SPECTRAL ANALYSIS OF STATIONARY GAUSSIAN PROCESSES

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1. Gauss functions

Let $(\Omega, \mathcal{E}, P)$ be a probability space, that is, $\Omega = \{\omega\}$ is a set of elements ω, and $\mathcal{E} = \{E\}$ is a sigma field of subsets E of Ω, and $P(E)$ is a countably additive measure defined on $\mathcal{E}$ with $P(\Omega) = 1$. We denote by $L^2(\Omega)$ the real L^2-space over $(\Omega, \mathcal{E}, P)$, that is, the real linear space of all real-valued $\mathcal{E}$-measurable functions $f(\omega)$ defined on Ω such that

$$(1) \qquad ||f||^2 = \int_\Omega |f(\omega)|^2 P(d\omega) < \infty.$$

Two functions from $L^2(\Omega)$ which coincide almost everywhere on Ω are identified in $L^2(\Omega)$. For any two functions $f(\omega)$ and $g(\omega)$ from $L^2(\Omega)$, their inner product (f, g) is defined by

$$(2) \qquad (f, g) = \int_\Omega f(\omega)g(\omega)\, P(d\omega).$$

A function $x(\omega)$ from $L^2(\Omega)$ is called a *Gauss function* if either (i) $x(\omega) = 0$ almost everywhere on Ω, or (ii) there exists a positive number $\sigma > 0$ such that

$$(3) \qquad P\{\omega | \alpha < x(\omega) < \beta\} = \frac{1}{\sqrt{2\pi\sigma}} \int_\alpha^\beta \exp\left(-\frac{u^2}{2\sigma}\right) du$$

for any real numbers α and β with $\alpha < \beta$. In the second case (ii), the function $x(\omega)$ is said to have a *Gaussian distribution* with mean 0 and variance $\sigma > 0$.

2. Gauss systems

Let $\mathcal{G} = \{x_1(\omega), \cdots, x_n(\omega)\}$ be a finite set of functions from $L^2(\Omega)$. Then $\mathcal{G}$ is called a *Gauss system* if the linear combination $\sum_{k=1}^n c_k x_k(\omega)$ is a Gauss function for any real numbers $c_1, \cdots, c_n$. Further $\mathcal{G}$ is said to have an *n-dimensional Gaussian distribution* with mean 0 if there exists a real positive definite matrix $A = (a_{k,l} | k, l = 1, \cdots, n)$ such that

$$(4) \qquad P\{\omega | \alpha_k < x_k(\omega) < \beta_k, k = 1, \cdots, n\} = \left(\frac{\det A}{(2\pi)^n}\right)^{1/2} \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_n}^{\beta_n} \exp\left[-\frac{1}{2}(Au, u)\right] du,$$

for any real numbers α_k and β_k with $\alpha_k < \beta_k$ and $k = 1, \cdots, n$, where $u = (u_1, \cdots, u_n)$ and $du = du_1 \cdots du_n$,

$$(5) \qquad (Au, u) = \sum_{k=1}^{n} \sum_{l=1}^{n} a_{k,l} u_k u_l,$$

and det A is the determinant of the matrix A. It is easy to see that $\mathcal{S} = \{x_1(\omega), \cdots, x_n(\omega)\}$ has an n-dimensional Gaussian distribution if and only if the following two conditions are satisfied: (i) $\mathcal{S}$ is a Gauss system; (ii) $\mathcal{S}$ is linearly independent, that is, $\sum_{k=1}^{n} c_k x_k(\omega) = 0$ almost everywhere on Ω only if $c_1 = \cdots = c_n = 0$. We quote one result concerning Gauss systems: A Gauss system $\mathcal{S} = \{x_1(\omega), \cdots, x_n(\omega)\}$ is *independent*, that is,

$$(6) \qquad P\{\omega | \alpha_k < x_k(\omega) < \beta_k, k = 1, \cdots, n\} = \prod_{k=1}^{n} P\{\omega | \alpha_k < x_k(\omega) < \beta_k\}$$

for any real numbers α_k and β_k with $\alpha_k < \beta_k$ and $k = 1, \cdots, n$, if and only if $\mathcal{S}$ is *orthogonal*, that is, $(x_k, x_l) = 0$ for any integers k and l with $1 \leqq k < l \leqq n$.

