ON SEQUENCES OF SUMS OF INDEPENDENT RANDOM VECTORS

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1. Introduction and summary

This paper is concerned with certain properties of the sequence $S_1, S_2, \cdots$ of the sums $S_n = X_1 + \cdots + X_n$ of independent, identically distributed, k-dimensional random vectors $X_1, X_2, \cdots$, where $k \geqq 1$. Attention is restricted to vectors X_n with integer-valued components. Let $A_1, A_2, \cdots$ be a sequence of k-dimensional measurable sets and let N denote the least n for which $S_n \in A_n$. The values $S_0 = 0, S_1, S_2, \cdots$ may be thought of as the successive positions of a moving particle which starts at the origin. The particle is absorbed when it enters set A_n at time n, and N is the time at which absorption occurs. Let M denote the number of times the particle is at the origin prior to absorption (the number of integers n, where $0 \leqq n < N$, for which $S_n = 0$). For the special case $P\{X_n = -1\} = P\{X_n = 1\} = 1/2$ it is found that

$$E(M) = E(|S_N|) \tag{1.1}$$

whenever $E(N) < \infty$. Thus the expected number of times the particle is at the origin prior to absorption equals its expected distance from the origin at the moment of absorption, for any time-dependent absorption boundary such that the expected time of absorption is finite. Some restriction like $E(N) < \infty$ is essential. Indeed, if N is the least $n \geqq 1$ such that $S_n = 0$, equation (1.1) would imply $1 = 0$. In this case $E(N) = \infty$.

The primary concern of this paper is to show that a result analogous to equation (1.1) is true for one-dimensional random variables under rather general conditions, and to obtain a similar result in two dimensions. The proof of equation (1.1) and its generalizations is based on an extension by Blackwell and Girshick [1] of an equation of Wald, the following special case of which is used (see theorem 2.1). If X_n is k-dimensional with $E(|X_n|) < \infty$, where, for $a = (a_1, \cdots, a_k)$, $|a| = (a_1^2 + \cdots + a_k^2)^{1/2}$, and $E(N) < \infty$, then

$$E(M) = E[g(S_N)], \tag{1.2}$$

where $g(s)$ is a solution of the equation

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$$\int g(s+x)\,dF(x) - g(s) = \chi_0(s), \tag{1.3}$$

$F(x)$ being the distribution function of X_n and $\chi_0(s) = 1$ or 0 according as $s = 0$ or $s \neq 0$, provided that $|g(s)| \leqq A + B|s|$ and $g(0) = 0$. The range of s in (1.3) is restricted to values such that $P\{S_n = s\} > 0$ for some n. Equation (1.1) is easily deduced from this theorem.

A result analogous to (1.1) can be expected only if $E(|X_n|) < \infty$ and $E(X_n) = 0$; for if $E(X_n) \neq 0$, then, in the absence of absorption, the expected number of returns to the origin is finite and therefore $E(M)$ is bounded, whereas $E(|S_N|)$ may be arbitrarily large.

First let X_n be one-dimensional, $E(X_n) = 0$, $0 < \sigma^2 = E(X_n^2) < \infty$, and let a denote the greatest common divisor of the integers x for which $P\{X_n = x\} > 0$. Then equation (1.3) has a solution $g_0(s)$ such that

$$g_0(s) \sim a\sigma^{-2}|s|, \qquad |s| \to \infty \tag{1.4}$$

(theorem 4.1). This implies that if $|S_N|$ is large with high probability, then $E(M)$ is approximately proportional to the expected distance of the particle from the origin at the moment of absorption, under the conditions stated above. If, in addition, the distribution of X_n is symmetric, then $g_0(s) \geqq a\sigma^{-2}|s|$ (theorem 4.1). Also, without assuming that $\sigma^2 < \infty$, $E(M) \leqq E(|S_N|)/E(|X_1|)$ (theorem 4.4).

Now let X_n be a two-dimensional random vector with $E(X_n) = 0$ and nonsingular second-moment matrix Σ. Then equation (1.3) has a solution $g_0(s)$ such that

$$g_0(s) \sim b \log |s|, \qquad |s| \to \infty, \tag{1.5}$$

where b is a positive constant which depends only on Σ and the set of points x for which $P\{X_n = x\} > 0$ (theorem 5.1).

A formal solution of equation (1.3) is

$$g(s) = \sum_{n=0}^{\infty} [P\{S_n = 0\} - P\{S_n = -s\}]. \tag{1.6}$$

The problem of the convergence or divergence of the sum in (1.6) is of independent interest. The sum $Q_n(s) = \sum_{m=0}^{n} P\{S_m = s\}$ is the expected number of times the particle is at the point s up to time n. It is known that $\lim_{n\to\infty} Q_n(s) = \infty$ if and only if $P\{S_n = s \text{ infinitely often}\} = 1$. If $k = 1$ and $E(X_n) = 0$, or if $k = 2$, $E(X_n) = 0$, and $E(|X_n|^2) < \infty$, then $\lim_{n\to\infty} Q_n(s) = \infty$ for any possible value s, the value s being possible if $P\{S_n = s\} > 0$ for some n (Chung and Fuchs [5]). Thus if the difference $Q_n(s) - Q_n(t)$ converges as $n \to \infty$, the two expected values remain close to each other even though each of them tends to infinity. The following related results are known: if s and t are possible values, then $Q_n(s)/Q_n(t) \to 1$ whenever $Q_n(s) \to \infty$ (Doeblin [6]); also, if $k = 1$ and $E(X_n) = 0$, then $P\{S_n = s\}/P\{S_n = t\} \to 1$, provided that the two probabilities are positive for n large enough (Chung and Erdös [4]). The problem of the convergence of sums like that in (1.6) has been posed by K. L. Chung [2] in the more general setting of Markov chains with stationary transition probabilities.

