ON THE PROBABILITY OF LARGE DEVIATIONS FOR THE SUMS OF INDEPENDENT VARIABLES

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1. Introduction

The classical theory of the summation of independent random variables as expounded in the book [8] in its simplest case considers the increasing sums $S_n = X_1 + \cdots + X_n$. For the properly normed and centered sums $Z_n = S_n/B_n - A_n$ the behavior in the limit of the probability measures generated by $\{Z_n\}$ on the real axis is studied.

The most general theorems are the integral theorems on the limit behavior of

$$P\{Z_n < x\}. \tag{1.1}$$

Although the theory of local limit theorems is rather well developed [8], it is not yet of such finished character as that of integral limit theorems. Limit theorems for the expression (1.1) usually suppose that $n \to \infty$ and x is a fixed number.

However, many problems occurring in such different fields as mathematical statistics [4], [2], information theory [5], [19], statistical physics of polymers [18], rubber chemistry [17], and even analytical arithmetics [11] require certain information on the limit behavior of (1.1) not contained in the classical limit theorems. The information required concerns the asymptotic behavior of

$$P\{Z_n > x\} \tag{1.2}$$

for "large values" of x, that is for $x = x_n$ increasing as n increases; the corresponding problems will be called problems on the probability of large deviations. As probabilities of events of this kind are generally small, in general, the usual methods of establishing the limit theorems (characteristic functions, partial differential equations) are too rough to give satisfactorily general results and the desired asymptotic results were considered in the literature under certain very stringent conditions imposed upon the variables X_j.

The first theorem on the probability of large deviations was published by A. I. Khinchin [10] in 1929 and related to the particular case of the Bernoulli variables. The same case was treated more completely by N. V. Smirnov [16]. In 1938 appeared the fundamental paper [4] of H. Cramér containing the first result of a general nature in the theory of large deviations. It was improved by W. Feller [7] and by V. V. Petrov [12].

Now, we shall formulate V. V. Petrov's result, restricting ourselves to the case of identically distributed variables for the sake of simplicity. Let $X_1, \cdots, X_n$ be a sequence of independent identically distributed variables with

$$E(X_j) = 0, \qquad D(X_j) = \sigma^2,$$

$$S_n = X_1 + \cdots + X_n, \qquad Z_n = \frac{S_n}{\sigma\sqrt{n}}, \tag{1.3}$$

$$F_n(x) = P\{Z_n < x\}, \qquad G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du.$$

H. Cramér's [3] condition (C)

$$E \exp (a|X_j|) < \infty \tag{1.4}$$

must hold for some $a > 0$. Then for $x \geqq 1$, $x = o(\sqrt{n})$ and $n \to \infty$ we have

$$\frac{1 - F_n(x)}{1 - G(x)} = \exp \left[\frac{x^3}{\sqrt{n}} \lambda \left(\frac{x}{\sqrt{n}}\right)\right]\left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right], \tag{1.5}$$

$$\frac{F_n(-x)}{G(x)} = \exp \left[\frac{x^3}{\sqrt{n}} \lambda \left(\frac{-x}{\sqrt{n}}\right)\right]\left[1 + O\left(\frac{x}{\sqrt{n}}\right)\right], \tag{1.6}$$

where $\lambda(z)$ is a power series involving the cumulants of the variables X_j and convergent for $|z| \leqq \epsilon_0$, $\epsilon_0 > 0$. Later on, W. Richter [13], [14], [15] introduced systematically the saddlepoint method into the theory of large deviations. For a particular case, this was done earlier by H. Daniels [4]. Under Cramér's condition (C), Richter deduced several local limit theorems for large deviations, established the connection of Cramér's method with the saddlepoint method, and investigated the necessity of condition (C) for the formulas (1.5) and (1.6) to hold for $x = o(\sqrt{n})$.

All the results hitherto obtained used Cramér's condition (C). The analytical meaning of the condition (C) is that the characteristic function (ch.f.) of the X_j is analytical in some neighborhood of zero, and so in the corresponding strip. This enables us to apply complex function theory and the saddlepoint method.

But if the condition (C) is violated, the methods hitherto applied fail. The purpose of this paper is to give some applications of a new approach which enables us to obtain rather general results. Of the class of problems subject to this method we shall treat here only the problem of normal convergence and the problem of limit theorems valid for all values of x for $n \to \infty$.

2. Zone of normal convergence: integral limit theorems

We consider here the normal convergence problem for large deviations for the sake of simplicity only for independent identically distributed variables.

Let $X_1, \cdots, X_n, \cdots$ be independent identically distributed variables with $E(X_j) = 0$, $D(X_j) = 1$, $Z_n = (X_1 + \cdots + X_n)/\sqrt{n}$. Let $\Psi(n) \to \infty$ be any monotone function. The sequence of the segments $[0, \Psi(n)]$ will be called a zone of normal convergence (z.n.c.) if, for $n \to \infty$,

$$\frac{P\{Z_n > x\}}{\frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2}\, du} \to 1 \tag{2.1}$$

for any $x \in [0, \Psi(n)]$. A z.n.c. $[-\Psi(n), 0]$ is defined similarly. The definition does not require the convergence to be uniform, although in all the theorems obtained it will be.

Under Cramér's condition (C) for any $\Psi(n) = o(n^{1/6})$, both $[0, \Psi(n)]$ and $[-\Psi(n), 0]$ will be z.n.c. The zones with $\Psi(n) = o(n^{1/6})$ will be called narrow zones. Our two first theorems relate to the zones with $\Psi(n) = n^\alpha$, where $\alpha > 0$ is a constant.

THEOREM 1. *If for any $\alpha < 1/2$, the zone $[0, n^\alpha]$ and the zone $[-n^\alpha, 0]$ are z.n.c., then all the variables X_j are normal.*

Of course, this is also sufficient for the zones $[0, n^\alpha]$ and $[-n^\alpha, 0]$ to be z.n.c. This last fact is trivial. Thus, we see that it is sufficient to investigate the values of $\alpha < 1/2$.

