TRANSFORMATIONS OF MARKOV PROCESSES CONNECTED WITH ADDITIVE FUNCTIONALS

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1. Introduction

Direct probabilistic constructions that allow us to build one Markov process from another are of interest in the theory of Markov processes as well as in a number of problems in mathematical analysis. In fact, if a Markov process $\tilde{X}$ can be obtained by means of a sufficiently simple transformation of a Markov process X, then it is possible to derive properties of the trajectory of the process $\tilde{X}$ from those of the trajectory of the process X. On the other hand, the solution of many problems in the theory of differential equations, as well as more general operator equations, can be expressed by actual formulas in terms of probability distributions connected with Markov processes. Therefore, by making use of transformations of Markov processes, it is possible to reduce problems of this type for more complicated operators to analogous problems for simpler operators.

In the present paper a general class of transformations of Markov processes is introduced and discussed whose brief description (for stationary Markov processes) is contained in the survey article [5] and in the note [8]. This class of transformations includes as special cases a number of special transformations that were discussed earlier, such as the formation of subprocesses [4], the transformation of the Wiener process which produces a drift (see for example [9]), and others. In the construction of this general class of transformations an important role is played by the concept of an *additive functional* of a Markov process. An additive functional of a Markov process X is a collection of random variables φ_t^s with $s \leqq t$ having the following two properties: (a) φ_t^s is defined in terms of the process in the time $[s, t]$ (a more exact formulation of this condition is given in 2.1A) and (b) $\varphi_t^s + \varphi_u^t = \varphi_u^s$ for all $s \leqq t \leqq u$.

The main results of the paper are given in sections 4 through 6 while section 2 is of an introductory nature. In it are given fundamental definitions and notations of the theory of Markov processes following the monograph [4]. In section 3 are given definitions and examples of additive functionals and some other allied subjects (multiplicative functionals, almost additive functionals, and so forth). The general construction giving transformations of Markov processes is described in sections 4 and 5. In section 6 conditions are studied under which the homogeneity of the process is preserved. In section 7 the results of Volkonsky [12] are developed in a modified form, enabling us to describe wide classes of additive functionals. Finally, the concluding section 8 is a survey of a number of results about additive functionals and transformations of Markov processes, which have been obtained recently by Volkonsky, Seregin, and the author.

2. Transition functions and Markov processes

2.1. Let $\mathcal{B}$ be a σ-algebra of the subsets of the set E containing all subsets consisting of a single point. Then the pair $(E, \mathcal{B})$ is called a *state space*. Let T be any positive number or $+\infty$. The function $P(s, x; t, \Gamma)$, with $0 \leqq s \leqq t < T$, $x \in E$, and $\Gamma \in \mathcal{B}$, is called a *transition function* if the following conditions are satisfied.

2.1A. For fixed s, t, and x the function $P(s, x; t, \Gamma)$ is a measure over the σ-algebra $\mathcal{B}$;

2.1B. for fixed s, t, and Γ the function $P(s, x; t, \Gamma)$ is a $\mathcal{B}$-measurable function of the point x;

2.1C. $P(s, x; t, E) \leqq 1$;

2.1D. $P(s, x; s, E \setminus x) = 0$;

2.1E. $P(s, x; u, \Gamma) = \int_E P(s, x; t, dy)\, P(t, y; u, \Gamma)$, for $s \leqq t \leqq u$.

The transition function is called *normal* if for any x and s, $P(s, x; s, E) = 1$.

As an example of a transition function we can take the so-called *Wiener transition function*, which is defined in n-dimensional Euclidean space by the formula

$$(1)\quad P(s, x; t, \Gamma) = \begin{cases} [2\pi(t-s)]^{-n/2} \displaystyle\int \exp\left[-\frac{(y-x)^2}{2(t-s)}\right] dy, & 0 \leqq s < t < T, \\ \chi_\Gamma(x), & 0 \leqq s = t < T, \end{cases}$$

where χ_Γ is the indicator of the set Γ, that is the function defined by

$$(2)\qquad \chi_\Gamma(x) = \begin{cases} 1, & x \in \Gamma \\ 0, & x \notin \Gamma. \end{cases}$$

Here $\mathcal{B}$ is the system of all Borel subsets of the space E, while $(y - x)^2$ is the scalar square of the vector $y - x$, and the integration proceeds with respect to the Lebesgue measure in E.

2.2. Suppose we are given

(a) a function $\zeta(\omega)$ on a space Ω, which assumes values in a segment $[0, T]$;

(b) a function $x(t, \omega) = x_t(\omega)$, defined for $0 \leqq t < \zeta(\omega)$ and assuming values in the state space $(E, \mathcal{B})$;

(c) for every $0 \leqq s \leqq t < T$, a σ-algebra $\mathfrak{M}_t^s$ in the space $\Omega_t = \{\omega: \zeta(\omega) > t\}$;

(d) for every $s \in [0, T]$ and $x \in E$ a function $\mathbf{P}_{s,x}(A)$ over some σ-algebra $\mathfrak{M}^s$ in the space Ω, which contains $\mathfrak{M}_t^s$ for all $t \in [s, T]$.

We shall say that these elements define a *Markov process* $X = (x_t, \zeta, \mathfrak{M}_t^s, \mathbf{P}_{s,x})$, if the following conditions are satisfied.

2.2A. If $s \leqq t \leqq u$, and $A \in \mathfrak{M}_t^s$, then $\{A, \zeta > u\} \in \mathfrak{M}_u^s$.

2.2B. $\{x_t \in \Gamma\} \in \mathfrak{M}_t^s$ for every $0 \leqq s \leqq t$ and $\Gamma \in \mathfrak{B}$.

2.2C. $\mathbf{P}_{s,x}$ is a probability measure over the σ-algebra $\mathfrak{M}^s$.

2.2D. For every $0 \leqq s \leqq t < T$ and $\Gamma \in \mathfrak{B}$,

$$P(s, x; t, \Gamma) = \mathbf{P}_{s,x}\{x_t \in \Gamma\} \tag{3}$$

is a $\mathfrak{B}$-measurable function of x.

2.2E. $P(s, x; s, E \setminus x) = 0$.

2.2F. If $0 \leqq s \leqq t \leqq u < T$, $x \in E$, $\Gamma \in \mathfrak{B}$, then

$$\mathbf{P}_{s,x}\{x_u \in \Gamma | \mathfrak{M}_t^s\} = P(t, x_t; u, \Gamma), \qquad \text{a.s. } \Omega_t, \mathbf{P}_{s,x}, \tag{4}$$

where the notation a.s. Ω_t, $\mathbf{P}_{s,x}$ stands for "almost sure over the set Ω_t with respect to the measure $\mathbf{P}_{s,x}$," that is, for all $\omega \in \Omega_t$, except for a set of measure zero.

The function $P(s, x; t, \Gamma)$ defined in (3) is a transition function in the space $(E, \mathfrak{B})$. It is called the transition function of the Markov process X. The quantity ζ is called the terminal time (or lifetime) of the process X. For a fixed ω the function $x_t(\omega)$ defines in the space E a trajectory of the process which corresponds to the elementary event ω. The system $\mathfrak{M}_t^s$ can be thought of as the totality of events which are observed during the time interval $[s, t]$. The value of $\mathbf{P}_{s,x}(A)$ can be interpreted as the probability of the event A under the condition that at the moment s the trajectory is at the point x. The integral of the function $\xi(\omega)$ with respect to the measure $\mathbf{P}_{s,x}$ over the whole domain in which it is defined is denoted by $\mathbf{M}_{s,x}(\xi)$.

It has been proved (see [4], theorem 4.2) that every normal transition function in the state space $(E, \mathfrak{B})$ corresponds to some Markov process, if it is possible to introduce into the space $(E, \mathfrak{B})$ a metric which satisfies the two following conditions

(a) E is representable as the sum of a denumerable number of compact subsets;

(b) the σ-algebra $\mathfrak{B}$ is generated by the system of all open sets.

It has been proved (see, for example, [4], chapter 6, section 7) that the Wiener transition function (see subsection 2.1) corresponds to some Markov process with continuous trajectories and $\zeta = T$. This process is called the *n-dimensional Wiener process in the time interval* $[0, T)$.

2.3. A number of important allied concepts are connected with every Markov process.

We say that $A \in \overline{\mathfrak{M}}^s$ if for every $x \in E$ there exist sets A_1 and A_2 in $\mathfrak{M}^s$ such that $A_1 \subseteq A \subseteq A_2$ and $\mathbf{P}_{s,x}(A_1) = \mathbf{P}_{s,x}(A_2)$. Taking $\mathbf{P}_{s,x}(A) = \mathbf{P}_{s,x}(A_1) = \mathbf{P}_{s,x}(A_2)$ we can extend the probability measure $\mathbf{P}_{s,x}$ to the σ-algebra $\overline{\mathfrak{M}}^s$. The σ-algebras $\overline{\mathfrak{M}}_t^s$ in the space Ω_t are defined by $\mathfrak{M}_t^s$ in the same way as $\overline{\mathfrak{M}}^s$ by $\mathfrak{M}^s$.

The σ-algebra $\mathfrak{N}^s$ is defined as the minimal σ-algebra of the space Ω, containing the sets $\{\omega: x_t(\omega) \in \Gamma\}$, with $t \geqq s$, $\Gamma \in \mathfrak{B}$. It may be proved that for every $B \in \mathfrak{N}^s$, the function $\mathbf{P}_{s,x}(B)$ is a $\mathfrak{B}$-measurable function of x.

If μ is an arbitrary finite measure over the σ-algebra $\mathfrak{B}$, then the formula

$$\mathbf{P}_{s,\mu}(B) = \int_E \mathbf{P}_{s,x}(B)\mu(dx), \qquad B \in \mathfrak{N}^s \tag{5}$$

defines over the σ-algebra $\mathfrak{N}^s$ a certain measure $\mathbf{P}_{s,\mu}$. If $\mu(E) = 1$, then $\mathbf{P}_{s,\mu}(\Omega) = 1$ and the value of $\mathbf{P}_{s,\mu}(B)$ can be naturally interpreted as the probability of the event B under the condition that the probability distribution of the state at the moment s is equal to μ.

We say that $a \in \overline{\mathfrak{N}}^s$ if for every finite measure μ in $\mathfrak{B}$ we can construct an A_1 and an A_2 of $\mathfrak{N}^s$ such that $A_1 \subseteq A \subseteq A_2$, and $\mathbf{P}_{s,\mu}(A_1) = \mathbf{P}_{s,\mu}(A_2)$. Analogously, we say that $\Gamma \in \overline{\mathfrak{B}}$ if for every finite measure μ in B there exist a Γ_1 and Γ_2 of $\mathfrak{B}$ such that $\Gamma_1 \subseteq \Gamma \subseteq \Gamma_2$ and $\mu(\Gamma_1) = \mu(\Gamma_2)$. The function $\mathbf{P}_{s,x}(B)$ is a $\overline{\mathfrak{B}}$-measurable function of x if $B \in \overline{\mathfrak{N}}^s$.

