RECENT DEVELOPMENTS IN THE THEORY OF CHARACTERISTIC FUNCTIONS

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1. Introduction

The study of characteristic functions has several aspects. Characteristic functions were introduced to permit the application of powerful analytical methods in probability theory. They were first used as a tool to study limit theorems, however the scope of their applications has constantly widened and includes now a large variety of problems in probability theory and in mathematical statistics. More recently mathematicians began to investigate problems concerning characteristic functions for their intrinsic mathematical interest. In some of these problems their probabilistic or statistical origin is still apparent, in others, the analytical character becomes dominant.

The present survey deals, therefore, with a variety of loosely connected topics. In the first part, we study certain frequency functions whose characteristic functions are known. These include the stable distributions, Pólya type distributions, and a related family. These results are interesting from the probabilistic as well as from the analytic viewpoint. The second part deals with problems motivated by certain statistical questions. The third part treats the arithmetic of distribution functions and related analytical problems.

Let $F(x)$ be a distribution function, that is a nonnegative, right-continuous function such that $F(-\infty) = 0$ while $F(+\infty) = 1$. The Fourier-Stieltjes transform of $F(x)$, that is the function

$$f(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x) \tag{1.1}$$

is called the characteristic function of $F(x)$. In this paper we denote distribution functions by capital letters, as $F(x)$, and the characteristic function of $F(x)$ by the corresponding small letter, as $f(t)$. If subscripts are used on the symbol for a distribution function then the same subscripts are attached to its characteristic function.

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PART I. APPLICATION OF CHARACTERISTIC FUNCTIONS TO THE STUDY OF DISTRIBUTION FUNCTIONS

It is often of interest to express properties of distribution functions in terms of their characteristic functions. In this part, we deal mainly with stable and with unimodal distributions. However, we mention here three interesting isolated results. J. R. Blum and M. Rosenblatt [1] gave conditions which assure that an infinitely divisible distribution is discrete or continuous or a mixture. J. Shapiro [52] derived a necessary and sufficient condition for the existence of moments of an infinitely divisible distribution. E. J. G. Pitman [45] studied characteristic functions which have a derivative of odd order at the origin.

2. Stable distributions and their frequency functions

A distribution function $F(x)$ is said to be stable if to every $b_1 > 0$, $b_2 > 0$, c_1 and c_2 there corresponds a positive number b and a real c such that

$$F\left(\frac{x - c_1}{b_1}\right) * F\left(\frac{x - c_2}{b_2}\right) = F\left(\frac{x - c}{b}\right) \tag{2.1}$$

holds. The asterisk denotes here the operation of convolution. Stable distributions were first investigated by P. Lévy, who showed that they are limit laws of normed sums of independently and identically distributed random variables. The characteristic function of a stable distribution is called a stable characteristic function. The relation (2.1) can be expressed in terms of characteristic functions as

$$f(b_1 t)f(b_2 t) = f(bt)e^{i\gamma t}, \tag{2.2}$$

where $\gamma = c - c_1 - c_2$.

It is possible to determine all stable characteristic functions $f(t)$; they are given by

$$\log f_a(t; \alpha, \beta, c) = iat - c|t|^\alpha \left\{1 + i\beta \frac{t}{|t|}\omega(|t|, \alpha)\right\}, \tag{2.3}$$

where

$$\omega(|t|, \alpha) = \begin{cases} \tan\dfrac{\pi\alpha}{2}, & \alpha \neq 1 \\ \dfrac{2}{\pi}\log|t|, & \alpha = 1 \end{cases} \tag{2.4}$$

and where $c \geqq 0$, $|\beta| \leqq 1$, $0 < \alpha \leqq 2$ while a is a real number. The number α is called the exponent of the stable distribution.

It is known that all stable distributions are absolutely continuous; we denote the frequency functions of the stable distribution with parameters a, α, β, c by $p_a(x; \alpha, \beta, c)$ and write $p(x; \alpha, \beta, c)$ for $p_0(x; \alpha, \beta, c)$ and $f(t; \alpha, \beta, c)$ for $f_0(t; \alpha, \beta, c)$. Then

$$p(x; \alpha, \beta, c) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} f(t; \alpha, \beta, c)\, dt. \tag{2.5}$$

It follows easily from (2.3) and (2.5) that

$$p(x; \alpha, \beta, c) = p(-x; \alpha, -\beta, c). \tag{2.6}$$

Explicit expressions for the frequency functions of stable distributions are only known in a few isolated cases. Thus (2.5) yields for $\alpha = 2$ the normal frequency function, for $\alpha = 1$ and $\beta = 0$ the density function of the Cauchy distribution. P. Lévy and N. V. Smirnov determined also the stable distribution with $\alpha = 1/2$; it belongs to the system of Pearson curves (Type V). Apart from these three cases, no stable distributions are known whose frequency functions are elementary functions.

The analytical behavior of the frequency functions of stable laws was first studied by A. I. Lapin (see [9], p. 183), who showed that $p(x; \alpha, \beta, c)$ is an entire function if $\alpha > 1$ while for $\alpha = 1$ the radius of convergence of the Taylor series for $p(x; \alpha, \beta, c)$ in the neighborhood of a point x of the real axis is at least equal to c. The case $\alpha < 1$ was investigated by A. V. Skorohod [55] who obtained the following result.

THEOREM 2.1. *The frequency function of a stable distribution with characteristic exponent $\alpha < 1$ has the form*

$$p(x; \alpha, \beta, c) = \begin{cases} \dfrac{1}{x} \Phi_1(x^{-\alpha}), & x > 0 \\ \dfrac{1}{|x|} \Phi_2(|x|^{-\alpha}), & x < 0, \end{cases} \tag{2.7}$$

where $\Phi_1(z)$ and $\Phi_2(z)$ are entire functions.

V. M. Zolotarev [68] found an expression for the density of a stable distribution with exponent α greater than one in terms of a density with exponent $1/\alpha$. In view of (2.6) it is sufficient to study the case where $x > 0$ while $-1 \leqq \beta \leqq 1$. Zolotarev obtained the following result.

THEOREM 2.2. *Let $v = -\beta \tan(\pi\alpha/2)$ and $a_v = (1 + v^2)^{1/2\alpha}$ then*

$$a_v x p(-a_v x; \alpha, \beta, 1) = a_{\tilde{v}} x^{-\alpha} p\left(a_{\tilde{v}} x^{-\alpha}; \frac{1}{2}, \tilde{\beta}, 1\right) \tag{2.8}$$

for all $x > 0$, $|\beta| \leqq 1$ and any $\alpha > 1$. Here $\tilde{v} = -\tilde{\beta} \tan(\pi\alpha/2)$ and

$$\tilde{\beta} = \left[\tan \frac{\pi}{2\alpha}\right]^{-1} \tan\left\{\frac{\pi}{2\alpha}\left[\frac{2}{\pi} \arctan v + (\alpha - 1)\right]\right\}. \tag{2.9}$$

V. M. Zolotarev [69] studied analytical relations between stable laws with different parameters and also used [70] systematically the Mellin transform to investigate analytical properties of stable laws. In the first of these two papers he considered the case where the exponent $\alpha < 1$ is a rational number and showed that $p(x; \alpha, \beta, 1)$ is the real part of a function which satisfies an ordinary differential equation. Zolotarev used these results, and some known formulas, to

express the frequency functions of stable distributions with parameters $\alpha = 2/3$, $\beta = 1$; $\alpha = 3/2$, $\beta = 1$; $\alpha = 2/3$, $\beta = 0$; $\alpha = 1/3$, $\beta = 1$; $\alpha = 1/2$, β arbitrary, in terms of higher transcendental functions.

Somewhat related problems were treated by P. Medgyessy [43]. He assumed that the exponent $\alpha = m/n$ is a rational number (m, n relatively prime integers) and that $\beta = 0$ in case $\alpha = 1$. He showed that $p = p(x; \alpha, \beta, c)$ satisfies a linear partial differential equation with constant coefficients,

$$K_1 \frac{\partial^{b_1+a_1} p}{\partial c^{b_1} \partial x^{a_1}} + K_2 \frac{\partial^{b_2+a_2} p}{\partial c^{b_2} \partial x^{a_2}} + K_3 \frac{\partial^{b_3+a_3} p}{\partial c^{b_3} \partial x^{a_3}} = 0. \tag{2.10}$$

Here the a_i, b_i, K_i, with $i = 1, 2, 3$, depend on m, n and β but are not uniquely determined. This fact can be used to simplify the partial differential equation for p. In case $\alpha \neq 1$ (but not necessarily rational) Medgyessy [44] obtained two partial integro-differential equations for p. Medgyessy used his results to study the decomposition of mixtures of stable distributions.

Among the analytical properties of stable frequency functions that were thoroughly investigated, we must finally mention their asymptotic behavior. A. V. Skorohod [54] gave a comprehensive survey of asymptotic formulas; he considers the cases shown in table I. More recently V. M. Zolotarev [70] gave

TABLE I

	$\alpha < 1$	$\alpha = 1$	$\alpha > 1$
$\beta = 1$	$x \to \infty$ $x \to 0 (x > 0)$	$x \to \infty$ $x \to -\infty$	$x \to \infty$ $x \to -\infty$
$-1 < \beta < 1$	$x \to \infty$ $x \to -\infty$	$x \to \infty$ $x \to -\infty$	$x \to \infty$ $x \to -\infty$
$\beta = -1$	$x \to -\infty$ $x \to 0 (x < 0)$	$x \to -\infty$ $x \to \infty$	$x \to -\infty$ $x \to \infty$

asymptotic expansions for stable distributions as the exponent α tends to the points $\alpha = 0$ and $\alpha = 1$.