THEOREM 2. *Let $\rho(n) \to \infty$ be a monotone function increasing as slowly as we please and let $0 < \alpha < 1/2$. If $\alpha < 1/6$, a necessary condition for the zones $[0, n^\alpha \rho(n)]$ and $[-n^\alpha \rho(n), 0]$ to be z.n.c. is*

$$E \exp |X_j|^{\frac{4\alpha}{2\alpha+1}} < \infty. \tag{2.2}$$

This condition is sufficient for the zones $[0, n^\alpha/\rho(n)]$ and $[-n^\alpha/\rho(n), 0]$ to be z.n.c. and the convergence is then uniform. If $1/6 \leqq \alpha < 1/2$, consider the sequence of the critical numbers

$$\frac{1}{6}, \frac{1}{4}, \frac{3}{10}, \cdots, \frac{1}{2}\frac{s+1}{s+3} \cdots \to \frac{1}{2}. \tag{2.3}$$

Let

$$\frac{1}{2}\frac{s+1}{s+3} \leqq \alpha < \frac{1}{2}\frac{s+2}{s+4}. \tag{2.4}$$

If the zones $[0, n^\alpha \rho(n)]$ and $[-n^\alpha \rho(n), 0]$ are z.n.c., the condition (2.2) must hold and moreover all the moments of X_j up to $(s + 3)$ must coincide with the moments of the normal law. These two conditions are sufficient for the zones $[0, n^\alpha/\rho(n)]$ and $[-n^\alpha/\rho(n), 0]$ to be z.n.c. This convergence is then uniform.

As the normal law is completely determined by the sequence of the corresponding moments, theorem 1 is an immediate consequence of theorem 2.

We consider now the narrow zones with $\Psi(n) = o(n^{1/6})$ other than $[0, n^\alpha]$. A condition necessary for the zones $[0, \Psi(n)\rho(n)]$ and $[-\Psi(n)\rho(n), 0]$ and sufficient for the zones $[0, \Psi(n)/\rho(n)]$ and $[-\Psi(n)/\rho(n), 0]$ to be z.n.c. is of the type

$$E \exp h(|X_j|) < \infty, \tag{2.5}$$

where $h(x)$ is a monotone function depending upon $\Psi(n)$.

It is simpler, however, to describe $\Psi(n)$ in terms of $h(x)$. To this end we consider several classes of functions of $h(x)$. The functions $h(x)$ will be assumed to be positive, monotone, and differentiable.

— Class I will denote the functions $h(x)$ satisfying the condition

$$(\log x)^{2+\zeta_0} \leqq h(x) \leqq x^{1/2}, \qquad x \geqq 1. \tag{2.6}$$

Here $\zeta_0 > 0$ is any small fixed number. Functions increasing faster than $x^{1/2}$ will not be required for the narrow zone investigations.

— Class II consists of the functions $h(x)$ under the condition

$$\rho_1(x) \log x \leqq h(x) \leqq (\log x)^{2+\zeta_0}, \tag{2.7}$$

where $x \geqq 1$, $\rho_1(x)$, $\rho_2(x)$, $\cdots$ in what follows are given positive monotone functions increasing as slowly as we please.

— Class III consists of functions $h(x)$ such that

$$3 \log x \leqq h(x) \leqq M \log x, \tag{2.8}$$

where $M \geqq 3$ is a given constant. The inequality $h(x) \geqq 3 \log x$ is connected with the existence of the third moment.

Consider now the functions of class I as defined by (2.6). We put

$$h(x) = \exp [H (\log x)]. \tag{2.9}$$

Then $H(z)$ is a monotone differentiable function. We introduce the following supplementary conditions

$$H'(z) \leqq 1, \tag{2.10}$$

$$H'(z) \exp H(z) \to \infty, \qquad z \to \infty. \tag{2.11}$$

These conditions follow from (2.6) if $H'(z)$ is assumed to be monotone; otherwise we adopt them to simplify the results.

Given a function $h(x)$ of class I, under the supplementary conditions (2.10) and (2.11), or of class II, we determine new functions $\Lambda(n)$ by means of the equation

$$h[\sqrt{n}\, \Lambda(n)] = [\Lambda(n)]^2. \tag{2.12}$$

THEOREM 3. *The condition*

$$E \exp h(|X_j|) < \infty, \tag{2.13}$$

where $h(x)$ *belongs to class* I [*with* (2.10) *and* (2.11)] *or class* II, *is necessary for the zones* $[0, \Lambda(n)\rho(n)]$ *and* $[-\Lambda(n)\rho(n), 0]$ *to be z.n.c. and sufficient for the zones* $[0, \Lambda(n)/\rho(n)]$ *and* $[-\Lambda(n)/\rho(n), 0]$ *to be z.n.c. The convergence in this case is uniform.*

We pass now to the functions $h(x)$ belonging to class III. This case can be studied by classical means [8]. For the sake of completeness we formulate

THEOREM 4. *Condition* 2.13 *where* $h(x)$ *belongs to class* III, *is necessary for the zones* $[0, \rho(n)(\log n)^{1/2}]$ *and* $[-\rho(n)(\log n)^{1/2}, 0]$ *to be z.n.c. and sufficient for the zones* $[0, (\log n)^{1/2}/\rho(n)]$ *and* $[-(\log n)^{1/2}/\rho(n), 0]$ *to be z.n.c.*

Thus, if $E|X_j|^M = \infty$ for a fixed M, the z.n.c. cannot be essentially wider than $[0, (\log n)^{1/2}]$. It is, roughly speaking, of this size if $E|X_j|^M < \infty$ with $M \geqq 3$. The case $E|X_j|^3 = \infty$ (nonexistence of the third moment) was studied by several authors (see [8] for the literature).

The case (2.8), which is class III of the functions $h(x)$, corresponds to slowly decreasing "probability tails" $P\{X_j > x\}$. In this case, as we shall show later, a new type of limit theorems holds: limit theorems valid for the whole x-axis.

3. Local limit theorems

In the preceding section we considered integral limit theorems. We pass now to the local limit theorems relating to normal convergence. These theorems are usually considered for variables possessing a probability density or for integer valued variables. We can consider also the probability measures on the ring of integers of an algebraic number field. We shall restrict ourselves to the class (d) of all random variables possessing a continuous bounded density $g(x)$. Then Z_n (see section 2) will also have a continuous density $p_{Z_n}(x)$. The zone $[0, \Psi(n)]$ will be called a zone of uniform local normal convergence (z.u.l.n.c.) if

$$\frac{p_{Z_n}(x)}{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}} \to 1, \tag{3.1}$$

as $n \to \infty$, uniformly for $x \in [0, \Psi(n)]$. The z.u.l.n.c. $[-\Psi(n), 0]$ are defined similarly.

THEOREM 5. *For the variables X_j belonging to the class* (d) *the z.u.l.n.c. behave with respect to the necessary and sufficient conditions indicated in theorems* 1 *to* 4 *in the same way as the z.n.c. for the general random variables in theorems* 1 *to* 4.