3. Gauss linear spaces

Let us denote by $\mathcal{G}$ the set of all Gauss functions in $L^2(\Omega)$. It is clear that $\mathcal{G}$ is a nowhere dense closed subset of $L^2(\Omega)$. It is also easy to see that $\mathcal{G}$ consists only of one function $x(\omega) = 0$ if and only if the probability space $(\Omega, \mathcal{E}, P)$ has an atomic element, that is, if and only if there exists a set E in $\mathcal{E}$ with $P(E) > 0$ such that $P(F) = 0$ or $P(E - F) = 0$ for any subset F of E in $\mathcal{E}$. On the other hand, if $(\Omega, \mathcal{E}, P)$ has no atomic element, then $\mathcal{G}$ is a reasonably large subset of $L^2(\Omega)$, and, in fact, contains an infinite dimensional linear subspace of $L^2(\Omega)$, while $\mathcal{G}$ itself is not a linear subspace of $L^2(\Omega)$.

A linear subspace $\mathfrak{X}$ of $L^2(\Omega)$ is called a *Gauss linear subspace* of $L^2(\Omega)$ if it is contained in $\mathcal{G}$. It is obvious from definition that a finite set $\mathcal{S} = \{x_1(\omega), \cdots, x_n(\omega)\}$ of functions from $L^2(\Omega)$ is a Gauss system if and only if it is contained in a Gauss linear subspace of $L^2(\Omega)$. There are two cases in which Gauss linear subspaces of $L^2(\Omega)$ play important rôles in the theory of stochastic processes.

EXAMPLE 1 (*Stationary Gaussian process*). Let $\mathcal{S} = \{x_n(\omega) | n = 0, \pm 1, \pm 2, \cdots\}$ be a two-sided infinite sequence of functions from $L^2(\Omega)$ with the following two properties: (i) $\sum_{k=-n}^{n} c_k x_k(\omega)$ is a Gauss function for any real numbers c_k with $k = -n, \cdots, n$ and $n = 1, 2, \cdots$; (ii) the inner product (x_k, x_l) depends only on the difference $k - l$. Here $\mathcal{S}$ is called a *stationary Gaussian process.*

Let $\mathfrak{X}$ be the closed linear subspace of $L^2(\Omega)$ spanned by $\mathcal{S}$. Obviously $\mathfrak{X}$ is a Gauss linear subspace of $L^2(\Omega)$, and it is easy to see that there exists a unitary transformation V of $\mathfrak{X}$ onto itself such that $Vx_n = x_{n+1}$ for $n = 0, \pm 1, \pm 2, \cdots$.

Conversely, let $\mathfrak{X}$ be a closed Gauss linear subspace of $L^2(\Omega)$, and let V be a unitary transformation of $\mathfrak{X}$ onto itself. Assume further that V is *monogenic* on $\mathfrak{X}$, that is, that there exists an element x_0 in $\mathfrak{X}$ such that the two-sided infinite sequence $\mathcal{S} = \{x_n = V^n x_0 | n = 0, \pm 1, \pm 2, \cdots\}$ spans $\mathfrak{X}$. It is then clear that $\mathcal{S}$

is a stationary Gaussian process. Thus, a stationary Gaussian process is nothing but a Gauss linear subspace of $L^2(\Omega)$ with a monogenic unitary transformation defined on it.

EXAMPLE 2 (*Generalized Brownian motion*). Let $(S, \mathfrak{B}, m)$ be a measure space, that is, $S = \{s\}$ is a set of elements s, while $\mathfrak{B} = \{B\}$ is a sigma field of subsets B of S, and $m(B)$ is a countably additive measure defined on $\mathfrak{B}$ with $0 < m(S) \leqq \infty$. We do not assume that $m(S) < \infty$, but it is assumed that there exists a sequence $\{B_n | n = 1, 2, \cdots\}$ of sets from $\mathfrak{B}$ such that $m(B_n) < \infty$ for $n = 1, 2, \cdots$ and $\bigcup_{n=1}^{\infty} B_n = S$. We denote by $\mathfrak{B}_0$ the subfamily of $\mathfrak{B}$ consisting of all sets B from $\mathfrak{B}$ with $m(B) < \infty$. Let $x(B, \omega)$ be a real-valued function defined on $\mathfrak{B}_0 \times \Omega$ with the following two properties: (i) for each set B from $\mathfrak{B}_0$, we have that $x_B(\omega) = x(B, \omega)$ is a Gauss function of ω with variance $\sigma = m(B)$; (ii) if the sets $B_1, \cdots, B_n$ from $\mathfrak{B}_0$ are disjoint, then the functions $x_{B_1}(\omega), \cdots, x_{B_n}(\omega)$ are independent. The function $x(B, \omega)$ is called a *generalized Brownian motion*.