It may be proved that if $A \in \overline{\mathfrak{N}}^t$, then

$$\mathbf{P}_{s,x}(A|\mathfrak{M}_t^s) = \mathbf{P}_{t,x_t}(A), \qquad \text{a.s. } \Omega_t, \mathbf{P}_{s,x}. \tag{6}$$

If the function ξ is $\overline{\mathfrak{N}}^t$-measurable and $\mathbf{P}_{s,x}$-integrable then

$$\mathbf{M}_{s,x}(\xi|\mathfrak{M}_t^s) = \mathbf{M}_{t,x_t}(\xi), \qquad \text{a.s. } \Omega_t, \mathbf{P}_{s,x}. \tag{7}$$

Here the measure $\mathbf{P}_{s,x}$ is assumed to apply as above to the σ-algebra $\overline{\mathfrak{M}}^s$. In what follows an important role is played by the σ-algebra

$$\mathfrak{R}_t^s = \mathfrak{M}_t^s \cap \overline{\mathfrak{N}}^s. \tag{8}$$

3. Additive and multiplicative functionals

3.1. A function $\varphi_t^s(\omega)$, with $0 \leqq s \leqq t < \zeta(\omega)$, assuming values in the interval $(-\infty, +\infty]$, is called an *additive functional* of the Markov process $X = (x_t, \zeta, \mathfrak{M}_t^s, \mathbf{P}_{s,x})$ if the following conditions are satisfied.

3.1A. φ_t^s is $\mathfrak{R}_t^s$-measurable.

3.1B. $\varphi_t^s(\omega) + \varphi_u^t(\omega) = \varphi_u^s(\omega), \qquad 0 \leqq s \leqq t \leqq u < \zeta(\omega).$

If instead of 3.1B a weaker condition

3.1*B*′. $\varphi_t^s(\omega) + \varphi_u^t(\omega) = \varphi_u^s(\omega), \qquad \text{a.s. } \Omega_u, \mathbf{P}_{s,x},$

is satisfied for any $0 \leqq s \leqq t \leqq u < T$ and $x \in E$. then we shall say that φ_t^s is an *almost additive functional* of X.

The functional φ_t^s is called *almost continuous on the right* if for any $s \in [0, T)$, $x \in E$ and for any sequence $t_n \downarrow t$

$$\varphi_{t_n}^s \to \varphi_t^s, \qquad \text{a.s. } \Omega_t, \mathbf{P}_{s,x}. \tag{9}$$

If for each $s \in [0, T)$ and $\omega \in \Omega$ the function $\varphi_t^s(\omega)$ is continuous on the right with respect to $t \in [s, \zeta(\omega))$, then we say that the functional φ_t^s is *continuous on the right*. Analogously we define the notions of an *almost continuous* and of a *continuous functional*.

In these definitions convergence is understood in the sense of the topology of the extended segment $(-\infty, +\infty]$, so that the requirement of continuity of φ_t^s does not preclude the possibility of infinite values of φ_t^s.

The functional φ_t^s is called *almost nonnegative* if for any $s \in [0, T)$ and $x \in E$

$$\varphi_t^s(\omega) \geqq 0, \qquad \text{a.s. } \Omega_t, \mathbf{P}_{s,x}. \tag{10}$$

If $\varphi_t^s(\omega) \geqq 0$ for all $0 \leqq s \leqq t < \zeta(\omega)$, then the functional φ_t^s is called *nonnegative*. For such a functional $\varphi_u^s - \varphi_t^s = \varphi_u^t \geqq 0$, with $s \leqq t \leqq u$ and therefore φ_t^s is a nondecreasing function of t.

Two almost additive functionals φ_t^s and $\tilde{\varphi}_t^s$ are called *equivalent* if for any $0 \leqq s \leqq t < T$ and any $x \in E$

(11) $$\varphi_t^s(\omega) = \tilde{\varphi}_t^s(\omega), \qquad \text{a.s. } \Omega_t, \mathbf{P}_{s,x}.$$

THEOREM 1. *Let φ_t^s be an almost additive functional satisfying the condition*

3.1C. $\varphi_s^s(\omega) = 0,$ a.s. Ω_s, $\mathbf{P}_{s,x}$, for all $s \in [0, T)$, $x \in E$.

If φ_t^s is almost nonnegative and almost continuous on the right then there exists an equivalent $\tilde{\varphi}_t^s$ which is a nonnegative, continuous on the right additive functional.

If φ_t^s is almost continuous on the right (continuous), then there exists an equivalent $\tilde{\varphi}_t^s$ which is a continuous on the right (continuous) additive functional.

It follows from condition 3.1B with $u = t = s$ that for all $0 \leqq s < T$, and $x \in E$, the function $\varphi_s^s(\omega) = 0$ or $+\infty$, a.s. Ω_s, $\mathbf{P}_{s,x}$. Therefore condition 2.1C is equivalent to the requirement that for any $0 \leqq s < T$, $x \in E$

(12) $$P_{s,x}\{\varphi_s^s = +\infty\} = 0.$$

The first conclusion of the theorem is proved in [4], chapter 3, theorem 3.2. The second statement can be proved analogously.

REMARK. It is possible to weaken somewhat the definition of the additive functional by asking only that the function $\varphi_t^s(\omega)$ be defined in the domain $\{0 \leqq s \leqq t < \bar{\zeta}(\omega)\}$, where $\bar{\zeta}(\omega)$ is an arbitrary function satisfying the inequality $\zeta(\omega) \leqq \bar{\zeta}(\omega) \leqq T$. In doing so condition 3.1A is replaced by the requirement that the function φ_t^s should be $\mathfrak{N}^s$-measurable and that it should induce an $\mathfrak{M}_t^s$-measurable function on Ω_t. In conditions 3.1B and 3.1B′ we replace ζ by $\bar{\zeta}$ and Ω_u by $\bar{\Omega}_u = \{\omega: \bar{\zeta}(\omega) > u\}$. Analogous changes must be made in the definitions of continuity, almost continuity, equivalence, and so forth. In doing so theorem 1 remains true.

This remark will be used in section 7, where we shall let $\bar{\zeta} = T$, that is, we shall suppose that the function $\varphi_t^s(\omega)$ is defined for all $\omega \in \Omega$. We note in this connection that if φ_t^s is an additive or almost additive functional given in the domain $\{0 \leqq s \leqq t < \zeta(\omega)\}$, and if for every $\omega \in \Omega_s$ there exists the limit

(13) $$\varphi_{\zeta-0}^s = \lim_{t \uparrow \zeta} \varphi_t^s,$$

then φ_t^s can be extended over the whole of Ω by the formula

(14) $$\varphi_t^s(\omega) = \begin{cases} 0, & \zeta \leqq s \\ \varphi_{\zeta-0}^s, & s < \zeta \leqq t. \end{cases}$$

This extension retains all the properties: an additive functional remains additive, a continuous one remains continuous, a nonnegative one remains nonnegative, and so on.

3.2. We shall now consider some examples of additive functionals.

3.2.1. If $h(t, x)$ is an arbitrary $\mathcal{B}_T^0 \times \mathcal{B}$-measurable function, where $\mathcal{B}_t^s$ is the σ-algebra of all Borel subsets of the segment $[s, t]$, then

$$\varphi_t^s(\omega) = h(t, x_t) - h(s, x_s) \tag{15}$$

is an additive functional.

3.2.2. Let the function $\tau_s(\omega)$, where $0 \leqq s < \zeta(\omega)$ satisfy the conditions.

3.2.2A. $s \leqq \tau_s(\omega) \leqq \zeta(\omega)$,

3.2.2B. $\{\tau_s > t\} \in \mathcal{R}_t^s, \qquad 0 \leqq s \leqq t < T,$

3.2.2C. $\{\tau_s > t\} \subseteq \{\tau_s = \tau_t\}, \qquad 0 \leqq s \leqq t < T.$

Then the formula

$$\varphi_t^s(\omega) = \begin{cases} 0, & s \leqq t < \tau_s(\omega) \\ +\infty, & \tau_s(\omega) \leqq t < \zeta(\omega) \end{cases} \tag{16}$$

defines a right-continuous, nonnegative additive functional. This is the general form of right-continuous additive functionals which assume only the two values 0 and ∞.

3.2.3. Let $V(u, x)$ be a $\mathcal{B}_T^0 \times \mathcal{B}$-measurable function on $[0, T] \times E$, let μ be a measure over the σ-algebra $\mathcal{B}_T^0$ and let us suppose that for all $s \in [0, T]$, $x \in E$, the integral $\int_s^\zeta V(u, x_u)\, \mu(du)$ converges or diverges to $+\infty$ almost surely with respect to $\mathbf{P}_{s,x}$. Then

$$\varphi_t^s(\omega) = \int_{(s,t]} V(u, x_u)\mu(du), \qquad 0 \leqq s \leqq t < \zeta(\omega) \tag{17}$$

defines a continuous on the right additive functional of X. If the measure μ is continuous, then this functional is continuous. If $V(u, x) \geqq 0$, then it is nonnegative.

3.2.4. Let $X = (x_t, T, \mathfrak{M}_t^s, \mathbf{P}_{s,x})$ be an n-dimensional Wiener process, given in the time interval $[0, T)$. Let $f(t, x)$ for $t \in [0, T)$, $x \in E$ be a function with values in E, satisfying the conditions

3.2.4A. for every $\Gamma \in \mathcal{B}$

$$\{(t, x) : f(t, x) \in \Gamma\} \in \mathcal{B}_T^0 \times \mathcal{B}, \tag{18}$$

3.2.4B. for every $t \in [0, T)$

$$\sup_{\substack{0 \leqq u \leqq t \\ x \in E}} f^2(u, x) < \infty. \tag{19}$$

It has been proved [6] that it is possible to select a value of a stochastic integral such that

$$\varphi_t^s = \int_s f(u, x_u) dx_u \tag{20}$$

defines a continuous additive functional of X. If, moreover, the following condition is fulfilled

3.2.4C. for every $s \in [0, T)$, $x \in E$,

$$\int_s^T \mathbf{M}_{s,x} f^2(u, x_u)\,du < \infty, \tag{21}$$

then the continuous additive functional (20) can be selected so that for every $\omega \in \Omega$ and for every $s \in [0, T)$ there exists the limit

$$\varphi^s_{T-0}(\omega) = \lim_{t \uparrow T} \varphi^s_t(\omega). \tag{22}$$

3.3. A function $\alpha^s_t(\omega)$ where $0 \leqq s \leqq t < \zeta(\omega)$, which assumes values in the interval $[0, +\infty)$, is called a multiplicative functional of the Markov process X if

3.3A. α^s_t is $\mathcal{R}^s_t$-measurable,

3.3B. $\alpha^s_t(\omega)\alpha^t_u(\omega) = \alpha^s_u(\omega), \qquad 0 \leqq s \leqq t \leqq u < \zeta(\omega).$

The formulas

$$\alpha^s_t = e^{-\varphi^s_t}, \qquad \varphi^s_t = -\log \alpha^s_t \tag{23}$$

establish a one-to-one correspondence between the set of all additive and all multiplicative functionals of the process X. In particular, to the additive functionals described in subsection 3.2 correspond the multiplicative functionals

$$\alpha^s_t = \exp\{-\int_{(s,t]} V(u, x_u)\mu(du)\}, \tag{24}$$

$$\alpha^s_t = \chi_{\tau_s > t}, \tag{25}$$

$$\alpha^s_t = \frac{g(t, x_t)}{g(s, x_s)}, \qquad g(t, x) = e^{-h(t,x)}. \tag{26}$$

To continuous on the right (continuous) additive functionals correspond continuous on the right (continuous) multiplicative functionals. The analogous assertions about almost continuous on the right and almost continuous functionals are also valid. To nonnegative additive functionals correspond multiplicative functionals which satisfy the condition $\alpha^s_t \leqq 1$, with $0 \leqq s \leqq t < T$. The notion of almost multiplicative functionals arises naturally and theorem 1 can be carried over to such functionals.