The local limit theorems for large deviations are easier to prove than the corresponding integral ones, by the method proposed here. In fact, the existence of the probability density $g(x)$ greatly facilitates the proof.

4. Proof of necessity of condition (2.2)

We shall be able to expound here the proofs for only the simplest cases so as to present the new approach in its most transparent form; the proofs of all the theorems 1 to 5, although not basically different, are more involved and will be published elsewhere. In particular, we shall treat only the zones $[0, n^\alpha]$ and $[-n^\alpha, 0]$ with $0 < \alpha < 1/2$ and only local limit theorems. However, since a part of the necessary conditions for the integral theorems is almost trivial, we shall begin by dwelling upon it.

Let the zones $[0, n^\alpha \rho(n)]$ and $[-n^\alpha \rho(n), 0]$ be z.n.c. We shall prove that

$$E \exp\left[|X_j|^{\frac{4\alpha}{2\alpha+1}}\right] < \infty. \tag{4.1}$$

Suppose (4.1) does not hold. Then it is easy to see that there exists either a sequence $x_m \to \infty$ such that

$$P\{X_1 > x_m\} > \exp\left[-2x_m^{\frac{4\alpha}{2\alpha+1}}\right] \tag{4.2}$$

or a sequence $-x_m \to -\infty$ such that

$$P\{X_1 < -x_m\} > \exp\left[-2x_m^{\frac{4\alpha}{2\alpha+1}}\right]. \tag{4.3}$$

Suppose that (4.2) holds. For a sufficiently large m, choose n such that $x_m = n^{\alpha+1/2}\rho(n)$. The zone $[0, n^\alpha\rho(n)]$ being a z.n.c., we must have

$$P\left\{Z_n > \frac{n^\alpha\rho(n)}{2}\right\} < \exp\left\{-\frac{n^{2\alpha}[\rho(n)]^2}{16}\right\}. \tag{4.4}$$

But the event $Z_n > n^\alpha\rho(n)/2$ will surely occur if the two independent events $X_1 > n^{\alpha+1/2}\rho(n)$ and $|(X_2 + X_3 + \cdots + X_n)/\sqrt{n}| < 1$ occur simultaneously. Hence, by the central limit theorem, where $c_0 > 0$ is a constant,

$$\begin{aligned} P\left\{Z_n > \frac{n^\alpha\rho(n)}{2}\right\} &> c_0\, P\left\{X_1 > n^{\alpha+1/2}\rho(n)\right\} \\ &> c_0 \exp\left\{-2n^{2\alpha}[\rho(n)]^{\frac{4\alpha}{2\alpha+1}}\right\} \end{aligned} \tag{4.5}$$

by (4.2). Since $\alpha < 1/2$, it follows that $4\alpha/(2\alpha+1) < 1$ and (4.5) contradicts (4.4). The case (4.3) is treated similarly. The proof of the necessity of (4.1) for the zones $[0, n^\alpha\rho(n)]$ and $[-n^\alpha\rho(n), 0]$ to be z.u.l.n.c. is constructed in a similar way.

5. Sufficiency with $0 < \alpha < 1/6$

We now pass to the local limit theorem. Let the variables X_j possess the bounded continuous density $g(x)$. We introduce some notation. By the letter B we shall denote a bounded function of the parameters considered, not always the same. $\delta_0, \delta_1, \cdots$; $\epsilon_0, \epsilon_1, \cdots$ will be small positive constants; $C_0, C_1, \cdots$; $c_0, c_1, \cdots$ positive constants. Write

$$\varphi(t) = Ee^{itX_j} = \int_{-\infty}^{\infty} e^{itu}g(u)\, du. \tag{5.1}$$

The function $|\varphi(t)|^2$ is a nonnegative Fourier transform and therefore (compare [1], p. 20) we have $|\varphi(t)|^2 \in L_1(-\infty, \infty)$, so that

$$\int_{-\infty}^{\infty} |\varphi(t)|^2\, dt < \infty. \tag{5.2}$$

Hence we have

$$p_{Z_n}(x) = \frac{\sqrt{n}}{2\pi}\int_{-\infty}^{\infty} [\varphi(t)]^n e^{-\sqrt{n}\, itx}\, dt. \tag{5.3}$$

Let (4.1) be satisfied. Suppose first that $\alpha < 1/6$. We must prove that

$[0, n^\alpha/\rho(n)]$ and $[-n^\alpha/\rho(n), 0]$ are z.u.l.n.c. We shall study only the first zone, the second one being treated analogously. Take x such that

$$0 \leqq x \leqq \frac{n^\alpha}{\rho(n)}. \tag{5.4}$$

In view of (4.1) the function $\varphi(t)$ is infinitely differentiable on the whole axis and so, for any $T > 0$, $|t| \leqq T$, and $p > 0$ an integer,

$$\varphi(t) = \varphi(0) + \frac{t\varphi'(0)}{1!} + \cdots + \frac{t^{p-1}}{(p-1)!}\varphi^{(p-1)}(0) + \frac{t^p}{p!}R_p(t), \tag{5.5}$$

where

$$|R_p(t)| \leqq 2 \sup_{|t| \leqq T} |\varphi^{(p)}(t)|. \tag{5.6}$$

Moreover, $\varphi'(0) = 0$; $\varphi''(0) = -1$. From this it follows that for $|t| \leqq \epsilon_0$,

$$\varphi(t) = 1 - \frac{t^2}{2} + Bt^3; \qquad |\varphi(t)| \leqq 1 - \frac{t^2}{4}. \tag{5.7}$$

Further, in view of the existence of the bounded continuous density $g(x)$, we have, for $|t| > \epsilon_0$,

$$|\varphi(t)| < 1; \qquad \varphi(t) \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{5.8}$$

Combining (5.2), (5.3), and (5.8) we get

$$P_{Z_n}(x) = \frac{\sqrt{n}}{2\pi}\int_{-\epsilon_0}^{\epsilon_0} [\varphi(t)]^n e^{-\sqrt{n}\,itx}\,dt + Be^{-c_0 n}. \tag{5.9}$$

Put

$$\mu = \frac{1}{2} - \alpha. \tag{5.10}$$

In view of (5.7) for $n^{-\mu} \leqq |t| \leqq \epsilon_0$, we obtain

$$\begin{aligned} |\varphi(t)|^n \leqq \left(1 - \frac{n^{-2\mu}}{4}\right)^n &= B\exp(-c_1 n^{1-2\mu}) \\ &= B\exp(-c_1 n^{2\alpha}). \end{aligned} \tag{5.11}$$