Chung obtained certain results on this problem under assumptions which exclude the case of sums of independent random variables. (K. L. Chung informed the author that the proof of theorem 5 in [2] is incorrect and that a correct proof will be found in [3].)

In the present paper the following results are obtained. The sum $\sum [P\{S_n = s\} - P\{S_n = t\}]$ is shown to converge for any possible values s, t in the following cases: For $k = 1$, if either $E(X_n^2) < \infty$ or the distribution of X_n is symmetric (theorem 4.5); and for $k = 2$ always (theorem 5.3). Under the assumptions made to derive equations (1.4) and (1.5), the sum in (1.6) is a solution of equation (1.3) and the former equations give the asymptotic values of the corresponding sums. Some of these results have also been obtained by F. Spitzer in [9] and [10], who kindly made his manuscripts available to the author before publication; for details see remark 2 at the end of section 4 and the remarks after the proofs of theorems 5.1 and 5.3.

The case of $k \geqq 3$ dimensions will not be considered in this paper. We only remark that in this case $\sum P\{S_n = s\}$ is known to converge for any s [5]; the sum in (1.6) is easily seen to be a solution of equation (1.3), and to be bounded as a function of s.

Note that the left side of (1.3) is a version of the Laplace operator acting on g. It is therefore not surprising that the solutions (1.4) and (1.5) behave asymptotically like the solutions of the classical Poisson equation corresponding to (1.3). See also Spitzer [10].

2. Application of a theorem of Blackwell and Girshick

Let $\{X_n\}$ and N be defined as in the beginning of section 1, except that the components of X_n need not be integers. For each positive integer j let ϕ_j be a real measurable function of $X_1, \cdots, X_j$ such that

$$E(\phi_j) = 0 \tag{2.1}$$

and, with the usual notation for a conditional expectation,

$$E(\phi_j | X_1, \cdots, X_i) = \phi_i, \qquad i \leqq j. \tag{2.2}$$

Suppose that there exists a function $f(x) \geqq 0$ such that

$$E[f(X_1)] < \infty; \qquad |\phi_j| \leqq \sum_{i=1}^{j} f(X_i) \qquad \text{if} \quad j \leqq N. \tag{2.3}$$

Blackwell and Girshick [1] have shown that if the above conditions are satisfied and $E(N) < \infty$, then $E(\phi_N)$ exists and equals 0.

We shall need the following corollary of this theorem.

THEOREM 2.1. *Let $g(s)$ be a real measurable function of $s \in R^k$ such that, for some constants a and b,*

$$|g(s)| \leqq a + b|s|. \tag{2.4}$$

Suppose that $E(|X_1|) < \infty$, $E(N) < \infty$, and that the function

$$h(s) = \int g(s + x)\, dF(x) - g(s) \tag{2.5}$$

is bounded. Then $E[g(S_N)]$ *exists and*

$$E[g(S_N)] - g(0) = E\left[\sum_{n=0}^{N-1} h(S_n)\right]. \tag{2.6}$$

PROOF. Let

$$\phi_j = g(S_j) - g(0) - \sum_{i=0}^{j-1} h(S_i) \tag{2.7}$$

or equivalently,

$$\phi_j = \sum_{i=1}^{j} \left[g(S_i) - \int g(S_{i-1} + x)\, dF(x)\right]. \tag{2.8}$$

It is easy to verify that ϕ_j satisfies conditions (2.1) and (2.2). Also, if c is an upper bound for $|h(s)|$, we have by (2.4)

$$|\phi_j| \leqq a + b|S_j| + a + cj \leqq \sum_{i=1}^{j} (b|X_i| + 2a + c). \tag{2.9}$$

Hence condition (2.3) is also satisfied. Thus $E(\phi_N) = 0$. Since $E\ [|\sum_{i=0}^{N-1} h(S_i)|] \leqq cE(N) < \infty$, therefore $E[g(S_N)]$ exists and equation (2.6) follows.

REMARK. Equation (2.6) holds under less restrictive conditions on g and h if more stringent assumptions on the distribution of N are imposed. For related results see also Doob [7], chapter 7, section 2.

Theorem 2.1 will be applied to the case where X_n has integer-valued components and $h(s) = \chi_0(s)$, where $\chi_0(s) = 1$ or 0 according as $s = 0$ or $s \neq 0$. In this case the right side of equation (2.6) is $E(M)$, with M as defined in the introduction. Thus it will be necessary to show that equation (2.5) with $h(s) = \chi_0(s)$ has a solution $g(s)$ which satisfies (2.4). The range of s in equation (2.5) may be taken as the set of the possible values s.

If $g_0(s)$ is a solution of (2.5) with $h(s) = \chi_0(s)$, and $\chi_T(s)$ is the characteristic function of a set T which consists of finitely many points with integer-valued coordinates, then $g_T(s) = \sum_{t \in T} g_0(s - t)$ is a solution of (2.5) with $h(s) = \chi_T(s)$, and then the right side of (2.6) is the expected number of times the particle is in the set T prior to absorption.