4. Transformations of transition functions

4.1. THEOREM 2. *Let* $X = (x_t, \zeta, \mathfrak{M}^s_t, \mathbf{P}_{s,x})$ *be a Markov process in a state space* $(E, \mathcal{B})$. *Let* α^s_t *be a multiplicative functional of the process* X *satisfying the condition*

4.1A. $\mathbf{M}_{s,x}\alpha^s_t \leqq 1$ *for all* $0 \leqq s \leqq t < T, x \in E$. *Then the formula*

$$\tilde{P}(s, x; t, \Gamma) = \mathbf{M}_{s,x}[\chi_\Gamma(x_t)\alpha^s_t] \tag{27}$$

defines a transition function in the state space $(E, \overline{\mathcal{B}})$.

PROOF. The function $\tilde{P}(s, x; t, \Gamma)$ obviously satisfies conditions 2.1A and 2.1D. Condition 2.1C follows from 4.1A. Condition 2.1B is satisfied in view of subsection 2.3. There remains to check condition 2.1E.

It is obvious from (27) that for any $\mathcal{B}$-measurable bounded function $f(x)$

$$\int_E f(y)\tilde{P}(s, x; t, dy) = \mathbf{M}_{s,x}[f(x_t)\alpha^s_t]. \tag{28}$$

Using conditions 3.3A and 3.3B and formula (7) we have

$$\tilde{P}(s, x; u, \Gamma) = \mathbf{M}_{s,x}[\chi_\Gamma(x_u)\alpha_u^s] = \mathbf{M}_{s,x}[\alpha_t^s\chi_\Gamma(x_u)\alpha_u^t] \tag{29}$$
$$= \mathbf{M}_{s,x}\{\alpha_t^s \mathbf{M}_{s,x}[\chi_\Gamma(x_u)\alpha_u^t|\mathfrak{M}_t^s]\}$$
$$= \mathbf{M}_{s,x}\{\alpha_t^s \mathbf{M}_{t,x_t}[\chi_\Gamma(x_u)\alpha_u^t]\} = \mathbf{M}_{s,x}[\alpha_t^s \tilde{P}(t, x_t; u, \Gamma)].$$

Comparing (28) and (29) we see that the function $\tilde{P}(s, x; t, \Gamma)$ satisfies condition 2.1E. This completes the proof of the theorem.

Condition 4.1A is always satisfied for the functional (25), and also for the functional (24) with $V(u, y) \geqq 0$.

For a multiplicative functional $\alpha_t^s = g(t, x_t)/g(s, x_s)$ the function $\tilde{P}(s, x; t, \Gamma)$ can be directly expressed in terms of the transition function $\tilde{P}(s, x; t, \Gamma)$ by the formula

$$\tilde{P}(s, x; t, \Gamma) = \frac{1}{g(s, x)} \int_\Gamma P(s, x; t, dy)\, g\,(t, y). \tag{30}$$

In doing so, condition 4.1A becomes

4.1A′. $\mathbf{M}_{s,x} g(t, x_t) \leqq g(s, x)$ for all $0 \leqq s \leqq t < T$, $x \in E$.

Let X be an n-dimensional Wiener process in the time interval $[0, T)$. Let us set

$$\alpha_t^s = \exp\left[-\int_s^t V(u, x_u)du - \int_s f(u, x_u)dx_u\right], \tag{31}$$

where $V(u, y)$ satisfies condition 3.2.3 and $f(u, y)$ satisfies 3.2.4A and 3.2.4B. It can be shown [6] that condition 4.1A′ is satisfied for α_t^s with $V \geqq f^2/2$. If $V = f^2/2$, then for any $s \in [0, T)$ and $x \in E$,

$$\mathbf{M}_{s,x}\, \alpha_t^s = 1. \tag{32}$$

If the function f satisfies condition 3.2.4C as well as 3.2.4A and 3.2.4B then for $V \geqq f^2/2$

$$M_{s,x}\, \alpha_{T-0}^s \leqq 1. \tag{33}$$

If, moreover

$$\sup_{\substack{0 \leqq u < T \\ x \in E}} f^2(u, x) < \infty, \tag{34}$$

then, for $V = f^2/2$,

$$\mathbf{M}_{s,x}\, \alpha_{T-0}^s = 1. \tag{35}$$

5. Transformations of Markov processes

5.1. In section 4 we constructed a transition function $\tilde{P}(s, x; t, \Gamma)$ starting with a Markov process X and a multiplicative functional α_t^s. By subsection 2.2 it is possible to construct a Markov process $\tilde{X}$ from the transition function $\tilde{P}(s, x; t, \Gamma)$ under some general conditions. However, in doing so the connection between the characters of the trajectories of the processes X and $\tilde{X}$ has not been made clear. Under some additional conditions this connection is clarified in theorem 3. It turns out that a Markov process with the transition function $\tilde{P}(s, x; t, \Gamma)$ can be selected in such a way that its trajectory coincides with the beginning of some trajectory of the process X.

THEOREM 3. *Let $X = (x_t, \zeta, \mathfrak{M}_t^s, \mathbf{P}_{s,x})$ be a normal Markov process given in the time interval $[0, T)$ in the state space $(E, \mathfrak{B})$ and in the space of elementary events Ω. Let the multiplicative functional α_t^s of the process X and the nonnegative function $\xi_t(\omega)$, where $\omega \in \Omega$, $0 \leqq t < \zeta(\omega)$, satisfy the conditions*

5.1A. $\alpha_t^s \xi_t \leqq \xi_s, \qquad 0 \leqq s \leqq t < \zeta(\omega)$

5.1B. $\lim_{t \downarrow s} \alpha_t^s \xi_t = \alpha_s^s \xi_s,$

5.1C. ξ_t is $\mathfrak{N}^t$-measurable,

5.1D. $M_{s,x}\xi_s = 1$ *for every* $0 \leqq s < T$ *and* $x \in E$.

Let $\psi_t^s = \alpha_t^s \xi_t$, for $0 \leqq s \leqq t < \zeta(\omega)$. For every $\omega \in Q_s = \{\omega: \alpha_s^s(\omega) = 1\}$ we construct a measure ψ over the σ-algebra $\mathfrak{B}_T^0$ on the interval $(s, \zeta(\omega)]$, such that for all $t \in [s, \zeta]$,

$$\psi^s(t, \zeta] = \psi_t^s \tag{36}$$

(such a measure exists and is unique). For $\omega \notin Q_s$ we denote by ψ^s the unit measure concentrated at the point s.

Let $\tilde{\Omega} = \Omega \times [0, T]$, $\tilde{\mathfrak{M}}^s = \mathfrak{M}^s \times \mathfrak{B}_T^0$, $\tilde{\zeta}(\omega, u) = \min [\zeta(\omega), u]$, $\tilde{x}_t(\omega, u) = x_t(\omega)$ for $0 \leqq t < \tilde{\zeta}(\omega, u)$. For every $C \in \tilde{\mathfrak{M}}^s$ we let

$$\tilde{\mathbf{P}}_{s,x}(C) = \mathbf{M}_{s,x}\psi^s(C_\omega), \tag{37}$$

where C_ω denotes the ω-section of the set C, that is the totality of numbers u such that $(\omega, u) \in C$. In particular it follows from (37) that for $A \in \mathfrak{M}^s$

$$\tilde{\mathbf{P}}_{s,x}(A) = \int_A \xi_s(\omega)\mathbf{P}_{s,x}(d\omega). \tag{38}$$

Let $\tilde{\mathfrak{M}}_t^s$ be the totality of all subsets of the space $\tilde{\Omega}$ of the form $A \times (t, T]$, where $A \in \mathfrak{M}_t^s$.

Then $\tilde{X} = (\tilde{x}_t, \tilde{\zeta}, \tilde{\mathfrak{M}}_t^s, \tilde{\mathbf{P}}_{s,x})$ defines a Markov process in the state space $(E, \overline{\mathfrak{B}})$ with the transition function

$$\tilde{P}(s, x; t, \Gamma) = \mathbf{M}_{s,x}[\chi_\Gamma(x_t)\alpha_t^s]. \tag{39}$$

In general if η is a $\mathfrak{M}_t^s$-measurable function such that $\eta\alpha_t^s$ is $\mathbf{P}_{s,x}$-integrable, then

$$\tilde{\mathbf{M}}_{s,x}[\eta\chi_{\tilde{\zeta}>t}] = \mathbf{M}_{s,x}[\eta\alpha_t^s]. \tag{40}$$

PROOF. (1) Let $\omega \in \Omega_s$. We note that for any $0 \leqq s \leqq t \leqq u < \zeta(\omega)$

$$\psi_u^s = \alpha_u^s \xi_u = \alpha_t^s \alpha_u^t \xi_u \leqq \alpha_t^s \xi_t = \psi_t^s. \tag{41}$$

Therefore, ψ_t^s for $s \leqq t < \zeta$ is a nonincreasing function of t. Comparing (41) with 5.1B we conclude that $\psi_u^s \uparrow \psi_t^s$ for $u \downarrow t$, and hence that ψ_t^s for $s \leqq t < \zeta$ is continuous on the right. Therefore there exists a measure (which is unique) over the

σ-algebra of all Borel subsets of the interval $(s, \zeta]$ such that $\psi^s(t, \zeta] = \psi^s_t$ for all $t \in [s, \zeta)$. We extend this measure over the σ-algebra $\mathcal{B}^0_T$, letting

$$\psi^s[0, s] = \psi^s(\zeta, T] = 0. \tag{42}$$

(2) Consider the system $\mathfrak{F}$ of all subsets A of the set $\Omega \times [s, T]$, such that $A_\omega \in \mathcal{B}^s_T$ for every $\omega \in \Omega$ and the function $\psi^s(A_\omega)$ is $\mathfrak{M}^s$-measurable. This system satisfies the conditions

(a) the whole space $\Omega \times [s, T]$ is an elenemts of $\mathfrak{F}$;

(b) if two nonintersecting sets belong to $\mathfrak{F}$ then their sum belongs to $\mathfrak{F}$;

(c) if $A, B \in \mathfrak{F}$, and $A \subseteq B$, then $B \setminus A \in \mathfrak{F}$;

(d) if $A_1, \cdots, A_n, \cdots \in \mathfrak{F}$ and $A_n \uparrow A$, then $A \in \mathfrak{F}$.

The system $\mathfrak{F}$ contains the totality $\mathcal{C}$ of all sets of the form $A \times (t, T]$ for $t \in [s, T]$, $A \in \mathfrak{M}^s$.

Inasmuch as the intersection of two sets of $\mathcal{C}$ belongs to $\mathcal{C}$, it follows that $\mathfrak{F}$ contains the σ-algebra $\tilde{\mathfrak{M}}^s = \mathfrak{M}^s \times \mathcal{B}^s_T$, generated by $\mathcal{C}$ (see, for example, [4], lemma 1.1). Therefore formula (37) has a meaning.