Hence from (5.9) we obtain

$$p_{Z_n}(x) = \frac{\sqrt{n}}{2\pi}\int_{-n^{-\mu}}^{n^{-\mu}} [\varphi(t)]^n \exp(-\sqrt{n}\,itx)\,dt + B\exp(-\delta_0 n^{2\alpha}). \tag{5.12}$$

6. Approximation to $\varphi(t)$

The function (5.5) is not analytic in general and so the Taylor series for it diverges. We must choose an appropriate approximation to it in the segment $|t| \leqq n^{-\mu}$, that is, choose a convenient p in the formula (5.5). We need estimates of $\varphi^{(q)}(0)$ for $q \leqq p$. We get

$$\varphi^{(q)}(t) = \int_{-\infty}^{\infty} e^{itx}(ix)^q g(x)\,dx. \tag{6.1}$$

Hence

(6.2) $$|\varphi^{(q)}(t)| \leqq \int_{-\infty}^{\infty} |x|^q g(x)\, dx.$$

Putting $k = (1 + 2\alpha)/4\alpha$, we obtain from (4.1)

(6.3) $$\int_{-\infty}^{\infty} \exp(|x|^{1/k}) g(x)\, dx < \infty,$$

whence we easily obtain

(6.4) $$|\varphi^{(q)}(t)| = B^q \Gamma(kq).$$

Let us write $K(t) = \log \varphi(t)$ where $K(0) = 0$, $|t| \leqq n^{-\mu}$. From (5.12) we obtain

(6.5) $$p_{Z_n}(x) = \frac{\sqrt{n}}{2\pi} \int_{-n^{-\mu}}^{n^{-\mu}} \exp[nK(t) - \sqrt{n}\, itx]\, dt + B \exp(-\delta_0 n^{2\alpha}).$$

Moreover, from (5.7) we conclude that, for $|t| \leqq \epsilon_0$,

(6.6) $$K(t) = B.$$

We need estimates for $K^{(q)}(t_0)$, for $|t_0| \leqq \epsilon_0/2$. Putting

(6.7) $$\tilde{\varphi}(t + t_0) = \varphi(t_0) + \frac{t\varphi'(t_0)}{1!} + \cdots + \frac{t^q \varphi^{(q)}(t_0)}{q!}$$

we have

(6.8) $$K^{(q)}(t_0) = [\log \tilde{\varphi}(t + t_0)]^{(q)}.$$

In fact, to compute $K^{(q)}(t_0) = [\log \varphi(t)]^{(q)}$, with $t = t_0$, we need only the derivatives $\varphi^{(p)}(t_0)$, with $p \leqq q$, and these coincide with $\tilde{\varphi}^{(p)}(t_0)$. If C_ρ is a circle $|t| \leqq \rho$ where $\tilde{\varphi}(t + t_0)$ has no zeros, then $\log \tilde{\varphi}(t + t_0)$ is analytic for $|t| \leqq \rho$ and

(6.9) $$K^{(q)}(t_0) = \frac{q!}{2\pi i} \oint_{C_\rho} \frac{\log \tilde{\varphi}(t + t_0)}{t^{q+1}}\, dt.$$

Consider now (6.7). From (6.4) we have

(6.10) $$\frac{\varphi^{(p)}(t_0)}{p!} = B \exp[Bp + (k - 1)p \log p].$$

We choose now for C_ρ the circle

(6.11) $$|t| \leqq \exp[-C_0 - (k - 1)\log q].$$

For sufficiently large C_0, we have inside C_ρ

(6.12) $$\left| \sum_{p=1}^{q} \frac{\varphi^{(p)}(t_0)}{p!} t^p \right| < \frac{1}{4}$$

as is easily seen from (6.10) and (6.11). Hence, on C_ρ we have

(6.13) $$\frac{1}{2} < |\tilde{\varphi}(t + t_0)| < \frac{3}{2}$$

in view of (5.7). Hence, from (6.9), for $|t_0| \leqq \epsilon_0/2$,

(6.14) $$K^{(q)}(t_0) = B \exp(Bq + kq \log q).$$

We write now the expansion (5.5) for the function $K(t)$ instead of $\varphi(t)$ and we put

$$p = m = \left[\frac{n^{2\alpha}}{\rho_1(n)}\right], \tag{6.15}$$

where the function $\rho_1(n)$ will be fixed later on depending upon $\rho(n)$. Hence, for $|t| \leqq n^{-\mu}$, as in (5.6),

$$\begin{aligned}\left|\frac{t^m R_m(t)}{m!}\right| &= B \exp m[B + (k - 1) \log m - \mu \log n] \\ &= B \exp m[B - (k - 1) \log \rho_1(n)]\end{aligned} \tag{6.16}$$

by definition of m, μ, and k. Moreover, $k = (2\alpha + 1)/4\alpha > 1$ because $\alpha < 1/2$; in fact, $\alpha < 1/6$. Hence, for $|t| \leqq n^{-\mu}$,

$$nK(t) = -n\frac{t^2}{2} + n\sum_{r=3}^{m} \Psi_r \frac{t^r}{r!} + B \exp\left(-\delta_1 \frac{n^{2\alpha}}{\rho_1(n)}\right). \tag{6.17}$$

Now, Re $nK(t) \leqq 0$ for $|t| \leqq n^{-\mu}$. Putting

$$K_3(t) = \sum_{r=3}^{m} \Psi_r \frac{t^r}{r!} \tag{6.18}$$

we obtain

(6.19)

$$p_{Z_n}(x) = \frac{\sqrt{n}}{2\pi} \int_{-n^{-\mu}}^{n^{-\mu}} \exp\left[-\frac{nt^2}{2} + nK_3(t) - \sqrt{n}\, itx\right] dt + B \exp\left[-\delta_2 \frac{n^{2\alpha}}{\rho_1(n)}\right].$$

We consider now the entire function

$$\exp\,[nK_3(t)] = 1 + \sum_{r=3}^{\infty} \frac{\chi_r t^r}{r!}. \tag{6.20}$$

With $\alpha < 1/6$, $\mu = 1/2 - \alpha > 1/3$, and for $|t| \leqq n^{-\mu}$, we have $n\Psi_3(t^3/3!) = Bn^{-\epsilon_1}$. Taking this into account, and using the estimate, as in (6.14),

$$\Psi_r = K^{(r)}(0) = B \exp\,(Br + kr \log r), \qquad r \geqq 3, \tag{6.21}$$

we obtain, after an easy computation,

$$nK_3(t) = B, \qquad |t| \leqq 2n^{-\mu}, \tag{6.22}$$

$$\chi_r = \frac{r!}{2\pi i} \oint_{|t| \leqq 2n^{-\mu}} \exp\,[nK_3(t)] \frac{dt}{t^{r+1}} = Br!\, 2^{-r} n^{r\mu}, \tag{6.23}$$

$$\exp\,[nK_3(t)] = \sum_{r=3}^{m} \frac{\chi_r t^r}{r!} + B.2^{-m}, \qquad |t| \leqq n^{-\mu}. \tag{6.24}$$