3. Unimodality of certain families of distribution functions

A distribution function $F(x)$ is said to be unimodal if there exists at least one value $x = a$ such that $F(x)$ is convex for $x < a$ and concave for $x > a$.

A. Wintner [60], [61] was one of the first to investigate unimodal distributions. He obtained

THEOREM 3.1. *The limiting distribution of a sequence of unimodal distributions is unimodal.*

THEOREM 3.2. *The convolution of two symmetric and unimodal distributions is also symmetric and unimodal.*

Another important result is due to A. I. Khinchin [13].

THEOREM 3.3. *The function $f(t)$ is the characteristic function of a unimodal distribution if, and only if, it can be represented in the form $f(t) = (1/t)\int_0^t g(u)\,du$ where $g(u)$ is some characteristic function.*

The study of unimodal distributions was recently stimulated by the discovery of an error in a proof by A. I. Lapin. Lapin asserted that the convolutions of unimodal distributions (not only of symmetric unimodal distributions) are unimodal. Subsequently Lapin's (erroneous) statement was used to show that the distributions of the L-class, and hence also all stable distributions, are unimodal. A distribution function $F(x)$, and its characteristic function $f(t)$, are said to belong to the L-class if for every c, with $0 < c < 1$, there exists a characteristic function $f_c(t)$ such that the relation $f(t) = f(ct)f_c(t)$ holds for all t. The distributions (characteristic functions) of the L-class also are called self-decomposable distributions (characteristic functions). The error in Lapin's proof was pointed out by K. L. Chung [3] (see also appendix II of Gnedenko and Kolmogorov's book [9]) who gave also counterexamples. The question whether the self-decomposable and stable distributions are unimodal became therefore once more an open problem.

I. A. Ibragimov [10] introduced the concept of strongly unimodal distributions. He calls a distribution strongly unimodal if its convolution with any unimodal distribution is unimodal. Ibragimov showed that a (nondegenerate) distribution function $F(x)$ is strongly unimodal if, and only if, $F(x)$ is continuous and the function $\log F'(x)$ is concave on the set of points on which neither the right nor the left derivative of $F(x)$ vanishes.

I. A. Ibragimov [11] constructed an example of a characteristic function of the L-class which does not belong to a unimodal distribution.

A. Wintner [60] showed that all symmetric stable distributions are unimodal, he [62] also succeeded in proving the unimodality of the symmetric and self-decomposable distributions. Finally I. A. Ibragimov and K. E. Cernin [12] showed that all stable distributions are unimodal. It would be interesting to obtain a characterization of all unimodal distributions of the L-class.

R. G. Laha [18] recently obtained two sufficient conditions which insure that a real valued, even function $f(t)$ is the characteristic function of a unimodal distribution. These conditions are similar to Pólya's condition [46] and are easily applicable.

THEOREM 3.4. *Let $f(t)$ be a real valued and even function which satisfies the conditions*

(i) $f(0) = 1$,

(ii) $\lim_{|t|\to\infty} f(t) = 0$,

(iii) *the function $tf'(t)$ exists and is continuous for all real t, moreover* $\lim_{t\to 0} tf'(t) = \lim_{|t|\to\infty} tf'(t) = 0$,

(iv) *the function* $g(t) = f(t) + tf'(t)$ *is convex for* $t > 0$.

Then $f(t)$ *is the characteristic function of a symmetric and unimodal distribution.*

The function $g(t)$ satisfies the conditions of Pólya's theorem (see [46], theorem 1) and is therefore a characteristic function. It follows from the definition of $g(t)$ that $f(t) = (1/t)\int_0^t g(u)\,du$; according to theorem 3.3 the function $f(t)$ is then the characteristic function of a unimodal distribution.

THEOREM 3.5. *Let* $f(t)$ *be a real valued, continuous and even function of the real variable* t *such that* $f(0) = 1$. *Suppose that there exists a function* $A(z)$ *of the complex variable* z, *where* $z = t + iy$ *with* t, y *real, that satisfies the conditions*

(i) $f(t) = A(t)$ *for real* $t > 0$,

(ii) $A(z)$ *is regular in the region* $-\epsilon_1 < \arg z < \pi/2 + \epsilon_2$, $\quad \epsilon_1 > 0, \epsilon_2 > 0$,

(iii) $|A(z)| = O(1)$ *as* $|z| \to 0$, $|A(z)| = O(|z|^{-\alpha})$ *as* $|z| \to \infty$ *with* $\alpha > 1$,

(iv) $\operatorname{Im} A(iy) \leqq 0$ *for* $y > 0$,

where $A(z)$ *may also assume complex values. Then* $f(t)$ *is the characteristic function of an absolutely continuous, symmetric and unimodal distribution.*

It follows from assumption (iii) that $f(t)$ is absolutely integrable; the integral

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} f(t)\,dt = \frac{1}{\pi}\int_0^{\infty} \cos tx\, f(t)\,dt \tag{3.1}$$

exists and is a real valued, continuous and even function of x. We use assumption (i) to write (3.1) in the form

$$p(x) = \frac{1}{\pi}\operatorname{Re}\int_0^{\infty} e^{itx} f(t)\,dt = \frac{1}{\pi}\operatorname{Re}\int_0^{\infty} e^{itx} A(t)\,dt. \tag{3.2}$$

To evaluate the last integral we consider the integral

$$\int_C e^{izx} A(z)\,dz \tag{3.3}$$

along a contour C which consists of the following four parts: (1) the segment $r \leqq t \leqq R$ of the real axis, (2) the circular arc Γ of radius R with center at the origin located in the first quadrant, (3) the segment $r \leqq y \leqq R$ of the imaginary axis, (4) the circular arc γ of radius r with center at the origin located in the first quadrant. We see from assumption (ii) that

$$\int_C e^{izx} A(z)\,dz = 0, \tag{3.4}$$

we also deduce from (iii) that, for $x \geqq 0$,

$$\lim_{r\to 0}\int_\gamma e^{izx} A(z)\,dz = \lim_{R\to\infty}\int_\Gamma e^{izx} A(z)\,dz = 0, \tag{3.5}$$

so that

$$\int_0^{\infty} e^{itx} A(t)\,dt = i\int_0^{\infty} e^{-yx} A(iy)\,dy. \tag{3.6}$$

We conclude therefore from (3.2), (3.6) and assumption (iv) that

$$p(x) = -\frac{1}{\pi}\int_0^\infty e^{-yx}\,\mathrm{Im}\,[A(iy)]\,dy, \tag{3.7}$$

so that $p(x) \geqq 0$ for all real x. Let $0 < x_1 < x_2$, it follows from (3.7) and (iv) that

$$p(0) > p(x_1) > p(x_2), \tag{3.8}$$

hence $p(x)$ has a unique maximum at $x = 0$. This completes the proof of the theorem.

The application of these theorems yields

THEOREM 3.6. *The function*

$$f(t) = \frac{1}{1 + |t|^\alpha} \tag{3.9}$$

is, for any α in the interval $0 < \alpha \leqq 2$, the characteristic function of a unimodal distribution.

The fact that the function $f(t)$, as defined in theorem 3.6 is a characteristic function was already established by Yu. V. Linnik [26]. If $0 < \alpha \leqq 1$, theorem 3.6 follows from theorem 3.4; if $1 < \alpha < 2$ we obtain it from theorem 3.5. For $\alpha = 2$ formula (3.9) yields the characteristic function of the Laplace distribution which is known to be unimodal. It is easy to show that $f(t)$, as given by (3.9), cannot be a characteristic function if $\alpha < 0$ or if $\alpha > 2$.

A. Wintner's [60] result, that all symmetric stable distributions are unimodal, follows easily from theorem 3.6. Let n be a positive integer and write $a_n = n^{-1/\alpha}$. The function

$$g_n(t) = [f(a_n t)]^{-n} = \frac{1}{\left(1 + \frac{|t|^\alpha}{n}\right)^n} \tag{3.10}$$

is, according to theorem 3.2, the characteristic function of a symmetric, unimodal distribution. We see from theorem 3.1 that the same statement holds for the function

$$\lim_{n\to\infty} g_n(t) = e^{-|t|^\alpha}. \tag{3.11}$$

PART II. CHARACTERIZATION PROBLEMS

Let $X_1, X_2, \cdots, X_n$ be a sample from a given population or, more generally, n independently but not necessarily identically distributed random variables. For the sake of convenience we shall use in the following the term statistic for a single valued and measurable function of the $X_1, X_2, \cdots, X_n$ even in case these random variables are not identically distributed. In this part we consider the problem of characterizing the distributions of the random variables X_1, $X_2, \cdots, X_n$ by properties of certain statistics. The work done in this direction up to 1955 was surveyed at the Third Berkeley Symposium [38]. However,

research on characterization problems continued during the past five years. As a result of these recent investigations the problem has changed its character. Up to 1950 one studied in this connection mainly a few specific populations, while the more recent developments deal primarily with the analytical properties of the characteristic functions of the random variables $X_1, X_2, \cdots, X_n$.

