_+ into the space

$N(A)$ of real-valued functions on A. We denote by G the subset of E^2 of the mappings g such that

(a) $0 \leqq g_t < \infty$ on A, $\quad t > 0$,
(b) $g_{t+s} = P_t g_s$, $\quad t, s > 0$.

We denote by G_0 the subset of $N(A)$ of the positive functions g_0 on A such that $0 \leqq P_t g_0 < \infty$ on A for $t > 0$.

The cones G and G_0 have properties entirely analogous to the properties of the cones F and F_0 which are stated in theorems 2.1.1, 2.1.2, and 2.1.3. In order to prove them, we need only remark that the proofs of these theorems did not use the inequality $P_t e \leqq e$, with $t > 0$, so that they are valid for any semigroup $\{P_t, t > 0\}$ of positive matrices on A such that the t-functions $P_t(i, j)$ are continuous and equal to $I(i, j)$ at $t = 0$. In particular, these theorems apply to the adjoint semigroup $P' = \{P'_t(i, j) = P_t(j; i)\}$ of a continuous semigroup of submarkovian matrices. We state for further reference,

THEOREM 2.2.1. *For every $g \in G$, the t-functions $g_t(i)$, with $i \in A$, are continuous on $[0, \infty)$; moreover, $g_0 = \lim_{t\to 0} g_t$ is an element of G_0 such that $P_t g_0 \leqq g_t$ where $t > 0$.*

THEOREM 2.2.2. *The set G is an L-cone in E^2; the lattice operations are defined by*

$$(2.2.1) \qquad \begin{aligned} (\vee_\beta g^\beta)_t &= \lim_{s \downarrow 0} \uparrow P_{t-s}(\sup_\beta g^\beta_s), \\ (\wedge_\beta g^\beta)_t &= \lim_{s \downarrow 0} \downarrow P_{t-s}(\inf_\beta g^\beta_s), \end{aligned} \qquad t > 0.$$

The set G_0 is an L-cone in $N(A)$ with the lattice operations defined as in $N(A)$.

The set of elements $g = \{g_t, t > 0\}$ of G such that $g_t = P_t g_0$ where $t > 0$, for $g_0 = \lim_{t\to 0} g_t$ is a positive band of G which is isomorphic to G_0, and will be identified with G_0; the complementary positive band of G_0 in G is composed of the elements g such that $g_0 = 0$.

COROLLARY. *Every element $g \in G$ may be written in one and only one way as $g_t = P_t g_0 + g'_t$, with $t > 0$, where $g_0 \in G_0$, and $g_0 = \lim_{t\to 0} g_t = \lim_{t\to 0} P_t g_0$ and where $\{g'_t, t > 0\} \in G$, $\lim_{t\to 0} g'_t = 0$.*

THEOREM 2.2.3. *The most general family $\{\mu(\cdot\,; i); i \in A\}$ of positive measures on the open interval $R_+ = (0, \infty)$ such that*

$$(2.2.2) \qquad \mu(t + \cdot\,; j) = \sum_j P_t(i, j)\mu(\cdot\,; j), \qquad t > 0, i \in A,$$

is given by $\mu(ds; i) = g_s(i)\, ds$ where $\{g_s, s > 0\}$ is an arbitrary element of G.

Let G_b designate the convex subcone of the elements of G such that for every $T > 0$ (or equivalently for some $T > 0$)

$$(2.2.3) \qquad \sup_i \left[\int_0^T dt\, g_t(i)\right] < \infty.$$

Clearly, G_b is thick in G.

The counterpart of theorems 2.1.4 and 2.1.5 is now the following; its proof is left to the reader.

THEOREM 2.2.4. *The Laplace transform,*

$$\hat{g}_x = \int_0^\infty dt\, e^{-xt} g_t, \qquad 0 < x < \infty, \tag{2.2.4}$$

establishes one-to-one correspondence between the elements $\hat{g} = \{\hat{g}_t, t > 0\}$ *of* G_b *and the families* $g = \{g_x, x < 0\}$ *of bounded positive functions on* A *such that*

$$\hat{g}_x = [I + (y - x)R_x]\hat{g}_y, \qquad 0 < x, y < \infty. \tag{2.2.5}$$

THEOREM 2.2.5. *On the partially ordered vector space* $[L^\infty(A)]^{R_+}$ *the subset of the elements* $\hat{g} = \{\hat{g}_x, x > 0\}$ *such that*

(a) $\hat{g}_x \in L_+^\infty(A)$, $\qquad 0 < x < \infty$,
(b) $\hat{g}_x = [I + (y - x)R_x]\hat{g}_y$, $\qquad 0 < x, y < \infty$,

forms an L-cone. The least upper bound of an upper-bounded family $\{\hat{g}^\beta\}$ *in this cone is given by*

$$(\vee \hat{g}^\beta)_x = \lim_{y\uparrow\infty} \uparrow [I + (y - x)R_x](\sup_\beta g_y^\beta). \tag{2.2.6}$$

This L-cone is isomorphic to the L-cone G_b *by the Laplace transform and will be designated by* $\hat{G}_b$.

Let p be the element of G_b such that

$$e - P_s e = \int_0^s dt\, p_t, \qquad s > 0. \tag{2.2.7}$$

The existence of such an element is a consequence of theorem 2.2.3. In fact, since $1 - P_s e(i)$ is, for every $i \in A$, a position nondecreasing function of s on R_+ which tends to 0 as $s \downarrow 0$, there exists a family $\{p(\cdot\,; i); i \in A\}$ of positive measures on R_+ such that $1 - P_s e(i) = \int_{(0,s)} p(dt; i)$, with $i \in A$, $s > 0$. Then the hypothesis of theorem 2.2.3 is satisfied as it follows from

$$P_t(e - P_s e) = (e - P_{t+s}e) - (e - P_t e), \qquad s, t > 0. \tag{2.2.8}$$

It may be remarked immediately that, as a consequence of

$$\sum_j \left[\int_0^\infty ds\, P_s(\cdot, j)\right] p_t(j) = P_t e - \lim_{s\uparrow\infty} \downarrow P_s e \leqq e, \tag{2.2.9}$$

the following implication holds for any $j \in A$

$$\sup_{t>0} p_t(j) > 0 \Rightarrow \int_0^\infty ds\, P_s(\cdot, j) < \infty \tag{2.2.10}$$

on A.

2.3. *The forms C.* Let P and P' be two submarkovian processes defined on the denumerable sets A and A' respectively. We shall call a family $\{C_{s,t}; s, t > 0\}$ of positive matrices on $A \times A'$ such that

$$C_{u+s,v+t} = P_u C_{s,t} P'_v, \qquad u, v, s, t > 0, \tag{2.3.1}$$

a "form relative to P and P'." An example of such a form is given by $C_{s,t} = g_s \otimes f_t$, where $s, t > 0$, if g is an element of $G(P)$ and f an element of $F(P')$.

THEOREM 2.3.1. *For any form $C = \{C_{s,t};\, s, t > 0\}$ relative to the processes P and P' the (s, t)-functions $C_{s,t}(i, j)$, with $i \in A$, $j \in A'$, may be extended to continuous functions on $\{s, t \geqq 0\}$. If we still denote this extension by $C_{s,t}$ we have*

$$C_{u+s,v+t} \geqq P_u C_{s,t} P'_v, \qquad u, v, s, t, \geqq 0, \tag{2.3.2}$$

where the equality sign holds except perhaps if $s = 0$ and $u > 0$ or if $t = 0$ and $v > 0$.

PROOF. The first part of the proof of this theorem is similar to the proof of theorem 2.1.1. In an analogous way it can be proved that

(1) the (s, t)-functions $C_{s,t}(i, j)$ are lower semicontinuous on $(s, t > 0)$,

(2) these functions have right limits on $(s, t \geqq 0)$, say $C^+_{s,t}$, such that

$$C^+_{s,t}(i, j) = \lim_{u \downarrow s, v \downarrow t} C_{u,v}(i, j) = \limsup_{u \to s, v \to t} C_{u,v}(i, j), \qquad s, t \geqq 0, \tag{2.3.3}$$

(3) the functions $C_{s,t}$ coincide with $C^+_{s,t}$ on $(s, t > 0)$ and are continuous there,

(4) $C^+_{u+s,v+t} \geqq P_u C^+_{s,t} P'_v, \qquad s, t, u, v \geqq 0.$

By the continuity we have just proved and by the properties of $F_0(P')$, we get, when $u, s, t > 0$,

$$C_{u+s,t} = \lim_{v \to 0} C_{u+s,t+v} = \lim_{v \to 0} [P_u C_{s,t}] P'_v = P_u C_{s,t}. \tag{2.3.4}$$

It can be proved similarly that $C_{s,v+t} = C_{s,t} P'_v$ for $s, t, v > 0$.

By fixing $u, s > 0$ and $i \in A$, consider the first identity as a relation in $F(P')$:

$$C_{u+s,t}(i, \cdot) = \sum_{k \in A} P_u(i, k) C_{s,t}(k, \cdot), \qquad t > 0. \tag{2.3.5}$$

By equating the projections in $F_0(P')$ of both sides of this relation, we obtain

$$C^+_{u+s,0} = P_u C^+_{s,0}, \qquad s, u > 0, \tag{2.3.6}$$

since $C^+_{s,0} = \lim_{t \to 0} C_{s,t}$ for $s > 0$. Similarly, we can show that

$$C^+_{0,v+t} = C^+_{0,t} P'_v, \qquad t, v > 0. \tag{2.3.7}$$

It remains to prove that the functions $C^+_{s,t}$ are continuous at every point $(u, 0)$ and $(0, u)$ for $u > 0$; but as a consequence of $C^+_{u+s,v} = P_u C^+_{s,v}$, where $u, s > 0$; $v \geqq 0$, we have

$$\liminf_{u \to 0, v \to 0} C^+_{u+s,v} \geqq P_u C^+_{s,0} = C^+_{u+s,0}, \qquad u, s > 0, \tag{2.3.8}$$

so that the functions $C^+_{s,t}$ are lower semicontinuous and, from what has been proved above, are continuous.

COROLLARY. *When $F(P') = F_0(P')$, $[G(P) = G_0(P)]$, every form C relative to P and P' satisfies $C_{s,t} = C_{s,0} P'_t$, $[C_{s,t} = P_s C_{0,t}]$, for $s, t > 0$.*

PROOF. This corollary is immediate since the hypothesis implies that for every $s > 0$ and $i \in A$ the element $\{C_{s,t}(i, \cdot);\, t > 0\}$ of $F(P)$ belongs to $F_0(P')$.

A form C relative to P and P' will be said to be *bounded* when for every $T > 0$ (or equivalently for some $T > 0$)

$$\sup_{i \in A} \sum_{j \in A'} \int_0^T ds \int_0^T dt\, C_{s,t}, \qquad i, j < \infty. \tag{2.3.9}$$

THEOREM 2.3.2. *The Laplace transform*

$$\hat{C}_{x,y} = \int_0^\infty \int_0^\infty ds\, dt\, e^{-(xs+yt)} C_{s,t}, \qquad 0 < x, y < \infty, \tag{2.3.10}$$

establishes a one-to-one correspondence between bounded forms C relative to P and P' and families $\{\hat{C}_{x,y}; 0 < x, y < \infty\}$ of positive matrices on $A \times A'$ such that

(a) $\sup_{i \in A} \sum_{j \in A'} \hat{C}_{x,y}(i, j) < \infty, \qquad 0 < x, y < \infty;$

(b) $\hat{C}_{x,y} = [I + (z - x)R_x]\hat{C}_{z,y},$
$\hat{C}_{x,y} = \hat{C}_{x,z}[I + (z - y)R'_y], \qquad 0 < x, y, z < \infty.$

PROOF. The deduction of the properties of $\hat{C}$ from the properties of the bounded form C is easy and similar to the first part of the proof of theorem 2.1.4.

To prove the converse, we first remark that for fixed $x > 0$, $i \in A$, we have $\{\hat{C}_{x,y}(i, \cdot); y > 0\}$ is an element of $F_b(P')$, so that by 2.1.4 there exists a family $\{\Gamma_{x,t}; t > 0\}$ of positive matrices on $A \times A'$ such that

$$\hat{C}_{x,y} = \int_0^\infty dt\, e^{-yt}\Gamma_{x,t}; \qquad \Gamma_{x,t+v} = \Gamma_{x,t}P'_v, \qquad v, t > 0; x > 0. \tag{2.3.11}$$

Moreover, since $\sum_j [I + (y - x)R_x](i, j)\hat{C}_{y,z}(j, \cdot) = \hat{C}_{x,z}(i, \cdot)$ holds in $\hat{F}_b(P')$, for every fixed $x, u > 0$ and $i \in A$, we must have in $F_b(P')$

$$\sum_j [I + (y - x)R_x](i, j)\Gamma_{y,t}(j, \cdot) = \Gamma_{x,t}(i, \cdot), \qquad t > 0. \tag{2.3.12}$$

Let us also remark that

$$\sup_i \sum_j \Gamma_{x,t}(i, j) < \infty, \qquad x > 0, t > 0 \tag{2.3.13}$$

since the left side is nonincreasing in t, so that

$$\left(\int_0^t ds\, e^{-zs}\right) \sup_i \sum_j \Gamma_{x,t}(i, j) \leqq \sup_i \sum_j \hat{C}_{x,z}(i, j) < \infty. \tag{2.3.14}$$

For fixed $t > 0$ and $j \in A'$, we have that $\{\Gamma_{x,t}(\cdot, j); x > 0\}$ is, by what has already been proved, an element of $\hat{G}_b(P)$; by theorem 2.2.4, there exists a family $\{C_{s,t}; s, t > 0\}$ of positive matrices on $A \times A'$ such that

$$\Gamma_{x,t} = \int_0^\infty ds\, e^{-xs}C_{s,t}, \qquad P_uC_{s,t} = C_{u+s,t}, \qquad u, s, t > 0. \tag{2.3.15}$$

Moreover, since $\Gamma_{x,t+v}(\cdot, j) = \sum_k \Gamma_{x,t}(\cdot, k)P'_v(k, j)$ holds in $\hat{G}_b(P)$ for every fixed $t, v > 0$, $j \in A'$, we must have in $G_b(P)$

$$C_{s,t+v}(\cdot, j) = \sum_k C_{s,t}(\cdot, k)P'_v(k, j), \qquad s > 0. \tag{2.3.16}$$

It is thus proved that $\{C_{s,t}; s, t > 0\}$ is a form relative to P and P' whose

Laplace transform is $\{C_{x,y}; x, y > 0\}$; this form is obviously bounded, since for $T > 0$

$$\sup_i \sum_j \int_0^T ds \int_0^T dt\, C_{s,t}(i,j) \leqq e^{(x+z)T} \sup_i \sum_j \hat{C}_{x,z}(i,j) < \infty. \tag{2.3.17}$$

3. Dominance

3.1. *The dominance relation.*

DEFINITION 3.1.1. *Given two semigroups P and P' of submarkovian matrices defined on the denumerable sets A and A', respectively, the s.g. P' will be said to dominate the s.g. P (in notation $P \subset P'$) if*

(a) $A \subset A'$,

(b) $P_t(i,j) \leqq P'_t(i,j)$, $\qquad i, j \in A; t > 0.$

It will often be found convenient to extend the domain of definition of the matrices $P_t (t > 0)$ and $R_x (x > 0)$ to $A' \times A'$ by letting $P_t(i,j) = 0 = R_x(i,j)$ when $(i,j) \in (A' \times A') - (A \times A)$ and for $t, x > 0$.