7. Conclusion of the proof

Putting (6.24) into (6.19) and taking (6.15) into account, we find that

$$(7.1)\qquad p_{Z_n}(x) = \frac{\sqrt{n}}{2\pi}\int_{-n^{-\mu}}^{n^{-\mu}} \exp\left(-\frac{nt^2}{2}\right)\left(1+\sum_{r=3}^{m}\frac{\chi_r}{r!}t^r\right)\exp(-itx\,n)\,dt + B\exp\left(-\delta_3\frac{n^{2\alpha}}{\rho_1(n)}\right).$$

The next step is to extend the limits of integration in (7.1) from $-\infty$ to ∞. Putting $\xi = t\sqrt{n}$, we obtain

$$(7.2)\qquad p_{Z_n}(x) = \frac{1}{2\pi}\int_{-n^{1/2-\mu}}^{n^{1/2-\mu}} \exp\left(-\frac{\xi^2}{2}\right)\left[1+\sum_{r=3}^{m}\frac{\chi_r}{r!}\left(\frac{\xi}{\sqrt{n}}\right)^r\right]\exp(-i\xi x)\,d\xi + B\exp\left[-\delta_3\frac{n^{2\alpha}}{\rho_1(n)}\right].$$

In order to obtain the required extension of the limit of integration, we estimate, for $3 \leqq r \leqq m$,

$$(7.3)\qquad \int_{n^{1/2-\mu}}^{\infty} e^{-\xi^2/2}\frac{\chi_r}{r!}n^{-r/2}\xi^r\,d\xi = B^r\exp\left(-\frac{1}{4}n^{1-2\mu}\right)\Gamma\left(\frac{r}{2}\right)n^{(\mu-1/2)r}.$$

Now

$$(7.4)\qquad \begin{aligned} B^r\Gamma\left(\frac{r}{2}\right)n^{(\mu-1/2)r} &= B\exp\left[Br+\frac{r}{2}\log r - r\left(\frac{1}{2}-\mu\right)\log n\right] \\ &= B\exp r\left[B+\frac{1}{2}\log m-\left(\frac{1}{2}-\mu\right)\log n\right] \\ &= B\exp\left[-\frac{r}{2}\log\rho_1(n)\right]. \end{aligned}$$

Summing over $r \leqq m$ we obtain from (7.3)

$$(7.5)\qquad \sum_{3\leqq r\leqq m}\int_{n^{1/2-\mu}}^{\infty} e^{-\xi^2/2}\frac{\chi_r}{r!}\left(\frac{1}{\sqrt{n}}\right)^r\xi^r\,d\xi = B\exp(-\delta_4 n^{2\alpha}).$$

Of course, the integrals $\int_{-\infty}^{-n^{1/2-\mu}}$ can be subjected to the same treatment, and we get

$$(7.6)\qquad P_{Z_n}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{2}\right)\left[1+\sum_{r=3}^{m}\frac{\chi_r}{r!}\left(\frac{\xi}{\sqrt{n}}\right)^r\right]\exp(-i\xi x)\,d\xi + B\exp\left(-\delta_3\frac{n^{2\alpha}}{\rho_1(n)}\right).$$

We are thus led to the expressions

$$(7.7)\qquad n^{-r/2}\left[\int_{-\infty}^{\infty} e^{-\xi^2/2}e^{-i\xi x}\xi^r\,d\xi\right] = n^{-r/2}\left[H_r^{(0)}(x)e^{-x^2/2}\right]$$

for $3 \leqq r \leqq m$. The polynomials $H_r^{(0)}(x)$ are Hermite polynomials in ax for suitable $a > 0$. Let now

$$(7.8)\qquad |x| \leqq \frac{n^\alpha}{\rho(n)}.$$

For $3 \leqq r < C_1$, when C_1 is any constant, we have for (7.7) the value

$$Be^{-x^2/2}\left[\frac{1}{\rho(n)}\right]^r. \tag{7.9}$$

Summing these for $r = 3, 4, \cdots ; r \leqq C_1$, we get the estimate

$$Be^{-x^2/2}\frac{1}{[\rho(n)]^3}. \tag{7.10}$$

Consider now $C_1 < r \leqq m$. We use the following expression for Hermite's polynomials $H_q(x)$ (see [7], p. 193).

$$H_q(x) = q! \sum_{s=0}^{[q/2]} \frac{(-1)^s(2x)^{q-2s}}{s!\,(q-2s)!}. \tag{7.11}$$

Hence ($x \geqq 0$, 0^0 assumed to be equal to 1),

$$|H_q^{(0)}(x)| = B^q q! \max_{s \leqq q/2} \frac{x^{q-2s}}{s!\,(q-2s)!}. \tag{7.12}$$

Let $s = q\rho$ with $0 \leqq \rho \leqq 1/2$ where $0 \leqq x \leqq n^\alpha/\rho(n)$ and $C_1 < q \leqq m$. Then (7.12) has the value

$$\begin{aligned} B \max_\rho \exp q\{B &+ (1-2\rho)[\alpha \log n - \log \rho(n)] \\ &- \rho \log q - \rho \log \rho - (1-2\rho)\log q - (1-2\rho)\log(1-2\rho) + \log q\}. \end{aligned} \tag{7.13}$$

Multiplying (7.13) by $n^{-q/2} = \exp[-(q \log n)/2]$ and by

$$\frac{\chi_q}{q!} = Bn^{q\mu} = B \exp(q\mu \log n), \tag{7.14}$$

we get, after an elementary computation,

$$B \max_\rho \exp q[B - (1-2\rho)\log \rho(n) + \rho \log q - \rho 2\alpha \log n]. \tag{7.15}$$

As $\log q \leqq \log m = 2\alpha \log n - \log \rho_1(n) + B$, we can replace (7.15) by

$$B \max_\rho \exp q[B - (1-2\rho)\log \rho(n) - \rho \log \rho(n)]. \tag{7.16}$$