(3) Formula (37) obviously defines a measure over the σ-algebra $\tilde{\mathfrak{M}}^s$. We prove that formula (40) holds for this measure. We denote by $\mathcal{H}$ the totality of all functions $\eta(\omega)$ for which (40) holds. We note first of all that $\mathcal{H}$ contains the indicators of all the sets A of $\mathfrak{M}^s_t$. In fact by (37), (36), (7), 3.3A, 5.1C, and 5.1D, for any $A \in \mathfrak{M}^s_t$,

$$\begin{aligned} \tilde{\mathbf{M}}_{s,x}(\chi_A \chi_{\tilde{\zeta} > t}) &= \tilde{\mathbf{P}}_{s,x}\{A \times (t, T]\} = \mathbf{M}_{s,x}(\psi^s_t \chi_A) \\ &= \mathbf{M}_{s,x}[\chi_A \alpha^s_t \mathbf{M}_{s,x}(\xi_t | \mathfrak{M}^s_t)] \\ &= \mathbf{M}_{s,x}[\chi_A \alpha^s_t \mathbf{M}_{t,x_t} \xi_t] = \mathbf{M}_{s,x}(\chi_A \alpha^s_t). \end{aligned} \tag{43}$$

It is obvious that the system $\mathcal{H}$ contains together with any two functions the linear combination of these functions and together with any nondecreasing sequence of nonnegative functions the limit of this sequence. Therefore (see, for example, [4], lemma 1.2) $\mathcal{H}$ contains all $\mathfrak{M}^s_t$-measurable functions η for which $\eta \alpha^s_t$ is $\mathbf{P}_{s,x}$-summable. This proves formula (40). If in (40) we let $\eta = \chi_\Gamma(x_t)$ we obtain (39).

(4) We now show that $(\tilde{x}_t, \tilde{\zeta}, \tilde{\mathfrak{M}}^s_t, \tilde{\mathbf{P}}_{s,x})$ satisfies conditions 2.2A to 2.2F. Conditions 2.2A and 2.2B are obviously satisfied. Further, by formulas (40) and (37) and taking into consideration that α^s_s can be equal only to either 0 or 1, we have

$$\begin{aligned} \tilde{\mathbf{P}}_{s,x}(\tilde{\Omega}) &= \tilde{\mathbf{M}}_{s,x}\{\chi_{\tilde{\zeta} > s} + \chi_{\zeta \leqq s}\} = \mathbf{M}_{s,x}\, \alpha^s_s + \mathbf{M}_{s,x}\, \psi^s[0, s] \\ &= \mathbf{M}_{s,x}\, \alpha^s_s + \mathbf{P}_{s,x}(\Omega \setminus Q_s) = \mathbf{M}_{s,x}\, \alpha^s_s + 1 - \mathbf{P}_{s,x}\, \{\alpha^s_s = 1\} = 1. \end{aligned} \tag{44}$$

Therefore condition 2.2C is satisfied. Conditions 2.2D and 2.2E follow readily from formula (39). It remains to check condition 2.2F. In order to do this we must show that for any $0 \leqq s \leqq t \leqq u < T$, with $x \in E$, $\Gamma \in \mathcal{B}$, and $\tilde{A} \in \tilde{\mathfrak{M}}^s_t$, when

$$\tilde{\mathbf{P}}_{s,x}(\tilde{A}, \tilde{x}_u \in \Gamma) = \tilde{\mathbf{M}}_{s,x}[\chi_{\tilde{A}} \tilde{P}(t, \tilde{x}_t; u, \Gamma)], \tag{45}$$

we have $\tilde{A} = A \times (t, T]$, where $A \in \mathfrak{M}_T^s$, and

(46) $$\{\tilde{A}, \tilde{x}_u \in \Gamma\} = A, x_u \in \Gamma\} \times (u, T].$$

By (40), 3.3B, 3.3A, and (39) we have

(47) $$\begin{aligned}\tilde{\mathbf{P}}_{s,x}(\tilde{A}, \tilde{x}_u \in \Gamma) &= \mathbf{M}_{s,x}[\chi_A\chi_\Gamma(x_u)\alpha_u^s] = \mathbf{M}_{s,x}[\chi_A\alpha_t^s\chi_\Gamma(x_u)\alpha_u^t] \\ &= \mathbf{M}_{s,x}\{\chi_A\alpha_t^s\mathbf{M}_{s,x}[\chi_\Gamma(x_u)\alpha_u^t|\mathfrak{M}_t^s]\} \\ &= \mathbf{M}_{s,x}\{\chi_A\alpha_t^s\mathbf{M}_{t,x_t}[\chi_\Gamma(x_u)\alpha_u^t]\} \\ &= \mathbf{M}_{s,x}\{\chi_A\alpha_t^s\tilde{P}(t, x_t; u, \Gamma)\}.\end{aligned}$$

On the other hand,

(48) $$\chi_{\tilde{A}}\tilde{P}(t, \tilde{x}_t; u, \Gamma) = \chi_A\chi_{\tilde{\zeta}>t}\tilde{P}(t, x_t; u, \Gamma)$$

and by (40)

(49) $$\tilde{\mathbf{M}}_{s,x}[\chi_{\tilde{A}}\tilde{P}(t, \tilde{x}_t; u, \Gamma)] = \mathbf{M}_{x,s}\{\chi_A\tilde{P}(t, x_t; u, \Gamma)\alpha_t^s\}.$$

Now (45) follows from (47) and (49).

5.2. The Markov process $\tilde{X}$ constructed in theorem 3 will be called a (α_t^s, ξ_t)-subprocess of the Markov process X. We shall consider the most important special classes of the (α_t^s, ξ_t)-subprocesses.

Suppose that X is a normal Markov process. Let the multiplicative function α_t^s and the function ξ_t satisfy conditions 5.1C and 5.1D and the two conditions

5.1A′. $\alpha_t^s\xi_t = \alpha_s^s\xi_s$, $\quad 0 \leqq s \leqq t < \zeta(\omega)$,

5.1B′. $\mathbf{P}_{s,x}\{\alpha_s^s = 1\} = 1$ $\quad$ for all $0 \leqq s < T; x \in E$.

It is obvious that conditions 5.1A and 5.1B follow from 5.1A′ and therefore it is possible to construct the (α_t^s, ξ_t)-subprocesses of X.

Consider the mapping γ of the space Ω into $\tilde{\Omega}$, defined by the formula $\gamma(\omega) = \{\omega, \zeta(\omega)\}$. We note that for every $C \in \tilde{\mathfrak{M}}^s = \mathfrak{M}^s \times \mathfrak{B}_T^0$

(50) $$\tilde{\mathbf{P}}_{s,x}(C) = \int_{\gamma^{-1}C} \xi_s(\omega)\mathbf{P}_{s,x}(d\omega).$$

This formula follows from (37) if we note that for every $\omega \in Q_s$ the whole measure ψ^s equal to $\xi_s(\omega)$ is concentrated in the point $\zeta(\omega)$ and that by condition 5.1B′ we have $\mathbf{P}_{s,x}(Q_s) = 1$.

It follows from formula (50) that $\tilde{\mathbf{P}}_{s,x}\{\tilde{\zeta} = \zeta\} = 1$ for all $0 \leqq s < T, x \in E$.

In the example under consideration it is natural to alter the process $\tilde{X}$ slightly in order to return from the space of elementary events $\tilde{\Omega}$ to the space Ω. We note that

(51) $$\tilde{\zeta}[\gamma(\omega)] = \zeta(\omega),$$

(52) $$\tilde{x}_t[\gamma(\omega)] = x_t(\omega),$$

(53) $$\gamma^{-1}\tilde{\mathfrak{M}}_t^s = \mathfrak{M}_t^s,$$

(54) $$\gamma^{-1}\tilde{\mathfrak{M}}^s = \mathfrak{M}^s.$$

Let

(55) $$\mathbf{P}'_{s,x}(\gamma^{-1}A) = \tilde{\mathbf{P}}_{s,x}(A), \qquad A \in \tilde{\mathfrak{M}}^s.$$

Then $X' = (x_t, \zeta, \mathfrak{M}_t^s, \mathbf{P}'_{s,x})$ defines a Markov process. This follows from theorem 2.5 of [4] or can be checked easily independently. In the terminology of [4] the process X' is obtained from the process $\tilde{X}$ by means of a transformation γ of the space of elementary events. The elements of this process are the same as those of the original process X with the exception of $\mathrm{P}'_{s,x}$. The latter, in view of (50), can be expressed by the formula

(56) $$\mathbf{P}'_{s,x}(A) = \int_A \xi_s(\omega)\mathbf{P}_{s,x}(d\omega), \qquad A \in \mathfrak{M}_s.$$

In this way the process X' is obtained from the process X by means of the transformation of the original measures $\mathbf{P}_{s,x}$ by formula (56). The transition function of the process X' is the same as of the process $\tilde{X}$ and can be expressed by formula (39).

We will suppose that the multiplicative function α_t^s satisfies condition 5.1B′ and the following conditions.

5.2A. For every $\omega \in \Omega_s$ there exists the limit

(57) $$\alpha_{\zeta-0}^s = \lim_{t \uparrow \zeta} \alpha_t^s.$$

5.2B. For all $s \in [0, T)$, $x \in E$,

(58) $$\mathbf{M}_{s,x}\alpha_{\zeta-0}^s = 1.$$

It is easily seen that in this case the function $\xi_s = \alpha_{\zeta-0}^s$ satisfies the conditions 5.1A′, 5.1B′, 5.1C, and 5.1D. Therefore we can obtain a Markov process with the transition function (39) by transforming the measures $\mathbf{P}_{s,x}$ by the formula

(59) $$\mathbf{P}'_{s,x}(A) = \int_A \alpha_{\zeta-0}^s(\omega)\mathbf{P}_{s,x}(d\omega), \qquad A \in \mathfrak{M}^s.$$

As an example of a multiplicative functional for which the transformation (59) is possible we can take the functional discussed in 4.1, where

(60) $$\alpha_t^s = \exp\left(-\frac{1}{2}\int_s^t f^2(u, x_u)du - \int_s^t f(u, x_u)dx_u\right)$$

for an n-dimensional Wiener process X given in the time interval $[0, T)$. Here we assume that f satisfies conditions 3.2.4A, 3.2.4B, 3.2.4C, and (34). According to the remarks at the end of subsections 3.2 and 4.1, the functional α_t^s satisfies conditions 5.2A and 5.2B.

5.3. If $\xi_s(\omega) = 1$ for all $0 \leqq s < \zeta(\omega)$, then we shall denote the (α_t^s, ξ_s)-subprocesses of X by the α_t^s-subprocesses. In order to be able to form the α_t^s-subprocesses it is necessary and sufficient that the following conditions be satisfied.

5.3A. For every $0 \leqq s \leqq t < \zeta(\omega)$, we have $\alpha_t^s(\omega) \leqq 1$.

5.3B. $\lim_{t \downarrow s} \alpha_t^s(\omega) = \alpha_s^s(\omega)$.

Conditions 5.3A and 5.3B are obviously equivalent to the requirement that the corresponding additive functional to α_t^s be nonnegative and continuous on the right. In particular these conditions are satisfied for the functional (24) with $V \geqq 0$, and for the functional (25).