3. Some lemmas for multidimensional distributions

In this section some lemmas are derived which are needed in the sequel. Here $F(x)$ will denote a k-dimensional distribution function with $k \geqq 1$, and $f(u) = \int \exp (ix'u)\, dF(x)$ its characteristic function, where x and u denote column vectors with k components and $x'u$ is the matrix product, x' denoting the transpose of x. A distribution function F is nondegenerate if no $k - 1$ dimensional hyperplane has F probability one. We write $\mathrm{R}f(u)$ and $\mathrm{I}f(u)$ for the real and the imaginary part of $f(u)$.

LEMMA 3.1. *If F is any nondegenerate k-dimensional distribution function, then there exist positive constants c and δ such that*

$$1 - \mathrm{R}f(u) \geqq c|u|^2, \qquad |u| < \delta. \tag{3.1}$$

PROOF. Since F is nondegenerate, we can choose $\delta > 0$ so small that the distribution function G_δ defined by $\int_A dG_\delta = \int_A dF / \int_{|x| < \delta^{-1}} dF$, with $A \subset \{|x| < \delta^{-1}\}$, is nondegenerate. Then $\int_{|x| < \delta^{-1}} |x'u|^2 \, dF$ is a positive definite quadratic form and hence is $\geqq c_1 |u|^2$ for some $c_1 > 0$. Now if $|u| < \delta$,

$$\begin{aligned} 1 - \mathrm{R}f(u) &= \int (1 - \cos x'u) \, dF \geqq \int_{|x'u| < 1} \left(\frac{1}{2}|x'u|^2 - \frac{1}{24}|x'u|^4\right) dF \\ &\geqq \frac{11}{24} \int_{|x||u| < 1} |x'u|^2 \, dF \geqq \frac{11}{24} \int_{|x| < \delta^{-1}} |x'u|^2 \, dF \\ &\geqq \frac{11}{24} c_1 |u|^2. \end{aligned} \tag{3.2}$$

LEMMA 3.2. *If F is a nondegenerate k-dimensional distribution function, there exist positive constants c and δ such that*

$$|f(u)| \leqq 1 - c|u|^2, \qquad |u| < \delta, \tag{3.3}$$

and for all positive integers n

$$|\mathrm{I}([f(u)]^n)| \leqq |\mathrm{I}f(u)| n (1 - c|u|^2)^{n-1}, \qquad |u| < \delta, \tag{3.4}$$

PROOF. Since $|f(u)|^2 = f(u)f(-u)$ is a real characteristic function, (3.3) follows easily from lemma 3.1. Also,

$$\begin{aligned} \mathrm{I}\{[f(u)]^n\} &= (2i)^{-1}\{[f(u)]^n - [f(-u)]^n\} \\ &= \mathrm{I}f(u)[f(u)^{n-1} + f(u)^{n-2}f(-u) + \cdots + f(-u)^{n-1}]. \end{aligned} \tag{3.5}$$

Hence $|\mathrm{I}\{[f(u)]^n\}| \leqq |\mathrm{I}f(u)| \, n \, |f(u)|^{n-1}$. With (3.3) this yields (3.4).

LEMMA 3.3. *If the k-dimensional random vector X with integral components has a nondegenerate k-dimensional distribution, there exists a nonsingular $k \times k$ matrix C such that the random vector $Y = CX$ has integral components and its characteristic function $f(t)$ is equal to 1 if and only if the components of the vector t are integral multiples of 2π.*

PROOF. Let $f_1(u) = E(\exp iX'u)$ be the characteristic function of X. By assumption the values of X belong to the set $\mathfrak{N}$ of vectors with integral components. Let $\mathcal{W}$ be the set of vectors w for which $f_1(w) = 1$. Then $w \in \mathcal{W}$ if and only if $x'w$ is an integral multiple of 2π whenever $P\{X = x\} > 0$. Since the distribution of X is nondegenerate, there exist k linearly independent vectors $x_1, \cdots, x_k$ in $\mathfrak{N}$ such that $P\{X = x_j\} > 0$ for $j = 1, \cdots, k$. Hence if $w \in \mathcal{W}$, the numbers $x_1'w, \cdots, x_k'w$ are integral multiples of 2π. It follows that the components of w are rational multiples of π, and that any bounded subset of $\mathcal{W}$ contains only a finite number of points. Also, if $w \in \mathcal{W}$, then $f_1(u + w) = f_1(u)$. By a known property of continuous periodic functions, this implies that $\mathcal{W}$ consists of the points $2\pi C'n$, with $n \in \mathfrak{N}$, where C is a fixed nonsingular $k \times k$ matrix. The elements of C are rational.

Let d denote a common denominator of the elements of C. Then the components of the random vector $Z = dCX$ are integers. The characteristic function $f_2(v)$ of Z satisfies the equation $f_2(d^{-1}C'^{-1}u) = f_1(u)$. Hence $f_2(v) = 1$ if and only if $v = d^{-1}C'^{-1}2\pi C'n = 2\pi d^{-1}n$, $n \in \mathfrak{N}$. In particular, $f_2(v) = 1$ at the k points $v_1 = (2\pi d^{-1}, 0, \cdots, 0), \cdots, v_k = (0, \cdots, 0, 2\pi d^{-1})$. If $P\{Z = z\} > 0$, then $z'v_j$ is an integral multiple of 2π, for $j = 1, \cdots, k$, and hence the components of Z are multiples of d. Therefore the random vector $Y = d^{-1}Z = CX$ has integral components, and its characteristic function $f(t) = f_2(d^{-1}t)$ equals 1 if and only if $t = 2\pi n$, with $n \in \mathfrak{N}$, as was to be proved.