If $\rho > 1/4$, then $\rho \log \rho(n) > \log \rho(n)/4$; if $\rho \leqq 1/4$, then $(1-2\rho)\log \rho(n) \geqq \log \rho(n)/2$. Hence (7.16) can be replaced by

$$B[\rho(n)]^{-\epsilon_2 q}, \qquad C_1 < q \leqq m. \tag{7.17}$$

Inserting this into (7.6) and summing over q, we obtain

$$p_{Z_n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\left[1 + \frac{B}{\rho(n)}\right] + B \exp\left[-\delta_3 \frac{n^{2\alpha}}{\rho_1(n)}\right]. \tag{7.18}$$

Note that C_1 is supposed to be chosen $> 1/\epsilon_2$. The relation (7.18) holds for $0 \leqq x \leqq n^\alpha/\rho(n)$. We take now $\rho_1(n) = \rho(n)$, from which

$$p_{Z_n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\left[1 + \frac{B}{\rho(n)}\right], \qquad 0 \leqq x \leqq \frac{n^\alpha}{\rho(n)}, \tag{7.19}$$

which was to be proved. The case where $-n^\alpha/\rho(n) \leqq x \leqq 0$ is treated similarly.

8. Proof of sufficiency when $1/6 \leqq \alpha < 1/2$

We pass now to the case $1/6 \leqq \alpha < 1/2$. The principal scheme of reasoning remains the same as in the previous case, for which $\alpha < 1/6$; we shall indicate only the points where it differs substantially from the previous scheme.

Let $1/6 \leqq \alpha < 1/2$ and assume condition (4.1) holds. In view of (5.7) and (4.1) we know that $K(t) = \log \varphi(t)$ possesses all derivatives for $|t| \leqq \epsilon_0$, and that $\Psi_r = K^{(r)}(0)$, when multiplied into a suitable power of $i = \sqrt{-1}$, becomes the rth cumulant of X_j. As is well known ([9], pp. 61–63), the rth cumulant depends only upon the moments up to the rth, and conversely. If the third, fourth, $\cdots$, rth cumulant vanish, while $EX_j = 0$ and $D(X_j) = 1$, then all the moments up to the rth are the moments of the normal law $N(0, 1)$. Consider the sequence of the critical numbers (2.3) and suppose that

$$\frac{1}{2}\frac{s+1}{s+3} \leqq \alpha < \frac{1}{2}\frac{s+2}{s+4}. \tag{8.1}$$

Consider the numbers $\Psi_3, \Psi_4, \cdots, \Psi_{s+3}$. If all these numbers vanish (and so the first $s + 3$ moments of X_j are normal), we proceed as in the previous case. We put $\mu = 1/2 - \alpha$, and hence $\mu \leqq 1/3$, and carry out the above computations. Instead of (6.17) we have now

$$nK(t) = -\frac{nt^2}{2} + n \sum_{r=s+4}^{m} \Psi_r \frac{t^r}{r!} + B \exp\left[-\delta_1 \frac{n^{2\alpha}}{\rho_1(n)}\right] \tag{8.2}$$

for $|t| \leqq n^{-\mu}$. Since $\mu = 1/2 - \alpha \geqq 1/(s+3)$, we have

$$n\Psi_{s+3} \frac{t^{s+4}}{(s+4)!} = Bn^{1-(s+4)/(s+3)} = Bn^{-1/(s+3)} \tag{8.3}$$

Reasoning now as in section 7, we obtain the local limit theorem for $0 \leqq x \leqq n^\alpha/\rho(n)$ and $-n^\alpha/\rho(n) \leqq x \leqq 0$.

9. Proof of necessity when $1/6 \leqq \alpha < 1/2$

Suppose now that not all the numbers $\Psi_3, \Psi_4, \cdots, \Psi_{s+3}$ vanish; let Ψ_{s_0+3} be the first nonvanishing number for $s_0 \leqq s$. We then set

$$\mu = \frac{1}{s_0 + 3}. \tag{9.1}$$

We can show that in this case there will be no local normal convergence even in the zone $[0, n^{1/2-\mu}]$ which is obviously smaller than the zone $[0, n^\alpha\rho(n)]$. We choose $m = [n^{1-2\mu}/C_2]$ with a sufficiently large C_2. We have then, proceeding as in sections 6 and 7,

$$(9.2)\qquad p_{Z_n}(x) = \frac{\sqrt{n}}{2\pi}\int_{-n^{-\mu}}^{n^{-\mu}} \exp\left[-\frac{nt^2}{2} + K_{s_0+3}(t) - i\sqrt{n}\,tx\right]dt + B\exp(-\delta_5 n^{1-2\mu}).$$

In this case

$$(9.3)\qquad K_{s_0+3}(t) = \sum_{r=s_0+3}^{m} \frac{\Psi_r t^r}{r!}.$$

Further,

$$(9.4)\qquad \exp[nK_{s_0+3}(t)] = 1 + \sum_{r=s_0+3}^{\infty} \frac{\chi_r t^r}{r!}.$$

It is important to notice that

$$(9.5)\qquad \chi_{s_0+3} = n\Psi_{s_0+3}, \qquad \Psi_{s_0+3} \neq 0.$$

Taking now

$$(9.6)\qquad x_0 = \eta n^{1/2-\mu}, \qquad \eta > 0,$$

with a sufficiently small η, and proceeding as in section 7, we obtain, after some computations similar to that of section 7,

$$(9.7)\qquad p_{Z_n}(x) = \frac{1}{\sqrt{2\pi}}e^{-x_0^2/2} + \frac{1}{2\pi}\frac{\chi_{s_0+3}}{(s_0+3)!}H_{s_0+3}^{(0)}(x_0)e^{-x_0^2/2}\frac{1}{(\sqrt{n})^{s_0+3}} + Be^{-x_0^2/2}\eta^{s_0+4},$$

for a sufficiently small $\eta > 0$. Taking (9.5) into account we see that the second term on the right side is of the form

$$(9.8)\qquad a_0\Psi_{s_0+3}\frac{1}{(s_0+3)!}e^{-x_0^2/2}\eta^{s_0+3}[1 + o(1)]$$

as $n \to \infty$, where $a_0 \neq 0$. Hence, there is no local normal convergence in the zone $[0, n^{1/2-\mu}]$ and likewise in the zone $[0, n^{\alpha}\rho(n)]$.

Hence, if there is uniform local normal convergence in the zones $[0, n^{\alpha}]$ and $[-n^{\alpha}, 0]$ for all $\alpha < 1/2$, then all Ψ_r must vanish and so all the X_j are normal.

10. An integral theorem uniform for the whole x-axis

There is an interesting class of probability densities $g(x)$ for which an integral limit theorem for the normal sum Z_n holds for the whole x-axis.