THEOREM 3.1.1. *Given two submarkovian processes P, P' such that P' dominates P, there exist two bounded forms $C = \{C_{s,t}; s, t \geqq 0\}$ and $D = \{D_{s,t}; s, t \geqq 0\}$ defined relative to P and P', and to P' and P, respectively, such that*

$$\begin{aligned} P'_t &= P_t + \int_0^t ds\, C_{s,t-s} \text{ on } A \times A', \\ P'_t &= P_t + \int_0^t ds\, D_{s,t-s} \text{ on } A' \times A \text{ for } t > 0. \end{aligned} \tag{3.1.1}$$

As a consequence, the following limits exist for $s, t \geqq 0$:

$$\begin{aligned} &\lim_{u \downarrow 0} P_s \frac{1}{u} (P'_u - P_u) P'_t = C_{s,t} \text{ on } A \times A' \\ &\lim_{u \downarrow 0} P'_s \frac{1}{u} (P'_u - P_u) P_t = D_{s,t} \text{ on } A' \times A. \end{aligned} \tag{3.1.2}$$

PROOF. This theorem will be shown to be implied by theorems 2.3.1 and 2.3.2, by means of the following computations. Since the proofs of the two first formulas of this theorem are similar, we shall content ourselves with proving the first of these formulas.

Since property (c') of section 1.2 immediately implies that the two expressions of the following definition are both equal on $A \times A'$ to $R'_z - R_x + (z - x) R_x R'_z$, we are allowed to introduce a family $\{\hat{C}_{xz}; x, z > 0\}$ of matrices on $A \times A'$ by the two equivalent formulas

$$\begin{aligned} \hat{C}_{x,z} &= \sum_{A'} [R'_z - R_z](\cdot, i)[I + (x - z)R'_z](i, \cdot) \\ &= \sum_{A} [I + (z - x)R_x](\cdot, j)[R'_z - R_z](j, \cdot) \quad \text{on } A \times A'. \end{aligned} \tag{3.1.3}$$

The hypothesis $P \subset P'$ and the first (second) of these formulas imply that the matrix $\hat{C}_{x,z}$ is positive when $x \geqq z$ ($x \leqq z$). Since this family $\hat{C}$ obviously satisfies the relations

(3.1.4) $$\hat{C}_{x,z} = \hat{C}_{x,y}[I + (y - z)R'_z] = [I + (y - x)R_x]\hat{C}_{y,z}, \quad 0 < x, y, z < \infty$$

[compare again property (c') of section 1.2], there remains only to show that

(3.1.5) $$\sup_{i \in A} \sum_{j \in A'} \hat{C}_{x,z}(i, j) < \infty, \qquad x, z > 0,$$

in order to be able to apply theorem 2.3.2. But, if $x \geqq z$, we have

(3.1.6) $$\begin{aligned} z\hat{C}_{x,z}e' &= z \sum_{A'} [R'_x - R_x](\cdot, i)[1 + (x - z)R'_z e'(i)] \\ &\leqq x \sum_{A'} [R'_x - R_x](\cdot, i) \leqq e, \end{aligned}$$

whereas if $x \leqq z$,

(3.1.7) $$\begin{aligned} x\hat{C}_{x,z}e' &= x \sum_{A} [I + (z - x)R_x](\cdot, j)[R'_z e' - R_z e'](j) \\ &\leqq \frac{x}{z} \sum_{A} [I + (z - x)R_x](\cdot, j) \leqq e. \end{aligned}$$

Let $\{C_{s,t};\, s, t \geqq 0\}$ be the bounded form relative to P and P' whose Laplace transform is $\hat{C}$; we then remark that on $A \times A'$, for every $x > 0$,

(3.1.8) $$\begin{aligned} \int_0^\infty dt\, e^{-xt}(P'_t - P_t) &= R'_x - R_x = \hat{C}_{xx} \\ &= \int_0^\infty \int_0^\infty ds\, dt\, e^{-x(s+t)}C_{s,t} \\ &= \int_0^\infty dt\, e^{-xt} \int_0^t ds\, C_{s,t-s}. \end{aligned}$$

Since for every fixed $i \in A$ and $j \in A'$, the t-functions $[P'_t - P_t](i, j)$ and $\int_0^t C_{s,t-s}(i, j)$ are continuous by theorem 2.3.1, they must be equal, as was to be shown.

To show the last formulas of this theorem, we remark that one has on $A \times A'$ when $u \downarrow 0$ and for $s, t \geqq 0$,

(3.19) $$\begin{aligned} P_s \frac{1}{u}(P'_u - P_u)P'_t &= P_s \frac{1}{u} \int_0^u dv\, C_{v,u-v}P'_t \\ &= \frac{1}{u} \int_0^u dv\, C_{v+s,u-v+t} \rightarrow C_{s,t} \end{aligned}$$

as well as a similar relation involving D.

3.2. *Kolmogorov's equations.* It is well known that for any submarkovian process P defined on the denumerable set A, the limit

(3.2.1) $$q_i = \lim_{s \downarrow 0} \frac{1}{s}[1 - P_s(i, i)], \qquad i \in A,$$

exists, that $0 \leqq q_i \leqq \infty$ and that, at least when $q_i < \infty$, one has

$$P_s(i, i) \geqq e^{-q_i s}, \qquad s > 0. \tag{3.2.2}$$

These results are indeed elementary consequences of the following properties of the function $\varphi(s) = -\log P_s(i, i)$ defined on R_+ for any fixed $i \in A$,

$$0 \leqq \varphi(s) \to 0(s \downarrow 0);\ \varphi(s + t) \leqq \varphi(s) + \varphi(t), \qquad s, t > 0. \tag{3.2.3}$$

If the subset $A^\infty = \{i : q_i < \infty\}$ of A is nonvoid, the submarkovian process P^∞ defined on A^∞ by

$$P_t^\infty(i, j) = e^{-q_i t} I_{A^\infty}(i, j), \qquad t > 0, \tag{3.2.4}$$

is clearly dominated by P, so that theorem 3.1.1 associates to the processes P^∞ and P two bounded forms C and D. Observe, however, that $F(P^\infty) = F_0(P^\infty)$ and that $G(P^\infty) = G_0(P^\infty)$, so that the corollary of theorem 2.3.1 shows that the bounded forms C and D are such that

$$\begin{aligned} C_{s,t}(i, j) &= e^{-q_i s} C_{0,t}(i, j), && i \in A^\infty; j \in A; s, t > 0,\\ D_{s,t}(i, j) &= D_{s,0}(i, j) e^{-q_j t}, && i \in A; j \in A^\infty; s, t > 0. \end{aligned} \tag{3.2.5}$$

Theorem 3.1.1 thus gives on A, for $t > 0$ and $i \in A^\infty$,

$$\begin{aligned} P_t(i, \cdot) &= e^{-q_i t} I(i, \cdot) + \int_0^t ds\, e^{-q_i(t-s)} C_{0,s}(i, \cdot),\\ P_t(\cdot, i) &= I(\cdot, i) e^{-q_i t} + \int_0^t ds\, D_{s,0}(\cdot, i) e^{-q_i(t-s)}. \end{aligned} \tag{3.2.6}$$

The t-functions $P_t(i, j)$ are then necessarily continuously differentiable on $[0, \infty)$ for $j \in A_\infty$ and

$$\begin{aligned} \frac{d}{dt}[e^{q_i t} P_t(i, \cdot)] &= e^{q_i t} C_{0,t}(i, \cdot),\\ \frac{d}{dt}[P_t(\cdot, i) e^{q_i t}] &= D_{0,t}(\cdot, i) e^{q_i t} \end{aligned} \tag{3.2.7}$$

are valid on A for $i \in A_\infty$ and $t \geqq 0$. We remark that these formulas imply that $C_{0,0} = D_{0,0}$ on $A^\infty \times A^\infty$, a result which will be stated in a more general form in theorem 3.3.4.

Let us restate the preceding results in the following theorem, which also takes into account the following properties of any form

$$C_{0,t} \geqq C_{0,0} P_t; \qquad D_{t,0} \geqq P_t D_{0,0}. \tag{3.2.8}$$

This theorem was first proved by D. G. Austin [1]; its integral form, that is, formulas (3.2.6), was proven by probabilistic methods in a paper due to Chung [5]. However, the hypothesis used by these authors is slightly more restrictive.

THEOREM 3.2.1. *Let P be a submarkovian process on A and suppose that the subset A^∞ of A on which $q_i = \lim_{s\to 0} 1/s[1 - P_s(i, i)] < \infty$ is nonvoid. Then the t-functions $P_t(i, j)$ are continuously differentiable on $[0, \infty)$ when i or $j \in A^\infty$; moreover they satisfy the following relations on A for every $i \in A^\infty$,*

$$
\begin{aligned}
&\frac{d}{dt} P_{t+s}(i, \cdot) = \left[\frac{d}{dt} P_t(i, \cdot)\right] P_s, \\
&\frac{d}{dt} P_{s+t}(\cdot, i) = P_s \left[\frac{d}{dt} P_t(\cdot, i)\right], \qquad s, t > 0; \\
&\frac{d}{ds} P_s(i, \cdot) \geqq \left[\frac{d}{dt} P_t(i, \cdot)\right]_{t=0} P_s, \\
&\frac{d}{ds} P_s(i, \cdot) \geqq P_s \left[\frac{d}{dt} P_t(\cdot, i)\right]_{t=0}, \qquad s > 0.
\end{aligned} \tag{3.2.9}
$$

3.3. *Further study of the dominance relation.* Doob [8] has shown that a semigroup P of permutation matrices (and a fortiori of submarkovian matrices) on a denumerable infinite set A need not have any continuity property. If, however, the semigroup P of submarkovian matrices dominates a submarkovian process P' defined on the same set A, it follows from

$$1 \geqq P_t(i, i) \geqq P'_t(i, i) \to 1, \qquad 0 < t \downarrow 0; i \in A, \tag{3.3.1}$$

that P is itself a submarkovian process on A. A similar result also holds if the semigroup P is dominated by a submarkovian process P', as is shown in the following theorem.

THEOREM 3.3.1. *Let $\{P'_t, t > 0\}$ be a submarkovian process on A'. Then any semigroup $\{P_t, t > 0\}$ of submarkovian matrices on A' (or on a subset of A') such that $P_t \leqq P'_t$ for $t > 0$, is necessarily a submarkovian process on a certain subset A of A', that is,*

(a) *$P_t(i, j) = 0$ if i or $j \notin A$ and $t > 0$,*

(b) *the t-functions $P_t(i, j)$ are continuous on $[0, \infty)$ and $\lim_{t\to 0} P_t(i, j) = I_A(i, j)$.*

This theorem remains true if the semigroup $\{P_t, t \in K\}$ of submarkovian matrices is supposed to be defined only for values of t in a dense semigroup K of R_+.

To avoid repetition, we shall prove this theorem and the following result together.

THEOREM 3.3.2. *Let P and P' be two submarkovian processes such that $P \subset P'$. Then there exists a convex subcone $\overline{F}$ of $F(P)$ which is thick in $F(P)$ and a positive band $\overline{F}'$ of $F(P')$ with the following properties:*

(a) *the L-cones $\overline{F}$ and $\overline{F}'$ are isomorphic by the mappings u and u',*

$$
\begin{aligned}
&[u(f)]_t = \lim_{s\downarrow 0} \uparrow \sum_A f_s(i) P'_{t-s}(i, \cdot) \text{ on } A', \qquad f \in \overline{F}, \\
&[u'(f')]_t = \lim_{s\downarrow 0} \downarrow \sum_A f'_s(i) P_{t-s}(i, \cdot) \text{ on } A, \qquad f' \in \overline{F}';
\end{aligned} \tag{3.3.2}
$$

(b) *for every $f \in F(P)$ not belonging to $\overline{F}$, for at least one $j \in A'$ and one $t > 0$, we have*

$$\lim_{s\downarrow 0} \uparrow \sum_A f_s(i) P'_{t-s}(i, j) = +\infty \tag{3.3.3}$$

for every f' in $(\overline{F}')^\perp$, we have, for all $t > 0$,

$$\lim_{s\downarrow 0} \downarrow \sum_A f'_s(i) P'_{t-s}(i, \cdot) = 0 \text{ on } A. \tag{3.3.4}$$

In particular, $F_e(P)$ is contained in $\bar{F}$ and is isomorphic to a positive band of $F_e(P')$ by u and u'. The restriction of u to $F_{e,0}(P) = L^1_+(A)$ is identical to the canonical imbedding of $L^1_+(A)$ in $L^1_+(A') = F_{e,0}(P')$, that is, we have

$$(3.3.5)\qquad \begin{aligned} &\lim_{s \downarrow 0} \uparrow P_s(i, \cdot)P'_{t-s} = P'_t(i, \cdot) \text{ on } A', && i \in A;\\ &\lim_{s \downarrow 0} \downarrow P'_s(i, \cdot)P_{t-s} = P_t(i, \cdot) \text{ [or 0] on } A, && i \in A \text{ [or } i \in A' - A]. \end{aligned}$$

Except for the assertion concerning $F_e(P)$, an analogous theorem could be proved relative to the L-cones $G(P)$ and $G(P')$; it suffices to replace expressions like $\lim_{s\to 0} \uparrow f_sP'_{t-s}$ or $\lim_{s\to 0} \downarrow f'_sP_{t-s}$ in the preceding formulation by their analogues $\lim_{s\downarrow 0} \uparrow P'_{t-s}g_s$ or $\lim_{s\to 0} \downarrow P_{t-s}g'_s$.