For the α_t^s-subprocess the measure $\tilde{\mathbf{P}}_{s,x}$ defined by formula (37) is an extension of the measure $\mathbf{P}_{s,x}$; that is, $\tilde{\mathbf{P}}_{s,x}(A \times [0, T]) = \mathbf{P}_{s,x}(A)$ for $A \in \mathfrak{M}^s$. (The subset $A \times [0, T]$ of the space $\Omega \times [0, T]$, as usual is identified with the subset A of the space Ω.) By (40) for any $A \in \mathfrak{M}_t^s$

$$\tilde{\mathbf{P}}_{s,x}(A, \tilde{\zeta} > t) = \int_A \alpha_t^s \mathbf{P}_{s,x}(d\omega) = \int_A \alpha_t^s \tilde{\mathbf{P}}_{s,x}(d\omega). \tag{61}$$

Starting with (37) it is not hard to verify that this formula holds for any $A \in \overline{\mathfrak{M}}^s$. Since the function α_t^s is $\mathfrak{R}_t^s$-measurable it is obvious that

$$\begin{aligned} \alpha_t^s &= \tilde{\mathbf{P}}_{s,x}\{\tilde{\zeta} > t | \mathfrak{M}^s\} = \tilde{\mathbf{P}}_{s,x}\{\tilde{\zeta} > t | \mathfrak{M}_t^s\} \\ &= \tilde{\mathbf{P}}_{s,x}\{\tilde{\zeta} > t | \mathfrak{R}_t^s\}, \qquad \text{a.s. } \Omega_t, \mathbf{P}_{s,x}. \end{aligned} \tag{62}$$

In view of (62) an intuitive picture of the formation of the α_t^s-subprocess can be given as follows: the trajectories of the original process terminate with a certain probability distribution; $\alpha_t^s(\omega)$ denotes the probability that the trajectory $x_u(\omega)$ will not terminate in the time interval $[s, t]$.

The α_t^s-subprocesses of the Markov processes are studied in detail in chapter 3 of monograph [4].

5.4. The third important class of (α_t^s, ξ_t)-subprocesses is connected with the class of functions $\eta_t(\omega)$, where $0 \leqq t < \zeta(\omega)$, which satisfy the conditions

5.4A. $\eta_t(\omega)$ is $\mathfrak{N}^t$-measurable;

5.4B. $0 \leqq \eta_t(\omega) \leqq \eta_s(\omega)$ for $0 \leqq s \leqq t < \zeta(\omega)$;

5.4C. for every $\omega \in \Omega$ the function $\eta_t(\omega)$ is continuous on the right in t;

5.4D. for any $s \in [0, T)$, $x \in E$, $0 < \mathbf{M}_{s,x}\eta_s < \infty$.

It is easily seen that under these conditions the function $f(t, x) = \mathbf{M}_{t,x}\eta_t$ satisfies condition 4.1A′, and that the pair

$$\alpha_t^s = \frac{f(t, x_t)}{f(s, x_s)}, \qquad \xi_t = \frac{\eta_t}{f(t, x_t)} \tag{63}$$

satisfies conditions 5.1A to 5.1D. The transition function of the (α_t^s, ξ_t)-subprocess that corresponds to this pair can be expressed in terms of the transition function of the original process by means of formula (30).

6. The stationary case

6.1. We shall now assume that the fundamental interval $[0, T)$ coincides with the half line $[0, \infty)$. The transition function $P(s, x; t, \Gamma)$ is called stationary if it depends only on the difference $t - s$:

$$P(s, x; t, \Gamma) = P(t - s, x, \Gamma). \tag{64}$$

The Markov process $X = (x_t, \zeta, \mathfrak{M}_t^s, \mathbf{P}_{s,x})$ is called stationary if its transition function is stationary and if for any $0 \leqq t < \zeta(\omega)$ there exists an $\omega' \in \Omega$ such that

$$x_{t+h}(\omega) = x_h(\omega'), \qquad \text{for all } 0 \leqq h < \zeta(\omega') = \zeta(\omega) - t. \tag{65}$$

Let X be a stationary process. We denote by $\mathfrak{N}^*$ the minimal system of subsets $\Omega_0 = \{\omega: \zeta(\omega) > 0\}$, that contains all the sets $\{\omega: x_t(\omega) \in \Gamma\}$, with $t \geqq 0$, $\Gamma \in \mathcal{B}$, and is invariant with respect to the operations of addition, intersection, and complementation. For every $\mathfrak{N}^*$-measurable function $\xi(\omega)$ we set

$$\theta_t \xi(\omega) = \xi(\omega'), \tag{66}$$

where ω' is connected with ω by relation (65). Note that this definition is correct since for all ω' connected with a given ω the value of $\xi(\omega')$ remains unaltered. The operators θ_t preserve all algebraic operations and the operation of passing to the limit. We have

$$\begin{aligned} \theta_h f(x_t) &= f(x_{t+h}), \\ \theta_h \int_{(s,t]} V(u, x_u)\mu(du) &= \int_{(s,t]} V(u, x_{u+h})\mu(du). \end{aligned} \tag{67}$$

It may be proved that if the process X is stationary, then for any $(\overline{\mathfrak{N}}^s \cap \mathfrak{N}^*)$-measurable function $\xi(\omega)$, with $\omega \in \Omega_s$, and any $h \geqq 0$, $x \in E$, the function $\theta_h \xi$ is $\overline{\mathfrak{N}}^{s+h}$-measurable and

$$\mathbf{M}_{s,x}\xi = \mathbf{M}_{s+h,x}(\theta_h \xi). \tag{68}$$

6.2. An additive functional φ_t^s of a stationary Markov process X is called stationary if for any $h \geqq 0$, with $0 \leqq s \leqq t < T$, and $x \in E$,

$$\theta_h \varphi_t^s = \varphi_{t+h}^{s+h} \qquad \text{a.s. } \Omega_{t+h}, \mathbf{P}_{s+h,x}. \tag{69}$$

The stationary property of multiplicative, almost additive, and almost multiplicative functionals is similarly defined. A functional that is equivalent to a stationary function is also stationary. A multiplicative (almost multiplicative) functional α_t^s is stationary if and only if the corresponding additive (almost additive) functional φ_t^s is stationary.

It follows from (67) that the additive functional

$$\varphi_t^s = \int_s^t V(x_u)du \tag{70}$$

is stationary. By [6] the functional

$$\varphi_t^s = \int_s^t f(x_u)dx_u \tag{71}$$

is stationary also. It is easily seen that the functional described in 3.2.1 is stationary, if the function $h(t, x)$ does not depend on t; the functional described in 3.2.2 is stationary if and only if for any h, $s \geqq 0$, $x \in E$

$$\theta_h \tau_s = \tau_{s+h} - h, \qquad \text{a.s. } P_{s+h,x} \{\tau_s > s + h\}. \tag{72}$$

THEOREM 4. *Let X be a stationary Markov process and let α_t^s be a stationary* *multiplicative functional of this process. Then the transition function formed by* (27) *is stationary. Every* (α_t^s, ξ_t)*-subprocess of the process* X *is a stationary Markov process.*

PROOF. We have

$$(73)\qquad \tilde{P}(s, x; t, \Gamma) = \mathbf{M}_{s,x}\{\chi_\Gamma(x_t)\alpha_t^s\} = \mathbf{M}_{s,x}\{\theta_s[\chi_\Gamma(x_{t-s})\alpha_{t-s}^0]\} = \mathbf{M}_{0,x}\{\chi_\Gamma(x_{t-s})\alpha_{t-s}^0\} = P(0, x; t - s, \Gamma),$$

which proves that the transition function $\tilde{P}(s, x; t, \Gamma)$ is stationary. Since the transition function of the (α_t^s, ξ_t)-subprocesses $\tilde{X}$ of the process X is given by formula (39), in order to check that $\tilde{X}$ is stationary it remains to show that for every $\tilde{\omega} \in \tilde{\Omega} = \Omega \times [0, T]$ there exists an $\tilde{\omega}'$ such that

$$(74)\qquad \tilde{x}_{t+h}(\tilde{\omega}) = \tilde{x}_h(\tilde{\omega}'), \qquad \text{for all } 0 \leqq h < \tilde{\zeta}(\tilde{\omega}') = \tilde{\zeta}(\tilde{\omega}) - t.$$

Let $\tilde{\omega} = (\omega, u)$, where $\omega \in \Omega$, $u \in [0, T]$. Since X is stationary there exists an ω' satisfying condition (65). It is easily seen that $\tilde{\omega}' = (\omega', u - t)$ satisfies (74).

7. Characteristics of almost additive functionals

7.1. The significance of additive functionals in the theory of Markov processes should be sufficiently clear from the foregoing sections. It is natural therefore to propose the problem of finding the most general form of additive and almost additive functionals for the most important classes of Markov processes. In this section we present the results first obtained by Volkonsky [12] in a somewhat different form. We will consider additive and almost additive functionals $\varphi_t^s(\omega)$ defined for all $\omega \in \Omega$ (see remark at the end of section 4).

Let φ_t^s be an almost additive functional of the Markov process $X = (x_t, \zeta, \mathfrak{M}_t^s, \mathbf{P}_{s,x})$. We shall call the function

$$(75)\qquad m_t^s(x) = \mathbf{M}_{s,x}\varphi_t^s$$

the characteristic of the functional φ_t^s. We assume that the mathematical expectation of the right side exists.

We denote by $W(X)$ the totality of all almost additive functionals φ_t^s of the process X that satisfy the following condition.

7.1A. For every $[s, t] \subset [0, T)$, and $x \in E$

$$(76)\qquad \lim_{\|S\| \to 0} \mathbf{M}_{s,x} \sum_{k=0}^{n-1} [\varphi_{s_{k+1}}^{s_k}]^2 = 0.$$

(We denote by S the subdivision $s = s_0 < s_1 < \cdots < s_n = t$ of the segment $[s, t]$ and set $\|S\| = \max_{0 \leqq k < n} (s_{k+1} - s_k)$.)

It is obvious that the class $W(X)$ contains together with every almost additive functional all the functionals equivalent to it. It is not difficult to see that the following condition is satisfied for all functionals of the class $W(X)$: for any $[s, t] \subset [0, T)$, $x \in E$,

$$(77)\qquad \mathbf{M}_{s,x}[\varphi_t^s]^2 < \infty.$$

THEOREM 5. *Let φ_t^s be a nonnegative almost additive functional satisfying condition (77). In order that φ_t^s belong to class $W(X)$ it is necessary and sufficient that there exist for φ_t^s an equivalent nonnegative continuous additive functional.*

PROOF. *Sufficiency:* let φ_t^s be a nonnegative continuous additive functional. Let

$$\gamma_s^t(h, \omega) = \sup_{s \leqq u \leqq v \leqq u+h \leqq t} \varphi_v^u(\omega). \tag{78}$$

Since φ_t^s is continuous and by (77) we have

$$\lim_{h \downarrow 0} \gamma_t^s(h, \omega) = 0, \qquad \text{a.s. } \Omega, \mathbf{P}_{s,x}. \tag{79}$$

Since φ_t^s is nonnegative we have

$$\sum_{k=0}^{n-1} [\varphi_{s_{k+1}}^{s_k}]^2 \leqq \gamma_t^s(||S||, \omega)\varphi_t^s \leqq [\varphi_t^s]^2. \tag{80}$$

Hence $\sum_{k=0}^{n-1}[\varphi_{s_{k+1}}^{s_k}]^2$ is majorized by a $\mathbf{P}_{s,x}$-summable function and tends to zero a.s. Ω, $\mathbf{P}_{s,x}$ as $||S|| \to 0$. Therefore condition 7.1A is satisfied.