Consider the class of all even continuous probability densities $g(x)$ such that for $x \geqq 1$

$$(10.1)\qquad P\{X_1 > x\} = \int_x^{\infty} g(u)\,du = \frac{A_a}{x^a} + \frac{A_{a+1}}{x^{a+1}} + \cdots + \frac{A_{4a+5}}{x^{4a+5}} + O\left(\frac{1}{x^{4a+5+\epsilon}}\right)$$

where $a \geqq 3$, a being an integer, A_j being constants. Note that the evenness condition is assumed for simplicity only. Let $X_1, X_2, \cdots, X_n$ be random variables with probability density $g(x)$ of this class. Then $E(X_j) = 0$. Write $D(X_j) = \sigma^2$, $Z_n = (X_1 + \cdots + X_n)/\sigma n$.

THEOREM 6. *For $x \geqq 1$ and as $n \to \infty$ we have uniformly with respect to x,*

$$\frac{P\{Z_n > x\}}{\frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-u^2/2}\, du + r(x, \sqrt{n})} \to 1, \tag{10.2}$$

where $r(x, \sqrt{n})$ is a rational function of both variables determined by the coefficients $A_a, \cdots, A_{4a+5}$ in (10.1). *Moreover, for $x \geqq n^{3/2+1/a} \log n$ we have*

$$r(x, \sqrt{n}) \sim nP\{X_1 > \sigma x\sqrt{n}\} = n \int_{\sigma x\sqrt{n}}^{\infty} g(u)\, du. \tag{10.3}$$

Of course, $g(x)$ being even, an analogous relation holds for $x \leqq -1$, while for $-1 < x < 1$ the classical normal convergence theorem acts.

Simple examples of theorem 6 are given by rational densities. For instance, if $g(x) = 2/\pi(x^2 + 1)^2$, so that $D(X_j) = 1$, we have

$$\frac{P\{Z_n > x\}}{\frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-u^2/2}\, du + \frac{2}{3\pi\sqrt{n}} \frac{1}{x^3}} \to 1, \tag{10.4}$$

for $x \geqq 1$, $n \to \infty$ and

$$\frac{2}{3\pi\sqrt{n}} \frac{1}{x^3} \sim nP\{X_1 > x\sqrt{n}\} \tag{10.5}$$

for $x \geqq n^{3/2+1/4} \log n$.

We shall indicate here briefly the principal points of the corresponding proof. We take $\sigma = 1$. The case

$$x \geqq n^{3/2+1/a} \log n \tag{10.6}$$

is treated in an elementary manner. If $y > n$, then the event $S_n > y$ implies at least one of the events $X_j > y/n$. Denote by $H_{k,\alpha_1,\alpha_2,\cdots,\alpha_k}$, for $k \leqq n$, the hypothesis that $X_{\alpha_1} > y/n$, $X_{\alpha_2} > y/n$, $\cdots$, $X_{\alpha_k} > y/n$, while this is not true for any other X_j. Hence

$$\begin{aligned} P\{S_n > y\} = &\sum_{(\alpha_1)} P\{H_{1,\alpha_1}\} P\{S_n > y | H_{1,\alpha_1}\} \\ &+ \sum_{k \geqq 2} \sum_{(\alpha_1, \cdots, \alpha_k)} P\{H_{k,\alpha_1,\cdots,\alpha_k}\} P\{S_n > y | H_{k,\alpha_1,\cdots,\alpha_k}\}, \end{aligned} \tag{10.7}$$

the summation being extended to all ordered sets of distinct numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_k$. Now, in view of (10.1) it is easy to compute that for

$$y \geqq n^{2+1/a} \log n, \tag{10.8}$$

the double sum on the right side of (10.7) is of order smaller than the first term, so

$$P\{S_n > y\} \sim \sum_{(\alpha_1)} P\{H_{1,\alpha_1}\} P\{S_n > y | H_{1,\alpha_1}\} = nP\{H_{1,1}\} P\{S_n > y | H_{1,1}\}. \tag{10.9}$$

Moreover, it is easy to see that

$$P\{S_n > y|H_{1,1}\} \sim P\left\{X_1 > y|X_1 > \frac{y}{n}\right\}. \tag{10.10}$$

Inserting this into (10.9) we get

$$P\{S_n > y\} \sim nP\{X_1 > y\}, \qquad y > n^{2+1/a}\log n,$$

or

$$P\{Z_n > x\} \sim nP\{X_1 > x\sqrt{n}\} \tag{10.11}$$

for

$$x \geqq n^{3/2+1/a}\log n. \tag{10.12}$$

We must now investigate the behavior of $P\{Z_n > x\}$ for $1 \leqq x < n^{3/2+1/a}\log n$. It is possible to do so by the method expounded in sections 6 and 7.

Let $\varphi(t)$ be a characteristic function. A function $\gamma(t)$ will be called a radial continuation of $\varphi(t)$ for the ray $t \geqq 0$, if it is defined in some neighborhood of $t = 0$ and coincides with $\varphi(t)$ in this neighborhood for $t \geqq 0$ (a radial continuation for $t \leqq 0$ is similarly defined). The function $\varphi(t) = (|t| + 1)\exp(-|t|)$, for instance, corresponding to the density $g(x) = 2/\pi(x^2 + 1)^2$, has two radial continuations: $\gamma(t) = (t + 1)\exp(-t)$ for $t \geqq 0$, and $\gamma(t) = (-t + 1)\exp(t)$ for $t \leqq 0$. Both continuations are entire functions (though different ones). Note that they are not even, while $\varphi(t)$ is even.