In section 4, we report on some developments which are still closely connected with the earlier work. In section 5, we treat some of the recent results which have a more analytical character.

4. Characterization of certain populations

In this section we characterize populations by means of certain regression properties. Let Y and X be two random variables and assume that the first moment of Y and the kth moment of X exist and write $E(Y|X)$ for the conditional expectation of Y, given X. We introduce

DEFINITION 4.1. *The random variable Y is said to have polynomial regression of order k on X if the relation*

$$E(Y|X) = \beta_0 + \beta_1 X + \cdots + \beta_k X^k, \qquad \beta_k \neq 0, \tag{4.1}$$

holds almost everywhere.

If $k = 2$ we use the term quadratic regression, if $k = 1$ we speak about linear regression. If $k = 0$, that is if $E(Y|X) = E(Y)$ almost everywhere, then we say that Y has constant regression on X. The following lemma is a simple generalization of a result (lemma 6.1) in [38] and is proven in the same manner.

LEMMA 4.1. *Let X and Y be two random variables and assume that the expectations $E(Y)$ and $E(X^k)$ exist where k is a nonnegative integer. The random variable Y has polynomial regression of order k on X if, and only if, the relation*

$$E(Ye^{itX}) = \sum_{j=0}^{k} \beta_j E(X^j e^{itX}) \tag{4.2}$$

holds for all real t.

In an earlier paper [38] the author characterized the Poisson population by making the following two assumptions: (i) the population distribution function has the point $x = 0$ as its left extremity, (ii) the statistic $S = k_{p+2} - k_p$ has constant regression on k_1. Here $p \geqq 1$ is a positive integer and k_j denotes the k-statistic of order j, a symmetric and homogeneous polynomial statistic of order j such that $E(k_j) = \kappa_j$ where κ_j is the jth cumulant of the population distribution function. This result was recently generalized [40] by considering instead of S the difference of two k-statistics of arbitrary order and by removing the restriction that the population was one sided. The following result was obtained.

THEOREM 4.1. *Let $X_1, X_2, \cdots, X_n$ be a sample from a population with population distribution function $F(x)$ and denote by $p \geqq 1$, $r \geqq 1$ two positive integers. Assume that the moment of order $p + r$ of $F(x)$ exists. The population distribution $F(x)$ has the characteristic functions*

$$f(t) = \exp\left[\lambda_1(e^{it} - 1) + \delta_r\lambda_2(e^{-it} - 1) + c_1it + \frac{1}{2}\epsilon_p c_2 t^2\right] \tag{4.3}$$

if, and only if, the statistic $k_{p+r} - k_p$ *has constant regression on* k_1. *Here* $\lambda_1, \lambda_2, c_1, c_2$ *are real constants such that* $\lambda_1 \geqq 0$, $\lambda_2 \geqq 0$, $c_2 \geqq 0$ *while* $\delta_r = [r/2] - [(r-1)/2]$ *and* $\epsilon_p = 1$ *if* $p > 1$ *and* $\epsilon_1 = 0$.

The symbol $[x]$ denotes the largest integer not exceeding x.

This theorem characterizes a family of distributions which consists of convolutions of a Poisson distribution, the conjugate to a Poisson distribution and a normal distribution. For certain values of r and p one or two of these factors must be absent. This theorem shows that the assumption that $k_{p+r} - k_p$ has constant regression on k_1 is not sufficient to characterize the Poisson population. If one wishes to characterize the Poisson population one must impose some additional restriction.

THEOREM 4.2. *Let* $X_1, X_2, \cdots, X_n$ *be a sample from a population with population distribution function* $F(x)$ *and denote by* $p \geqq 1$, $r \geqq 1$ *two positive integers. Assume that*

(i) *the* $(p + r)$*th moment of* $F(x)$ *exists*

(ii) $F(x) = 0$ *for* $x < 0$ *while* $F(x) > 0$ *for* $x \geqq 0$.

The population is a Poisson population if, and only if $k_{p+r} - k_p$ *has constant regression on* k_1.

For the proof of these results certain extensions of the theorem of Marcinkiewicz are needed (see [39]). These provide necessary conditions which an entire function must satisfy in order to be a characteristic function.

M. C. K. Tweedie [59] investigated the regression of the sample variance on the sample mean. He characterized several populations by assuming that the sample variance has quadratic or linear regression on the sample mean. More recently, R. G. Laha and E. Lukacs [19] considered a general quadratic statistic

$$Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}X_iX_j + \sum_{j=1}^{n} b_jX_j \tag{4.4}$$

and determined all populations which have the property that Q has quadratic regression on the statistic $X_1 + X_2 + \cdots + X_n$. In this study it is necessary to distinguish several cases which are defined in terms of relations between the coefficients a_{ij} and b_j of Q and the regression coefficients $\beta_0, \beta_1, \beta_2$. The following distributions were obtained: (i) the normal distribution, (ii) Poisson-type distributions (that is, Poisson distributions with scale and location parameters), (iii) binomial and negative-binomial distributions, (iv) Gamma distributions, (v) distributions with characteristic function

$$f(t) = e^{i\mu t}[\cosh at + i\lambda \sinh at]^{-\rho}, \tag{4.5}$$

where a, λ, μ, ρ are real and $a \neq 0$, $\rho > 0$.

We conclude this section by mentioning two characterizations of the Wiener process. V. P. Skitovich [53] obtained the following result.

THEOREM 4.3. *Let* $X(t)$ *be a homogeneous stochastic process with independent*

increments which is defined in a closed interval $[A, B]$. *Let* $a(t)$ *and* $b(t)$ *be two functions which are continous in* $[A, B]$ *such that* $\int_A^B [a(t)b(t)]^2\,dt \neq 0$ *and that at least one of the integrals*

$$\int_A^B \frac{a^2(t)}{b^2(t)}\,dt \quad or \quad \int_A^B \frac{b^2(t)}{a^2(t)}\,dt \tag{4.6}$$

exists. Suppose that the two stochastic integrals $\int_A^B a(t)\,dX(t)$ *and* $\int_A^B b(t)\,dX(t)$ *are independently distributed, then* $X(t)$ *is a Wiener process.*

The stochastic integrals are here stochastic limits of Riemann-Stieltjes sums; however, the mode of stochastic convergence differs somewhat from the usual definitions. V. P. Skitovich requires that the distribution function of the Riemann sums should converge uniformly to the distribution of a random variable. This random variable is then called the stochastic integral. V. P. Skitovich also extended his earlier result concerning the independence of linear forms to linear forms in infinitely many, identically distributed random variables.

R. G. Laha and E. Lukacs [20] characterized the Wiener process by means of the following regression property.

THEOREM 4.4. *Let* $X(t)$ *be a stochastic process which satisfies the conditions*

(a) $X(t)$ *is defined in a finite, closed interval* $[A, B]$,

(b) $X(t)$ *is homogeneous and has independent increments,*

(c) $X(t)$ *is of second order and its mean value function and covariance function are of bounded variation in* $[A, B]$.

Suppose that $a(t)$ *and* $b(t)$ *are two continuous functions defined in* $[A, B]$ *such that* $a(t)b(t) \neq 0$ *for all* $t \in [A_1, B_1]$, *where* $A \leqq A_1 < B_1 \leqq B$. *Suppose further that* $a(t)$ *is not proportional to* $b(t)$. *Let*

$$Y = \int_A^B a(t)\,dX(t), \quad and \quad Z = \int_A^B b(t)\,dX(t) \tag{4.7}$$

be two stochastic integrals, defined as limits in the mean. The process $X(t)$ *is a Wiener process if, and only if,*

(i) Y *has linear regression on* Z

(ii) *The conditional variance of* Y, *given* Z, *does not depend on* Z.

5. Analytical aspects of the characterization problem

The characterization problems which we studied in the preceding section, and also those treated in part II of [38], dealt with specific statistics, such as the sample mean and k-statistics. Properties of the distributions of these statistics were used to characterize completely certain populations, except for the numerical values of some parameters. The solution of these problems usually was carried out in three steps: (1) The assumptions concerning the distribution of the statistics are used to derive a differential equation for the characteristic function of the population or the characteristic functions of the random variables in case

they are not identically distributed, (2) this differential equation is solved, (3) those solutions are determined that are characteristic functions. Frequently the last step is the most difficult, sometimes the differential equation is so complicated that it cannot be solved readily. These difficulties lead then to investigations concerning the analytical properties of the solutions of those differential equations that are characteristic functions.

The first results in this direction were obtained by A. A. Zinger and Yu. V. Linnik [66], [67]. To formulate these results it is necessary to introduce the following terminology.

We consider an ordinary differential equation

$$\sum A_{j_1 \cdots j_n} f^{(j_1)}(t) \cdots f^{(j_n)}(t) = c[f(t)]^n \tag{5.1}$$

and denote the order of the differential equation by m. The $A_{j_1 \cdots j_n}$ are real constants and the sum is taken over all nonnegative integers which satisfy the condition

$$j_1 + j_2 + \cdots + j_n \leqq p. \tag{5.2}$$

Here p is an integer such that at least one of the coefficients $A_{j_1 \cdots j_n}$ with $j_1 + \cdots + j_n = p$ is different from zero. We adjoin to the differential equation (5.1) a polynomial

$$A(x_1, x_2, \cdots, x_n) = \frac{1}{n!} \sum{}^* \sum A_{j_1 \cdots j_n} x_{k_1}^{j_1} \cdots x_{k_n}^{j_n}, \tag{5.3}$$

where the first summation $\sum^*$ runs over all permutations $(k_1, k_2, \cdots, k_n)$ of the first n positive integers while the second summation is taken over all subscripts satisfying (5.2).