Let $\mathfrak{X}$ be the closed linear subspace of $L^2(\Omega)$ spanned by the functions $x_B(\omega) = x(B, \omega)$ for $B \in \mathfrak{B}_0$. Then $\mathfrak{X}$ is obviously a Gauss linear subspace of $L^2(\Omega)$. Let further $L^2(S)$ be the real L^2-space over the measure space $(S, \mathfrak{B}, m)$, that is, $L^2(S)$ is the real linear space of all real-valued $\mathfrak{B}$-measurable functions $\xi(s)$ defined on S such that

$$(7) \qquad ||\xi||^2 = \int_S |\xi(s)|^2 \, m(ds) < \infty$$

Again, two functions from $L^2(S)$ which coincide almost everywhere on S are identified in $L^2(S)$. For any two functions $\xi(s)$ and $\eta(s)$ from $L^2(S)$, their inner product (ξ, η) is defined by

$$(8) \qquad (\xi, \eta) = \int_S \xi(s)\eta(s) \, m(ds).$$

It is then easy to see that there exists a unitary transformation $\overline{W}$ which maps $L^2(S)$ onto $\mathfrak{X}$ such that $\overline{W}\xi_B = x_B$ for any set B from $\mathfrak{B}_0$, where $\xi_B(s)$ is the characteristic function of a set B and $x_B(\omega) = x(B, \omega)$. In fact, if we put

$$(9) \qquad W\left(\sum_{k=1}^{n} c_k \xi_{B_k}\right) = \sum_{k=1}^{n} c_k x_{B_k}$$

for any sets $B_1, \cdots, B_n$ from $\mathfrak{B}_0$ (disjoint) and for any real numbers $c_1, \cdots, c_n$, then W is a linear and isometric mapping of the set $L_0^2(S)$ of all simple functions from $L^2(S)$ onto a dense linear subspace of $\mathfrak{X}$, and the required unitary transformation $\overline{W}$ of $L^2(S)$ onto $\mathfrak{X}$ is obtained as the continuous extension of W from $L_0^2(S)$ to $L^2(S)$. We observe that if $\mathfrak{S} = \{\xi_1, \cdots, \xi_n\}$ is an orthogonal system in $L^2(S)$, then the image $\overline{W}(\mathfrak{S}) = \{\overline{W}\xi_1, \cdots, \overline{W}\xi_n\}$ of $\mathfrak{S}$ by $\overline{W}$ is an independent system in $\mathfrak{X}$.

Conversely, if we start from a unitary transformation $\overline{W}$ of $L^2(S)$ onto a Gauss linear subspace $\mathfrak{X}$ of $L^2(\Omega)$, and if we put $x(B, \omega) = x_B(\omega)$, where $x_B = \overline{W}\xi_B$ and

$\xi_B(s)$ is the characteristic function of a set B from $\mathcal{B}_0$, then $x(B, \omega)$ is a generalized Brownian motion. Thus a generalized Brownian motion is nothing but a unitary transformation of $L^2(S)$ onto a Gauss linear subspace of $L^2(\Omega)$, or equivalently, an isometric embedding of $L^2(S)$ into $L^2(\Omega)$ as a Gauss linear subspace of $L^2(\Omega)$.

4. Fundamental Gauss linear spaces

Let $\mathfrak{X}$ be a Gauss linear subspace of $L^2(\Omega)$. We denote by $\mathcal{R}(\mathfrak{X})$ the set of all functions $f(\omega)$ of the form

$$f(\omega) = P\{x_1(\omega), \cdots, x_p(\omega)\} = \sum a_{n_1,\cdots,n_p} \prod_{k=1}^{p} [x_k(\omega)]^{n_k}, \tag{10}$$

where p is a positive integer, $x_1(\omega), \cdots, x_p(\omega)$ are functions from $\mathfrak{X}$, and

$$P(u_1, \cdots, u_p) = \sum a_{n_1,\cdots,n_p} u_1^{n_1} \cdots u_p^{n_p} \tag{11}$$

is a polynomial of p variables $u_1, \cdots, u_p$ with real coefficients $a_{n_1,\cdots,n_p}$. It is easy to see that $\mathcal{R}(\mathfrak{X})$ is a linear subspace of $L^2(\Omega)$ and is a ring with respect to the multiplication $(fg)(\omega) = f(\omega)g(\omega)$. Here $\mathcal{R}(\mathfrak{X})$ is called the *ring of functions generated by* $\mathfrak{X}$.