4. One-dimensional random variables

In this section $X_1, X_2, \cdots$ will be independent, integer-valued random variables with common distribution function $F(x)$ and characteristic function $f(u)$. We shall assume that the greatest common divisor of the saltuses of F is 1; the general case is easily reduced to this special case. Then $f(u) = 1$ if and only if u is a multiple of 2π. Since $f(u)$ is continuous, this implies that for any $\delta > 0$ there is a $c > 0$ such that $|1 - f(u)| > c$ if $\delta < |u| < \pi$.

THEOREM 4.1. *Let $F(x)$ be a distribution on the integers such that the greatest common divisor of its saltuses is* 1 *and*

$$\int x\, dF(x) = 0, \qquad 0 < \int x^2\, dF(x) = \sigma^2 < \infty. \tag{4.1}$$

Then the limit

$$g_0(s) = \lim_{\gamma \to 0+} (2\pi)^{-1} \int_{\gamma < |u| < \pi} (1 - e^{isu})[1 - f(u)]^{-1}\, du \tag{4.2}$$

exists (and clearly is real) for all real s; $g_0(s)$ satisfies the equation

$$\int g(s + x)\, dF(x) - g(s) = \chi_0(s), \qquad s = 0, \pm 1, \pm 2, \cdots; \tag{4.3}$$

and

$$g_0(s) = \sigma^{-2}|s| + o(|s|), \qquad |s| \to \infty. \tag{4.4}$$

If, in addition, F is symmetric, that is,

$$\int_{x<y} dF(x) = \int_{x>y} dF(x), \qquad -\infty < y < \infty, \tag{4.5}$$

then

$$g_0(s) \geqq \sigma^{-2}|s|, \qquad s = 0, \pm 1, \pm 2, \cdots, \tag{4.6}$$

where the sign of equality holds if -1, 0 *and* 1 *are the only saltuses of F.*

For the proof we require the following known lemma (see Hsu [7]), which is easy to verify.

LEMMA 4.2. *If* $\int x\,dF = 0$ *and* $\int x^2\,dF < \infty$, *then*

$$\int_0^\infty u^{-3}|If(u)|\,du < \infty. \tag{4.7}$$

PROOF OF THEOREM 4.1. We have, for $0 < \gamma < \pi$,

$$\int_{\gamma<|u|<\pi} (1 - e^{isu})[1 - f(u)]^{-1}\,du \tag{4.8}$$
$$= 2\int_\gamma^\pi \{(1 - \cos su)[1 - Rf(u)] + (\sin su)If(u)\}|1 - f(u)|^{-2}\,du.$$

Since $1 - Rf(u) = (1/2)\sigma^2u^2 + o(u^2)$ and $1 - f(u) = (1/2)\sigma^2u^2 + o(u^2)$ as $u \to 0$, it follows from lemma 4.2 that the limit of the integral, as $\gamma \to 0$, exists, and

$$g_0(s) = \pi^{-1}\int_0^\pi \{(1 - \cos su)[1 - Rf(u)] + (\sin su)If(u)\}|1 - f(u)|^{-2}\,du. \tag{4.9}$$

If we replace s by $s + x$ in (4.9), the integrand is absolutely integrable with respect to $du\,dF(x)$. Hence the order of integration may be interchanged and after simplifying we obtain

$$\int_{-\infty}^\infty g_0(s + x)\,dF(x) - g_0(s) = (2\pi)^{-1}\int_{-\pi}^\pi e^{isu}\,du. \tag{4.10}$$

The right side equals $\chi_0(s)$ for any integer s. Thus $g_0(s)$ satisfies equation (4.3).

To prove the asymptotic equation (4.4), we write $g_0(s) = A(s) + B(s)$ corresponding to the two terms in (4.8). The term $A(s)$ can be written

$$A(s) = (2\pi)^{-1}\int_{-\pi}^\pi (1 - \cos su)[1 - f(u)]^{-1}\,du. \tag{4.11}$$

For an arbitrary $\epsilon > 0$, with $\epsilon < 1$, we can choose $\delta > 0$ so that $1 - f(u) = (1 + \theta\epsilon)2^{-1}\sigma^2u^2$ for $|u| < \delta$ where $|\theta| \leqq 1$. Since the integrand is bounded for $\delta \leqq |u| \leqq \pi$,

$$A(s) = (2\pi)^{-1}(1 + \theta_1\epsilon)^{-1}2\sigma^{-2}\int_{|u|<\delta} u^{-2}(1 - \cos su)\,du + O(1), \tag{4.12}$$

where $|\theta_1| \leqq 1$. We have

$$\pi^{-1}\int_{|u|<\delta} u^{-2}(1 - \cos su)\,du = \pi^{-1}\int_{-\infty}^\infty u^{-2}(1 - \cos su)\,du + O(1) \tag{4.13}$$
$$= |s| + O(1).$$

Hence $A(s) = (1 + \theta_1\epsilon)^{-1}\sigma^{-2}|s| + O(1)$ and therefore

$$A(s) = \sigma^{-2}|s| + o(|s|). \tag{4.14}$$

Now consider $B(s)$. Given $\epsilon > 0$, we can, by lemma 4.2, choose $\delta > 0$ so that $\int_0^\delta u^{-3}|If(u)|\,du < \epsilon$, and also $|1 - f(u)| \geqq cu^2$ for $|u| < \delta$, where $c = 4^{-1}\sigma^2$, say.