The differential equation (5.1) is said to be positive definite if the adjoint polynomial (5.3) is nonnegative.

A. A. Zinger and Yu. V. Linnik obtained

THEOREM 5.1A. *Suppose that the function $f(t)$ is, in a certain neighborhood of the origin, a solution of the positive definite differential equation* (5.1) *and assume that $m \geqq n - 1$. If the solution is a characteristic function, then it is necessarily an entire function.*

We indicate here also the motivation for this study. Let

$$P = P(X_1, X_2, \cdots, X_n) = \sum A_{j_1 \cdots j_n} X_1^{j_1} \cdots X_n^{j_n} \tag{5.4}$$

be a polynomial statistic of degree p. We adjoin to P the statistic

$$P^* = \frac{1}{n!} \sum{}^* \sum A_{j_1 \cdots j_n} X_{k_1}^{j_1} \cdots X_{k_n}^{j_n}, \tag{5.5}$$

where the summations $\sum^*$ and $\sum$ are taken in the same way as in formula (5.3).

The statistic P is said to be a regular polynomial statistic of degree p and order m if the following three conditions are satisfied:

(i) The statistic P^*, adjoint to P, is a nonnegative polynomial,

(ii) no exponent in P exceeds m,

(iii) at least one variable in P has the exponent m.

The assumption that a regular polynomial statistic P has constant regression on the sample mean $\overline{X} = (X_1 + X_2 + \cdots + X_n)/n$ leads then to a positive definite differential equation for the characteristic function of the population. In this case, one can apply theorem 5.1a and obtain

THEOREM 5.1B. *Let $X_1, X_2, \cdots, X_n$ be a sample from a population with distribution function $F(x)$ and assume that $F(x)$ has moments up to order m. Let Λ be the sum $X_1 + X_2 + \cdots + X_n$ and let*

$$P = \sum A_{j_1 \cdots j_n} X_1^{j_1} \cdots X_n^{j_n} \tag{5.6}$$

be a regular polynomial statistic of degree p and order m, where $m \leqq p$. If

(i) *P has constant regression on Λ,*

(ii) *$m \geqq n - 1$,*

then the characteristic function $f(t)$ of $F(x)$ is an entire function.

Theorem 5.1A is a very interesting result concerning the analytic properties of the solutions of certain ordinary differential equations. If one wishes to use it in connection with characterization problems one obtains theorem 5.1B. However, in applying theorem 5.1B one is greatly handicapped by the severe restrictions contained in its assumptions. This is illustrated by the following facts. The positive definiteness of the differential equation excludes the k-statistics k_p of order $p > 2$, whereas we know that the normal distribution is characterized by the property that k_p has constant regression on Λ for any $p \geqq 2$. Moreover, in the case of k_2, condition (ii) of theorem 5.1B restricts the sample size n to 3. It would therefore be desirable to derive similar results under modified conditions.

In the early attempts to characterize populations by the independence of two statistics it was always assumed that certain moments of the population distribution function exist. Later it was possible to relax or remove this assumption in special cases. For example, K. C. Chanda [2] and Yu. V. Linnik [29] study the independence of a polynomial and a linear statistic and show that it is sufficient to suppose that the moment of order $\delta > 0$ exists, where δ is not necessarily an integer and can be arbitrarily small.

A. A. Zinger [63], [64] investigated the independence of two polynomial statistics and eliminated the assumption concerning the existence of moments. For the formulation of his results we need the following definition.

A polynomial $P = P(x_1, x_2, \cdots, x_n)$ of degree m is said to be admissible, if the coefficients of the terms x_j^m, where $j = 1, 2, \cdots, n$, are not zero. Here and in the following we assume that similar terms have been collected in every polynomial which we consider.

THEOREM 5.2. *Let $X_1, X_2, \cdots, X_n$ be n independently (but not necessarily identically) distributed random variables. Let $P_1(X_1, \cdots, X_n)$ and $P_2(X_1, \cdots, X_n)$ be two admissible polynomials. If P_1 and P_2 are independent, then each X_j, where $j = 1, 2, \cdots, n$, has finite moments of all orders.*

A stronger result is obtained if one of the polynomials is a linear statistic.

THEOREM 5.3. *Let $X_1, X_2, \cdots, X_n$ be n independently (but not necessarily identically) distributed random variables with characteristic functions $f_1(t), f_2(t), \cdots, f_n(t)$ respectively. Let $P = P(X_1, X_2, \cdots, X_n)$ be an admissible polynomial statistic and $\Lambda = \Lambda(X_1, X_2, \cdots, X_n) = \sum_{j=1}^{n} a_j X_j$, with $a_j \neq 0$ for $j = 1, 2, \cdots, n$, be a linear form. If P and Λ are independent then the characteristic functions $f_j(t)$, for $j = 1, 2, \cdots, n$, are entire functions of finite order.*

Theorems 5.2 and 5.3 are due to A. A. Zinger who showed that they are valid also for admissible quasipolynomial statistics. A function $S = S(x_1, x_2, \cdots, x_n)$ is called a quasipolynomial if there exists a continuous function $T(x)$ and two nonnegative polynomials $P_1 = P_1(x_1, x_2, \cdots, x_n)$ and $P_2 = P_2(x_1, x_2, \cdots, x_n)$ of the same degree such that the inequality

$$P_1(x_1, x_2, \cdots, x_n) \leqq T[S(x_1, x_2, \cdots, x_n)] \leqq P_2(x_1, x_2, \cdots, x_n) \tag{5.7}$$

is satisfied for all $x_1, x_2, \cdots, x_n$. A quasipolynomial is said to be admissible if P_1 is an admissible polynomial. We give next a condition which assures that $f_j(t)$ is the characteristic function of a normal distribution.

COROLLARY TO THEOREM 5.3. *Suppose that the conditions of theorem 5.3 are satisfied and that the characteristic function $f_j(z)$ has no zeros in the entire complex plane, then the random variable X_j is normally distributed.*

According to theorem 5.3, the function $f_j(z)$ is an entire function of finite order m without zeros. We apply Hadamard's factorization theorem and see that $f_j(z) = \exp[P_m(z)]$ where P_m is a polynomial of degree m. The statement of the corollary follows then from Marcinkiewicz' theorem.

This corollary is of no immediate use in characterization problems since one of its conditions is expressed in terms of the characteristic function $f_j(t)$. It is therefore desirable to find a condition which the polynomial statistic $P(X_1, \cdots, X_n)$ of theorem 5.3 must satisfy in order that $f_j(z)$ should have no (real or complex) zeros. Before proceeding further we must introduce a special class of polynomials. Let

$$P(x_1, \cdots, x_n) = \sum_{j_1 + \cdots + j_n \leqq p} A_{j_1 \cdots j_n} x_1^{j_1} \cdots x_n^{j_n} \tag{5.8}$$

be a polynomial of degree p; it can be written as the sum

$$P(x_1, \cdots, x_n) = P_0(x_1, \cdots, x_n) + P_1(x_1, \cdots, x_n), \tag{5.9}$$

where

$$P_0(x_1, \cdots, x_n) = \sum_{j_1 + \cdots + j_n = p} A_{j_1 \cdots j_n} x_1^{j_1} \cdots x_n^{j_n} \tag{5.10}$$

is a homogeneous polynomial of degree p, while $P_1(x_1 \cdots, x_n)$ is a polynomial of degree less than p. We say that the polynomial $P(x_1, \cdots, x_n)$ is nonsingular if the following two conditions are satisfied:

(i) $P_0(x_1, \cdots, x_n)$ contains the pth power of at least one variable,

(ii) $\pi_0(\nu) \neq 0$ for all integers $\nu > 0$. Here $\pi_0(\nu)$ is the polynomial formed by replacing each positive power x_s^j by $\nu^{(j)} = \nu(\nu - 1) \cdots (\nu - j + 1)$ in $P_0(x_1, \cdots, x_n)$.

We call $\pi_0(\nu)$ the adjoint polynomial to $P(x_1, \cdots, x_n)$.

THEOREM 5.4. *Let $X_1, X_2, \cdots, X_n$ be n independently and identically distributed random variables and assume that the characteristic function $f(z)$ of their common distribution is an entire function. Suppose that a nonsingular polynomial statistic*

$$P = P(X_1, \cdots, X_n) = \sum_{j_1+\cdots+j_n \leqq p} A_{j_1 \cdots j_n} X_1^{j_1} \cdots X_n^{j_n} \tag{5.11}$$

of degree p has constant regression on $\Lambda = X_1 + \cdots + X_n$. The characteristic function $f(z)$ has then no zero in the whole complex plane.

We emphasize that the characteristic function $f(t)$ is defined for all complex arguments $z = t + iy$, where t, y real, by writing it as $f(z)$. Theorem 5.4 is due to Yu. V. Linnik [29]; we give here a somewhat simplified proof.