PROOF. In order to prove theorem 3.3.1, its generalization and theorem 3.3.2, we consider a submarkovian process P' on A' and a semigroup $\{P_t, t \in K\}$ of submarkovian matrices on A' such that $P_t \leqq P'_t$ for $t \in K$, where K is a dense semigroup of R_+. We first remark that the two L-cones $F(P)$ and $G(P)$ may still be defined for the semigroup P; if the general element of $F(P)$ is a family $\{f_t, t \in K\}$ of positive finite measures on A' such that $f_{t+s} = f_tP_s$ for $s, t \in K$, the first part of theorem 2.1.2 showing that $F(P)$ is an L-cone and using no continuity property of P remains valid; similarly for $G(P)$.

For any element $f \in F(P)$, the inequality

$$(3.3.6)\qquad f_uP'_{s+t} \geqq f_uP_sP'_t = f_{u+s}P'_t, \qquad u, s \in K;\ t \geqq 0,$$

shows that for fixed $t > 0$, the positive (perhaps infinite) measures $f_sP'_{t-s}$ on A' decrease with $s \in K$ on $(0, t]$. We designate by $\bar{F}$ the subset of elements f of $F(P)$ such that $\lim_{K\ni s\downarrow 0} \uparrow f_sP'_{t-s}$ is a finite measure for every $t > 0$ and denote by $u(f) = \{[u(f)]_t, t > 0\}$ this family of measures which clearly is an element of $F(P')$. The subset $\bar{F}$ of $F(P)$ is moreover a convex subcone thick in $F(P)$; let us also remark that $[u(f)]_t \geqq f_t$ on A' for $t \in K$.

For any element $f' \in F(P')$, the inequality

$$(3.3.7)\qquad f'_uP_{s+t} \leqq f'_uP'_sP_t = f'_{u+s}P_t, \qquad u > 0;\ s, t \in K,$$

shows that for fixed $t \in K$, the positive finite measures f'_sP_{t-s} increase with $s \in K$ on $(0, t]$. As a consequence, the formula $[u'(f')]_t = \lim_{K\ni s\downarrow 0} \downarrow f'_sP_{t-s}$ defines, when t varies in K, an element $u'(f')$ of $F(P)$ such that $[u'(f')]_t \leqq f'_t$ on A' for $t \in K$. This last inequality implies that $[u'(f')]_sP'_{t-s} \leqq f'_t$ when $s \in K$ and $t \geqq s$ in R_+, so that $u'(f')$ is an element of $\bar{F}$ and that $uu' \leqq I'$ on $F(P')$. Similarly it follows from $[u(f)]_t \geqq f_t$ for $f \in F$ and $t \in K$ that $[u(f)]_sP_{t-s} \geqq f_t$ when $t \geqq s$ in K, so that the general inequality $u'u \geqq I$ holds on F.

Since the mappings u and u' are obviously additive and increasing on $\bar{F}$ and $F(P')$, we have just proved that the hypotheses of theorem 1.3.2 are satisfied. The first part of theorem 3.3.2 is thus proved under a weaker hypothesis on P. A similar result holds for the L-cones $G(P)$ and $G(P')$. We shall now prove that P may be extended to a submarkovian process on a subset of A'.

For any $i \in A'$, the u-image in $F(P')$ of $\{P_t(i, \cdot), t \in K\}$ exists and is bounded above by $\{P'_t(i, \cdot), t > 0\}$, that is, by $\epsilon_i \in F_0(P')$. This is possible only if there exists a real constant $c(i) \in [0, 1]$ such that on A'

$$P_t(i, \cdot) \leqq \lim_{K \in s \downarrow 0} \uparrow \sum_j P_s(i, j) P'_{t-s}(j, \cdot) = c(i) P'_t(i, \cdot) \tag{3.3.8}$$

for $t > 0$ (the first inequality only when $t \in K$). This relation and the inequality

$$0 \leqq \sum_{j \neq i} P_s(i, j) P'_{t-s}(j, i) \leqq P'_t(i, i) - P'_s(i, i) P'_{t-s}(i, i) \rightarrow 0 \tag{3.3.9}$$

as $K \rightarrow s \downarrow 0$ valid for $t > 0$, $i \in A'$, imply that $P_s(i, i) P'_{t-s}(i, i)$ converges to $c(i) P'_t(i, i)$ as $K \ni s \downarrow 0$, that is, that

$$\lim_{K \ni s \downarrow 0} P_s(i, i) = c(i), \qquad i \in A. \tag{3.3.10}$$

Similar reasoning shows that for every $i \in A'$ there exists a real constant $d(i) \in [0, 1]$ such that on A', for $t > 0$,

$$P_t(\cdot, i) \leqq \lim_{K \ni s \downarrow 0} \uparrow \sum_j P'_{t-s}(\cdot, j) P_s(j, i) = P'_t(i, i)\, d(i). \tag{3.3.11}$$

This constant $d(i)$ being also equal to $\lim_{s \downarrow 0} P_s(i, i)$ does not differ from $c(i)$. Moreover, from

$$\begin{aligned} P_{u+v}(i, i) &= \sum_j P_u(i, j) P_v(j, i) \\ &\leqq c(i) \Big[\sum_j P'_u(i, j) P'_v(j, i) \Big] d(i) = c^2(i) P'_{u+v}(i, i) \end{aligned} \tag{3.3.12}$$

follows, when $u, v \in K$, and $(u + v) \downarrow 0$, that $c(i) \leqq c^2(i)$, that is, that $c(i) = 0$ or 1.

If the subset A of A' is defined as the one on which $c(\cdot) = 1$, we have now shown that

(1) $P_t(i, j) = 0$ if i or $j \in A$, $t \in K$;

(2) $\lim_{K \ni s \downarrow 0} P_s(i, i) = i$ if $i \in A$.

This proves that the t-functions $P_t(i, j)$ are uniformly continuous on K, since by the submarkovian property of the matrices P_t we have

$$\begin{aligned} \sum_j |P_{t+s}(i, j) - P_s(i, j)| &\leqq \sum_{j,K} |P_t(i, K) - I(i, K)| P_s(K, j) \\ &\leqq 2[1 - P_t(i, i)] \rightarrow 0 \end{aligned} \tag{3.3.13}$$

when $s \in K$, with $K \in t \downarrow 0$, for every $i \in A$. It is clear by this inequality that the unique continuous extension of the t-functions $P_t(i, j)$ to R_+ gives a submarkovian process on A. Theorem 3.3.1 and its generalization are thus proved.

To conclude the proof of theorem 3.3.2, we remark that if $f \in F_e(P)$, then for $t > 0$

$$\sum_{A'} [u(f)]_t(k) = \sum_{A'} \lim_{s \downarrow 0} \uparrow f_s P'_{t-s}(k) \leqq \lim_{s \downarrow 0} \uparrow \sum_A f_s(j) < \infty. \tag{3.3.14}$$

This proves that $F_e(P) \subset \overline{F}$ and, since $[u(f)]_t \geqq f_t$, even that

$$\lim_{t\downarrow 0}\uparrow\langle[u(f)]_t, e'\rangle = \lim_{t\downarrow 0}\uparrow\langle f_t, e\rangle. \tag{3.3.15}$$

Finally, to prove that if $f_0 \in L^1_+(A)$, then $u(f_0) = f_0$, where f_0 denotes the element of $L^1_+(A)$ as well as its canonical images in $F_e(P)$ and $F_e(P')$, we deduce from what precedes that for $t > 0$

$$\begin{aligned}[u(f_0)]_t &= \lim_{s\downarrow 0}\uparrow f_0 P_s P'_{t-s} \\ &= \sum_A f_0(i)\,[\lim_{s\downarrow 0}\uparrow P_s(i,\cdot)P'_{t-s}] \\ &= \sum_A f_0(i)P'_t(i,\cdot).\end{aligned} \tag{3.3.16}$$

From theorem 3.3.2 we deduce the following result, which will be needed later.

THEOREM 3.3.3. *Let C' be a form relative to the two given submarkovian processes P and P'. If P^* is a third submarkovian process and if $P^* \subset P'$, then the formula*

$$C^*_{s,t} = \lim_{u\downarrow 0}\downarrow C'_{s,u}P^*_{t-u}, \qquad s, t > 0, \tag{3.3.17}$$

defines a form C relative to P and P^*. If $P' \subset P^*$ (instead of the converse relation), then the formula

$$C^*_{s,t} = \lim_{u\downarrow 0}\uparrow C'_{s,u}P^*_{t-u}, \qquad s, t > 0, \tag{3.3.18}$$

defines, if this limit is everywhere finite, a form C relative to P and P^*. Moreover, one has in the first case

$$\lim_{u\downarrow 0}\uparrow C^*_{s,u}P'_{t-u} \leqq C'_{s,t}, \qquad s, t > 0, \tag{3.3.19}$$

whereas in the second case equality holds, that is,

$$\lim_{u\downarrow 0}\downarrow C^*_{s,u}P'_{t-u} = C'_{s,t}, \qquad s, t > 0. \tag{3.3.20}$$

PROOF. This theorem is obtained immediately from theorem 3.3.2, by fixing $s > 0$ and $i \in A$ in the forms $C'_{s,t}(i,\cdot)$ and $C^*_{s,t}(i,\cdot)$, since we obtain in this way elements of $F(P')$ and of $F(P^*)$.

This theorem allows us to state the following complement to theorem 3.1.1.

THEOREM 3.3.4. *Let the hypotheses and notation be those of theorem 3.1.1. Then the common values of the limit matrices on $A \times A$*

$$Q_{s,t} = \lim_{u\downarrow 0}\downarrow C_{s,u}P_{t-u} = \lim_{u\downarrow 0}\downarrow P_{s-u}D_{u,t}, \qquad s, t > 0, \tag{3.3.21}$$

define a form $Q = \{Q_{s,t}; s, t > 0\}$ relative to P and P. Moreover, the following relations hold on $A \times A$

$$\begin{aligned}&\lim_{u\downarrow 0} P_s\left[\frac{1}{u}(P'_u - P_u)\right]P_t = Q_{s,t}, && s, t \geqq 0;\\ &Q_{s,0} = C_{s,0}, \qquad Q_{0,s} = D_{0,s}, && s \geqq 0,\end{aligned} \tag{3.3.22}$$

and

$$C_{s,0}(j, \cdot) = 0, \ D_{0,s}(\cdot, j) = 0 \tag{3.3.23}$$

on A *when* $j \in A' - A$.

PROOF. From theorem 3.3.3, we deduce that $Q_{s,t} = \lim_{u \downarrow 0} \downarrow C_{s,u}P_{t-u}$ defines, when $s, t \in (0, \infty)$, a form relative to P and P'. From theorem 3.1.1, we deduce that for $u, t > 0$,

$$\begin{aligned}(P'_u - P_u)P_t &= \lim_{v \downarrow 0} \downarrow (P'_u - P_u)P'_vP_{t-v} \\ &= \lim_{v \downarrow 0} \downarrow \left(\int_0^u dw\, C_{u-w,v+w}\right) P_{t-v} \\ &\geqq \int_0^u dw\, Q_{u-w,t+w},\end{aligned} \tag{3.3.24}$$

so that we obtain for $s, u > 0$ and $t > v > 0$,

$$\begin{aligned}\left(\frac{1}{u}\int_0^u dw\, C_{s+u-w,v+w}\right) P_{t-v} &\geqq P_s \frac{1}{u}(P'_u - P_u)P_t \\ &\geqq \frac{1}{u}\int_0^u dw\, Q_{s+u-w,t+w}.\end{aligned} \tag{3.3.25}$$

Finally, by letting $u \downarrow 0$ and then $v \downarrow 0$, we obtain

$$Q_{s,t} = \lim_{u \downarrow 0} P_s \frac{1}{u}(P'_u - P_u)P_t, \qquad s \geqq 0, t > 0. \tag{3.3.26}$$

An analogous argument shows that $Q'_{s,t} = \lim_{u \downarrow 0} \downarrow P_{s-u}C_{u,t}$ defines, for $s, t \in (0, \infty)$, a form relative to P and P', and that

$$Q'_{s,t} = \lim_{u \downarrow 0} P_s \frac{1}{u}(P'_u - P_u)P_t, \qquad s \geqq 0, t > 0. \tag{3.3.27}$$

This implies that $Q = Q'$ and that $Q_{s,0} = C_{s,0}$, $Q_{0,s} = D_{0,s}$ for $s > 0$. In particular, $C_{s,0}(\cdot, j) = 0$ when $j \in A' - A$ and similarly $D_{s,0}(j, \cdot) = 0$ when $j \in A' - A$. From theorem 3.1.1 we conclude finally that on $A \times A$

$$C_{0,0} = D_{0,0} = Q_{0,0} = \lim_{u \downarrow 0} \frac{1}{u}(P'_u - P_u). \tag{3.3.28}$$

3.4. *Topology on the set of states.* In this section we shall use the results of theorems 3.3.1 and 3.3.2, to prove

THEOREM 3.4.1. *Given a submarkovian process* P *defined on* A, *there exists a topology on* A *and for every closed set* B *of* A *a submarkovian process* P^B *on* B^c, *with the following probabilistic interpretation: for every separable submarkovian function* $\{X_t, t \geqq 0\}$ *with transition probabilities* P, *we have for* $t > 0$ *and for every* $B \subset A$

$$P_t^{\bar{B}}(i, j) = P\{X_t = j; X_s \in B^c \text{ for } s < t | X_0 = i\}, \tag{3.4.1}$$

where $\bar{B}$ *is the closure of the subset* B *of* A. *More generally,*

$$(3.4.2) \qquad 0 \leqq P_t^B(i,j) - \sum_{m=1}^{q} P_t^{\overline{B \cup B_m}}(i,j) + \cdots + (-1)^q P_t^{\overline{B \cup B_1 \cup \cdots \cup B_q}}(i,j)$$
$$= P\{\epsilon_1\epsilon_2\epsilon_3 | X = i\},$$

where, for brevity, we denote

ϵ_1: $X_t = j$,

ϵ_2: $X_s \in B^c$, $s < t$,

ϵ_3: $X_{s_m} \in B_m - B$ *for at least one* $s_m < t$, $\qquad m = 1, \cdots, q$,

with B and B_m, for $m = 1, \cdots, q$, arbitrary subsets of A.

PROOF. Let K_n, where $n > 1$, be the set of strictly positive real numbers t such that $2^n t$ is an integer and let K be the union of the K_n in R_+; each of the sets K_n is a semigroup under addition and so is K; moreover, K is dense in R_+.