11. Continuation of the proof

From (10.1) we deduce by integration by parts that $\varphi(t)$ has a radial continuation $\gamma(t)$, coinciding with it for $t \geqq 0$, which is differentiable at least $b = 4a + 2$ times. We proceed now to calculate $p_{Z_n}(x)$ (the integral theorem can be obtained later by integration). As $g(x)$ is even, $\varphi(t)$ is real and we obtain

$$p_{Z_n}(x) = \frac{\sqrt{n}}{\pi}\operatorname{Re}\int_0^\infty [\varphi(t)]^n e^{-\sqrt{n}\,itx}\,dt. \tag{11.1}$$

Applying the reasoning of section 5 [compare (5.9)] we get

$$\begin{aligned} p_{Z_n}(x) &= \frac{\sqrt{n}}{\pi}\operatorname{Re}\int_0^{\epsilon_0} [\varphi(t)]^n e^{-\sqrt{n}\,itx}\,dt + Be^{-c_0 n} \\ &= \frac{\sqrt{n}}{\pi}\operatorname{Re}\int_0^{\epsilon_0} [\gamma(t)]^n e^{-\sqrt{n}\,itx}\,dt + Be^{-c_0 n}. \end{aligned} \tag{11.2}$$

Further

$$\varphi(t) = \gamma(t) = 1 - \frac{t^2}{2} + O(t^3), \qquad 0 \leqq t \leqq \epsilon_0. \tag{11.3}$$

From this,

$$p_{Z_n}(x) = \frac{\sqrt{n}}{\pi}\operatorname{Re}\int_0^{\log n/\sqrt{n}} [\gamma(t)]^n e^{-\sqrt{n}\,itx}\,dt + Be^{-\delta_0\log^2 n}. \tag{11.4}$$

Also

$$(11.5)\qquad \begin{aligned}\gamma(t) &= \gamma_0(t) + B|t|^b,\\ \gamma_0(t) &= -\frac{t^2}{2} + \sum_{q=3}^{b-1}\frac{\gamma^{(q)}(0)}{q!}t^q + Bt^b,\end{aligned}$$

for $|t| \leqq \epsilon_0$. For $|t| \leqq \log n/\sqrt{n}$, we have

$$(11.6)\qquad \gamma(t) = \gamma_0(t) + B_\epsilon n^{-b/2+\epsilon}$$

for any $\epsilon > 0$, B_ϵ depending upon ϵ. Further

$$(11.7)\qquad K(t) = \log\gamma(t) = \log\gamma_0(t) + B_\epsilon n^{-b/2+\epsilon}, \qquad 0 \leqq t \leqq \frac{\log n}{\sqrt{n}}.$$

Further, for $0 \leqq t \leqq \log n/\sqrt{n}$,

$$(11.8)\qquad \log\gamma_0(t) = -\frac{t^2}{2} + \sum_{q=3}^{b-1}\Psi_q\frac{t^q}{q!} + Bt^b.$$

For $0 \leqq t \leqq \log n/\sqrt{n}$, we have $Bnt^b = B_\epsilon n^{-b/2+1+\epsilon}$.

Thus

$$(11.9)\qquad p_{Z_n}(x) = \frac{\sqrt{n}}{\pi}\operatorname{Re}\int_0^{\log n/\sqrt{n}}\exp n\left(-\frac{t^2}{2} + \sum_{q=3}^{b-1}\Psi_q\frac{t^q}{q!} - \sqrt{n}\,itx\right)dt + B_\epsilon n^{-b/2+1+\epsilon}.$$

If we write $K_3(t) = \sum_{q=3}^{b-1}\Psi_q t^q/q!$ we obtain

$$(11.10)\qquad nK_3(t) = B_\epsilon n^{-1/2+\epsilon}, \qquad 0 \leqq t \leqq \frac{\log n}{\sqrt{n}}.$$

Hence

$$(11.11)\qquad \exp[nK_3(t)] = 1 + K_4(t, n) + B_\epsilon n^{-b/2+1+\epsilon},$$

where

$$(11.12)\qquad K_4(t, n) = \sum_{q=1}^{b}\frac{[nK_3(t)]^q}{q!}.$$

Hence, taking $t = \xi/\sqrt{n}$, we get

$$(11.13)\qquad p_{Z_n}(x) = \frac{1}{\pi}\operatorname{Re}\int_0^{\log n}\exp\left(-\frac{\xi^2}{2}\right)\left[1 + K_4\left(\frac{\xi}{\sqrt{n}}, n\right)\right]\exp(-i\xi x)\,d\xi + B_\epsilon n^{-b/2+1+\epsilon}.$$

Extending the integration limit to $\xi = \infty$ (the error is estimated trivially),

$$(11.14)\qquad p_{Z_n}(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} + \frac{1}{\pi}\operatorname{Re}\int_0^\infty\exp\left(-\frac{\xi^2}{2}\right)K_4\left(\frac{\xi}{\sqrt{n}}, n\right)\exp(-i\xi x)\,d\xi + B_\epsilon n^{-b/2+1+\epsilon}.$$

The function $K_4(\xi/\sqrt{n}, n)$ is a polynomial with respect to $\xi/\sqrt{n}$. The evaluation of (11.14) is thus reduced to the evaluation of the integrals

$$E(x, r) = \operatorname{Re} \int_0^\infty e^{-\xi^2/2} e^{i\xi x} \xi^r \, d\xi \tag{11.15}$$

for large values of x. If r is an even number,

$$E(x, r) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\xi^2/2} e^{i\xi x} \xi^r \, d\xi \tag{11.16}$$

is easily expressed in terms of exp $(-x^2/2)$ and Hermite's polynomials [compare (7.7)]. If r is odd there is apparently no expression in terms of the elementary function but for large values of x, (11.15) can be easily evaluated by integration by parts. Thus, for $r = 1$, we have $E(x, r) \sim 1/x^2$.

We must now show that the formula (11.14) holds for the values of x satisfying

$$1 \leqq x \leqq n^{3/2+1/a} \log n. \tag{11.17}$$

Comparing $nP\{X_1 > x\sqrt{n}\}$ [compare (10.12)] to the remainder term $B_\epsilon n^{-b/2+1+\epsilon}$ in (11.14) we deduce that $n^{-b/2+1+\epsilon}$ must be smaller than the product $n(n^{2+1/a+\epsilon})^{-a} = n^{-2a-\epsilon'}$; from this $b/2 - 1 > 2a$ or $b > 4a + 2$. Under this condition we obtain the local theorem valid up to $n^{3/2+1/a+\epsilon}$ uniformly and hence, by integration, the integral theorem. The relation (10.12) enables us to obtain it for the whole x-axis.

12. Concluding remarks

The condition of evenness was assumed only to simplify the final formulas; if it is not fulfilled, (11.14) will only involve $\int_{-\infty}^0$ in addition to $\int_0^\infty$. Moreover, the analogous limit theorem on the whole axis can be obtained for the variables X_j such that

$$P\{X_j > x\} = \int_{\nu=a}^{a_1} \frac{dG(\nu)}{x^\nu} + O\left(\frac{1}{x^{a_1-\epsilon}}\right), \qquad x \geqq 1, \tag{12.1}$$

where $a_1 > 4a + 5$ and $G(\nu)$ is a function of bounded variation. A similar relation must hold for negative values of x.

The new approach expounded here is applicable also to independent variables which are not identically distributed and to the investigation of nonnormal convergence.

The asymptotic behavior of the large deviations of order statistics can be also studied by this method.

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