It easily follows from lemma 4.1 that the relation

$$E(Pe^{iz\Lambda}) = E(P)\, E(e^{iz\Lambda}) \tag{5.12}$$

holds for all complex z. We write

$$f^{(j)} = f^{(j)}(z) = \frac{d^j}{dz^j} f(z) = i^j E(X^j e^{izX}) \tag{5.13}$$

and note that $f^{(0)}(z) = f(z)$. We see from (5.12) that

$$\sum_{j_1+\cdots+j_n \leqq p} i^{-(j_1+\cdots+j_n)} A_{j_1 \cdots j_n} f^{(j_1)} \cdots f^{(j_n)} = E(P)[f(t)]^n. \tag{5.14}$$

We give an indirect proof of the theorem and assume therefore that the function $f(z)$ has zeros. Let the point $z = z_0$ be one of the zeros of $f(z)$ which are nearest to the origin and denote the order of z_0 by ν, where ν is a positive integer.

Since $f(z)$ does not vanish in the circle $|z| < |z_0|$ we may divide (5.14) by $[f(z)]^n$. We write

$$R_0(z) = \sum_{j_1+\cdots+j_n = p} A_{j_1 \cdots j_n} \frac{f^{(j_1)} \cdots f^{(j_n)}}{f^n} \tag{5.15}$$

and

$$R_1(z) = \sum_{j_1+\cdots+j_n < p} i^{p-(j_1+\cdots+j_n)} A_{j_1 \cdots j_n} \frac{f^{(j_1)} \cdots f^{(j_n)}}{f^n} \tag{5.16}$$

and see that

$$R_0(z) + R_1(z) = C, \qquad |z| < |z_0|, \tag{5.17}$$

where $C = i^p E(P)$. Let

$$\phi = \phi(z) = \log f(z); \tag{5.18}$$

it is then easily verified that

$$\frac{f^{(j)}}{f} = \phi^{(j)} + \theta_j(\phi', \phi'', \cdots, \phi^{(j-1)}), \qquad j = 1, 2, \cdots, \tag{5.19}$$

where θ_j is a polynomial in $\phi', \phi'', \cdots, \phi^{(j-1)}$. We also write $\phi^{(0)} \equiv 1$ and $\theta_0 \equiv 0$.

We substitute (5.19) into (5.17) and get for $|z| < |z_0|$

$$S_0(z) + S_1(z) = C, \tag{5.20}$$

where

$$S_0(z) = \sum_{j_1+\cdots+j_n=p} A_{j_1\cdots j_n}[\phi^{(j_1)} + \theta_{j_1}]\cdots[\phi^{(j_n)} + \theta_{j_n}] \tag{5.21}$$

and

$$S_1(z) = \sum_{j_1+\cdots+j_n<p} i^{p-(j_1+\cdots+j_n)} A_{j_1\cdots j_n}[\phi^{(j_1)} + \theta_{j_1}]\cdots[\phi^{(j_n)} + \theta_{j_n}]. \tag{5.22}$$

According to our assumptions

$$f(z) = (z - z_0)^\nu g(z), \tag{5.23}$$

where $g(z)$ is an entire function such that $g(z_0) \neq 0$. It is easy to verify that

$$\phi'(z) = \frac{\nu}{z - z_0} + h_1(z), \tag{5.24}$$

and in general

$$\phi^{(j)}(z) = \frac{(-1)^{j-1}(j-1)!\nu}{(z - z_0)^j} + h_j(z), \qquad j = 1, 2, \cdots. \tag{5.25}$$

Here the functions $h_j(z)$ are regular at $z = z_0$. We substitute (5.25) into (5.20) and see that

$$\frac{\gamma_p}{(z - z_0)^p} + \frac{\gamma_{p-1}}{(z - z_0)^{p-1}} + \cdots + \frac{\gamma_1}{z - z_0} + H(z) = C, \tag{5.26}$$

where $H(z)$ is regular at $z = z_0$.

We note that relation (5.26) leads to a contradiction if at least one of the coefficients $\gamma_1, \gamma_2, \cdots, \gamma_p$ is different from zero. We complete the proof of the theorem by showing that $\gamma_p \neq 0$.

We remark that γ_p depends only on ν and on the coefficients of the homogeneous polynomial $P_0(x_1, x_2, \cdots, x_n)$. We see that γ_p is the coefficient of $(z - z_0)^{-p}$ in the expression which we obtain by substituting (5.25) into (5.21). We get the same value for the coefficient of $(z - z_0)^{-p}$ if we substitute (5.23) into (5.15). The coefficient γ_p can also be obtained by substituting $\psi(z) = C_1(z - z_0)^\nu$ instead of $f(z)$ into (5.15). Here $C_1 \neq 0$ is a constant. We see then that

$$\psi^{(j)}(z) = C_1\nu^{(j)}(z - z_0)^{\nu-j}, \qquad j = 1, 2, \cdots, \nu. \tag{5.27}$$

Therefore

$$\gamma_p = \sum_{j_1+\cdots+j_n=p} A_{j_1\cdots j_n}\nu^{(j_1)}\cdots\nu^{(j_n)}. \tag{5.28}$$

Thus $\gamma_p = \pi_0(\nu) \neq 0$ for all positive integer values of ν and theorem 5.4 is proved. We can use theorems 5.3 and 5.4 and the corollary to theorem 5.4 to get the following characterization of the normal distribution.

THEOREM 5.5. *Let $X_1, X_2, \cdots, X_n$ be n independently and identically distributed random variables. Let $P = P(X_1, X_2, \cdots, X_n)$ be an admissible, nonsingular polynomial statistic and $\Lambda = X_1 + X_2 + \cdots + X_n$. If P and Λ are independent then the common distribution of the random variables X_j is normal.*

We consider applications of theorem 5.5.

THEOREM 5.6. *Let $X_1, X_2, \cdots, X_n$ be n independently and identically distri-*

buted random variables with common distribution function $F(x)$. *Let* $\Lambda = X_1 + X_2 + \cdots + X_n$ *and* $P = P(X_1, X_2, \cdots, X_n)$ *be an admissible, homogeneous polynomial statistic of degree* p *satisfying*

(i) P *and* Λ *are independently distributed,*

(ii) $E(P) = \kappa_p$ *where* κ_p *is the* p*th cumulant of* $F(x)$. *Then* $F(x)$ *is normal.*

Note that the existence of the cumulants of $F(x)$ is insured by the fact that $F(x)$ has, according to theorem 5.4, an entire characteristic function. Let $f(t)$ be the characteristic function of $F(x)$. Then $\phi(t) = \log f(t)$ exists in a certain neighborhood of the origin. It follows from the assumptions of our theorem that $\phi^{(p)}(t) = d^p\phi(t)/dt^p$ exists. It is easily seen by induction that $\phi^{(p)}(t)$ is a polynomial in $f'/f, f''/f, \cdots, f^{(p)}/f$ which has the form

$$\phi^{(p)}(t) = \sum \lambda_{s_1 s_2 \cdots s_p} \left(\frac{f'}{f}\right)^{s_1} \left(\frac{f''}{f}\right)^{s_2} \cdots \left(\frac{f^{(p)}}{f}\right)^{s_p} = G\left(\frac{f'}{f}, \frac{f''}{f}, \cdots, \frac{f^{(p)}}{f}\right). \tag{5.29}$$

The summation is here extended over all nonnegative integers $s_1, s_2, \cdots, s_p$ which satisfy the relation

$$s_1 + 2s_2 + \cdots + ps_p = p. \tag{5.30}$$

We put $t = 0$ in (5.29) and obtain

$$\kappa_p = \sum \lambda_{s_1 s_2 \cdots s_p} \alpha_1^{s_1} \alpha_2^{s_2} \cdots \alpha_p^{s_p} = G(\alpha_1, \alpha_2, \cdots, \alpha_p), \tag{5.31}$$

where α_j is the jth moment of $F(x)$ and where the summation is extended over all subscripts satisfying (5.30). Let

$$P = \sum_{j_1 + \cdots + j_n = p} A_{j_1 \cdots j_n} X_1^{j_1} \cdots X_n^{j_n}; \tag{5.32}$$

it is always possible to choose coefficients $b_{s_1 \cdots s_p; j_1 \cdots j_n}$ so that

$$P = \sum \lambda_{s_1 \cdots s_p} \sum{}^* b_{s_1 \cdots s_p; j_1 \cdots j_n} X_1^{j_1} \cdots X_n^{j_n}. \tag{5.33}$$

Here, and in the following, the summation $\sum$ is extended over all $s_1, \cdots, s_p$ satisfying (5.30) while the summation $\sum^*$ runs over all permutations of $(j_1, \cdots, j_n)$ such that s_r of these exponents equal r for $r = 1, 2, \cdots, p$ and the remaining $n - s_1 - \cdots - s_p$ exponents are zero.