Let us denote by $\mathcal{E}_0(\mathfrak{X})$ the family of all subsets E of Ω of the form $E = \{\omega | \alpha < x(\omega) < \beta\}$, where $x(\omega)$ is a function from $\mathfrak{X}$, and α and β are two real numbers with $\alpha < \beta$. Let further $\mathcal{E}(\mathfrak{X})$ be the field of subsets of Ω generated by $\mathcal{E}_0(\mathfrak{X})$. Then $\mathcal{E}(\mathfrak{X})$ is called the *field of sets generated by* $\mathfrak{X}$.

THEOREM 1. *Let* $\mathfrak{X}$ *be a Gauss linear subspace of* $L^2(\Omega)$. *Then the ring* $\mathcal{R}(\mathfrak{X})$ *of functions generated by* $\mathfrak{X}$ *is dense in* $L^2(\Omega)$ *if and only if the field* $\mathcal{E}(\mathfrak{X})$ *of sets generated by* $\mathfrak{X}$ *is dense in* $\mathcal{E}$.

This last condition means that, for any set E from $\mathcal{E}$ and for any positive number $\epsilon > 0$, there exists a set E' from $\mathcal{E}(\mathfrak{X})$ such that $P(E \cup E' - E \cap E') < \epsilon$. A Gauss linear subspace $\mathfrak{X}$ of $L^2(\Omega)$ is said to be *fundamental* if the conditions of theorem 1 are satisfied.

5. Multiplicative unitary transformations

A unitary transformation $\tilde{V}$ which maps $L^2(\Omega)$ onto itself is said to be *multiplicative* if $(\tilde{V}(fg))(\omega) = (\tilde{V}f)(\omega)(\tilde{V}g)(\omega)$ almost everywhere on Ω for any two bounded functions $f(\omega)$ and $g(\omega)$ from $L^2(\Omega)$. A one-to-one mapping φ of Ω onto itself is called a *measure preserving transformation* defined on the probability space $(\Omega, \mathcal{E}, P)$ if $E \in \mathcal{E}$ implies $\varphi(E) \in \mathcal{E}$, $\varphi^{-1}(E) \in \mathcal{E}$ and $P\{E\} = P\{\varphi(E)\} = P\{\varphi^{-1}(E)\}$. It was proved by J. von Neumann [7] that, if $(\Omega, \mathcal{E}, P)$ is the Lebesgue probability space, that is, if $\Omega = \{\omega | 0 \leqq \omega \leqq 1\}$ = the closed unit interval of real numbers, $\mathcal{E}$ = the sigma field of all Lebesgue measurable subsets E of Ω, and $P(E)$ = the ordinary Lebesgue measure of E, then, for any multiplicative unitary transformation $\tilde{V}$ of $L^2(\Omega)$ onto itself, there exists a measure preserving transformation φ defined on $(\Omega, \mathcal{E}, P)$ such that $(\tilde{V}f)(\omega) = f(\varphi^{-1}(\omega))$ almost everywhere on Ω for any function $f(\omega)$ from $L^2(\Omega)$.

THEOREM 2. *Let $\mathfrak{X}$ be a fundamental Gauss linear subspace of $L^2(\Omega)$, and let V be a unitary transformation of $\mathfrak{X}$ onto itself. Then there exists a multiplicative unitary transformation $\tilde{V}$ of $L^2(\Omega)$ onto itself such that $\tilde{V}x = Vx$ for any function $x(\omega)$ from $\mathfrak{X}$. If, in particular, the underlying probability space $(\Omega, \mathcal{E}, P)$ is the Lebesgue probability space, then there exists a measure preserving transformation φ_V defined on $(\Omega, \mathcal{E}, P)$ such that $(Vx)(\omega) = x(\varphi_V^{-1}(\omega))$ almost everywhere on Ω for any function $x(\omega)$ from $\mathfrak{X}$, and $(\tilde{V}f)(\omega) = f(\varphi^{-1}(\omega))$ almost everywhere on Ω for any function $f(\omega)$ from $L^2(\Omega)$.*

It is easy to see that $\tilde{V}$ is defined on $\mathfrak{R}(\mathfrak{X})$ by the formula,

$$(\tilde{V}f)(\omega) = P\{(Vx_1)(\omega), \cdots, (Vx_p)(\omega)\} = \sum a_{n_1,\cdots,n_p} \prod_{k=1}^{p} [(Vx_k)(\omega)]^{n_k}, \tag{12}$$

if a function $f(\omega)$ from $\mathfrak{R}(\mathfrak{X})$ has the form (10). Thus defined $\tilde{V}$ is obviously a linear and multiplicative mapping of $\mathfrak{R}(\mathfrak{X})$ onto itself, but it requires a nontrivial argument based on the properties of Gauss functions to show that $\tilde{V}$ is an isometry on $\mathfrak{R}(\mathfrak{X})$.