For any subset B of A and any $n \geqq 1$, we introduce by recurrence on $s \in K_n$ a family of positive matrices on $A \times A$

$$(3.4.3) \qquad \Pi_{2^{-n}}^{B,n} = P_{2^{-n}}; \qquad \Pi_{s+2^{-n}}^{B,n} = \sum_{k \notin B} \Pi_s^{B,n}(\cdot, k) P_{2^{-n}}(k, \cdot).$$

The following properties of these matrices are evident

(a) $0 \leqq \Pi_s^{B,n} \leqq P_s$, $\qquad s \in K_n$;

(b) $\Pi_{s+t}^{B,n} = \sum_{k \notin B} \Pi_s^{B,n}(\cdot, k)\, \Pi_t^{B,n}(k, \cdot)$, $\qquad s, t \in K_n$.

They imply, with $s + t = u$ and $s = 2^{-n}$ in (b), that

$$(3.4.4) \qquad \sum_{j \notin A} \Pi_u^{B,n}(i,j) \leqq 1 - P_{2^{-n}}(i,i), \qquad i \in B; 2^{-n} < u \in K_n.$$

Moreover the expression

$$(3.4.5) \qquad \Pi_t^{B,n}(i,j) - \sum_{m=1}^{q} \Pi_t^{B \cup B_m,n}(i,j) + \cdots + (-1)^q\, \Pi_t^{B \cup B_1 \cup \cdots \cup B_q}(i,j)$$

has a probabilistic interpretation similar to the one given by the last formula of the theorem provided that the parameter values are restricted to K_n; as a consequence this expression is always nonnegative.

The elementary inequality on $A \times A$

$$(3.4.6) \qquad \Pi_{2^{-n}}^{B,n+1}(i,j) = \sum_{k \in B} P_{2^{-(n+1)}}(i,k) P_{2^{-(n+1)}}(k,j)$$
$$\leqq P_{2^{-n}}(i,j) = \Pi_{2^{-n}}^{B,n}(i,j)$$

easily implies that the matrices $\Pi_t^{B,n}$ decrease as n increases; we are thus allowed to introduce the limit matrices

$$(3.4.7) \qquad \Pi_s^B(i,j) = \lim_{n \uparrow \infty} \downarrow \Pi_s^{B,n}(i,j)$$

on A, for $s \in K$, so that

(a) $0 \leqq \Pi_s^B \leqq P_s$ on A $\qquad$ for $s \in K$;

(b) $\Pi_{s+t}^B = \sum_{k \notin B} \Pi_s^B(\cdot, k)\, \Pi_t^B(k, \cdot)$ $\qquad$ for $s, t \in K$.

Moreover $\Pi_u^B(i, \cdot) = 0$ when $i \in B$ and $u \in K$, so that $\{\Pi_s^B, s \in K\}$ is a semigroup of submarkovian matrices on A dominated by the process P.

By the generalized version of theorem 3.3.1, there corresponds to B a subset $\overline{B}$ of A such that $\{\Pi_s^B, s \in K\}$ may be extended in a unique way to a submarkovian process $\{\Pi_s^B, s > 0\}$ on $(\overline{B})^c$; in the case where $\Pi_s^B \equiv 0$, we put $\overline{B} = A$. From $\Pi_s^B(i, \cdot) = 0$ for $i \in B$, it follows that $B \subset \overline{B}$.

We now show that $\Pi_s^B = \Pi_s^{\overline{B}}$ for $s \in K$ so that the Π^B process depends on B only through $\overline{B}$ and so that $\overline{\overline{B}} = \overline{B}$. We first remark that $B \subset \overline{B}$ implies by construction of the Π processes that $\Pi_s^{\overline{B}} \leqq \Pi_s^B$. Conversely, since $\Pi_s^B = 0$ outside of $(\overline{B})^c \times (\overline{B})^c$ and since Π_s^B is a semigroup dominated by P, it follows easily by recurrence on $s \in K_n$ that $\Pi_s^B \leqq \Pi_s^{\overline{B},n}$, so that, letting $n \uparrow \infty$, we have $\Pi_s^B \leqq \Pi_s^{\overline{B}}$ for $s \in K$.

We let P^B designate, for any subset B of A such that $B = \overline{B}$ the submarkovian process equal to $\Pi^{B'}$ for any B' such that $\overline{B'} = B$. Then the last inequality of theorem 3.4.1 follows from the similar inequality for the $\Pi^{B,n}$ by letting $n \to \infty$. We use this inequality to show that the subsets B such that $B = \overline{B}$ are the closed sets of a topology on A; since it has already been shown that $B \subset \overline{B} = \overline{\overline{B}}$ for any B, and that $\overline{\phi} = \phi$, $\overline{A} = A$, it is sufficient to show that $\overline{B_1 \cup B_2} = \overline{B_1} \cup \overline{B_2}$ or even that $\overline{B_1 \cup B_2} \subset \overline{B_1} \cup \overline{B_2}$ for any two subsets B_1, B_2 of A. But by the aforementioned inequality (with $B = \phi$) we have, for every $i \in A$,

$$P_t(i, i) + P_t^{\overline{B_1 \cup B_2}}(i, i) \geqq P_t^{\overline{B}_1}(i, i) + P_t^{\overline{B}_2}(i, i). \tag{3.4.8}$$

As $t \downarrow 0$, this shows that $\overline{B_1 \cup B_2} \subset \overline{B}_1 \cup \overline{B}_2$.

The following theorem gives a characterization of the process $\{P_t^B, t > 0\}$ as a maximal element for the dominance relation.

THEOREM 3.4.2. *Any semigroup $\{Q_t, t > 0\}$ (defined perhaps only for t in a dense semigroup of R_+) of submarkovian matrices on A which is dominated by a submarkovian process P on A and which is such that either $Q_t(i, \cdot) = 0$ on A for $i \in B$, $t > 0$ or such that $Q_t(\cdot, i) = 0$ on A for $i \in B$, $t > 0$ is a submarkovian process on a subset of $(\overline{B})^c$ which is dominated by $P^{\overline{B}}$.*

PROOF. By theorem 3.3.1, or its generalized version, Q is necessarily a submarkovian process on a subset of A which is dominated by P. The second hypothesis on Q then implies by induction on $s \in K_n$ that $Q_s \leqq \Pi_s^{B,n}$, where $s \in K_n$, so that, letting $n \uparrow \infty$, we have $Q_s \leqq \Pi_s^B$ $(s \in K)$. By continuity, this implies that $Q \subset P^{\overline{B}}$.

COROLLARY. *On the subset $A^\infty = \{i: q_i < \infty\}$ of A (compare section 3.2) the induced topology is the discrete topology of A^∞. For any subset B of A such that $(\overline{B})^c$ is a finite subset (a fortiori if B^c is finite), one has $\overline{B} = B \cup (A^\infty)^c$.*

PROOF. The first assertion is equivalent to $q_k < \infty$ and $k \notin B \Rightarrow k \notin \overline{B}$; this follows from theorem 3.4.2 by taking $Q_t = \exp(-q_k t) I_{\{k\}}$. This already implies for any B, that $\overline{B} \subset B \cup (A^\infty)^c$. To show the converse inclusion when $(\overline{B})^c$ is finite, we remark that $P^{\overline{B}}$ is then of the form $\exp(tQ)$ on $(\overline{B})^c$ so that

(3.4.9) $$P_t(i,i) \geqq P_t^{\bar{B}}(i,k) \geqq e^{tQ(i,i)}, \qquad i \notin B;\ t > 0,$$

so that $(\bar{B})^c \subset A^\infty$.

For any closed set $B = \bar{B}$ of the topology on A, let C^B and D^B denote the forms relative to the process P^B and P, or P and P^B, introduced by theorem 3.1.1. We proceed to evaluate these forms in terms of the matrices $\Pi^{B,n}$. Let us first introduce the following families of positive measures on R_+

(3.4.10) $$\begin{aligned} \rho^{B,n}(\cdot\,; i, j) &= \sum_{s \in K_n} \epsilon_s(\cdot)\, \Pi_s^{B,n}(i,j), \\ \sigma^{B,n}(\cdot\,; j, i) &= \sum_{s \in K_n} \epsilon_s(\cdot)\, \Pi_s^{B,n}(j,i) \end{aligned}$$

for any $B \subset A$, $n > 0$ and $i \in A$, $j \in B$. It is easily checked, and clear from the probabilistic interpretations (in terms of first and last passage time through B) of these measures, that we have on R_+

(3.4.11) $$\begin{aligned} \rho^{B,n}(u + \cdot\,; i, j) &= \sum_{k \notin B} \Pi_u^{B,n}(i,k)\rho^{B,n}(\cdot\,; k, j), \\ \sigma^{B,n}(u + \cdot\,; j, i) &= \sum_{k \notin B} \sigma^{B,n}(\cdot\,; j, k)\, \Pi_u^{B,n}(k,i) \end{aligned}$$

when $B \subset A$, $n > 0$, $u \in K_n$, and $i \in A$, $j \in B$, and that we have also

(3.4.12) $$\begin{aligned} P_v(i,j) - \Pi_v^{B,n}(i,j) &= \int_{s<v} \sum_{k \in B} \rho^{B,n}(ds; i, k)P_{v-s}(k,j) \\ &= \int_{s<v} \sum_{k \in B} P_{v-s}(i,k)\sigma^{B,n}(ds; k, j) \end{aligned}$$

when $B \subset A$, $n > 0$, $v \in K_n$, and $i, j \in A$. By passing to the limit as $n \uparrow \infty$, we obtain the following result.

LEMMA 3.4.1. *For every $B \subset A$, $j \in A$, and $t > 0$ the following vague convergences of positive measures on $(0, t)$ hold as $n \uparrow \infty$,*

(3.4.13) $$\sum_{k \in B} \rho^{B,n}(ds; i, k)P_{t-s}(k,j) \to \begin{cases} \epsilon_0(ds)P_t(i,j), & i \in \bar{B}; \\ ds\, C_{s,t-s}^{\bar{B}}(i,j), & i \in \bar{B}; \end{cases}$$

(3.4.14) $$\sum_{k \in B} P_{t-s}(j,k)\sigma^{B,n}(ds; k; i) \to \begin{cases} \epsilon_0(ds)P_t(j,i), & i \in \bar{B}; \\ ds\, D_{s,t-s}^{\bar{B}}(j,i), & i \in \bar{B}. \end{cases}$$

PROOF. These two formulas are proved in exactly the same way; we limit ourselves to proving the first one. Let us first remark that formula (3.4.12) implies that for $B \subset A$, $n > 0$, $v \in K_n$, $w > 0$ and $i, j \in A$, we have

(3.4.15) $$\int_{0<s<v} \sum_{k \in B} \rho^{B,n}(ds; i, k)P_{v-s+w}(k,j) = \sum [P(i,l) - \Pi_v^{B,n}(i,l)]P_w(l,j)$$

and that the second member increases to $[(P - P_v^{\bar{B}})P_w](i,j)$ as $n \uparrow \infty$. In particular, if $i \in B$, we have proved that for $v \in K$, $v < t$ in R_+, and $j \in A$, we have as $n \uparrow \infty$,

$$(3.4.16)\qquad \int_{0<s<v} \sum_{k\in B} \rho^{B,n}(ds; i, k)P_{t-s}(k, j) \to P_t(i, j).$$

This relation is equivalent to the first case of the formula of the lemma.

Let us now combine the above equality with formula (3.4.11) in order to obtain for any $B \subset A$, $n > 0$, $u \in K_n$, $v \in K_n$, $w > 0$, and $i, j \in A$

$$(3.4.17)\qquad \int_{0<s<v} \sum_{k\in B} \rho^{B,n}(u + ds; i, k)P_{v-s+w}(k, j) = \sum_{k\in B} \Pi_n^{B,n}(i, k)[(P_v - \Pi^{B,n})P_w](k, j).$$

The last member is bounded above by $P_{u+v+w}(i, j)$ and therefore converges to $[P_u^B(P_v - P^B)P_w](i, j)$ as $n \uparrow \infty$. Using the definition of C^B we have thus shown that when $i \notin B$

$$(3.4.18)\qquad \int_{u<s<u+v} \sum_{k\in B} \rho^{B,n}(ds; i, k)P_{t-s}(k, j) \to \int_{u<s<u+v} ds\, C_{s,t-s}^B(i, j)$$

for any $u \in K$, $v \in K$, $t > u + v$, and $j \in A$, as $n \uparrow \infty$. This relation is equivalent to the second case of the formula of the lemma.

In the particular case where B is a finite subset of A, the lemma 3.4.1 leads to the following result.

THEOREM 3.4.3. *Any finite subset B of A is closed. Moreover, for any finite subset B there exists a family $\{\Gamma_u^B; u \geqq 0\}$ of positive matrices on $B^c \times B$ and a family $\{\Delta_u^B; u \geqq 0\}$ of positive matrices on $B \times B^c$ with the following properties:*

(a) *the u-functions $\Gamma_u^B(i, j)$ and $\Delta_u^B(j, i)$ are continuous on $[0, \infty)$ and*

$$(3.4.19)\qquad \begin{aligned} \Gamma_{u+v}^B &= P_u^B\Gamma_v^B; & \Delta_{u+v}^B &= \Delta_u^B P_v^B, & u, v > 0;\\ \Gamma_u^B &\geqq P_u^B\Gamma_0^B; & \Delta_u^B &\geqq \Delta_0^B P_u^B, & u > 0\end{aligned}$$

on $B^c \times B$ and $B \times B^c$ respectively;

(b) *the following formulas hold on $B^c \times A$ and $A \times B^c$ respectively:*

$$(3.4.20)\qquad \begin{aligned} P_u &= P_u^B + \sum_{k\in B} \int_0^u dv\, \Gamma_v^B(\cdot, k)P_{u-v}(k, \cdot);\\ P_u &= P_u^B + \sum_{k\in B} \int_0^u dv\, P_{u-v}(\cdot, k)\, \Delta_v^B(k, \cdot), \qquad t > 0.\end{aligned}$$

PROOF. The following reasoning for the measures $\rho^{B,n}$ can be carried through, mutatis mutandis, for the measures $\sigma^{B,n}$; to avoid repetition we leave this to the reader.

Lemma 3.4.1 first implies that $\limsup_{n\to\infty} \int_0^t \rho^{B,n}(ds; i, j) < \infty$ for every $i \in A$, $j \in B$, and $t > 0$. This means that the sequence $\rho^{B,n}(\cdot; i, j)$ of positive measures on $[0, \infty)$ has at least one limit in the vague topology for every $B \subset A$, $i \in A$, $j \in B$ as $n \to \infty$.