Necessity: let $0 \leqq s \leqq t < T$, $t_k \downarrow t$. Since φ_t^s is nonnegative

$$\varphi_{t_1}^s \geqq \varphi_{t_2}^s \geqq \cdots \geqq \varphi_{t_k}^s \geqq \cdots \geqq \varphi_t^s, \qquad \text{a.s. } \Omega, \mathbf{P}_{s,x}. \tag{81}$$

Therefore there exists almost surely with respect to $\mathbf{P}_{s,x}$ the limit $\xi = \lim \varphi_{t_k}^s$. Let $s_0 < s_1 < \cdots < s_n$ be any subdivision containing as two consecutive points t and t_k. Then

$$(\xi - \varphi_t^s)^2 \leqq (\varphi_{t_k}^s - \varphi_t^s)^2 = [\varphi_{t_k}^t]^2 \leqq \sum_{i=0}^{n-1} [\varphi_{s_{i+1}}^{s_i}]^2 \tag{82}$$

and by condition 7.1A we have $\mathbf{M}_{s,x}(\xi - \varphi_t^s)^2 = 0$; that is, the functional φ_t^s is continuous on the right. Obviously condition 3.1C is satisfied and by theorem 1 a continuous nonnegative additive functional $\tilde{\varphi}_t^s$ can be found that is equivalent to φ_t^s.

We note that for any $s = s_0 < s_1 < \cdots < s_n = t$

$$\sup_{u \in (s,t)} [\tilde{\varphi}_u^{u-0}]^2 \leqq \sum_{k=0}^{n-1} [\tilde{\varphi}_{s_{k+1}}^{s_k}]^2, \qquad \text{a.s. } \Omega_u, \mathbf{P}_{s,x}. \tag{83}$$

Let

$$A^s = \{\omega: \tilde{\varphi}_u^{u-0} = 0 \quad \text{for all } u \in (s, T)\}. \tag{84}$$

It follows from (83) and 7.1A that $\mathbf{P}_{s,x}(A^s) = 1$ for all s and x. The formula

$$\hat{\varphi}_t^s = \begin{cases} \tilde{\varphi}_t^s, & \omega \in A^s \\ +\infty, & \omega \notin A^s \end{cases} \tag{85}$$

defines a nonnegative continuous additive functional equivalent to φ_t^s.

THEOREM 6. *Almost additive functionals φ_t^s of class $W(X)$ are defined by their characteristic, uniquely up to equivalence. More precisely, for any $0 \leqq s \leqq t < T$, $x \in E$*

$$\varphi_t^s = \underset{||S||\to 0}{\text{l.i.m.}} \sum_{k=0}^{n-1} m_{s_{k+1}}^{s_k}(x_{s_k}) \quad (\mathbf{P}_{s,x}). \tag{86}$$

The notation $\xi = \text{l.i.m.}\ \xi_n \quad (\mathbf{P}_{s,x})$ means that $\mathbf{M}_{s,x}(\xi_n - \xi)^2 \to 0$.

PROOF. Let

$$\begin{aligned}\delta_k = \varphi_{s_{k+1}}^{s_k} - m_{s_{k+1}}^{s_k}(x_{s_k}) &= \varphi_{s_{k+1}}^{s_k} - \mathbf{M}_{s_k,x_{s_k}}\varphi_{s_{k+1}}^{s_k} \\ &= \varphi_{s_{k+1}}^{s_k} - \mathbf{M}_{s,x}\{\varphi_{s_{k+1}}^{s_k}|\mathfrak{M}_{s_k}^s\}.\end{aligned} \tag{87}$$

It is easily seen that

$$\mathbf{M}_{s,x}\delta_k\delta_l = 0, \qquad k \neq l \tag{88}$$

and that

$$\begin{aligned}\mathbf{M}_{s,x}\delta_k^2 &= \mathbf{M}_{s,k}[\varphi_{s_{k+1}}^{s_k}]^2 - \mathbf{M}_{s,x}[m_{s_{k+1}}^{s_k}(x_k)]^2 \\ &\leqq \mathbf{M}_{s,x}[\varphi_{s_{k+1}}^{s_k}]^2.\end{aligned} \tag{89}$$

Furthermore,

$$\varphi_t^s - \sum_{k=0}^{n-1} m_{s_{k+1}}^{s_k}(x_{s_k}) = \sum_{k=0}^{n-1} \delta_k. \tag{90}$$

It follows from (88), (89), and (90) that

$$\mathbf{M}_{s,x}[\varphi_t^s - \sum_{k=0}^{n-1} m_{s_{k+1}}^{s_k}(x_{s_k})]^2 \leqq \mathbf{M}_{s,x} \sum_{k=0}^{n-1} [\varphi_{s_{k+1}}^{s_k}]^2. \tag{91}$$

Hence (86) follows from 7.1A.

7.2. What functions $m_t^s(x)$ are characteristics of almost additive functionals of the process X? It follows readily from 3.1A, 3.1B, and (7) that the following condition is necessary.

7.2A. For any $0 \leqq s \leqq t \leqq u \leqq T$,

$$m_t^s(x) + \mathbf{M}_{s,x} m_u^t(x_t) = m_u^s(x). \tag{92}$$

Condition 7.2A by itself is not sufficient. Some supplementary conditions are given by the following theorem.

THEOREM 7. *Suppose that a nonnegative function* $m_t^s(x)$ *satisfies condition* 7.2A, *as well as the following conditions.*

7.2B. *For every* $t \in [0, T)$ *the function* $m_t^s(x)$ *is a* $\mathcal{B}_t^0 \times \mathcal{B}$*-measurable function of* s *and* x.

7.2C. $\sup_{x\in E} \sup_{0\leqq s\leqq t\leqq s+h<T} m_t^s(x) = \beta(h) \to 0$ as $h \to 0$.

Then $m_t^s(x)$ *is a characteristic of some almost nonnegative almost additive functional* φ_t^s *of class* $W(X)$. *Moreover*

$$\varphi_t^s = \underset{h\downarrow 0}{\text{l.i.m.}} \int_s^t \frac{1}{h}\, m_{u+h}^u(x_u)du. \tag{93}$$

COROLLARY. *By theorem 5 the functional* φ_t^s *satisfying* (93) *can be selected in such a way as to be a nonnegative continuous additive functional of the process* X.

In the proof of theorem 7 we need the following lemma.

LEMMA 1. *If*

$$\varphi_t^s = \int_s^t f(u, x_u)du, \tag{94}$$

then for any $0 \leqq s \leqq t \leqq u < T$, *and* $x \in E$,

$$\mathbf{M}_{s,x}[\varphi_u^t]^2 = 2\mathbf{M}_{s,x} \int_t^u f(v, x_v) m_u^v(x_v) dv. \tag{95}$$

PROOF OF LEMMA 1. We have

$$[\varphi_u^t]^2 = 2 \int_t^u f(v, x_v)\varphi_u^v \, dv. \tag{96}$$

By (7)

$$\begin{aligned} \mathbf{M}_{s,x}[f(v, x_v)\varphi_u^v(x_v)] &= \mathbf{M}_{s,x}\{f(v, x_v)\mathbf{M}_{s,x}[\varphi_u^v|\mathfrak{M}_v^s]\} \\ &= \mathbf{M}_{s,x}[f(v, x_v)\mathbf{M}_{v,x_v}\varphi_u^v] \\ &= \mathbf{M}_{s,x}[f(v, x_v)m_u^v(x_v)]. \end{aligned} \tag{97}$$

PROOF OF THEOREM 7. Let

$$f_h(u, x) = \frac{1}{h} m_{u+h}^u(x), \qquad f_{h,l}(u, x) = f_h(u, x) - f_l(u, x), \tag{98}$$

$$\varphi_t^s(h) = \int_s^t f_h(u, x_u)du, \qquad \varphi_t^s(h, l) = \int_s^t f_{h,l}(u, x_u)du, \tag{99}$$

$$m_t^s(x, h) = M_{s,x}\varphi_t^s(h), \qquad m_t^s(x, h, l) = M_{s,x}\varphi_t^s(h, l). \tag{100}$$

Note that by 7.2A

$$m_t^s(x, h) = \frac{1}{h} \int_s^t M_{s,x} m_{u+h}^u(x_u)du = \frac{1}{h} \int_t^{t+h} m_u^s(x)du - \frac{1}{h} \int_s^{s+h} m_u^s(x)du, \tag{101}$$

and by 7.2C

$$|m_t^s(x, h) - m_t^s(x)| < 2\beta(h). \tag{102}$$

Moreover,

$$|m_t^s(x; t, l)| \leqq |m_t^s(x, h) - m_t^s(x)| + |m_t^s(x) - m_t^s(x, l)| \leqq 2[\beta(h) + \beta(l)]. \tag{103}$$

By lemma 1 and by estimates (102) and (103) we have

$$\begin{aligned} \mathbf{M}_{s,x}[\varphi_t^s(h, l)]^2 &= 2\mathbf{M}_{s,x} \int_s^t f_{h,l}(v, x_v)m_t^v(x_v; h, l)dv \\ &\leqq 4[\beta(h) + \beta(l)][m_t^s(x, h) + m_t^s(x, l)] \\ &\leqq 8[\beta(h) + \beta(l)][\beta(t - s) + \beta(h) + \beta(l)]. \end{aligned} \tag{104}$$

The inequality $\beta(a + b) \leqq \beta(a) + \beta(b)$, for $a, b \geqq 0$ follows readily from 7.2A, and by 7.2C we have that $\beta(a) < \infty$ for any $a > 0$. Therefore it folllows from (104) that there exists the limit $\varphi_t^s = \text{l.i.m.}_{h\downarrow 0}\varphi_t^s(h)$, where

$$\mathbf{M}_{s,x}[\varphi_t^s - \varphi_t^s(h)]^2 \leqq 8\beta(h)[\beta(t - s) + \beta(h)]. \tag{105}$$

It follows from (105) that φ_t^s satisfies condition 3.1B′. We shall show that the functions φ_t^s can be selected in such a way that they satisfy 3.1A. We note that the above construction determines each function φ_t^s only up to an addition of a function, which is equal to zero outside some set A satisfying the condition $\mathbf{P}_{s,x}(A) = 0$ for all $x \in E$. By (105) we can select a sequence $h_k \downarrow 0$ such that for all $x \in E$

$$\mathbf{P}_{s,x}\left\{|\varphi_t^s - \varphi_t^s(h_k)| > \frac{1}{2^k}\right\} \leqq \frac{1}{2^k}. \tag{106}$$

Then by the Borel-Cantelli lemma

$$\varphi_t^s(h_k) \to \varphi_t^s \qquad \text{a.s. } \Omega, \mathbf{P}_{s,x}, \tag{107}$$

and we can let

$$\varphi_t^s = \begin{cases} \lim \varphi_t^s(h_k), & \text{if the limit exists,} \\ +\infty, & \text{otherwise.} \end{cases} \tag{108}$$

It is clear that φ_t^s is an almost nonnegative, almost additive functional, satisfying the inequality (105) and therefore also (93).