The measure preserving transformation φ_V is uniquely determined (up to a set of measure zero) by the unitary transformation V. It is clear that the mapping $V \rightarrow \varphi_V$ is a homomorphism of the group of all unitary transformations of $\mathfrak{X}$ onto itself into the group of all measure preserving transformations defined on $(\Omega, \mathcal{E}, P)$. It is also clear that the mapping $V \rightarrow \tilde{V}$ is a homomorphism of the group of all unitary transformations of $\mathfrak{X}$ onto itself into the group of all unitary transformations of $L^2(\Omega)$ onto itself.

Thus the problem of the spectral analysis of a unitary transformation V of a fundamental Gauss linear subspace $\mathfrak{X}$ of $L^2(\Omega)$ onto itself is reduced to the problem of the measure preserving transformation φ_V defined on $(\Omega, \mathcal{E}, P)$ or to that of the multiplicative unitary transformation $\tilde{V}$ of $L^2(\Omega)$ onto itself.

6. Homogeneous chaos

We now consider the polynomial chaos and the homogeneous chaos introduced by N. Wiener [10]. Let $\mathfrak{X}$ be a Gauss linear subspace of $L^2(\Omega)$. Let us denote by $\mathcal{P}_n(\mathfrak{X})$ the set of all functions $f(\omega)$ of the form (10) in which $P(u_1, \cdots, u_p)$ is a polynomial of $u_1, \cdots, u_p$ of degree at most n, that is, $P(u_1, \cdots, u_p)$ has the form (11) in which $a_{n_1,\cdots,n_p} = 0$ if $n_1 + \cdots + n_p > n$. It is clear that $\mathcal{P}_n(\mathfrak{X})$ is a linear subspace of $L^2(\Omega)$, that $\mathcal{P}_n(\mathfrak{X}) \subset \mathcal{P}_{n+1}(\mathfrak{X})$ for $n = 0, 1, 2, \cdots$, and that $\bigcup_{n=1}^{\infty} \mathcal{P}_n(\mathfrak{X}) = \mathfrak{R}(\mathfrak{X})$. Then $\mathcal{P}_n(\mathfrak{X})$ is called the nth *polynomial chaos* over $\mathfrak{X}$. It is easy to see that $\mathcal{P}_0(\mathfrak{X})$ is the one-dimensional subspace of $L^2(\Omega)$ consisting of all constant functions, and that $\mathcal{P}_1(\mathfrak{X}) = \mathcal{P}_0(\mathfrak{X}) \oplus \mathfrak{X}$, where $\oplus$ denotes the direct sum of two mutually orthogonal linear subspaces of $L^2(\Omega)$.

Let us put $\mathcal{Q}_0(\mathfrak{X}) = \mathcal{P}_0(\mathfrak{X})$ and $\mathcal{Q}_1(\mathfrak{X}) = \mathfrak{X}$. Also let $\mathcal{Q}_n(\mathfrak{X}) = \mathcal{P}_n(\mathfrak{X}) \ominus \mathcal{P}_{n-1}(\mathfrak{X}) =$ the orthocomplement of $\mathcal{P}_{n-1}(\mathfrak{X})$ in $\mathcal{P}_n(\mathfrak{X})$, that is, $\mathcal{Q}_n(\mathfrak{X})$ is the set of all elements of $\mathcal{P}_n(\mathfrak{X})$ which are orthogonal to all elements of $\mathcal{P}_{n-1}(\mathfrak{X})$ for $n = 2, 3, \cdots$. Then $\mathcal{Q}_n(\mathfrak{X})$ is called the nth *homogeneous chaos* over $\mathfrak{X}$.