Then

$$\pi|B(s)| \leqq \left|\int_0^\pi (\sin su)\mathrm{I}f(u)|1 - f(u)|^{-2}\, du\right| \leqq \left|\int_0^\delta\right| + \left|\int_\delta^\pi\right| \tag{4.15}$$
$$\leqq c^{-2}\int_0^\delta u^{-4}|su\, \mathrm{I}f(u)|\, du + O(1)$$
$$\leqq c^{-2}|s|\epsilon + O(1).$$

Hence $B(s) = o(|s|)$. With (4.14) this implies (4.4).

Now assume that F is symmetric. Then

$$g_0(s) = \pi^{-1}\int_0^\pi (1 - \cos su)[1 - f(u)]^{-1}\, du, \tag{4.16}$$

where

$$1 - f(u) = \int (1 - \cos xu)\, dF(x). \tag{4.17}$$

If x is an integer,

$$1 - \cos xu = 2\sin^2\frac{xu}{2} = (1 - \cos u)\left[\frac{\sin (xu/2)}{\sin (u/2)}\right]^2 \tag{4.18}$$
$$= (1 - \cos u)|1 + e^{-iu} + e^{-2iu} + \cdots + e^{-(|x|-1)iu}|^2 \leqq (1 - \cos u)x^2,$$

with equality for $x = -1, 0, 1$. Therefore

$$0 \leqq 1 - f(u) \leqq (1 - \cos u)\sigma^2. \tag{4.19}$$

Hence

$$g_0(s) \geqq \pi^{-1}\sigma^{-2}\int_0^\pi (1 - \cos su)(1 - \cos u)^{-1}\, du = \sigma^{-2}|s| \tag{4.20}$$

for integral values s, with equality holding if $-1, 0, 1$ are the only saltuses of F. The proof of theorem 4.1 is complete.

If the conditions of theorem 4.1 are satisfied and $E(N) < \infty$, then the conditions of theorem 2.1 with $h(s) = \chi_0(s)$ and $g(s) = g_0(s)$ are satisfied. Condition (2.4) follows from $g_0(s) \sim \sigma^{-2}|s|$ and the fact that $g_0(s)$ is bounded in any finite interval. We have $\sum_{n=0}^{N-1} \chi_0(S_n) = M$ where M is defined in the introduction. Hence we can state

THEOREM 4.3. *If the conditions of theorem* 4.1 *are satisfied and* $E(N) < \infty$, *then*

$$E(M) = E[g_0(S_N)], \tag{4.21}$$

where $g_0(s) = \sigma^{-2}|s| + o(|s|)$. *If F is symmetric,* $g_0(s) \geqq \sigma^{-2}|s|$, *with equality holding if* $-1, 0, 1$ *are the only saltuses of F.*

The next theorem gives an upper bound for $E(M)$. Here F need not be a distribution on the integers, and the second moment need not be finite.

THEOREM 4.4. *If F is any one-dimensional distribution function such that*

$$\beta_1 = \int |x|\, dF < \infty, \qquad \int x\, dF = 0, \tag{4.22}$$

and if $E(N) < \infty$, *then*

$$E(M) \leqq \beta_1^{-1} E(|S_N|). \tag{4.23}$$

The sign of equality holds if F is constant except for saltuses at $-1, 0, 1$.

PROOF. Let $h(s) = \int |s + x|\, dF(x) - |s|$. We have $h(s) \leqq \beta_1 = h(0)$ and $h(s) \geqq \left|\int (s + x)\, dF(x)\right| - |s| = 0$. Hence $h(s) \geqq \beta_1 \chi_0(s)$. By theorem 2.1

$$E(|S_N|) = E\left[\sum_{n=0}^{N-1} h(S_n)\right] \geqq \beta_1 E\left[\sum_{n=0}^{N-1} \chi_0(S_n)\right] = \beta_1 E(M), \tag{4.24}$$

which implies (4.13). If $\int_{|x| \leqq 1} dF = 1$, then $h(s) = \beta_1 \chi_0(s)$ for all integers s. The condition for equality follows.

Theorems 4.3 and 4.4 imply that if the conditions of the former are satisfied and F is symmetric, then

$$\sigma^{-2} E(|S_N|) \leqq E(M) \leqq \beta_1^{-1} E(|S_N|), \tag{4.25}$$

where both equality signs hold if $-1, 0, 1$ are the only saltuses of F.

THEOREM 4.5. *If F is a distribution on the integers and either*

$$\int_{x<y} dF(x) = \int_{x>y} dF(x), \qquad -\infty < y < \infty, \tag{4.26}$$

or

$$\int x^2\, dF(x) < \infty, \tag{4.27}$$

then the sum

$$\sum_{n=0}^{\infty} (P\{S_n = s\} - P\{S_n = t\}) \tag{4.28}$$

converges for any two possible values s and t (not always absolutely). If the greatest common divisor of the saltuses of F is 1, *the sum* (4.28) *is equal to* $g_0(-t) - g_0(-s)$, *where* $g_0(s)$ *is defined in theorem* 4.1.