It follows from (105) that

$$\begin{aligned} \mathbf{M}_{s,x}[\varphi_t^s - \varphi_t^s(h)] &\leqq \{\mathbf{M}_{s,x}[\varphi_t^s - \varphi_t^s(h)]^2\}^{1/2} \\ &\leqq 3\{\beta(h)[\beta(t-s) + \beta(h)]\}^{1/2}. \end{aligned} \tag{109}$$

Comparing this formula with (102) we conclude that $m_t^s(x)$ is a characteristic of the functional φ_t^s.

There remains to check that the functional φ_t^s belongs to $W(X)$. Let $S = \{s = s_0 < s_1 < \cdots < s_n = t\}$ be a subdivision of the segment $[s, t]$. By lemma 1, inequality (102), and the fact that $m_t^s(x)$ is nonnegative we have

$$\begin{aligned} \mathbf{M}_{s,x}[\varphi_{s_{k+1}}^{s_k}(h)]^2 &= 2\mathbf{M}_{s,x}\int_{s_k}^{s_{k+1}} f_h(v, x_v)\, m_{s_{k+1}}^v(x_v; h)\,dv \\ &\leqq 2\mathbf{M}_{s,x}\int_{s_k}^{s_{k+1}} f_h(v, x_v)[m_{s_{k+1}}^v(x_v) + 2\beta(h)]\,dv \\ &\leqq 2[\beta(||S||) + 2\beta(h)]\mathbf{M}_{s,x}\int_{s_k}^{s_{k+1}} f_h(v, x_v)\,dv. \end{aligned} \tag{110}$$

From (110) and (102) we have

$$\begin{aligned} \mathbf{M}_{s,x}\sum_{k=0}^{n-1}[\varphi_{s_{k+1}}^{s_k}(h)]^2 &\leqq 2[\beta(||S||) + 2\beta(h)]m_t^s(x, h) \\ &\leqq 2[\beta(||S||) + 2\beta(h)][m_t^s(x) + 2\beta(h)]. \end{aligned} \tag{111}$$

Therefore,

$$\mathbf{M}_{s,x}\sum_{k=0}^{n-1}[\varphi_{s_{k+1}}^{s_k}]^2 \leqq 2\beta(||S||)\, m_t^s(x). \tag{112}$$

7.3. The characteristic of an almost additive functional is a function of three variables. With some additional conditions it can be expressed as a function of two variables. In fact, suppose that for any $s \in [0, T)$ and $x \in E$ there exists a.s. $\mathbf{P}_{s,x}$ the limit

(113) $$\varphi^s_{\zeta-0} = \lim_{t \uparrow \zeta} \varphi^s_t$$

and let $|\varphi^s_t|$ be majorized for all $t \in [s, T)$ by some $\mathbf{P}_{s,x}$-summable function. Then for any $0 \leqq s \leqq t \leqq u < T$,

(114) $$\lim_{u \uparrow T} \mathbf{M}_{s,x}\varphi^t_u = \mathbf{M}_{s,x}\varphi^t_{\zeta-0}.$$

Let

(115) $$m^s(x) = \mathbf{M}_{s,x}\varphi^s_{\zeta-0}.$$

By (114)

(116) $$\lim_{u \uparrow T} m^s_u(x) = m^s(x)$$

and from 7.2A we obtain the following expression for the characteristic of the functional φ^s_t in terms of the function $m^s(x)$,

(117) $$m^s_t(x) = m^s(x) - \mathbf{M}_{s,x}\, m^t(x_t).$$

It is not hard to show that every function $m^s_t(x)$ obtained from any function $m^s(x)$ by formula (117) satisfies condition 7.2A.

8. Survey of some results and problems concerning additive functionals and transformations of Markov processes

In what follows we consider only stationary Markov processes. We make use on several occasions of the notion of a strong Markov process. The definition of this concept will be found in [4], chapter 5.

8.1. *Random time change.* It is possible to connect with additive functionals a transformation of Markov processes having quite a different character from the transformations described in section 5.

Let $X = (x_t, \zeta, \mathfrak{M}^s_t, \mathbf{P}_{s,x})$ be a stationary Markov process and let φ^s_t be a continuous additive functional of X, satisfying the conditions

8.1A. for any $h \geqq 0$, $0 \leqq s \leqq t < T$, $\omega \in \Omega_{t+h}$,

(118) $$\theta_h\varphi^s_t(\omega) = \varphi^{s+h}_{t+h}(\omega);$$

8.1B. for any $0 \leqq s < t < T$, $\omega \in \Omega_t$,

(119) $$\varphi^s_t(\omega) > 0.$$

Condition 8.1A is a stronger variant of the stationary condition, while condition 8.1B is a stronger variant of the condition on nonnegativeness.

For every $\omega \in \Omega$ the function $\varphi^0_t(\omega)$, with $0 \leqq t < \zeta(\omega)$ is monotone increasing and continuous. The inverse function $\tau_t(\omega)$, with $0 \leqq t < \varphi^0_{\zeta-0}(\omega)$, is also continuous and monotone increasing. Under some general assumptions about the Markov process X it is possible to construct a Markov process $\tilde{X} = (\tilde{x}_t, \tilde{\zeta}, \tilde{\mathfrak{M}}^s_t, \tilde{\mathbf{P}}_{s,x})$ for which the state space and space of elementary events are the same as for the process X with

(120) $$\tilde{\zeta}(\omega) = \varphi^0_{\zeta-0}(\omega),$$

(121) $$\tilde{x}_t(\omega) = x_{\tau_t(\omega)}(\omega),$$

(122) $$\tilde{\mathfrak{M}}_t^s = \tilde{\mathfrak{N}}_t^s,$$

(123) $$\tilde{\mathbf{P}}_{0,x} = \mathbf{P}_{0,x},$$

where $\tilde{\mathfrak{N}}_t^s$ is defined with respect to $\tilde{x}_t$ in the same way as $\mathfrak{N}_t^s$ is defined with respect to x_t and the measures $\tilde{\mathbf{P}}_{s,x}$ for $s > 0$ are defined in terms of the measures $\tilde{\mathbf{P}}_{0,x} = \mathbf{P}_{0,x}$ by means of the formula $\tilde{\mathbf{P}}_{s,x}(\tilde{\theta}_s A) = \mathbf{P}_{0,x}(A)$, where the operators $\tilde{\theta}_s$ are constructed with respect to $\tilde{x}_t$ in the same way as the operators θ_s are constructed with respect to x_t. In 6.1 we defined the action of these operators θ_s on functions; their action on sets can be defined by the formula

(124) $$\theta_s A = \{\omega : \theta_s \chi_A = 1\}.$$

In fact it is sufficient to require that E be a metrizable, locally compact space, and that $\mathcal{B}$ be a σ-algebra of all Borel subsets of the space E, that X be a right-continuous strongly Markov process and that the σ-algebra $\mathfrak{M}_t^0$ contain every subset $A \subseteq \Omega_t$ such that for all $u > t$, we have $A \cap \Omega_u \in \mathfrak{N}_u^0$.

We say that the process $\tilde{X}$ is obtained from X by means of a random time change $\tau_t(\omega)$. Transformations of random time change are considered by Volkonsky in papers [9] and [11]. I am also informed from a private letter of K. Itô that random time change plays an important role in the monograph of Itô and H. P. McKean on the theory of diffusion processes that is to appear shortly.

8.2. *Transformations of infinitesimal operators.* To a stationary transition function $P(t, x, \Gamma)$ corresponds a family of operators

(125) $$T_t f(x) = \int_E P(t, x, dy) f(y),$$

in the space E of all bounded $\mathcal{B}$-measurable functions f. The infinitesimal operator of the transition function $P(t, x, \Gamma)$ is defined by the formula

(126) $$Af(x) = \lim_{t \downarrow 0} \frac{T_t f(x) - f(x)}{t}$$

for functions for which the limit of the right side exists uniformly with respect to $x \in E$. Sometimes it is more convenient to consider the weak infinitesimal operator which is also defined by formula (126), but instead of uniform convergence we require convergence for every x and boundedness of $[T_t f(x) - f(x)]/t$ for all $x \in E$ and $t \downarrow 0$.

Let E be a metrizable space, let $\mathcal{B}$ be a σ-algebra of all its Borel subsets. The transition function $P(t, x, \Gamma)$ in the space $(E, \mathcal{B})$ is called of Feller type if the operators T_t transform the space C of all continuous bounded functions into itself. For such transition functions it is natural to narrow down the domain of definition of the operator A by imposing the following conditions: $f \in C$, $Af \in C$. This restricted operator we shall call the *C-infinitesimal operator*.

Let X be a stationary Markov process, let $\alpha_t^s = \exp(-\varphi_t^s)$ be a stationary multiplicative functional of X. Then the transition function defined by (27) is stationary and its infinitesimal operator $\tilde{A}$ can be calculated from the formula

$$\tilde{A}f(x) = \lim_{t \downarrow 0} \frac{\mathbf{M}_{0,x}\, \alpha_t^s f(x_t) - f(x)}{t}. \tag{127}$$

How is this operator $\tilde{A}$ connected with the infinitesimal operator A of the process X? The answer to this question is known only for some special classes of functionals α_t^s.

In 1955, Dynkin [3] showed that if the transition function of the process X is Fellerian and stochastically continuous and if

$$\varphi_t^s = -\log \alpha_t^s = \int_s^t V(x_u)du, \tag{128}$$

where $V(x)$ is a continuous bounded function, then $\tilde{A} = A - V$, where V is the multiplication operator of the function V. Here we take for A and $\tilde{A}$ weak C-infinitesimal operators and say that the transition function $P(t, x, \Gamma)$ is stochastically continuous if $\lim_{t \downarrow 0} P(t, x, U) = 1$ for every neighborhood U of the point x.

Volkonsky [11] has shown that if φ_t^s is any stationary nonnegative continuous additive functional for which $\mathbf{M}_{0,x}\varphi_{\zeta-0}^0 < \infty$, then

$$\tilde{A} = A(E + S), \tag{129}$$

where

$$Sf(x) = \mathbf{M}_{0,x} \int_0^\zeta f(x_u)d\varphi_u^0. \tag{130}$$

The operator S can also be expressed in terms of the characteristics $m_t^s(x)$ of the functional φ_t^s. The above result is proved under the assumption that X is a standard continuous process, satisfying the condition $\sup_{x \in E}\mathbf{M}_{0,x}\zeta < \infty$. A process X is called standard if it satisfies the conditions shown after (129) as well as the condition of "quasi-continuity on the left," namely, if τ_n are quantities which do not depend on the future (or Markov times), then it follows from $\tau_n(\omega) \uparrow \tau(\omega)$, with $\omega \in \tilde{\Omega}$ that for every $x \in E$ the relation $x_{\tau n} \to x_\tau$, a.s. $\tilde{\Omega}$, $\mathbf{P}_{0,x}$ holds. This condition is satisfied for instance by all continuous processes.

If $\alpha_t^s = g(x_t)/g(x_s)$ where the functions $g(x)$ and $1/g(x)$ are bounded, then it follows readily from (30) that $\tilde{A}f = (1/g)A(gf)$.