Let $H_n(u)$ be the nth Hermite polynomial of u defined by

$$H_n(u) = (-1)^n \frac{1}{n!} e^{u^2/2} \frac{d^n}{du^n} e^{-u^2/2}, \tag{13}$$

$n = 0, 1, 2, \cdots$. It is then easy to show that $\mathcal{Q}_n(\mathfrak{X})$ consists exactly of all functions $f(\omega)$ of the form

$$f(\omega) = \sum a_{n_1, \cdots, n_p} \prod_{k=1}^{p} H_{n_k}[x_k(\omega)], \tag{14}$$

where p is a positive integer, $\{x_1(\omega), \cdots, x_p(\omega)\}$ is an orthonormal system from $\mathfrak{X}$, when $n_1, \cdots, n_p$ are nonnegative integers, and $a_{n_1, \cdots, n_p} = 0$ if $n_1 + \cdots + n_p \neq n$. We observe that all terms on the right side of (14) are mutually orthogonal and hence

$$||f||^2 = \sum |a_{n_1, \cdots, n_p}|^2 \frac{1}{n_1! \cdots n_p!}. \tag{15}$$

Let V be a unitary transformation which maps $\mathfrak{X}$ onto itself. We put

$$(V_{(n)}f)(\omega) = \sum a_{n_1, \cdots, n_p} \prod_{k=1}^{p} H_{n_k}[(Vx_k)(\omega)], \tag{16}$$

if a function $f(\omega)$ from $\mathcal{Q}_n(\mathfrak{X})$ has the form (14). Thus $V_{(n)}$ is actually the restriction of $\tilde{V}$ [defined by (12)] to $\mathcal{Q}_n(\mathfrak{X})$, and is a unitary transformation which maps $\mathcal{Q}_n(\mathfrak{X})$ onto itself. It is interesting to observe that (16) is uniquely determined (up to a set of measure zero) by $f(\omega)$ and V, although the expression (14) for a given function $f(\omega)$ from $\mathcal{Q}_n(\mathfrak{X})$ is not unique.

Thus our problem is reduced to the problem of the spectral analysis of the unitary transformation $V_{(n)}$ of $\mathcal{Q}_n(\mathfrak{X})$ onto itself which is defined by formula (16).

7. Symmetric tensor products

We next consider the tensor product of a Hilbert space. We follow the construction given by F. J. Murray and J. von Neumann [6]. Let $\mathfrak{H} = \{x, y, \cdots\}$ be a complete or incomplete Hilbert space with the inner product (x, y) and the norm $||x|| = [(x, x)]^{1/2}$. Let $\mathfrak{H}^{(n)}$ be the real linear space of all elements of the form

$$x^{(n)} = \sum_{k=1}^{p} a_k(x_1^k \otimes \cdots \otimes x_n^k), \tag{17}$$

where $x_1^k, \cdots, x_n^k$ are elements of $\mathfrak{H}$, and a_k is a real number, $k = 1, \cdots, p$; that is, $\mathfrak{H}^{(n)}$ is a real linear space algebraically spanned by elements $x^{(n)}$ of the form $x^{(n)} = x_1 \otimes \cdots \otimes x_n$, where $x_1, \cdots, x_n$ are elements of $\mathfrak{H}$. We put

$$(18) \qquad (x^{(n)}, y^{(n)}) = \sum_{k=1}^{p} \sum_{l=1}^{q} a_k b_l \prod_{i=1}^{n} (x_i^k, y_i^l),$$

if $x^{(n)}$ from $\mathcal{H}^{(n)}$ has the form (17) and $y^{(n)}$ from $\mathcal{H}^{(n)}$ has the form

$$(19) \qquad y^{(n)} = \sum_{l=1}^{q} b_l(y_1^l \otimes \cdots \otimes y_n^l),$$

where $y_1^l, \cdots, y_n^l$ are elements of $\mathcal{H}$, and b_l is a real number, $l = 1, \cdots, q$. Clearly $(x^{(n)}, y^{(n)})$ is a positive semidefinite bilinear form defined on $\mathcal{H}^{(n)}$, and $||x^{(n)}|| = [(x^{(n)}, x^{(n)})]^{1/2}$ is a seminorm on $\mathcal{H}^{(n)}$. Let $\mathcal{H}_0^{(n)}$ be the set of all elements $x^{(n)}$ of $\mathcal{H}^{(n)}$ such that $||x^{(n)}|| = 0$. Then $\mathcal{H}_0^{(n)}$ is a linear subspace of $\mathcal{H}^{(n)}$. The nth *tensor product* $\mathfrak{T}_n(\mathcal{H})$ of $\mathcal{H}$ is defined as the factor space $\mathcal{H}^{(n)}/\mathcal{H}_0^{(n)}$, that is, a real linear space obtained from $\mathcal{H}^{(n)}$ by identifying the elements $x^{(n)}$ and $y^{(n)}$ of $\mathcal{H}^{(n)}$ such that $x^{(n)} - y^{(n)} \in \mathcal{H}_0^{(n)}$. In order to avoid complicated notations, we use the same notation $x^{(n)}$ for the elements of $\mathfrak{T}_n(\mathcal{H})$. The inner product and the norm on $\mathfrak{T}_n(\mathcal{H})$ are defined in the obvious way from the bilinear form $(x^{(n)}, y^{(n)})$ and the seminorm $||x^{(n)}||$ on $\mathcal{H}^{(n)}$. These are again denoted by $(x^{(n)}, y^{(n)})$ and $||x^{(n)}||$ respectively. $\mathfrak{T}_n(\mathcal{H})$ is an incomplete Hilbert space with respect to the inner product $(x^{(n)}, y^{(n)})$ and the norm $||x^{(n)}||$ if $\mathcal{H}$ is infinite dimensional and $n > 1$.