It follows from (5.33) that

$$E(P) = \sum \lambda_{s_1 \cdots s_p} \alpha_1^{s_1} \cdots \alpha_p^{s_p} \sum{}^* b_{s_1 \cdots s_p; j_1 \cdots j_n}. \tag{5.34}$$

Hence we see from condition (i) and (5.31) that

$$\sum{}^* b_{s_1 \cdots s_p; j_1 \cdots j_n} = 1 \tag{5.35}$$

for all p-tuples $(s_1, \cdots, s_p)$ for which $\lambda_{s_1 \cdots s_p} \neq 0$. We form now the polynomial $\pi_0(\nu)$ corresponding to P. We get from (5.33)

$$\pi_0(\nu) = \sum \lambda_{s_1 \cdots s_p} \sum{}^* b_{s_1 \cdots s_p; j_1 \cdots j_n} [\nu^{(1)}]^{s_1} \cdots [\nu^{(p)}]^{s_p} \tag{5.36}$$

and thus in view of (5.35) and (5.31)

$$\pi_0(\nu) = G(\nu^{(1)}, \cdots, \nu^{(p)}). \tag{5.37}$$

Since κ_p is the coefficient of $t^p/p!$ in the expansion of $\log [\sum_j \alpha_j t^j/j!]$, we note

that $\pi_0(\nu)$ is the coefficient of $t^p/p!$ in the expansion of $\log [\sum_j \nu^{(j)}t^j/j!] = \log [(1 + t)^\nu]$. We obtain therefore

$$\pi_0(\nu) = (-1)^{p-1}(p - 1)!\,\nu. \tag{5.38}$$

Since $\pi_0(\nu) \not\equiv 0$ for all positive integers ν, we see that P is a nonsingular polynomial. Theorem 5.6 follows then immediately from theorem 5.2.

The conditions of theorem 5.6 are satisfied if P is the k-statistic of order p. This special case was already proved earlier by several authors (see [38], p. 201). However, it was then assumed that the pth moment of $F(x)$ exists. The more general methods used here make this assumption unnecessary.

A second class of polynomial statistics for which it is possible to compute the adjoint polynomial are the central moments.

LEMMA 5.1. *Let p be a positive integer and write*

$$P = P(x_1, \cdots, x_n) = \sum_{k=1}^{n} (x_k - \bar{x})^p, \qquad \bar{x} = \frac{1}{n}\sum_{k=1}^{n} x_k. \tag{5.39}$$

The adjoint polynomial of P is then

$$\pi_p(\nu) = \frac{(-1)^p p!}{n^{p-1}} \sum_{r=0}^{p} (-1)^r (n - 1)^r \binom{\nu}{r}\binom{n\nu - \nu}{p - r}. \tag{5.40}$$

For the proof of this lemma we refer to [22]. We can use the explicit form of $\pi_p(\nu)$ to derive a condition which assures that the sample moment of order p is a nonsingular polynomial statistic.

LEMMA 5.2. *Let p be a positive integer and write*

$$P = P(x_1, \cdots, x_n) = \sum_{k=1}^{n} (x_k - \bar{x})^p, \qquad \bar{x} = \frac{1}{n}\sum_{k=1}^{n} x_k. \tag{5.41}$$

If $(p - 1)!$ is not divisible by $n - 1$, the adjoint polynomial $\pi_p(\nu)$ of P has no nonzero integer roots.

Suppose that for some integer ν, with $\nu \not\equiv 0$, we have $\pi_p(\nu) = 0$. Then we see from lemma 5.1 that

$$\begin{aligned}\binom{n\nu - \nu}{p} &= (n - 1)\binom{\nu}{1}\binom{n\nu - \nu}{p - 1} \\ &\quad - (n - 1)^2\binom{\nu}{2}\binom{n\nu - \nu}{p - 2} + \cdots + (-1)^{p-1}(n - 1)^p\binom{\nu}{p}.\end{aligned} \tag{5.42}$$

Thus multiplying by $p!$ and cancelling the common factor $(n - 1)\nu$ we find that

$$(n\nu - \nu - 1)(n\nu - \nu - 2)\cdots(n\nu - \nu - p + 1) \equiv 0 \{\mathrm{mod}\ (n - 1)\} \tag{5.43}$$

so that

$$(p - 1)! \equiv 0 \{\mathrm{mod}\ (n - 1)\}. \tag{5.44}$$

We consider in the following a sample $X_1, X_2, \cdots, X_n$ from a certain population and write

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n), \qquad m_p = \frac{1}{n}\sum_{j=1}^{n} (X_j - \bar{X})^p \tag{5.45}$$

for the sample mean and the sample central moment of order p, respectively. We prove

THEOREM 5.7. *Let $X_1, X_2, \cdots, X_n$ be a sample of size n from a certain population. Let p be a positive integer such that $(p-1)!$ is not divisible by $(n-1)$. The population is normal if, and only if, the sample central moment m_p of order p is independently distributed of the sample mean $\bar{X}$.*

REMARK. The condition that $(p-1)!$ is not divisible by $(n-1)$ is satisfied if $n > (p-1)! + 1$.

The necessity of the condition of theorem 5.7 follows from the well known fact that in a normal population any translation-invariant statistic is independent of the sample mean. The sufficiency of the condition follows from theorem 5.5 and lemma 5.2.

We conclude this section by mentioning two other recent developments.

Yu. V. Linnik [28] studied the possibility of determining the family of distribution functions to which the population distribution function belongs from the distribution function of a statistic. The second problem is of a different nature since it concerns the characterization of distributions and not of populations. Let X and Y be two independently and identically distributed random variables and assume that their common distribution is normal with zero mean. It is then known that the quotient $Z = X/Y$ is distributed according to a Cauchy law. A number of authors investigated whether the normal distribution could be characterized by this property. Counterexamples were constructed by R. G. Laha [15], G. P. Steck [56], and J. G. Mauldon [42]. R. G. Laha [16], [17] undertook a systematic study of the family C of distributions $F(x)$ which have the following property: If X and Y are two independently and identically distributed random variables with common distribution $F(x)$, then the quotient X/Y has a Cauchy distribution. He obtained a number of properties of $F(x)$ and could also characterize the normal distribution, assuming $F(x) \in C$ and some additional restrictions which include the existence of all moments of $F(x)$.

PART III. THE ARITHMETIC OF DISTRIBUTION FUNCTIONS AND RELATED TOPICS

The arithmetic of distribution functions deals with the decomposition of characteristic functions into factors which are characteristic functions of non-degenerate distributions. Most of the recent developments in this area were stimulated by some results obtained by H. Cramér [4], P. Lévy [24] and D. A. Raikov [47] more than twenty years ago. We state here two of the classical results.

CRAMÉR'S THEOREM. *Let $f(t) = \exp(i\mu t - \sigma^2 t^2/2]$ be the characteristic function of the normal distribution and suppose that $f(t) = f_1(t)f_2(t)$ where $f_1(t)$ and $f_2(t)$ are characteristic functions. Then $f_1(t)$ and $f_2(t)$ are necessarily characteristic functions of normal distributions.*

This theorem was first conjectured by P. Lévy and later proved by H. Cramér

and it is therefore often called the theorem of Lévy-Cramér. A similar property of the Poisson distribution was established by D. A. Raikov.

RAIKOV'S THEOREM. *Let* $f(t) = \exp[\lambda(e^{it} - 1)]$ *be the characteristic function of the Poisson distribution and suppose that* $f(t) = f_1(t)f_2(t)$ *where* $f_1(t)$ *and* $f_2(t)$ *are characteristic functions. Then* $f_1(t)$ *and* $f_2(t)$ *are necessarily characteristic functions of Poisson distributions.*

We see immediately from the theorems of Cramér and Raikov that the normal distribution, as well as the Poisson distribution, belongs to the family of infinitely divisible distributions that have no indecomposable factors. We shall discuss this family in section 6, and we shall treat other problems which are closely connected with the investigations of Cramér, Lévy, and Raikov in sections 7 and 8.

6. Infinitely divisible characteristic functions that have no indecomposable factors

D. A. Raikov raised (in [47]) several questions concerning the structure of infinitely divisible characteristic functions that have no indecomposable factors. These problems are very difficult and the first advances in this area were made by Yu. V. Linnik approximately twenty years after the publication of Raikov's paper. Yu. V. Linnik's work on the factorization of infinitely divisible laws will be discussed in this section.

D. A. Raikov and P. Lévy established the following remarkable fact: The convolution of two Poisson type distributions has no indecomposable factors; however, convolutions of three Poisson type distributions may have indecomposable factors. This result made it desirable to investigate the factorization of the convolution of a normal and a Poisson distribution. Yu. V. Linnik [30], [31] obtained

THEOREM 6.1. *Let*

$$f(t) = \exp\left\{\lambda(e^{it} - 1) + i\mu t - \frac{1}{2}\sigma^2 t^2\right\}, \qquad \mu \text{ real}, \quad \sigma^2 \geqq 0, \quad \lambda \geqq 0, \tag{6.1}$$

be the characteristic function of the convolution of a normal and of a Poisson distribution. Suppose that $f(t)$ *admits the decomposition* $f(t) = f_1(t)f_2(t)$. *Then*

$$f_j(t) = \exp\left\{\lambda_j(e^{it} - 1) + i\mu_j t - \frac{1}{2}\sigma_j^2 t^2\right\}, \qquad j = 1, 2, \tag{6.2}$$

where

$$\lambda = \lambda_1 + \lambda_2, \qquad \sigma^2 = \sigma_1^2 + \sigma_2^2, \qquad \lambda_j \geqq 0, \qquad \sigma_j^2 \geqq 0. \tag{6.3}$$

Theorem 6.1 contains Cramér's theorem and Raikov's theorem as special cases. However, the method of proof is entirely different and requires more powerful analytical tools. This is explained by the fact that Cramér's theorem deals with an entire characteristic function of finite order while Raikov's theorem is concerned with a periodic characteristic function with real period (thus necessarily the characteristic function of a lattice distribution). Under the assump-

tions of theorem 6.1 both these advantages are lost, and the proof becomes much more complicated and requires some sharp estimates used in Vinogradov's study of trigonometric sums.