It is known (the case $n = 1$ is considered for instance in [9]) that if X is an n-dimensional Wiener process and if

$$\varphi_t^s = \frac{1}{2}\int_s^t h^2(x_u)du + \int_s^t h(x_u)dx_u \tag{131}$$

then the operator $\tilde{A}$ for doubly continuously differentiable functions is given by the formula

$$\tilde{A}f(x) = Af(x) - h(x)\,\mathrm{grad}\,f(x), \tag{132}$$

where

$$Af = \frac{1}{2}\Delta f = \frac{1}{2}\sum \left[\frac{\partial^2 f}{\partial x_i}\right]^2. \tag{133}$$

The question as to how the infinitesimal operator changes with the random time change is considered by Volkonsky [9]. He proved in particular that if X is a right-continuous process of Feller type in a compact state space, then under a time change corresponding to an additive functional

$$\varphi_t^s = \int_s^t V(x_u)du, \tag{134}$$

where V is a continuous positive function, the C-infinitesimal operator of the process X is multiplied by the function $1/V(x)$.

8.3. *Regular excessive functions. Integral representation of excessive functions.* Let the state space $(E, \mathcal{B})$ satisfy conditions (a) and (b) of subsection 2.2. Denote by $\Omega(E)$ the totality of all functions $\varphi(t)$ with values in E, defined on all intervals $[0, \lambda)$. Let

$$\begin{aligned} \zeta(y) &= \lambda, \\ x_t(\varphi) &= \varphi(t), \qquad 0 \leqq t < \zeta(\varphi). \end{aligned} \tag{135}$$

Denote by $\mathfrak{N}^s$ the σ-algebra of the space $\Omega(E)$ generated by $\{\varphi: x_u(\varphi) \in \Gamma\}$, where $\Gamma \in \mathcal{B}$, $u \geqq s$, and by $\mathfrak{N}_t^s$ the σ-algebra of the space $\Omega_t(E) = \{\varphi: \zeta(\varphi) > t\}$, generated by $\{\varphi: x_u(\varphi) \in \Gamma\}$ where $\Gamma \in \mathcal{B}$, $u \in [s, t]$. It is shown (see [4], theorems 4.2 and 2.6) that to every normal transition function in the space $(E, \mathcal{B})$ corresponds uniquely the Markov process $X = (x_t, \zeta, \mathfrak{N}_t^s, P_{s,x})$ for which $\Omega(E)$ is the space of elementary events; x_t and ζ are defined by formula (135) and the measures $\mathbf{P}_{s,x}$ are given over the σ-algebra $\mathfrak{N}^s$. We call this Markov process *canonical.* If the transition function is stationary, then the corresponding canonical process is also stationary.

Suppose that φ and ψ belong to $\Omega(E)$. We shall say that ψ is a contraction of φ, if $\zeta(\psi) \leqq \zeta(\varphi)$ and $\psi(t) = \varphi(t)$ for $0 \leqq t < \zeta(\psi)$. We shall call the subset $A \subseteq \Omega(E)$ *complete* if together with every element φ every contraction of φ is contained in A.

Let $P(t, x, \Gamma)$ be a stationary transition function in the state space $(E, \mathcal{B})$, let T_t be the operators defined by (125). The nonnegative function $f(x)$ is called excessive with respect to $P(t, x, \Gamma)$ if

8.3A1. $T_tf(x) \leqq f(x)$ for all $t \geqq 0$, $x \in E$.

8.3A2. $T_tf(x) \uparrow f(x)$ for $t \downarrow 0$.

Let the positive function $f(x)$ be excessive with respect to the normal transition function $P(t, x, \Gamma)$. Then

$$P^f(t, x, \Gamma) = \frac{1}{f(x)} \int_\Gamma P(t, x, dy) f(y) \tag{136}$$

is again some normal transition function in the space $(E, \mathcal{B})$. Let $X = (x_t, \zeta, \mathfrak{N}_t^s, \mathbf{P}_{s,x})$ and $X^f = (x_t, \zeta, \mathfrak{N}_t^s, \mathbf{P}_{s,x}^f)$ be two canonical Markov processes corresponding to the transition functions $P(t, x, \Gamma)$ and $P^f(t, x, \Gamma)$. We shall call the function f regular if the following condition is satisfied.

8.3B. If A is a complete subset of the space $\Omega(E)$ and if for some s and for all

$x \in E$ we have $P^{f*}_{s,x}(A) = 1$, then $P^{f*}_{s,x}(A) = 1$. By $P^*_{s,x}$ and $P^{f*}_{s,x}$ we denote the outer measures corresponding to the measures $P_{s,x}$ and $P^f_{s,x}$.

In order that there exist a Markov process with the transition function $P(t, x, \Gamma)$ and a set of trajectories $A \subseteq \Omega(E)$, it is necessary and sufficient that $P^*_{s,x}(A) = 1$ for all $s \geqq 0$ and $x \in E$. Therefore condition 7.3B means that every complete set A which can serve as a set of all trajectories for a process with the transition function $P(t, x, \Gamma)$ can also serve as a set of all trajectories for a process with the transition function $P^f(t, x, \Gamma)$.

The question of regularity of the excessive function f is closely connected with the question of the possibility of representing this function as a mathematical expectation. We shall say that the $\mathfrak{N}^0$-measurable nonnegative function $\xi(\omega)$ with $\omega \in \Omega_0$ is an *excessive random variable* of a stationary Markov process X if

8.3C1. $\theta_t\xi \leqq \xi$ for all $t \geqq 0, \omega \in \Omega_0$.

8.3C2. $\lim_{t \downarrow 0} \theta_t\xi = \xi$.

By (7), if ξ is an excessive random variable for X, then

$$f(x) = \mathbf{M}_{0,x}\xi \tag{137}$$

is an excessive function with respect to the transition function $P(t, x, \Gamma)$.

We note that the possibility of representing the function $f(x)$ in the form (137) depends only on $P(t, x, \Gamma)$, but does not depend on the choice of the process X having the transition function $P(t, x, \Gamma)$. This follows readily from the results of section 3 of chapter 2 of [4], where a description will be found of the class of all Markov processes corresponding to the same transition function. We shall show that the representability of the function $0 < f(x) < +\infty$ by the form (137) is sufficient for the function to be regular. In fact let A be any complete subset of the space Ω_E which can serve as the set of trajectories of a process with the transition function $P(t, x, \Gamma)$. Let X be such a process. Consider the representation (137) for the process X. The system of functions

$$\eta_t = \theta_t\xi, \qquad 0 \leqq t < \zeta(\omega) \tag{138}$$

satisfies conditions 5.4A to 5.4D. Let

$$f(x) = \mathbf{M}_{t,x}\eta_t = \mathbf{M}_{0,x}, \quad \alpha^s_t = \frac{f(x_t)}{f(x_s)}, \quad \xi_t = \frac{\eta_t}{f(x_t)}. \tag{139}$$

By 5.4, the (α^s_t, ξ_t)-subprocess $\tilde{X}$ of the process X has the transition function $P^f(t, x, \Gamma)$. But, obviously, the set of all trajectories of $\tilde{X}$ coincides with A.

One criterion for the representation of an excessive function by means of an excessive random variable follows from theorem 7. In fact, let

$$\sup_{x \in E} |T_t f(x) - f(x)| \to 0 \qquad \text{for } t \downarrow 0. \tag{140}$$

Then the function

$$m^s_t(x) = f(x) - T_{t-s}f(x) \tag{141}$$

satisfies conditions 7.2A to 7.2C, and by the corollary to theorem 7

(142) $$m_t^s(x) = \mathbf{M}_{s,x}\varphi_t^s,$$

where φ_t^s is some nonnegative additive functional of the process X. It follows from (141) and (142) with $s = 0$ that

(143) $$\mathbf{M}_{0,x}\varphi_t^0 = f(x) - T_t f(x) = f(x) - \mathbf{M}_{0,x}f(x_t).$$

As is known (see, for example, [2]) there exists a.s. $\mathbf{P}_{s,x}$ the limit

(144) $$\lim_{t\uparrow\infty} f(x_t) = \gamma(\omega).$$

This follows from a general theorem about semimartingales ([1], theorem 4.1). In fact if we let

(145) $$\xi_t = \begin{cases} f(x_t) & 0 \leqq t < \zeta(u), \\ 0, & t \geqq \zeta(\omega), \end{cases}$$

then it can be easily checked that for any $0 \leqq s \leqq t$,

(146) $$M_{0,x}\{\xi_t | \mathfrak{M}_s^0\} = \xi_s, \qquad \text{a.s. } \Omega, \mathbf{P}_{0,x}.$$

Therefore $(\xi_t, \mathfrak{M}_t^0)$ is a semimartingale.

From formula (143) with $t \uparrow \infty$, we have $f(x) = \mathbf{M}_{0,x}\xi$, where $\xi = \gamma + \varphi_{\zeta-0}^0$. It is not hard to check that this last formula defines an excessive random variable.

Hence condition (140) is sufficient for the representation of the excessive function f in the form (137) and therefore for the regularity of f. Every function satisfying condition (140) is bounded. L. V. Seregin found sufficient conditions of regularity applicable to nonbounded functions. These conditions are close to being necessary and make possible the construction of large classes of nonregular excessive functions (all these functions are unbounded).

8.4. *Transformations of one-dimensional regular diffusion processes.* Let $E = [r_1, r_2]$ be a line segment, let $\mathcal{B}$ be a σ-algebra of all Borel subsets of E. Every stationary continuous strongly Markov process in the state space $(E, \mathcal{B})$ is called a diffusion process. A diffusion process is called regular if for any points x, y of E

(147) $$\mathbf{P}_{0,x}\{x_t = y \downarrow \text{ for some } t \in [0, \zeta(\omega))\} = 1.$$

Let $\varphi(x)$, with $r_1 \leqq x \leqq r_2$ be an arbitrary continuous increasing function, let $\varphi_{-1}(y)$, with $\varphi(r_1) \leqq y \leqq \varphi(r_2)$ be the inverse function. Then, as can be easily seen, $X' = [\varphi(x_t), \zeta, \mathfrak{M}_t^s, \mathbf{P}_{s,\varphi_{-1}(x)}]$ defines a regular diffusion process in the segment $[\varphi(r_1), \varphi(r_2)]$. It is not hard to check that the class of one-dimensional regular diffusion processes is invariant also with respect to a random time change corresponding to an arbitrary continuous additive functional satisfying conditions 8.1A and 8.1B. Volkonsky [10] showed that with the help of these two transformations every regular diffusion process with infinite lifetime can be obtained from a Wiener process on the segment [0, 1] with reflecting screens at the points 0 and 1.

In order to obtain all processes with arbitrary lifetimes it is necessary to add to the two types of transformations above some of the other transformations described in section 5. As was shown in [7], it is sufficient to consider the (α_t^s, ξ_t)-subprocesses where α_t^s and ξ_t are defined in terms of some excessive random variable ξ by the formulas

$$\alpha_t^s = \frac{\mathbf{M}_{0,x_t}\xi}{\mathbf{M}_{0,x_s}\xi}, \qquad \xi_t = \frac{\theta_t\xi}{\mathbf{M}_{0,x_t}\xi}. \tag{148}$$

See 5.4 and (139).

Another class of additional transformations that is sufficient for obtaining all regular diffusion processes with arbitrary lifetimes from the regular diffusion processes with infinite lifetimes is proposed by Volkonsky [11] (see also [12]). Volkonsky has shown that such a class forms the α_t^s-subprocesses, where α_t^s is a multiplicative functional, satisfying conditions 5.3A–5.3B.

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