Let π be a permutation of the set $(1, \cdots, n)$ of positive integers. We put

$$(20) \qquad R_\pi x^{(n)} = \sum_{k=1}^{p} a_k(x_{\pi(1)}^k \otimes \cdots \otimes x_{\pi(n)}^k),$$

if an element $x^{(n)}$ from $\mathcal{H}^{(n)}$ has the form (17). Then R_π is a linear transformation of $\mathcal{H}^{(n)}$ onto itself which satisfies $(R_\pi x^{(n)}, R_\pi y^{(n)}) = (x^{(n)}, y^{(n)})$ for any two elements $x^{(n)}$ and $y^{(n)}$ from $\mathcal{H}^{(n)}$. From this follows that R_π maps $\mathcal{H}_0^{(n)}$ onto itself. Thus R_π may be considered as a unitary transformation of $\mathfrak{T}_n(\mathcal{H})$ onto itself. The nth *symmetric tensor product* $\mathfrak{S}_n(\mathcal{H})$ of $\mathcal{H}$ is defined as the linear supspace of $\mathfrak{T}_n(\mathcal{H})$ consisting of all elements of $\mathfrak{T}_n(\mathcal{H})$ which are invariant under R_π for any permutation π of $(1, \cdots, n)$. If we put

$$(21) \qquad M_n = \frac{1}{n!} \sum_\pi R_\pi,$$

where $\sum$ denotes the sum for all $n!$ permutations π of $(1, \cdots, n)$, then M_n is a projection of $\mathfrak{T}_n(\mathcal{H})$ onto $\mathfrak{S}_n(\mathcal{H})$.

Let V be a unitary transformation of $\mathcal{H}$ onto itself. We put

$$(22) \qquad V^{(n)} x^{(n)} = \sum_{k=1}^{p} a_k(V x_1^k \otimes \cdots \otimes V x_n^k),$$

if an element $x^{(n)}$ from $\mathcal{H}^{(n)}$ has the form (17). Then $V^{(n)}$ is a linear transformation of $\mathcal{H}^{(n)}$ onto itself which satisfies $(V^{(n)}x^{(n)}, V^{(n)}y^{(n)}) = (x^{(n)}, y^{(n)})$ for any two elements $x^{(n)}$ and $y^{(n)}$ from $\mathcal{H}^{(n)}$. From this follows that $V^{(n)}$ maps $\mathcal{H}_0^{(n)}$ onto itself. Thus $V^{(n)}$ may be considered as a unitary transformation of $\mathfrak{T}_n(\mathcal{H})$ onto itself. It is also easy to see that $R_\pi V^{(n)} = V^{(n)} R_\pi$ for any permutation π of $(1, \cdots, n)$. From this follows that $V^{(n)}$ may be considered as a unitary transformation of

$\mathfrak{S}_n(\mathfrak{H})$ onto itself. $V^{(n)}$ thus defined on $\mathfrak{S}_n(\mathfrak{H})$ is called the *unitary transformation induced on* $\mathfrak{S}_n(\mathfrak{H})$ *by* V.

8. Isomorphisms of homogeneous chaos and symmetric tensor products

Let $\mathfrak{X}$ be a Gauss linear subspace of $L^2(\Omega)$, and let V be a unitary transformation of $\mathfrak{X}$ onto itself. Let $V_{(n)}$ be the unitary transformation of the nth homogeneous chaos $\mathfrak{Q}_n(\mathfrak{X})$ over $\mathfrak{X}$ onto itself which was defined in section 6 by (16). On the other hand, consider $\mathfrak{X}$ as a complete or incomplete Hilbert space $\mathfrak{H}$, and let $V^{(n)}$ be the unitary transformation of the nth symmetric tensor product $\mathfrak{S}_n(\mathfrak{X})$ of $\mathfrak{X}$ onto itself which was defined in section 7. It is then possible to show that these two unitary transformations $V_{(n)}$ and $V^{(n)}$ are spectrally isomorphic, or more precisely.