The method developed for the proof of theorem 6.1 also permitted the attack of the more general factorization problems of infinitely divisible laws. A series of papers by Yu. V. Linnik [32], [33], [34], [35], [36] contains his investigations on this subject; the principal problem is the determination of the structure of the infinitely divisible laws that have no indecomposable factors. In the following, we call this class of characteristic functions the class I_0. In order to discuss Linnik's investigations of the class I_0 we must introduce certain terms and notation.

It is well known that every infinitely divisible characteristic function $f(t)$ can be written in the canonical form

$$\log f(t) = ita - \frac{1}{2}\sigma^2t^2 + \int_{-\infty}^{-0}\left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)dM(u) + \int_{+0}^{+\infty}\left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)dN(u), \tag{6.4}$$

where the constants a and σ^2 are real, $\sigma^2 \geqq 0$, and where $M(u)$ and $N(u)$ satisfy

(i) $M(u)$ and $N(u)$ are nondecreasing in the intervals $(-\infty, 0)$ and $(0, +\infty)$ respectively,

(ii) the integrals $\int_{-\epsilon}^{0} u^2\,dM(u)$ and $\int_0^{\epsilon} u^2\,dN(u)$ are finite for every $\epsilon > 0$,

(iii) $M(-\infty) = N(+\infty) = 0$.

It is convenient to call $M(u)$ the negative and $N(u)$ the positive Poisson spectrum of $f(t)$. We say that an infinitely divisible characteristic function has bounded negative Poisson spectrum if there exists a positive number A such that $\int_{-\infty}^{-A} dM(u) = 0$. Similarly we define infinitely divisible distributions with bounded positive Poisson spectrum. We say that the Poisson spectrum of an infinitely divisible characteristic function $f(t)$ is bounded, if its positive as well as its negative Poisson spectrum is bounded. An infinitely divisible characteristic function is said to have a finite spectrum if

$$\log f(t) = ait - \frac{1}{2}\sigma^2t^2 + \sum_{j=1}^{m}\lambda_j(e^{it\mu_j} - 1) + \sum_{j=1}^{n}\lambda_{-j}(e^{-it\nu_j} - 1). \tag{6.5}$$

Here m and n are nonnegative integers, $\lambda_j > 0$, $\lambda_{-j} > 0$, $\mu_j > 0$, $\nu_j > 0$. If either m or n is equal to zero then the corresponding sum is omitted. An infinitely divisible characteristic function $f(t)$ is said to have a denumerable Poisson spectrum if

$$\log f(t) = ait - \frac{1}{2}\sigma^2t^2 + \sum_{j=1}^{\infty}\lambda_j\left(e^{it\mu_j} - 1 - \frac{it\mu_j}{1+\mu_j^2}\right) + \sum_{j=1}^{\infty}\lambda_{-j}\left(e^{-it\nu_j} - 1 + \frac{it\nu_j}{1+\nu_j^2}\right), \tag{6.6}$$

where $\lambda_j > 0$, $\lambda_{-j} > 0$ and where the series

$$\sum_{j=1}^{\infty} \frac{\lambda_j \mu_j^2}{1+\mu_j^2} \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{\lambda_{-j} \nu_j^2}{1+\nu_j^2} \tag{6.7}$$

are convergent and where

$$\sum_{\mu_j < \epsilon} \lambda_j \mu_j^2 + \sum_{\nu_j < \epsilon} \lambda_{-j} \nu_j^2 \tag{6.8}$$

tends to zero as ϵ approaches zero. The numbers μ_j and ν_j are called the Poisson frequencies, the λ_j and λ_{-j} the energy parameters of $f(t)$. A characteristic function with finite or denumerable Poisson spectrum is said to have a rational spectrum if $\mu_j/\mu_k = r_{jk}$ and $\nu_j/\nu_k = s_{jk}$ are rational numbers. Let $f(t)$ be a characteristic function with a bounded Poisson spectrum which is contained in the interval $[-A, B]$. We say that the spectrum of $f(t)$ is rational to the right of the point α, where $0 < \alpha < B$, if

$$\begin{aligned} \log f(t) = ita - \frac{1}{2}\sigma^2 t^2 + \int_{-A}^{-0} \left(e^{itu} - 1 - \frac{itu}{1+u^2}\right) dM(u) \\ + \int_{+0}^{\alpha} \left(e^{itu} - 1 - \frac{itu}{1+u^2}\right) dN(u) + \sum_{j=1}^{m} \lambda_j \left[\exp\left(\frac{ita_j\mu}{q}\right) - 1\right], \end{aligned} \tag{6.9}$$

where $\mu > 0$, $\lambda_j > 0$, where $j = 1, \cdots, m$, while $0 < a_1 < \cdots < a_m$ are integers such that $a_1\mu/q > \alpha$. In a similar manner we can define a Poisson spectrum which is rational to the left of the point β, where $-A < \beta < 0$.

We can now state some of Linnik's results.

THEOREM 6.2. *In order that an infinitely divisible characteristic function with normal component, with $\sigma^2 > 0$, should have no indecomposable factor it is necessary that its Poisson spectrum be finite or denumerable. Moreover, the Poisson frequencies of the positive spectrum must have the form*

$$\cdots, k_{-2}k_{-1}\mu, k_{-1}\mu, \mu, \frac{\mu}{k_1}, \frac{\mu}{k_1k_2}, \cdots, \frac{\mu}{k_1k_2\cdots k_s}, \cdots \tag{6.10}$$

while the Poisson frequencies of the negative spectrum must be

$$\cdots, g_{-2}g_{-1}\nu, g_{-1}\nu, \nu, \frac{\nu}{g_1}, \frac{\nu}{g_1g_2}, \cdots, \frac{\nu}{g_1g_2\cdots g_s}, \cdots \tag{6.11}$$

where the $\cdots, k_{-2}, k_{-1}, k_1, k_2, \cdots$ and the $\cdots, g_{-2}, g_{-1}, g_1, g_2, \cdots$ are arbitrary integers (not necessarily all different) which are greater than one. If the Poisson spectrum of $f(t)$ is bounded then this condition is also sufficient for the absence of indecomposable factors.

This is probably one of the most important advances in the arithmetic of distribution function since the publication of the fundamental studies of P. Lévy, A. I. Khinchin, and D. A. Raikov. The necessity of the condition of theorem 6.2 is established in [34]. The proof is based on several lemmas which are of independent interest. As an example we mention the following theorem (this is Linnik's basic lemma I).

THEOREM 6.3. *Let γ, λ_1 and λ_2 be three positive numbers and let $\alpha = p/q$ be a rational number where p and q are two integers such that $1 < p < q$ and $(p, q) = 1$. Then*

$$f(t) = \exp\left[-\gamma t^2 + \lambda_1(e^{it} - 1) + \lambda_2(e^{it\alpha} - 1) - \nu(e^{it/q} - 1)\right] \tag{6.12}$$

is a characteristic function provided that $\nu > 0$ is sufficiently small.

REMARK. This theorem can be used to construct convolutions of a normal, a Poisson, and a Poisson type distribution that have indecomposable factors. This is similar to the result of D. A. Raikov and P. Lévy concerning the factors of a convolution of three Poisson type distributions (see the paragraph before theorem 6.1).

The sufficiency of the condition of theorem 6.2 for characteristic functions with bounded Poisson spectrum is established in [35]. In the last mentioned paper we find also the proofs of several theorems, stated already in [34]. We quote here only one of these results.

THEOREM 6.4. *Let $f(t)$ be a characteristic function with bounded Poisson spectrum, then all its factors have the form*

$$g(t) = \exp\{P_3(it) + t^4 \int_{-A}^{B} e^{itu}\, \Phi(u)\, du\}, \tag{6.13}$$

where $P_3(it)$ is a polynomial of degree not exceeding three and where $\Phi(u)$ is quadratically integrable (in the Lebesgue sense) over the interval $[-A, B]$ which contains the spectrum of $f(t)$.

Linnik [35] studied infinitely divisible characteristic functions that have a normal component and a bounded Poisson spectrum which is rational to the right (or left) of a point of $[-A, B]$. He determined the possible factors of such characteristic functions.

The investigation of infinitely divisible characteristic functions without a normal component or of infinitely divisible characteristic functions with unbounded Poisson spectrum is very difficult. We mention here one of the few results (see [36]) which are known at present.

THEOREM 6.5. *Suppose that an infinitely divisible characteristic function $f(t)$ has the form* (6.6) *and that its Poisson frequencies μ_j and ν_j satisfy the conditions* (6.10) *and* (6.11). *Assume further that there exist positive constants μ, ν, c and α such that*

$$\log\log\frac{1}{\lambda_j} > c\,\mu_j^{1+\alpha}, \qquad \log\log\frac{1}{\lambda_{-j}} > c\,\nu_j^{1+\alpha}, \tag{6.14}$$

for all $\mu_j > \mu$ and $\nu_j > \nu$, then $f(t)$ has only infinitely divisible components.