PROOF. If $\int |x|\, dF < \infty$ and $\int x\, dF \neq 0$, then $\sum_{n=0}^{\infty} P\{S_n = s\} < \infty$ for all s. Hence in the case (4.27) we shall make the additional assumption

$$\int x\, dF(x) = 0. \tag{4.29}$$

We also may and shall assume that $\int_{x=0} dF(x) \neq 1$ and that the greatest common divisor of the saltuses of F is 1. To prove the convergence of (4.28) it is sufficient to show that

$$\sum_{n=0}^{\infty} (P\{S_n = 0\} - P\{S_n = s\}) \tag{4.30}$$

converges for any integer s.

We have

$$(4.31)\qquad 2\pi(P\{S_n = 0\} - P\{S_n = s\}) = \int_{-\pi}^{\pi} (1 - e^{-isu})[f(u)]^n\, du = J_n + K_n,$$

$$J_n = \int_{|u|<\delta}, \quad K_n = \int_{\delta<|u|<\pi},$$

and δ is chosen as in lemma 3.2.

First consider J_n. We write $J_n = A_n + B_n$, where

$$(4.32)\qquad \begin{aligned} A_n &= \int_{|u|<\delta} (1 - \cos su)\, \mathrm{R}([f(u)]^n)\, du, \\ B_n &= -\int_{|u|<\delta} (\sin su)\, \mathrm{I}([f(u)]^n)\, du. \end{aligned}$$

By lemma 3.2, $|\mathrm{R}([f(u)]^n)| \leqq |f(u)|^n \leqq (1 - cu^2)^n$ for $|u| < \delta$. Hence if $|u| < \delta$

$$(4.33)\qquad \sum_{n=0}^{\infty} |(1 - \cos su)\, \mathrm{R}([f(u)]^n)| \leqq \sum_{n=0}^{\infty} \frac{1}{2} s^2u^2(1 - cu^2)^n = \frac{1}{2} c^{-1}s^2.$$

By (3.4), if $|u| < \delta$,

$$(4.34)\qquad \sum_{n=0}^{\infty} |(\sin su)\, \mathrm{I}([f(u)]^n)| \leqq \sum_{n=0}^{\infty} |su|\, |\mathrm{I}f(u)|\, n(1 - cu^2)^{n-1} = c^{-2}|s|\, |u|^{-3}\, |\mathrm{I}f(u)|.$$

This upper bound is integrable by lemma 4.2. Thus the sums of the absolute values of the integrands in A_n and B_n are bounded by integrable functions. It follows that $\sum J_n$ is absolutely convergent and

$$(4.35)\qquad \begin{aligned} \sum_{n=0}^{\infty} J_n &= \int_{|u|<\delta} \sum_{n=0}^{\infty} [(1 - \cos su)\, \mathrm{R}\{[f(u)]^n\} - (\sin su)\, \mathrm{I}\{[f(u)]^n\}]\, du \\ &= \lim_{\gamma\to 0+} \int_{\gamma<|u|<\delta} (1 - e^{-isu})[1 - f(u)]^{-1}\, du. \end{aligned}$$

Now consider K_n. We have

$$(4.36)\qquad \sum_{m=0}^{n-1} K_m = \int_{\delta<|u|<\pi} (1 - e^{-isu})[1 - f(u)]^{-1}\{1 - [f(u)]^n\}\, du.$$

Since $\sup_{\delta<|u|<\pi} |1 - f(u)|^{-1} = c_0$ is finite,

$$(4.37)\qquad \left| \int_{\delta<|u|<\pi} (1 - e^{-isu})[1 - f(u)]^{-1}\, du \right| \leqq 4\pi c_0.$$

Also,

$$(4.38)\qquad \left| \int_{\delta<|u|<\pi} (1 - e^{-isu})[1 - f(u)]^{-1}[f(u)]^n\, du \right| \leqq 2c_0 \int_{-\pi}^{\pi} |f(u)|^n\, du.$$

Since $|f(u)| \leqq 1$, with equality only at a finite number of points in $[-\pi, \pi]$, the integral on the right tends to 0 as $n \to \infty$. Thus $\sum K_m$ converges. Hence the sum (4.30) converges also.

The last part of the theorem follows from (4.35) and the preceding paragraph.

REMARK 1. If $|f(u)| \neq 1$ for $0 < |u| \leqq \pi$ (which means that the differences between the saltuses of F have the greatest common denominator 1), then the sum (4.30) converges absolutely. Indeed, in this case $\sum |K_m| < \infty$. But in general the convergence is not absolute. For example, if F is a distribution on the odd integers, then S_n is odd or even according as n is odd or even. Hence if s is odd and t is even,

$$\sum_{n=0}^{\infty} |P\{S_n = s\} - P\{S_n = t\}| = \sum_{n=0}^{\infty} (P\{S_n = s\} + P\{S_n = t\}), \tag{4.39}$$

which is infinite if $\int x\, dF = 0$.