Let us now suppose that B is a fixed finite subset of A and that $\rho(\cdot; i, j)$ is a vague limit of $\rho^{B,n}(\cdot; i, j)$ as $n \to \infty$, with $i \in A$, $j \in B$. Lemma 3.4.1 implies that on $[0, t]$ for $j \in A$,

$$(3.4.21) \qquad \sum_{k \in B} \rho(ds; i, k) P_{t-s}(k, j) = \begin{cases} \epsilon_0(ds P_t(i, j), & i \notin \overline{B}, \\ ds\, C^{\overline{B}}_{s,t-s}(i, j), & i \in \overline{B}. \end{cases}$$

When $i \in \overline{B}$, this relation implies first that the measures $\rho(\cdot\,; i, k)$ are concentrated at 0, and thus (3.4.21) reduces to

$$(3.4.22) \qquad \sum_{k \in B} \rho(\{0\}; i, k) P_t(k, j) = P_t(i, j), \qquad i \in \overline{B}, j \in A; t > 0.$$

This implies, by letting $t \downarrow 0$, that $B = \overline{B}$ and that $\rho(\cdot\,; i, k) = \epsilon_0(\cdot) I_B(i, k)$ when $i, k \in B$.

When $i \notin \overline{B}$, the above relation and the continuity properties of C^B imply that

$$(3.4.23) \qquad ds\, C^B_{s,0}(i, j) = \begin{cases} \rho(ds; i, j), & j \in B, \\ 0, & j \notin B, \end{cases}$$

and that $C^B_{s,t} = C^B_{s,0} P_t$. Letting Γ^B_s be the restriction of $C^B_{s,0}$ to $B^c \times B$, we obtain the formulas of theorem 3.4.3, relative to the matrices Γ as corollaries of theorem 2.3.1 and theorem 3.1.1.

Of the following two corollaries, the first is an easy consequence of the integral formulas of the theorem, whereas the second has already been proved.

COROLLARY 3.4.1. *As $u \downarrow 0$, the following limits hold:*

$$(3.4.24) \qquad \frac{2}{u^2}(P_u - P^B_u) \to \sum_{k \in B} \Gamma^B_0(\cdot, k)\, \Delta^B_0(k, \cdot) \quad on \quad B^c \times B^c,$$

$$(3.4.25) \qquad \frac{1}{u} P_u \to \Gamma^B_0 \quad on \quad B^c \times B,$$

$$(3.4.26) \qquad \frac{1}{u} P_u \to \Delta^B_0 \quad on \quad B \times B^c.$$

COROLLARY 3.4.2. *For any finite subset B of A, the following limits hold in the vague topology on measures on $[0, \infty)$, when $n \uparrow \infty$, for every $j \in A$,*

$$(3.4.27) \qquad \rho^{B,n}(ds; i, j) \to \begin{cases} \epsilon_0(ds) I_B(i, j), & i \in B, \\ ds\, \Gamma^B_s(i, j), & i \notin B; \end{cases}$$

$$(3.4.28) \qquad \sigma^{B,n}(ds; j, i) \to \begin{cases} \epsilon_0(ds) I_B(j, i), & i \in B, \\ ds\, \Delta^B_s(j, i), & i \notin B. \end{cases}$$

In another paper [2], we proved the following results among others as consequences of the preceding proposition:

(a) the positive matrices on $B \times B$ defined by

$$(3.4.29) \qquad A^B_u = \sum_{k \notin B} D^B_v(\cdot, k) C^B_{u-v}(k, \cdot),$$

for $u > v > 0$, do not depend on v and satisfy the inequality

$$(3.4.30) \qquad \int_0^\infty du\, A^B_u(i, j) < \infty$$

when $i \neq j$ in B;

(b) there exists a matrix Q^B on B such that

$$(3.4.31)\qquad \begin{aligned} Q^B(i,j) &\geqq 0, && i \neq j,\\ \sum_j Q^B(\cdot,j) &\leqq 0 && \text{on } B, \end{aligned}$$

with the properties

$$(3.4.32)\qquad \lim_{t\to 0}\frac{1}{t}P_t(i,j) = Q^B(i,j) - \int_0^\infty du\, A_u^B(i,j), \qquad i \neq j,$$

$$(3.4.33)\qquad \lim_{t\to 0}\frac{1-P_t(i,i)}{\int_0^t ds\, sA_s(i,i) + t\int_t^\infty ds\, A_s(i,i) - tQ^B(i,i)} = 1, \qquad i \in B.$$

Moreover a simple necessary and sufficient condition was found on the matrices C_u^B, D_u^B, where $u > 0$, and Q^B, given the process P^B on B^c, for a process P to exist with all the properties stated in theorem 3.5.1 and here above. This process is proved to be unique and in fact its restriction on B is given by

$$(3.4.34)\qquad \int_0^\infty dt\, e^{-xt}P_t(i,j) = \exp\left[\int_0^\infty dt\,(1-e^{-xt})A_t^B + xI_B - Q^B\right], \qquad i,j \in B.$$

3.5. *Transitivity of the dominance relation.* The purpose of this section is to study the consequences of the transitivity of the dominance relation. We designate by P^1, P^2, P^3 three submarkovian processes such that $P^1 \subset P^2$ and that $P^2 \subset P^3$, so that obviously we also have $P^1 \subset P^3$. For any couple $a < b$ in $\{1, 2, 3\}$ we designate by F^{ab} the convex subcone thick in $F(P^a)$ and by F^{ba} the positive band of $F(P^b)$ which were defined in theorem 3.3.2 relative to the processes $P^a \subset P^b$. It was shown there that F^{ab} and F^{ba} are isomorphic by the mapping u^{ab} and u^{ba} defined by

$$(3.5.1)\qquad \begin{aligned} [u^{ab}(f^a)]_t &= \lim_{s\downarrow 0}\uparrow f_s^a P_{t-s}^b, && t > 0; f^a \in F(P^a),\\ [u^{ba}(f^b)]_t &= \lim_{s\downarrow 0}\downarrow f_s^b P_{t-s}^a, && t > 0; f^b \in F(P^b). \end{aligned}$$

The best transitivity result that could be expected is valid as is shown in the following theorem.

THEOREM 3.5.1. *With the preceding notation we have*

$$(3.5.2)\qquad u^{21}(F^{21} \cap F^{23}) = F^{13}, \qquad u^{23}(F^{21} \cap F^{23}) = F^{31}$$

and

$$(3.5.3)\qquad u^{13} = u^{23}u^{12} \quad on \quad F^{13}, \qquad u^{31} = u^{21}u^{32} \quad on \quad F^{31}.$$

PROOF. Let f^2 be an element of $F^{21} \cap F^{23}$ with images $f^1 = u^{21}(f^2)$ and $f^3 = u^{23}(f^3)$ in F^{12} and F^{32} respectively. Then the inequality $f_t^1 \leqq f_t^2 \leqq f_t^3$, which holds for $t > 0$, implies that $f_t^1 \leqq [u^{31}(f^3)]_t \leqq [u^{32}(f^3)]_t = f_t^2$. This last inequality and the fact that $u^{31}(f^3) \in F(P^1)$ imply

$$(3.5.4)\qquad f_t^1 \leqq [u^{31}(f^3)]_t \leqq [u^{21}(f^2)]_t = f_t^1,$$

that is, $f^1 = u^{31}(f^3)$. A similar argument shows that $f^3 = u^{13}(f^1)$.

Conversely, if f^1 and f^3 are elements in F^{13} and F^{31}, respectively, such that $u^{13}(f^1) = f^3$ and $u^{31}(f^3) = f^1$, let us define f^2 as $u^{12}(f^1)$ and f'^2 as $u^{32}(f^3)$ in $F(P^2)$. Clearly, we have $f_t^1 \leqq f_t^2 \leqq f_t'^2 \leqq f_t^3$ for $t > 0$, so that $u^{13}(f^1) \leqq u^{23}(f^2) \leqq u^{23}(f'^2) \leqq f^3$ hold in $F(P^3)$. However, by hypothesis this triple inequality must be a triple equality and by applying u^{32} to the last three terms, one obtains $f^2 = f'^2 = u^{32}(f^3)$. Since we also have by construction $f^2 = u^{12}(f^1)$, we have shown that $f^2 \in F^{23} \cap F^{21}$. The proof of the theorem is concluded.

A theorem similar to theorem 3.5.1 is of course valid for the cones $G(P^a)$, with $a = 1, 2, 3$.

Let us denote by $C^{ab}(D^{ba})$, where $a < b$ in $\{1, 2, 3\}$, the bounded forms defined in theorem 3.1.1 relative to the processes P^a and P^b (P^b and P^a). These forms satisfy the following relations.

THEOREM 3.5.2. *Each of the forms C^{13} and D^{31} is the sum of two similar forms*

$$
\begin{aligned}
C_{s,t}^{13} &= \lim_{u \downarrow 0} \downarrow P_{s-u}^1 C_{u,t}^{23} + \lim_{u \downarrow 0} \uparrow C_{s,u}^{12} P_{t-u}^3, \\
D_{s,t}^{31} &= \lim_{u \downarrow 0} \downarrow D_{s,u}^{32} P_{t-u}^1 + \lim_{u \downarrow 0} \uparrow P_{s-u}^3 D_{u,t}^{21}, \qquad s, t > 0.
\end{aligned} \tag{3.5.5}
$$

PROOF. This theorem is easily deduced from theorem 3.3.3 by the following argument based on monotone convergence. Since the two given formulas are proved in exactly the same way, we content ourselves with proving the first of them.

By theorem 3.3.3 we are allowed to introduce the following two forms relative to P^1 and P^3

$$
C'_{s,t} = \lim_{u \downarrow 0} \downarrow P_{s-u}^1 C_{u,t}^{23}, \qquad C''_{s,t} = \lim_{u \downarrow 0} \uparrow C_{s,u}^{12} P_{t-u}^3. \tag{3.5.6}
$$

To show that the second of these two forms exists we remark that $C_{s,t}^{12} \leqq C_{s,t}^{13}$. Then from the following identity, which is valid when $s, t, u > 0$,

$$
\begin{aligned}
&P_s^1(P_u^3 - P_u^1)P_t^3 \\
&\quad = P_s^1(P_u^2 - P_u^1) (\lim_{w \downarrow 0} \uparrow P_w^2 P_{t-w}^3) + (\lim_{w \downarrow 0} \downarrow P_{s-w}^1 P_w^2) (P_u^3 - P_u^2) P_t^3
\end{aligned} \tag{3.5.7}
$$

or equivalently, from

$$
\begin{aligned}
&\int_0^u dv\, C_{s+v,t+u-v}^{13} \\
&\quad = \lim_{u \downarrow 0} \uparrow \left(\int_0^u dv\, C_{s+v,w+u-v}^{12} \right) P_{t-w}^3 + \lim_{w \downarrow 0} \downarrow P_{s-w}^1 \left(\int_0^u dv\, C_{w+v,t+u-v}^{23} \right),
\end{aligned} \tag{3.5.8}
$$

we deduce that

$$
\int_0^u dv\, C_{s+v,t+u-v}^{13} = \int_0^u dv\, C''_{s+v,t+u-v} + \int_0^u dv\, C'_{s+v,t+u-v} \tag{3.5.9}
$$

and by continuity, letting $u \downarrow 0$, that $C_{s,t}^{13} = C''_{s,t} + C'_{s,t}$, as was to be shown.

4. Construction of processes

4.1. *Perturbation forms and Feller's construction.*

DEFINITION 4.1.1. *A perturbation from* Π *relative to a submarkovian process* P

defined on the denumerable set A is a family $\{\Pi_{s,t}, s, t > 0\}$ of positive matrices on $A \times A$ such that

(a) $P_u \Pi_{s,t} = \Pi_{u+s,t}, \qquad \Pi_{s,t} P_u = \Pi_{s,u+t}, \qquad s, t, u > 0;$
(b) $\Pi_{s,t} e \leqq p_s, \qquad s, t > 0.$

Note that p is defined in section 2.2. We remark that a Markovian process admits no perturbation form other than 0.

Theorem 2.3.1 implies that the (s, t) functions $\Pi_{s,t}(i, j)$ may be extended to $\{s, t \geqq 0\}$ as continuous functions such that

$$\begin{aligned} &P_u \Pi_{s,0} = \Pi_{u+s,0}, \qquad \Pi_{0,s} P_u = \Pi_{0,s+u}, \qquad s, u > 0, \\ &P_u \Pi_{0,s} \leqq \Pi_{u,s}, \qquad \Pi_{s,0} P_u \leqq \Pi_{s,u}, \qquad s, u \geqq 0. \end{aligned} \tag{4.1.1}$$

Since the mapping of $(0, \infty)$ in G defined by $t \to \{\Pi_{s,t} e, s > 0\}$ is readily seen to be decreasing, we are allowed to introduce the element $\{\pi_s, s > 0\}$ of G by

$$\pi_s = \lim_{t \downarrow 0} \uparrow \Pi_{s,t}\, e \leqq p_s, \qquad s > 0. \tag{4.1.2}$$

In the very special case where the process P is of the form $P_t(i, j) = \exp(-q_i t) I(i, j)$ where $t > 0$; $i, j \in A$, for a finite nonnegative function q on A, all perturbation forms Π relative to P are given by

$$\Pi_{s,t}(i, j) = e^{-q_i s} Q(i, j) e^{-q_j t}, \qquad s, t \geqq 0; i, j \in A, \tag{4.1.3}$$

where Q is an arbitrary positive matrix on A such that $\sum_j Q(i, j) \leqq q_i$. In this particular case, a well-known construction due to Feller associates to $\{P_t, t > 0\}$ and to Q a "jump process" on A. Our first aim in this section is to extend this construction to general P and Π (such an extension has also been considered by Moyal [18]).