THEOREM 3. *Let* $\mathfrak{X}$ *be a Gauss linear subspace of* $L^2(\Omega)$. *Put*

$$W_n f = M_n \left\{ \sqrt{n!} \sum a_{n_1, \cdots, n_p} \frac{\overbrace{x_1 \otimes \cdots \otimes x_1}^{n_1 \text{ times}} \otimes \cdots \otimes \overbrace{x_p \otimes \cdots \otimes x_p}^{n_p \text{ times}}}{n_1! \cdots n_p!} \right\} \tag{23}$$

if a function $f(\omega)$ *from* $\mathfrak{Q}_n(\mathfrak{X})$ *has the form* (14). *Then* W_n *is a unitary transformation of* $\mathfrak{Q}_n(\mathfrak{X})$ *onto* $\mathfrak{S}_n(\mathfrak{X})$ *and satisfies*

$$W_n^{-1} V^{(n)} W_n = V_{(n)}, \qquad n = 0, 1, 2, \cdots \tag{24}$$

for any unitary transformation V *of* $\mathfrak{X}$ *onto itself.*

Thus our problem is reduced to the problem of the spectral analysis of the unitary transformations $V^{(n)}$ induced on the symmetric tensor products $\mathfrak{S}_n(\mathfrak{X})$ by V. It is easy to obtain the spectral resolution for $V^{(n)}$ in terms of the spectral resolution for V. In this way we can show, among other things, that $V^{(n)}$ (and hence $V_{(n)}$), for $n = 1, 2, \cdots$, have a continuous spectrum (or absolutely continuous spectrum) if and only if V has a continuous spectrum (or absolutely continuous spectrum). This fact, when applied to the case of example 1 of section 3, gives the known results of K. Itô [1] and G. Maruyama [5] concerning stationary Gaussian processes. Also, if we apply it to the case of example 2 of section 3, then we obtain the results concerning the flow of Brownian motions as announced in [4]. This isomorphism of $\mathfrak{Q}_n(\mathfrak{X})$ and $\mathfrak{S}_n(\mathfrak{X})$ also explains the relations between the theory of multiple Wiener integrals due to K. Itô [2], [3] and the theory of symmetric tensor algebras due to I. E. Segal [8], [9]. The details of arguments leading to the proofs of the theorems stated in this paper, as well as the discussion of further properties of homogeneous chaos and symmetric tensor products, will be given in a separate paper.

REFERENCES

[1] K. Itô, "A kinematic theory of turbulence," *Proc. Acad. Tokyo*, Vol. 20 (1944), pp. 120–122.

[2] ———, "Multiple Wiener integral," *J. Math. Soc. Japan*, Vol. 3 (1951), pp. 157–169.
[3] ———, "Complex multiple Wiener integral," *Jap. J. Math.*, Vol. 22 (1952), pp. 63–86.
[4] S. KAKUTANI, "Determination of the spectrum of the flow of Brownian motion," *Proc. Nat. Acad. Sci. U.S.A.*, Vol. 36 (1950), pp. 319–323.
[5] G. MARUYAMA, "The harmonic analysis of stationary stochastic processes," *Mem. Fac. Sci., Kyushu Univ., Ser. A.*, Vol. 4 (1949), pp. 45–106.
[6] F. J. MURRAY and J. VON NEUMANN, "On rings of operators," *Ann. of Math.*, Vol. 37 (1936), pp. 116–229.
[7] J. VON NEUMANN, "Zur Operatorenmethode in der klassischen Mechanik," *Ann. of Math.*, Vol. 33 (1932), pp. 587–642.
[8] I. E. SEGAL, "Tensor algebras over Hilbert spaces, I," *Trans. Amer. Math. Soc.*, Vol. 81 (1956), pp. 106–134.
[9] ———, "Tensor algebras over Hilbert spaces, II," *Ann. of Math.*, Vol. 63 (1956), pp. 160–175.
[10] N. WIENER, "The homogeneous chaos," *Amer. J. Math.*, Vol. 60 (1938), pp. 897–936.