REMARK 2. An inspection of the proof of theorem 4.5 shows that the sum

$$\begin{aligned} &\sum_{n=0}^{\infty} (P\{S_n = 0\} - \frac{1}{2} P\{S_n = s\} - \frac{1}{2} P\{S_n = -s\}) \\ &\qquad = \sum_{n=0}^{\infty} (2\pi)^{-1} \int_{-\pi}^{\pi} (1 - \cos su)\, \mathrm{R}\{[f(u)]^n\}\, du \end{aligned} \tag{4.40}$$

converges whenever 0, s, $-s$ are possible values, with no other restrictions. Assumption (4.27) is used only to prove the convergence of the sum involving $\mathrm{I}\{[f(u)]^n\}$, that is, the sum $\sum_{n=0}^{\infty} (P\{S_n = s\} - P\{S_n = -s\})$. The convergence of the sum (4.40) has also been proved by F. Spitzer [9], where an interesting probability interpretation of the reciprocal of this sum is given. Spitzer also obtains the asymptotic value of the sum (4.40) when $E(X_1) = 0$ and $0 < E(X_1^2) < \infty$, corresponding to (4.4), as well as under certain assumptions when $E(X_1^2) = \infty$.

5. Two-dimensional random vectors

The three theorems of this section are two-dimensional analogues of theorems 4.1, 4.3, and 4.5. Due to lemma 3.3, the assumption (5.2) of the following theorem is not restrictive; in particular, without (5.2) equation (5.5) remains true with det Σ replaced by det $(C\Sigma C')$.

THEOREM 5.1. *Let $F(x) = F(x_1, x_2)$ be the distribution function and $f(u) = f(u_1, u_2)$ the characteristic function of a two-dimensional random vector with integer-valued components such that*

$$\int x\, dF = 0, \qquad \int |x|^2\, dF < \infty, \qquad \det(\Sigma) > 0, \tag{5.1}$$

where $\Sigma = \left\|\int x_\mu x_\nu\, dF\right\|$, *and*

$$f(u) \neq 1, \qquad |u_1| \leqq \pi, \qquad |u_2| \leqq \pi, \qquad (u_1, u_2) \neq (0, 0). \tag{5.2}$$

Then the integral

$$g_0(s) = (2\pi)^{-2} \int_D (1 - e^{is'u})[1 - f(u)]^{-1}\, du, \qquad D = \{u \mid |u_1| < \pi,\ |u_2| < \pi\}, \tag{5.3}$$

exists (and is real) for all s with real components; $g_0(s)$ satisfies the equation

$$\int g(s + x)\, dF(x) - g(s) = \chi_0(s), \tag{5.4}$$

where s ranges over the vectors with integral components; and

$$g_0(s) = \pi^{-1} (\det \Sigma)^{-1/2} \log |s| + o\,(\log |s|) \qquad |s| \to \infty. \tag{5.5}$$

PROOF. It follows from (5.1) that $1 - f(u) = u'\Sigma u + o(u'\Sigma u)$ and that $u'\Sigma u \geqq c|u|^2$, where $c > 0$. Hence there are positive constants δ_1 and c_1 such that $|1 - f(u)|^{-1} \leqq c_1|u|^{-2}$ if $u'\Sigma u < \delta_1^2$. Also, by (5.2), $|1 - f(u)|^{-1}$ is bounded in the subset of D where $u'\Sigma u \geqq \delta_1^2$. Hence there is a constant c_2 such that $|1 - f(u)|^{-1} \leqq c_2|u|^{-2}$ if $u \in D$. Since $|1 - e^{is'u}| \leqq |s'u| \leqq |s|\,|u|$, we have

$$|(1 - e^{is'u})[1 - f(u)]^{-1}| \leqq c_2|s|\,|u|^{-1}, \qquad u \in D, \tag{5.6}$$

and $\int_D |u|^{-1}\, du < \infty$. Thus the integral in (5.3) exists. It is easy to verify that $g_0(s)$ satisfies equation (5.4), noting that $\int_D e^{is'u}\, du = \chi_0(s)$ when s has integral components.

It remains to prove equation (5.5). For a preassigned $\epsilon > 0$ with $\epsilon < 2^{-1}$, we can choose $\delta > 0$ so that

$$1 - f(u) = (1 + \theta\epsilon)2^{-1}u'\Sigma u \qquad \text{if} \quad u'\Sigma u < \delta^2, \tag{5.7}$$

where $|\theta| \leqq 1$, and so that $u'\Sigma u < \delta^2$ implies $u \in D$. Then $g_0(s) = g^*(s) + O(1)$ as $s \to \infty$, where

$$g^*(s) = (2\pi)^{-2} \int_{u'\Sigma u < \delta^2} (1 - e^{is'u})[1 - f(u)]^{-1}\, du. \tag{5.8}$$

Let Γ be a square matrix such that $\Sigma = \Gamma'\Gamma$. In (5.8) we first make the substitution $v = \Gamma u$ and put $t = \Gamma'^{-1}s$. Then $u'\Sigma u = |v|^2$, $s'u = t'v$, and $r^2 = s'\Sigma^{-1}s = |t|^2$. We next make an orthonormal substitution $w = Cv$ such that $w_1 = t'v/r$, and define $f_1(w) = f(\Gamma^{-1}C'w)$. Then

$$g^*(s) = (2\pi)^{-2} (\det \Sigma)^{1/2} \int_{|w| < \delta} (1 - e^{irw_1})[1 - f_1(w)]^{-1}\, dw \tag{5.9}$$

and, by (5.7), $1 - f_1(w) = (1 + \theta\epsilon)2^{-1}|w|^2$ if $|w| < \delta$. Since $\epsilon < 2^{-1}$, we have $[1 - f_1(w)]^{-1} = 2(1 + 2\theta_1\epsilon)|w|^{-2}$, where $|\theta_1| \leqq 1$. Hence, with $|\theta_2| \leqq 1$,

$$\begin{aligned} &2^{-1}(2\pi)^2 (\det \Sigma)^{1/2} g^*(s) \\ &\quad = \int_{|w| < \delta} (1 - e^{irw_1})|w|^{-2}\, dw + 2\theta_2\epsilon \int_{|w| < \delta} |1 - e^{irw_1}|\, |w|^{-2}\, dw. \end{aligned} \tag{5.10}$$