To every integer $n \geqq 1$, we associate a family of positive matrices on $A \times A$ depending on $n + 1$ positive real indices through the formula

$$\Pi^{(n)}_{s_0, s_1, \cdots, s_n} = \Pi_{s_0, s_1 - u_1} \Pi_{u_1, s_2 - u_2} \cdots \Pi_{u_{n-1}, s_n}, \tag{4.1.4}$$

where $s_0 > 0$, $s_1 > u_1 > 0$, $\cdots$, $s_{n-1} > u_{n-1} > 0$, $s_n > 0$. It is easily seen that the right member does not depend on the choice of the u_i. We then form the positive matrices

$$P^{(n)}_s = \int_{\Sigma t_i \leqq s} dt_1 \cdots dt_n\, \Pi^{(n)}_{s - \Sigma t_i, t_1, \cdots, t_n}, \qquad n \geqq 1; s > 0, \tag{4.1.5}$$

and put $P^{(0)}_s = P_s$ with $s > 0$. These matrices satisfy the following relations, which also permit us to define them by recurrence on n,

$$\begin{aligned} P^{(n+1)}_s &= \lim_{\epsilon \downarrow 0} \uparrow \int_0^{s-\epsilon} du\, \Pi_{u,\epsilon} P^{(n)}_{s-u-\epsilon}, \\ P^{(n+1)}_s &= \lim_{\epsilon \downarrow 0} \uparrow \int_\epsilon^{s} du\, P^{(n)}_{u-\epsilon} \Pi_{\epsilon, s-u}, \qquad n \geqq 0, s > 0. \end{aligned} \tag{4.1.6}$$

To show the validity of the second formula, for example, it is enough to remark that

$$(4.1.7)\qquad \int_0^s du\, P_{u-\epsilon}^{(n)}\, \Pi_{\epsilon,s-u} = \int_{\substack{\Sigma t_i \leqq s\\ t_n \geqq \epsilon}} dt_1 \cdots dt_n\, dt_{n+1}\, \Pi^{(n+1)}_{s-\Sigma t_i, t_1,\cdots,t_n,t_{n+1}},$$

and that the second member of this identity depends on ϵ only through the domain of integration.

We then show by induction on N that the matrices $\sum_0^N P_s^{(n)}$ are submarkovian for $s > 0$. This is certainly true for $N = 0$. If it is true for N and for all s, it is also true for $N + 1$ by the following argument

$$(4.1.8)\qquad \begin{aligned}\sum_0^{N+1} P_s^{(n)} e &= P_s e + \lim_{\epsilon \downarrow 0} \int_0^{s-\epsilon} du\, \Pi_{u,\epsilon} \left(\sum_0^N P_{s-u-\epsilon}^{(n)}\right) e\\ &\leqq P_s e + \int_0^s du\, \pi_u\\ &= e - \int_0^s du\, (p_u - \pi_u) \leqq e, \qquad s > 0.\end{aligned}$$

The dependence of the matrices $P_s^{(n)}$ on s is exhibited by

$$(4.1.9)\qquad P_{s+t}^{(n)} = \sum_{m=0}^n P_s^{(m)} P_t^{(n-m)}, \qquad s, t > 0;\ n \geqq 0.$$

To show this identity, we remark that

$$(4.1.10)\qquad P_s^{(m)} P_t^{(n-m)} = \int_{\substack{\Sigma s_i \leqq s\\ \Sigma t_i < t}} ds_1 \cdots ds_m\, dt_1 \cdots dt_{n-m}\, \Pi^{(n)}_{s_1,\cdots,s_m,s+t-\Sigma s_i - \Sigma t_j, t_1,\cdots,t_{n-m}},$$

so that the indicated formula results from the fact that in the cone of positive $(n + 1)$-dimensional vectors $x = \{x_0, \cdots, x_n\}$, we have

$$(4.1.11)\qquad \left\{x : \sum_0^n x_i = s + t\right\} = \sum_{m=0}^n \left\{x : \sum_0^{m=1} x_i \leqq s,\ \sum_{m+1}^n x_i \leqq s,\ \sum_0^n x_i = s + t\right\}.$$

THEOREM 4.1.1. *The formula*

$$(4.1.12)\qquad \tilde{P}_s = \sum_{n=0}^\infty P_s^{(n)}, \qquad s > 0,$$

defines a submarkovian process $\tilde{P}$ *on* $\tilde{A} = A$ *which dominates the process* P. *The forms* C *and* D *associated to these two processes by theorem* 3.2.1 *are such that*

$$(4.1.13)\qquad \begin{aligned}&\lim_{u\downarrow 0} \uparrow \Pi_{s,u} \tilde{P}_{t-u} = C_{s,t}, &&\lim_{u\downarrow 0} \uparrow \tilde{P}_{s-u} \Pi_{u,t} = D_{s,t}\\ &\lim_{u\downarrow 0} \downarrow C_{s,u} P_{t-u} = \Pi_{s,t}, &&\lim_{u\downarrow 0} \downarrow P_{s-u} D_{u,t} = \Pi_{s,t}\end{aligned}$$

for $s, t > 0$.

PROOF. From the construction above, it is clear that $\{\tilde{P}_s, s > 0\}$ is a semigroup of submarkovian matrices, since

(4.1.14) $$\tilde{P}_s\tilde{P}_t = \sum_{m,n=0}^{\infty} P_s^{(m)}P_t^{(n)} = \sum_{m=0}^{\infty} P_{s+t}^{(m)} = \tilde{P}_{s+t}, \qquad s, t > 0.$$

The continuity of this semigroup is an immediate consequence of $\tilde{P}_s \geqq P_s$, when $s > 0$, for then

(4.1.15) $$0 \leqq 1 - \tilde{P}_s(i, i) \leqq 1 - P_s(i, i) \to 0, \qquad s \downarrow 0; i \in A.$$

To establish the relations between Π and C (the proof of the relations between Π and D is similar) we first remark that for $s, t > 0$

(4.1.16) $$\begin{aligned} C_{s,t} &= \lim_{u\to 0} P_s \frac{1}{u}(\tilde{P}_u - P_u)\tilde{P}_t \\ &\geqq \lim_{u\to 0} P_s \frac{1}{u} P_u^{(1)} P_t \\ &= \lim_{u\to 0} \frac{1}{u}\int_0^u dv\, \Pi_{s+v,t+u-v} = \Pi_{s,t}. \end{aligned}$$

It then follows from theorem 3.3.2 that the formula

(4.1.17) $$C'_{s,t} = \lim_{\epsilon\downarrow 0} \uparrow \Pi_{s,\epsilon}\tilde{P}_{t-\epsilon} \leqq C_{s,t}$$

defines a form C' relative to P and P such that

(4.1.18) $$\Pi_{s,t} = \lim_{\epsilon\downarrow 0} \downarrow C'_{s,\epsilon}P_{t-\epsilon}, \qquad s, t > 0;$$

but

(4.1.19) $$\begin{aligned} \int_0^t ds\, C'_{s,t-s} &= \lim_{\epsilon\downarrow 0} \uparrow \int_0^{t-\epsilon} ds\, \Pi_{s,\epsilon}\left(\sum_0^{\infty} P_{t-\epsilon-s}^{(n)}\right) \\ &= \tilde{P}_t - P_t = \int_0^t ds\, C_{s,t-s} \end{aligned}$$

so that, by continuity, the two forms C and C' are equal.

THEOREM 4.1.2. *Suppose that the perturbation form* Π *defined relative to the submarkovian process* P *is such that* $\Pi_{s,t} = \Pi_{s,0} P_t$ *with* $s, t > 0$. *Then the Laplace transform establishes for every* $x > 0$ *an isomorphism between the positive band of* $G_b(\tilde{P})$ *on which* $\lim_{s\downarrow 0} \downarrow P_{t-s}\tilde{g}_s = 0$, *with* $t > 0$ *(compare theorem* 3.3.3), *and the L-cone of positive bounded solutions of the single equation*

(4.1.20) $$\left(\int_0^{\infty} ds\, e^{-xs}\, \Pi_{s,0}\right) h = h, \qquad h \in 1_+^{\infty}(A).$$

It follows from theorem 4.1.1 and from the special form of Π that

(4.1.21) $$\tilde{P}_t = P_t + \int_0^t du\, \Pi_{u,0}\, \tilde{P}_{t-u}, \qquad t > 0,$$

so that we have, for any element $\tilde{g} \in G_b(\tilde{P})$,

(4.1.22) $$\tilde{g}_t = \tilde{P}_{t-s}\tilde{g}_s = P_{t-s}\tilde{g}_s + \int_0^{t-s} du\, \Pi_{u,0}\, \tilde{g}_{t-u}, \qquad t > 0.$$

Then the condition $\lim_{s\to 0} \downarrow P_{t-s}\tilde{g}_s = 0$, with $t > 0$, is for any $\tilde{g} \in G_b(\tilde{P})$ equivalent to

$$\tilde{g}_t = \int_0^t du\, \Pi_{u,0}\, \tilde{g}_{t-u}, \qquad t > 0, \tag{4.1.23}$$

or, after a Laplace transform, to

$$\tilde{g}_x = Q_x \tilde{g}_x, \qquad x > 0, \tag{4.1.24}$$

introducing the notation $Q_x = \int_0^\infty du \exp(-xu)\, \Pi_{u,0}$ for the present proof.

But on the other hand, it follows from the construction of $\tilde{P}$, that

$$\tilde{R}_x = \left(\sum_0^\infty Q_x^n\right) R_x, \qquad x > 0, \tag{4.1.25}$$

so that we have, for any bounded positive solution h of $Q_{x_0} h = h$,

$$\begin{aligned}[I + (x_0 - x)\tilde{R}_x]h &= h + \left(\sum_0^\infty Q_x^n\right)(x_0 - x) R_x Q_{x_0} h \\ &= h + \left(\sum_0^\infty Q_x^n\right)(Q_x h - h) \\ &= \lim_{n \to \infty} Q_x^n h \geqq 0.\end{aligned} \tag{4.1.26}$$

Here we have used the fact that $[I + (x_0 - x)R_x]Q_{x_0} = Q_x$ for $x > 0$. The preceding computation shows that $\{[I + (x_0 - x)R_x]h, x > 0\}$ is an element of $\hat{G}_b(\tilde{P})$ since all the functions $[I + (x_0 - x)\tilde{R}_x]h$ are positive and bounded. If $\{\tilde{g}_s, s > 0\}$ is the corresponding element in $G_b(\tilde{P})$, it follows from $Q_x\tilde{g}_x = \tilde{g}_x$, with $x > 0$, that $\lim_{s\to 0} P_{t-s}\tilde{g}_s = 0$ for $t > 0$.

COROLLARY 4.1.1. *Suppose that the perturbation form* Π *defined relative to the submarkovian process* P *is such that* $\Pi_{s,t} = \Pi_{s,0} P_t$ *with* $s, t > 0$. *In order for the* $\tilde{P}$ *process constructed in theorem* 4.1.1 *to be Markovian, it is necessary that for all* $x > 0$ *and sufficient that for one* $x > 0$,

(a) $\left[\int_0^\infty e^{-xu}\, du\, \Pi_{u,0}\right] e = e - xR_x e,$

(b) $\left[\int_0^\infty e^{-xu}\, du\, \Pi_{u,0}\right] h = h$ *has no bounded positive solution other than* 0.

In the notation of the preceding paragraph we remark that

$$\begin{aligned}x\tilde{R}_x e + \left(\int_0^\infty Q_x^n\right)(\hat{p}_x - Q_x e) &= \sum_0^\infty Q_x^n(e - Q_x e) \\ &= \lim_{n\to\infty} Q_x^n e,\end{aligned} \tag{4.1.27}$$

where $\hat{p}_x = e - xR_x e$ (compare section 2.3). Since the condition $x\tilde{R}_x e = e$ is necessary (for all $x > 0$) and sufficient (for one $x > 0$) for the process $\tilde{P}$ to be Markovian, a similar condition is that $Q_x e = \hat{p}$ and that $\lim_{n\to\infty} Q_x^n e = 0$. To prove the corollary it is therefore sufficient to prove that its condition (b) is equivalent to $\lim_{n\to\infty} Q_x^n e = 0$; but this follows from the fact that $\lim_{n\to\infty} Q_x^n e$ is a bounded positive solution of $Q_x h = h$ and that conversely any such solution, being bounded above by ch for a constant $c > 0$, is also bounded above by $c \lim_{n\to\infty} Q_x^n e$.

THEOREM 4.1.3. *Given two submarkovian processes P and P' defined on A and A' and such that $P \subset P'$, the form Q defined in theorem 3.3.4, $Q_{s,t} = \lim_{u\downarrow 0} P_s(1/u)(P'_u - P_u)P_t$ is a perturbation form on A for the process P. The "perturbated process" $\tilde{P}$ on A constructed from P and Q as in theorem 4.1.1, is such that $P \subset \tilde{P} \subset P'$. Moreover, for any perturbation form Π on A relative to P and for the corresponding perturbated process $P^{(\Pi)}$ of P by Π the following equivalences hold*

$$P^{(\Pi)} \subset P' \Leftrightarrow P^{(\Pi)} \subset \tilde{P} \Leftrightarrow \Pi_{s,t} \leqq Q_{s,t}, \qquad s, t > 0. \tag{4.1.28}$$

PROOF. To show that Q is a perturbation form relative to P, we remark that

$$\begin{aligned} Q_{s,t}e &\leqq \lim_{u\downarrow 0} P_s \frac{1}{u}(P'_u - P_u)P_t e \leqq \lim_{u\downarrow 0} P_s \frac{1}{u}(e - P_u e) \\ &= p_s, \qquad s, t \geqq 0. \end{aligned} \tag{4.1.29}$$

The proof that $\tilde{P} \subset P'$ is made by induction on N in $\sum_{n<N} P_t^{(n)} \leqq P'_t$, with $t > 0$ (notation of section 4.1). This inequality is clearly satisfied for $N = 1$. If it is satisfied for the parameter value N, then

$$\begin{aligned} \sum_{n\leqq N} P_t^{(n)} &= P_t + \lim_{\epsilon\downarrow 0} \int_0^{t-\epsilon} du\, Q_{u,\epsilon}\Big(\sum_{n<N} P_{t-u-\epsilon}^{(n)}\Big) \\ &\leqq P_t + \lim_{\epsilon\downarrow 0} \int_0^{t-\epsilon} du\, C_{u,\epsilon} P'_{t-u-\epsilon} = P'_t. \end{aligned} \tag{4.1.30}$$

If Π is a perturbation form relative to P, it clearly follows from the construction of a perturbated process that $P^{(\Pi)} \subset \tilde{P}$ whenever $\Pi_{s,t} \leqq Q_{s,t}$ for $s, t > 0$. The converse implication $P^{(\Pi)} \subset \tilde{P} \Rightarrow \Pi \leqq Q$ follows from theorems 3.3.4 and 4.1.1 by means of

$$\begin{aligned} \Pi_{s,t} &= \lim_{u\downarrow 0} P_s \frac{1}{u}(P_u^{(\Pi)} - P_u)P_t \\ &\leqq \lim_{u\downarrow 0} P_s \frac{1}{u}(P'_u - P_u)P_t = Q_{s,t}, \qquad s, t > 0. \end{aligned} \tag{4.1.31}$$

This concludes the proof of theorem 4.1.3.