The condition of the theorem means that the energy parameters corresponding to high frequencies decrease rapidly.

The theorems discussed in this section indicate that the class I_0 forms a rather small subset of the class of all infinitely divisible characteristic functions. The

problem of finding the indecomposable factors of infinitely divisible characteristic functions not belonging to the class I_0 has not yet been attacked. For instance, it is known (as a consequence of a more general result of H. Cramér [5]) that the Gamma distribution has indecomposable factors; however, these factors have not yet been determined.

7. Factor-closed and strongly factor-closed families

The theorems of Cramér and Raikov which we stated in the introductory paragraph of part III can be formulated in a somewhat different manner.

We say (see H. Teicher [57]) that a family $\mathfrak{f}$ of characteristic functions is factor-closed if the relations

(i) $f \in \mathfrak{f}$

(ii) $f = f_1 f_2$, where f_1, f_2 are characteristic functions, imply that $f_1 \in \mathfrak{f}, f_2 \in \mathfrak{f}$.

The theorems of Cramér and Raikov admit then the following two, different formulations: (a) The characteristic functions of normal (respectively Poisson) distributions belong to the class I_0. (b) The characteristic functions of normal (respectively Poisson) distributions form a factor-closed family.

The first formulation leads to the studies discussed in section 6; in the following we treat problems which are related to the second formulation.

A. A. Zinger and Yu. V. Linnik [65] stimulated these investigations by proving the following, interesting extension of Cramér's theorem.

THEOREM 7.1. *Let $f_1(t), f_2(t), \cdots, f_n(t)$ be arbitrary characteristic functions and suppose that the relation*

$$\prod_{j=1}^{n} [f_j(t)]^{\alpha_j} = \exp\left[i\mu t - \frac{1}{2}\sigma^2 t^2\right] \tag{7.1}$$

holds for some positive real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ in some neighborhood of the origin. Then the characteristic functions $f_j(t)$, for $j = 1, 2, \cdots, n$, belong to normal distributions.

In order to give a concise formulation of certain results we introduce the following terminology.

A family $\mathfrak{f}$ of characteristic functions is said to be strongly factor-closed if the relations

(i) $f \in \mathfrak{f}$

(ii) $\prod_{j=1}^{n} f_j^{\alpha_j} = f$, where $f_1, \cdots, f_n$ are characteristic functions, valid for some positive $\alpha_1, \cdots, \alpha_n$ imply that $f_j \in \mathfrak{f}$, for $j = 1, 2, \cdots, n$.

We can then state the result of theorem 7.1 by saying that the family of normal distributions is strongly factor-closed.

Every strongly factor-closed family is also factor-closed; at present one has no example of a factor-closed family of characteristic functions which is not strongly factor-closed. The properties of being factor-closed or strongly factor-closed have a different character: The first can be regarded as a probabilistic property of a

family of characteristic functions and can also be expressed in terms of random variables; the second is essentially of an analytical nature. It is therefore not surprising that the proofs establishing these two properties use different techniques.

In conclusion, we give a table of references to papers which contain the proofs that certain families of characteristic functions are factor-closed, respectively, strongly factor-closed. In this table we denote the family of analytic characteristic functions by $\mathfrak{f}_A$, the family of entire characteristic functions $\mathfrak{f}_E$ and write $\mathfrak{M}_{2m}$ for the family of characteristic functions which have derivatives up to the order $2m$, where m is a positive integer. Actually, H. Teicher [57] discusses a somewhat more general family than the binomial.

TABLE II

Family	Factor-closed	Strongly factor-closed
Normal	Cramér [4]	Zinger-Linnik [65]
Poisson	Raikov [47]	Dugué [7], [8]
Binomial	Teicher [57]	Teicher [58]
$\mathfrak{f}_A$, $\mathfrak{f}_E$	Raikov [47], Lévy [23], [25]	Laha [14]
$\mathfrak{M}_{2m}$	Devinatz [6]	proof similar to the one in [65] makes use also of [6]

We mention also that the family of the characteristic functions of discrete distributions is factor-closed and that the characteristic functions of lattice distributions form a strongly factor-closed family. The proofs of these two statements are almost trivial. Yu. V. Linnik [37] showed also that a rather wide subset of the class I_0 is strongly factor-closed.

Yu. V. Linnik [27] questioned whether it is possible to extend theorem 7.1 to infinite products. This problem was solved recently by L. V. Mamay [41] who obtained

THEOREM 7.2. *Let $\{f_j(t)\}$, for $j = 1, 2, \cdots$, be an arbitrary sequence of characteristic functions and let $\{\alpha_j\}$, for $j = 1, 2, \cdots$, be a sequence of positive numbers such that $\alpha_j > \epsilon_0 > 0$. Suppose that $g(z)$ is a function of the complex variable $z = t + iy$ which is regular in the strip*

$$|\mathrm{Im}(z)| < \lambda \tag{7.2}$$

and which has no zeros. Assume further that there exists a $\Delta > 0$ such that

$$\prod_{j=1}^{\infty} |f_j(t)|^{\alpha_j} = g(t) \tag{7.3}$$

is valid for $|t| < \Delta$. Then the $f_j(t)$ are analytic characteristic functions which are regular at least in the strip (7.2) and the relation (7.3) holds also for complex values satisfying (7.2).

8. Stability theorems

We say that a family of distribution functions is factor-closed if the corresponding characteristic functions form a factor-closed family. The existence of factor-closed families leads to the following question. Suppose that a distribution function $F(x)$ is, in some sense, close to a distribution of a factor-closed family $\mathfrak{f}$ and assume that F is the convolution of two distributions F_1 and F_2, that is, $F = F_1 * F_2$. Is it then possible to assert that the components F_1 and F_2 are necessarily close to some distribution of the factor-closed family $\mathfrak{f}$? We call theorems which make this assertion stability theorems.

In the following we denote the normal distribution function by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy \tag{8.1}$$

and the Poisson distribution function by

$$F(x; \lambda) = e^{-\lambda} \sum_{\gamma=0}^{\infty} \frac{\lambda^j \epsilon(x - j)}{j!}. \tag{8.2}$$

Here $\epsilon(x) = 0$ for $x < 0$ but $\epsilon(x) = 1$ for $x \geqq 0$ is the degenerate distribution function, while λ is a positive constant.

The first stability theorem was derived by N. A. Sapogov [48] who obtained the following result.

THEOREM 8.1. *Let $X = X_1 + X_2$ be the sum of two independent random variables and assume that the distribution function $F(x)$ of X satisfies the condition*

$$\sup_{-\infty < x < \infty} |F(x) - \Phi(x)| < \epsilon, \tag{8.3}$$

where $\epsilon < 1$ is a given positive number. Let $F_1(x)$ be the distribution function of X_1 and write

$$a_1 = \int_{-N}^{N} x\, dF_1(x), \qquad \sigma_1^2 = \int_{-N}^{N} x^2\, dF_1(x) - a_1^2 > 0, \qquad N = \left(\log \frac{1}{\epsilon}\right)^{1/2}. \tag{8.4}$$

Then

$$\sup_{-\infty < x < \infty} \left| F_1(x) - \Phi\left(\frac{x - a_1}{\sigma_1}\right) \right| < C\sigma_1^{-3/4} \left(\log \frac{1}{\epsilon}\right)^{-1/8}; \tag{8.5}$$

here C is a constant which is independent of ϵ, σ_1, a_1. A similar inequality holds for the distribution function $F_2(x)$ of X_2.

Recently N. A. Sapogov [49], [50] improved this estimate and extended it also to random vectors. We state also a stability theorem for the family of Poisson type distributions which is due to O. V. Shalayevskiy [51].

THEOREM 8.2. *Let $X = X_1 + X_2$ be the sum of two independent random variables and assume that the distribution function $F(x)$ of X satisfies the condition*

$$\sup_{-\infty < x < \infty} |F(x) - F(x; \lambda)| < \epsilon, \tag{8.6}$$

where $\epsilon < 1$ and λ are positive numbers. Let $F_j(x)$ be the distribution function of

X_j, for $j = 1, 2$, and let a be the upper bound of those values y for which $P\{X_1 < y\} \leqq \sqrt{\epsilon}$. We write $\lambda_1 = \int_0^{N+1} x\, dF_1(x + a)$ and $\lambda_2 = \int_0^{N+1} x\, dF_2(x + a)$, where $1/\epsilon = N^N$. Then for sufficiently small ϵ

$$\begin{aligned} &\sup_{-\infty<x<\infty} |F_1(x) - F(x - a; \lambda_1)| < \left(\lambda + \frac{1}{\lambda}\right)\left[\log \frac{1}{\epsilon}\right]^{-\omega}, \\ &\sup_{-\infty<x<\infty} |F_2(x) - F(x + a; \lambda_2)| < \left(\lambda + \frac{1}{\lambda}\right)\left[\log \frac{1}{\epsilon}\right]^{-\omega}, \end{aligned} \tag{8.7}$$

where $\omega < 1/2$ is a constant.

A general stability theorem for a wide class of infinitely divisible distributions was given by Yu. V. Linnik [36].

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