We calculate

$$(5.11)\qquad \int\limits_{|w|<\delta} (1 - e^{irw_1})|w|^{-2}\,dw$$
$$= \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} (1 - \cos rw_1)(w_1^2 + w_2^2)^{-1}\,dw_1\,dw_2 + O(1)$$
$$= 4\int_0^{\delta} (1 - \cos rw_1)\,dw_1\,w_1^{-1}\int_0^{\delta w_1^{-1}} (1 + x^2)^{-1}\,dx + O(1)$$
$$= 4\int_0^{\delta} (1 - \cos rw_1)w_1^{-1}\left[\frac{1}{2}\pi + O(w_1)\right]dw_1 + O(1)$$
$$= 2\pi\int_0^{\delta r} (1 - \cos x)x^{-1}\,dx + O(1)$$
$$= 2\pi \log(\delta r) + O(1)$$

and

$$(5.12)\qquad \int\limits_{|w|<\delta} |1 - e^{irw_1}|\,|w|^{-2}\,dw = \int\limits_{|u|<r\delta} |1 - e^{iu_1}|\,|u|^{-2}\,du$$
$$\leqq 2\int\limits_{1<|u|<r\delta} |u|^{-2}\,du + O(1) = \pi\log(\delta r) + O(1).$$

Hence $g_0(s) = g^*(s) + O(1) = \pi^{-1}(\det\Sigma)^{-1/2}\log r + o(\log r)$. Since Σ^{-1} is positive definite, there exist positive constants c_1 and c_2 such that $c_1|s| \leqq r \leqq c_2|s|$. Hence $\log r \sim \log|s|$ as r (or $|s|$) $\to \infty$. This implies (5.5).

REMARK. According to lemma 2.4 of Spitzer [10], the term $o(\log|s|)$ in (5.5) can be replaced by const $+\ o(1)$ if certain additional assumptions are made.

From theorems 2.1 and 5.1 we obtain

THEOREM 5.2. *If the conditions of theorem* 5.1 *are satisfied and* $E(N) < \infty$, *then* $E(M) = E[g_0(S_N)]$, *where* $g_0(s)$ *is defined by* (5.3) *and*

$$(5.13)\qquad g_0(s) \sim \pi^{-1}(\det\Sigma)^{-1/2}\log|s|, \qquad |s| \to \infty.$$

THEOREM 5.3. *If* X_n *is a two-dimensional random vector with integer-valued components whose distribution function* $F(x)$ *is nondegenerate, then the sum*

$$(5.14)\qquad \sum_{n=0}^{\infty} (P\{S_n = s\} - P\{S_n = t\})$$

converges (not always absolutely) for any two possible values s, t. *If*

$$(5.15)\qquad f(u) \neq 1, \qquad |u_1| \leqq \pi, \qquad |u_2| \leqq \pi, \qquad (u_1, u_2) \neq (0, 0),$$

then the sum (5.14) *is equal to* $g_0(-t) - g_0(-s)$, *where* $g_0(s)$ *is defined by* (5.3).

PROOF. By lemma 3.3 we may assume that condition (5.15) is satisfied. It is sufficient to show that $\sum_{n=0}^{\infty}(P\{S_n = 0\} - P\{S_n = -s\})$ converges for any s with integer-valued components and equals $g_0(s)$. We have

$$(5.16)\qquad (2\pi)^2(P\{S_n = 0\} - P\{S_n = -s\}) = \int\limits_D (1 - e^{is'u})[f(u)]^n\,du$$
$$= J_n + K_n,$$

where

$$J_n = \int_{D_0}, \qquad K_n = \int_{D_1}, \qquad D = \{u \big| |u_j| < \pi, j = 1, 2\}, \qquad D_0 = \{u \big| |u| < \delta\}, \; D_1 = D - D_0. \tag{5.17}$$

Here δ is chosen as in lemma 3.2, so that $|f(u)| \leqq 1 - c|u|^2$ if $u \in D_0$. Hence for $u \in D_0$,

$$\sum_{n=0}^{\infty} |(1 - e^{is'u})[f(u)]^n| \leqq \sum_{n=0}^{\infty} |s'u|(1 - c|u|^2)^n \leqq |s| \, |u| c^{-1} |u|^{-2}. \tag{5.18}$$

This upper bound is integrable over D_0. Hence $\sum J_n$ converges absolutely and

$$\sum_{n=0}^{\infty} J_n = \int_{D_0} (1 - e^{is'u})[1 - f(u)]^{-1} \, du. \tag{5.19}$$

As in the proof of theorem 4.5 it is seen that

$$\sum_{n=0}^{\infty} K_n = \int_{D_1} (1 - e^{is'u})[1 - f(u)]^{-1} \, du, \tag{5.20}$$

where the convergence in general is not absolute. Equations (5.16), (5.19), and (5.20) imply the required conclusion.

REMARK. For the symmetric case $f(u) = f(-u)$, theorem 5.3 essentially coincides with lemma 2.1 of Spitzer [10].

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