The preceding results give a partial solution to the following fundamental problem: how can one characterize and construct all the submarkovian processes P' which dominate a given process P on A?

Let P' be a fixed submarkovian process on A'. Up to now we have constructed for any process P on A such that $P \subset P'$, a perturbated process $\tilde{P}$ on A such that $\tilde{P} \subset P$. This construction can obviously be repeated on $\tilde{P}$ to give a perturbated process $\tilde{P}^{(2)}$ of $\tilde{P}$ on A which is such that $P \subset \tilde{P} \subset \tilde{P}^{(2)} \subset P'$. In the general case the process $\tilde{P}^{(2)}$ is different from $\tilde{P}$, that is, the form $Q^{(2)}$ defined by $Q_{s,t}^{(2)} = \lim_{u\to 0} \tilde{P}_s(1/u)(P'_u - \tilde{P}_u)\tilde{P}_t$ is different from 0. An example where $Q^{(2)} \neq 0$ was first constructed by Kendall under the denotation "flash of flashes." We remark that theorem 3.5.1 shows that $\lim_{u\downarrow 0} \downarrow P_{s-u} Q_{u,t}^{(2)} = 0$,

$\lim_{u\downarrow 0}\downarrow Q^{(2)}_{s,u}P_{t-u} = 0$ for $s, t > 0$. Repeating the preceding construction over and over, we obtain a sequence $P \subset \tilde{P} \subset \tilde{P}^{(2)} \subset \cdots \subset \tilde{P}^{(n)}$ of processes which are all dominated by P'. Moreover, since the set of submarkovian processes on a fixed set A is easily seen to be inductive, the preceding sequence may be extended to a transfinite sequence and to a last process P^* on A which is still dominated by P'. In the case where $P^* = P'$, which is possible only if $A = A'$, one may consider the problem of the characterization and construction of P' from P solved; the solution has been given by successive perturbations.

In the general case, the process P^* on A, being the last in our perturbation procedure, has the further property that

$$\lim_{u\to 0} P^*_s \frac{1}{u}(P'_u - P^*_u)P^*_t = 0, \qquad s, t > 0. \tag{4.1.32}$$

We have thus reduced the general problem of characterization and construction of a process P' from a process P such that $P \subset P'$ to the analogous problem in the particular case where a relation such as the preceding (with P instead of P^*) is valid.

4.2. *Absolute dominance.* Given two submarkovian processes P on A and P' on A', we say P is *absolutely dominated by* P if

(1) $P \subset P'$, that is, if $A \subset A'$ and if $P_t \leqq P'_t$ on A for $t > 0$,

(2) $\lim_{u\to 0} P_s u^{-1}(P'_u - P_u)P_t = 0$ for $s, t > 0$.

In the following, we shall designate by $\mathfrak{F}$ the positive band of $F(P')$ composed of the elements f' such that $\lim_{s\downarrow 0}\downarrow f'_s P_{t-s} = 0$ for $t > 0$ (compare theorem 3.3.2); the positive band $\mathcal{G}$ of $G(P')$ is similarly defined as the set of g' for which $\lim_{s\downarrow 0}\downarrow P_{t-s}g'_s = 0$ for $t > 0$. We also let $\mathfrak{F}_b = \mathfrak{F} \cap F_b(P')$, and $\mathfrak{F}_e = \mathfrak{F} \cap F_e(P')$ while $\mathcal{G}_b = \mathcal{G} \cap G_b(P')$. The main result of this section will show that $\mathfrak{F} = \mathfrak{F}_b = \mathfrak{F}_e$, that $\mathcal{G} = \mathcal{G}_b$ and that $\mathfrak{F}$ and $\mathcal{G}$ are in duality whenever $\mathfrak{F}_e$ or $\mathcal{G}_b$ is a finite dimensional cone.

For every closed subset B of A, the process P^B constructed in section 3.4 is absolutely dominated by P; in this particular case, the following theorem reduces to theorems 4 and 5 of [20].

THEOREM 4.2.1. *Let P on A and P' on A' be two submarkovian processes such that P is absolutely dominated by P' and suppose that either $\mathfrak{F}_e$ or $\mathcal{G}_b$ is a finite dimensional cone. Then there exists a finite set Ω which contains $A' - A$ and for every $\omega \in \Omega$ there exist*

(1) *an element of $\mathfrak{F}_e$, to be denoted by $\{P'_t(\omega, \cdot), t > 0\}$ equal to $\{P'_t(i, \cdot), t > 0\}$ if $\omega = i \in A' - A$, for which $\lim_{t\downarrow 0}\uparrow \sum_{A'} P'_t(\omega, i) = 1$ and such that the most general element of $\mathfrak{F}$ is given by*

$$f'_t = \sum_{\Omega} \xi(\omega)P'_t(\omega, \cdot) \quad \textit{on} \quad A', \qquad t > 0, \tag{4.2.1}$$

where ξ is an arbitrary positive measure on Ω;

(2) *an element of $\mathcal{G}_b$, to be denoted by $\{P'_t(\cdot, \omega), t > 0\}$ equal to $\{P'_t(\cdot, i), t > 0\}$ if $\omega = i \in A' - A$, such that the most general element of $\mathcal{G}$ is given by*

$$g'_t = \sum_{\Omega} P'_t(\cdot, \omega)\eta(\omega) \quad on \quad A', \qquad t > 0, \tag{4.2.2}$$

where η is an arbitrary positive function on Ω;

(3) *an element of $F_b(P)$, to be denoted by $\{F_t(\omega, \cdot), t > 0\}$, and an element of $G_b(P)$, to be denoted by $\{G_t(\cdot, \omega), t > 0\}$, such that*

$$\begin{aligned} C_{s,t} &= \sum_{\Omega} G_s(\cdot, \omega)P'_t(\omega, \cdot) \quad on \quad A \times A', \qquad s, t > 0,\\ D_{s,t} &= \sum_{\Omega} P'_s(\cdot, \omega)F_t(\omega, \cdot) \quad on \quad A' \times A, \qquad s, t > 0, \end{aligned} \tag{4.2.3}$$

if C and D designate the forms defined by theorem 3.1.1 *relative to the given processes.*

Moreover if the two sets $\{\Pi_t, t > 0\}$ and $\{H_t, t > 0\}$ of positive matrices on Ω are defined (independently of s) by

$$\begin{aligned} \Pi_t(\omega_1, \omega_2) &= \sum_{i \in A'} P'_{t-s}(\omega_1, i)\, P'_s(i, \omega_2),\\ H_t(\omega_2, \omega_1) &= \sum_{i \in A} F_{t-s}(\omega_2, i)\, G_s(i, \omega_1), \qquad 0 < s < t, \end{aligned} \tag{4.2.4}$$

[so that $\Pi_t(\omega, \cdot) = P_t(\omega, \cdot)$ and $\Pi_t(\cdot, \omega) = P_t(\cdot, \omega)$ on $A' - A$] there exists a matrix Q on Ω such that for every $x > 0$

$$\left(\int_0^\infty dt\, e^{-xt}\, \Pi_t\right)^{-1} = xI_{(A'-A)} + \int_0^\infty dt\, (1 - e^{-xt})H_t - Q. \tag{4.2.5}$$

The following inequalities also hold

$$\begin{aligned} &\sum_{\Omega} G_t(\cdot, \omega) \leqq p_t \quad on \quad A,\\ &\int_0^\infty dt \min(1, t)H_t(\omega, \omega) < \infty, && \omega \in \Omega,\\ &\int_0^\infty dt\, H_t(\omega, \omega) = +\infty, && \omega \notin A' - A,\\ &0 \leqq \int_0^\infty dt\, H_t(\omega_1, \omega_2) \leqq Q(\omega_1, \omega_2), && \omega_1 \neq \omega_2,\\ &0 \leqq \lim_{u \downarrow 0} \uparrow \left\{\sum_A F_u(\omega_1, i)\left[1 - \int_0^\infty dt\, G_t(i, \omega_2)\right]\right\}\\ &\qquad \leqq - \sum_{\omega_2} Q(\omega_1, \omega_2), && \omega_1 \in \Omega. \end{aligned} \tag{4.2.6}$$

In what precedes, all the elements introduced are uniquely determined by the processes P and P′ except for $\{P'_t(\cdot, \omega), t > 0\}$, $\{F_t(\omega, \cdot), t > 0\}$, and $\{Q(\omega, \cdot)\}$ when $\omega \notin A' - A$, which can be replaced by $\{P'_t(\cdot, \omega)c_\omega, t > 0\}$, $\{c_\omega^{-1}F_t(\omega, \cdot), t > 0\}$, and $\{c_\omega^{-1}Q(\omega, \cdot)\}$, where $0 < c_\omega < \infty$.

PROOF. We shall prove this theorem under the hypothesis that $\mathfrak{F}_e$ is finite-dimensional; the case where $\mathcal{G}_b$ is assumed to be finite-dimensional is proved in the same way. Since any finite-dimensional L-cone is isomorphic to the cone of positive vectors of a finite-dimensional Cartesian space, the hypothesis immediately implies the existence of a basis in $\mathfrak{F}_e$, that is, of elements of $\mathfrak{F}_e$ with the properties in (1) of the statement of the theorem (with $\mathfrak{F}_e$ instead of $\mathfrak{F}$). A set

Ω is defined in this way; also $A' - A$ is isomorphic to a subset of Ω since $\{P'_t(i, \cdot), t > 0\}$ is, for every $i \in A' - A$, an element of $\mathfrak{F}_e$ and $A' - A$ is identified with this subset.

Since for every $s > 0$, $i \in A$, we have $\{C_{s,t}(i, \cdot), t > 0\}$ is an element of $\mathfrak{F}_e$ by the absolute dominance hypothesis, there exists a positive function $G_s(i, \cdot)$ on Ω such that

$$C_{s,t}(i, \cdot) = \sum_{\Omega} G_s(i, \omega)P'_t(\omega, \cdot) \quad \text{on} \quad A', \qquad s, t > 0. \tag{4.2.7}$$

Moreover, it follows from $P_u C_{s,t} = C_{u+s,t}$, with u, s, $t > 0$, and from the fact that C is bounded that $\{G_t(\cdot, \omega), t > 0)\} \in G_b(P)$ for every $\omega \in \Omega$. The first of formulas (4.2.3) has thus been shown to hold.

Let us prove that

$$\sum_{\Omega} G_s(\cdot, \omega) = p_s - \overline{p}_s \quad \text{on} \quad A, \qquad s > 0, \tag{4.2.8}$$

when $\overline{p}_s = \lim_{u \downarrow 0} \downarrow P_{s-u}p'_u$, an element of $G(P)$. This formula is in fact a direct consequence of the following formula, valid on A,

$$\begin{aligned} \int_0 du\, p_u - \int_0^t du\, p'_u &= P'_t e' - P_t e \\ &= \sum_{\Omega} \int_0 du\, G_u(\cdot, \omega)P'_{t-u}e'(\omega), \end{aligned} \tag{4.2.9}$$

as is seen by multiplying by $(1/t)P_s$ on the left and letting t tend to 0. This allows us to define a positive function γ on A by

$$\gamma(i) = 1 - \int_0^\infty du \sum_{\Omega} G_u(i, \omega) \geqq 1 - \int_0^\infty du\, p_u(i) \geqq 0. \tag{4.2.10}$$

To show that $\mathcal{G}_b$ is finite-dimensional whenever $\mathfrak{F}_e$ is finite-dimensional, we first extend the definition of $D_{s,t}$ from $A' \times A$ to $(A + \Omega) \times A$ by letting, independently of u,

$$D_{s,t}(\omega, \cdot) = P'_{s-u}(\omega, \cdot)D_{u,t}, \qquad s > u > 0; t > 0 \tag{4.2.11}$$

so that by theorem 3.1.1 the following formula holds on A, for $t > 0$,

$$\begin{aligned} P'_t(\omega, \cdot) &= \lim_{s \downarrow 0} \left[P'_s(\omega, \cdot)\left(P_{t-s} + \int_0^{t-s} du\, D_{t-s-u,u}\right)\right] \\ &= \lim_{s \downarrow 0} \left[P'_s(\omega, \cdot)P_{t-s} + \int_0^{t-s} du\, D_{t-u,u}(\omega, \cdot)\right] \\ &= \int_0^t du\, D_{t-u,u}(\omega, \cdot). \end{aligned} \tag{4.2.12}$$

On the other hand, we extend the domain of definition of any $g' = \{g'_t, t > 0\} \in \mathcal{G}$ to $A + \Omega$ by letting, independently of u,

$$g'_s(\omega) = \langle P'_{s-u}(\omega, \cdot), g'_u\rangle \tag{4.2.13}$$

so that by theorem 3.1.1, the following formula holds on A, for $s > 0$,

$$(4.2.14)\qquad \begin{aligned} g'_s &= \lim_{t\downarrow 0}\left[\left(P_{s-t} + \int_0^{s-t} du\, C_{u,s-t-u}\right) g'_t\right] \\ &= \lim_{t\downarrow 0}\left[P_{s-t}g'_t + \sum_{\Omega}\int_t^s du\, G_u(\cdot,\omega)g'_{s-u}(\omega)\right] \\ &= \sum_{\Omega}\int_0^s du\, G_u(\cdot,\omega)g'_{s-u}(\omega). \end{aligned}$$

Combining the two preceding formulas and letting, independently of u,

$$(4.2.15)\qquad D_{s,t}(\omega_1,\omega_2) = \langle D_{s,u}(\omega_1,\cdot),\, G_{t-u}(\cdot,\omega_2)\rangle,\quad s>0,\quad t>u>0,$$

we obtain, for $g'\in\mathcal{G}$, $\omega\in\Omega$, and $s, t>0$,

$$(4.2.16)\qquad \begin{aligned} g'_{t+s}(\omega) &= \langle P'_t(\omega,\cdot),\, g'_s\rangle \\ &= \sum_{A'-A} P'_t(\omega,i)g'_s(i) + \sum_{\omega_1}\int_0^t du\int_0^s dv\, D_{u,s+t-u-v}(\omega,\omega_1)g'_v(\omega_1) \end{aligned}$$

or equivalently, for any $g'\in\mathcal{G}_b$, after a double Laplace transform,

$$(4.2.17)\qquad \begin{aligned} \hat{g}'_x(\omega) &= \hat{g}'_y(\omega) + (y-x)\sum_{A'-A} R'_x(\omega,i)g'_y(i) \\ &\quad + \sum_{\omega_1}\left[\int_0^\infty ds\, e^{-xs}\int_0^\infty dt\,(e^{-xt}-e^{-yt})D_{s,t}(\omega,\omega_1)\right] g'_y(\omega_1) \end{aligned}$$

for $x, y>0$ and $\omega\in\Omega$.

This last equation together with (4.2.14) shows that the mapping of $\mathcal{G}_b$ into the cone of positive functions on Ω defined by $g\to\hat{g}'_y$ is one to one, for every fixed $y>0$. Hence, $\dim(\mathcal{G}_b)\leqq\dim(\mathfrak{F}_e)<\infty$. The beginning of the present proof may now be transposed from $\mathfrak{F}_e$ to $\mathcal{G}_b$ to obtain the existence of

(1) a finite set Ω', containing $A'-A$;

(2) elements $\{P'_t(\cdot,\omega'),\ t>0\}$ of $\mathcal{G}_b$ with $P'_t(\cdot,\omega) = P'_t(\cdot,i)$ when $\omega' = i\in A'-A$ such that $\{\sum_\Omega P'_t(\cdot,\omega')\eta(\omega'),\ t>0\}$ is the most general element of $\mathcal{G}_b$ when $\eta\geqq 0$;

(3) element $\{F_t(\omega',\cdot),\ t>0\}\in F_b(P)$ such that on $A'\times A$

$$(4.2.18)\qquad D_{s,t} = \sum_{\Omega'} P'_s(\cdot,\omega')F_t(\omega',\cdot),\qquad s,t>0.$$

With this notation and putting, independently of s,

$$(4.2.19)\qquad \Pi_t(\omega,\omega') = \sum_{A'} P'_s(\omega,i)P'_{t-s}(i,\omega'),\qquad \omega\in\Omega,\ \omega'\in\Omega';\ t>s>0,$$

we may rewrite equations (4.2.12) and (4.2.14) as

$$(4.2.20)\qquad \begin{aligned} P'_t(\omega,\cdot) &= \sum_{\Omega'}\int_0^t du\,\Pi_{t-u}(\omega,\omega')F_u(\omega',\cdot), \\ P'_s(\cdot,\omega') &= \sum_{\Omega}\int_0^t du\, G_u(\cdot,\omega)\,\Pi_{s-u}(\omega,\omega')\ \text{on } A,\qquad s,t>0, \end{aligned}$$

whereas equation (4.2.16) takes the form

$$\Pi_{t+s}(\omega, \omega') = \sum_{A'-A} \Pi_t(\omega, i)\, \Pi_s(i, \omega') + \sum_{\omega_1'} \sum_{\Omega_1} \int_0^t du \int_0^s dv\, \Pi_{t-u}(\omega, \omega_1')H_{u+v}(\omega_1', \omega_1)\, \Pi_{s-v}(\omega_1, \omega') \tag{4.2.21}$$

if $\{H_t, t > 0\}$ on $\Omega' \times \Omega$ is defined as in (4.2.4).

The Laplace transform $\{\hat{\Pi}_x, x > 0\}$ of $\{\Pi_t, t > 0\}$ exists and is finite since by the definition of Π we have $\hat{\Pi}_x(\omega, \omega') \leqq \|R_x'(\cdot, \omega')\|_\infty < \infty$. Let us also introduce on $\Omega' \times \Omega$ the positive matrices Z_x defined by

$$\begin{aligned} Z_x(\omega', \omega) &= xI_{A'-A}(\omega', \omega) + \int_0^\infty dt\, (1 - e^{-xt})H_t(\omega', \omega) \\ &= xI_{A'-A}(\omega', \omega) + x\left\langle \int_0^\infty dt\, e^{-xt}F_t(\omega', \cdot) \int_0^\infty dt\, G_t(\cdot, \omega)\right\rangle. \end{aligned} \tag{4.2.22}$$

These matrices are finite since $\{F_t(\omega', \cdot), t > 0\} \in F_b(P)$ for every $\omega' \in \Omega'$ and since $\sum_\Omega \int_0^\infty dt\, G_t(\cdot, \omega) \leqq e$ on A. An easy computation then shows that (4.2.21) is equivalent, by a double Laplace transform, to the following equation, valid for $\omega \in \Omega$, $\omega' \in \Omega'$, and $x, y > 0$,

$$\begin{aligned} &\hat{\Pi}_x(\omega, \omega') - \hat{\Pi}_y(\omega, \omega') \\ &\quad = \sum_{\omega_1'} \sum_{\omega_1} \hat{\Pi}_x(\omega, \omega_1')[Z_y(\omega_1', \omega_1) - Z_x(\omega_1', \omega_1)]\, \hat{\Pi}_y(\omega_1, \omega'). \end{aligned} \tag{4.2.23}$$

Let us remark further that the matrices $\hat{\Pi}_x$ defined on $\Omega \times \Omega'$ are invertible, since if there existed a function $\eta \not\equiv 0$ on Ω' [or Ω] such that for a fixed y, $\sum_{\omega'} \hat{\Pi}_y(\cdot, \omega')\eta(\omega') = 0$ on Ω [or $\sum_\omega \eta(\omega)\, \hat{\Pi}_x(\omega, \cdot) = 0$ on Ω'], the formulas (4.2.23) and (4.2.20) would imply

$$\begin{aligned} &\sum_{\Omega'} P_s'(\cdot, \omega')\eta(\omega') = 0, \\ &[\sum_{\Omega} \eta(\omega)P_s'(\omega, \cdot) = 0], \text{ on } A', \qquad s > 0, \end{aligned} \tag{4.2.24}$$

which is impossible.

This property of the matrices $\hat{\Pi}_x$ together with (4.2.23) implies the existence on $\Omega' \times \Omega$ of a matrice Q independent of x such that for every $x > 0$,

$$\begin{aligned} &\sum_{\Omega'} \hat{\Pi}_x(\cdot, \omega')[Z_x(\omega', \cdot) - Q(\omega', \cdot)] = I_\Omega, \\ &\sum_{\Omega} [Z_x(\cdot, \omega) - Q(\cdot, \omega)]\, \hat{\Pi}_x(\omega, \cdot) = I_{\Omega'}. \end{aligned} \tag{4.2.25}$$

A systematic study of equations such as (4.2.23) was given in [20]; in order to apply theorem 3 of this reference, we need only remark that

(a) $\lim_{x\to\infty} \downarrow \hat{\Pi}_x = 0$ on $\Omega \times \Omega'$;

(b) $\lim_{x\to\infty} \hat{\Pi}_x Z_x = \lim_{x\to\infty} (I_\Omega + \hat{\Pi}_x Q) = I_\Omega$ on $\Omega \times \Omega$,

$\lim_{x\to\infty} Z_x \hat{\Pi}_x = \lim_{x\to\infty} (Q\, \hat{\Pi}_x + I_{\Omega'}) = I_{\Omega'}$ on $\Omega' \times \Omega'$;

(c) $\sum_{\omega}\sum_{\omega'} \hat{\Pi}_x(\cdot, \omega')Z_x(\omega', \omega)$

$$= \int_0^\infty dt\, xe^{-xt}\left\{\sum_{A'-A} P'_t(\cdot, i) + \sum_A P'_t(\cdot, i)[1 - \gamma(i)]\right\}$$

$$\leqq 1 \quad \text{on} \quad \Omega.$$

As a consequence of the theorem cited, there exists a one-to-one correspondence between Ω and Ω' (which reduces to the identical mapping on $A' - A$) by which Ω and Ω' are identified so that the inequalities (4.2.6) relative to H are valid. These inequalities also show that the preceding correspondence between Ω and Ω' is uniquely determined.

To conclude the proof of theorem 4.2.1, there remains to be proved the last of the inequalities (4.2.6). We shall prove therefore that if $q' = \{q'_t, t > 0\}$ is the component of $p' = \{p'_t; t > 0\}$ in $\mathcal{G}_b$ so that

$$\begin{aligned} q'_t &= \sum_{\Omega'} P'_t(\cdot, \omega')q(\omega') \quad \text{on} \quad A', \\ q'_t(\omega) &= \sum_{\Omega'} \Pi_t(\omega, \omega')q(\omega') \quad \text{on} \quad \Omega \end{aligned} \tag{4.2.26}$$

for a positive function q on Ω', then this function q is given by

$$q(\omega') = -\sum_{\Omega} Q(\omega', \omega) - \lim_{u \downarrow 0} \uparrow \sum_A F_u(\omega', i)\gamma(i). \tag{4.2.27}$$

We first remark that by (4.2.8)

$$\gamma - P_t\gamma = \int_0^t du\, p_u - \int_0 du \sum_{\Omega} G_u(\cdot, \omega) = \int_0^t du \overline{p}_u \geqq 0 \tag{4.2.28}$$

so that $\langle F_u(\omega', \cdot), \gamma\rangle$ is a nonincreasing function of u for

$$\langle F_{u+v}(\omega', \cdot), \gamma\rangle = \langle F_u(\omega', \cdot), P_v\gamma\rangle \leqq \langle F_u(\omega', \cdot), \gamma\rangle, \qquad u, v > 0. \tag{4.2.29}$$

On the other hand, the relation

$$\sum_{\omega'}\sum_{\omega} \hat{\Pi}_x(\cdot, \omega')[Z_x(\omega', \omega) - Q(\omega', \omega)] = 1 \quad \text{on} \quad \Omega \tag{4.2.30}$$

is equivalent to the following equation on Ω,

$$\sum_{A-A'} P'_t(\cdot, i) + \sum_A P'_t(\cdot, i)[1 - \gamma(i)] = 1 + \sum_{\omega'}\sum_{\omega}\left[\int_0^t ds\, \Pi_s(\cdot, \omega')\right] Q(\omega', \omega), \tag{4.2.31}$$

that is, to

$$1 - P'_t e(\omega) = \sum_{\omega'} \int_0^t ds\, \Pi_s(\cdot, \omega')\left[-\sum_{\omega_1} Q(\omega', \omega_1) - \langle F_{t-s}(\omega', \cdot), \gamma\rangle\right]. \tag{4.2.32}$$

On the other hand, it follows from $p'_t = q'_t + \lim_{s\downarrow 0} \uparrow P'_{t-s}\overline{p}_s$ that

$$\begin{aligned} 1 - P'_t e'(\omega) &= \int_0^t ds\, q'_s(\omega) + \lim_{u \downarrow 0}\left[\int_0^t ds\, P'_s(\omega, \cdot)\right]\overline{p}_u \\ &= \sum_{\omega'} \int_0^t ds\, \Pi_s(\omega, \omega')\left[q(\omega') + \lim_{u\downarrow 0} \int_0^{t-s} dv\, \langle F_v(\omega', \cdot), \overline{p}_u\rangle\right]. \end{aligned} \tag{4.2.33}$$

However, since

$$\int_0^{t-s} dv\, \langle F_v(\omega', \cdot), \bar{p}_u\rangle = \int_0^{t-s} dv\, \langle F_u(\omega', \cdot), \bar{p}_v\rangle = \langle F_u(\omega', \cdot), \gamma - P_{t-s}\gamma\rangle, \tag{4.2.34}$$

we have obtained

$$1 - P'_t e'(\omega) = \sum_{\omega'} \int_0^t ds\, \Pi_s(\omega, \omega') \left[q(\omega') + \lim_{u \downarrow 0} \uparrow \langle F_u(\omega', \cdot), \gamma\rangle - \langle F_{t-s}(\omega', \cdot), \gamma\rangle\right]. \tag{4.2.35}$$

This equation compared with (4.2.32) gives (4.2.27).

Theorem 4.2.1 admits the following full converse.

THEOREM 4.2.2. *Suppose we are given,*

(a) *a submarkovian process P on A;*

(b) *a finite set Ω and a subset Ω' of Ω;*

(c) *elements $\{F_t(\omega, \cdot), t > 0\} \in F_b(P)$ and $\{G_t(\cdot, \omega), t > 0\} \in G_b(P)$ for every $\omega \in \Omega$;*

(d) *a matrix Q on Ω. Let us further suppose that the inequalities* (4.2.6) *are satisfied with Ω' instead of $A' - A$ and with $\{H_t, t > 0\}$ defined by* (4.2.4). *Then the following formulas uniquely define, for every $t > 0$, positive matrices P'_t on $A + \Omega$ whose restrictions to $A + \Omega'$ constitute a submarkovian process on $A + \Omega'$*

$$\left(\int_0^\infty dt\, e^{-xt} P'_t\right)^{-1} = xI_{\Omega'} + \int_0^\infty dt\, (1 - e^{-xt}) H_t - Q \quad on \quad \Omega \times \Omega, \quad x > 0,$$

$$P'_t = P_t + \int_0^t ds \sum_\Omega G_s(\cdot, \omega) P'_{t-s}(\omega, \cdot) \quad on \quad A \times (A + \Omega), \tag{4.2.36}$$

$$P'_t = P_t + \int_0^t ds \sum_\Omega P'_{t-s}(\cdot, \omega) F_s(\omega, \cdot) \quad on \quad (A + \Omega) \times A.$$

This process P' has moreover all the properties stated in theorem 4.2.1 *with the same notation.*

Proof. The verification that the preceding formulas define a submarkovian process P' with the stated properties will be left to the reader, who is also referred to [20] where an entirely similar verification is made in theorem 5.

COROLLARY 4.2.1. *With the same notation as in the preceding theorem,*

$$\lim_{t \to 0} \frac{1}{t} P_t(i, j) = Q(i, j) - \int_0^\infty ds\, H_s(i, j), \qquad i \neq j \quad in \quad A' - A,$$

$$\lim_{t \to 0} \{[1 - P_t(i, i)] - tQ(i, i) + \int_0^\infty ds \min(s, t) H_s(i, i)\} = 1, \qquad i \in A' - A, \tag{4.2.37}$$

and as a consequence

$$\lim_{t\to 0} \frac{1}{t} [1 - P_t(i, i)] = -Q(i, i) + \int_0^\infty ds\, H_s(i, i) \leqq \infty, \quad i \in A' - A. \tag{4.2.38}$$

COROLLARY 4.2.2. *For the process P' to be Markovian on A' it is necessary and sufficient that the first and last inequalities of* (4.2.6) *be equalities.*

PROOF. Immediate by formulas (4.2.8) and (4.2.27).

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