A STOCHASTIC TREATMENT OF SOME CLASSICAL INTERPOLATION PROBLEMS

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1. The nonstochastic theory

The interpolation problems to be discussed in this paper arise in the theory of approximation by rational functions in the complex domain. The problems are connected with the following basic question, here stated rather informally. Let B denote a bounded point set of the complex z-plane, let the function f be given on the boundary of B, and let $\{S_n : z_{1n}, z_{2n}, \cdots, z_{nn}\}$ be a sequence of point sets chosen somehow on the boundary of B. Let $L_n = L_n(f; z)$ denote the polynomial in z of degree at most $n - 1$ which is found by interpolation to the values of f at the points S_n. Under what circumstances will $\lim_{n\to\infty} L_n$ exist, and when it does what will the limit be? Preferably of course it will be related in some way to f.

If B is the unit disk, the question becomes one of a special kind of trigonometric interpolation, but not of a type which has been studied intensively as such. If B is the real interval $-1 \leqq z \leqq 1$ the question involves interpolation by real polynomials, or trigonometric interpolation by cosine polynomials. Such problems have been thoroughly investigated over the last fifty years (see Zygmund [1], chapter 10). Attention in the general complex case has been centered on convergence at interior points of B rather than on the boundary of B where the interpolation points are placed, and the required techniques appear to be quite different from those useful in the purely real case. It is the complex case with which this paper is solely concerned.

The history of the complex case might be said to go back to Méray [2], who in 1884 came up with a slightly disturbing example. He pointed out that if B is the unit disk, and S_n consists of the nth roots of unity, and $f(z) = 1/z$, then $L_n(f; z) = z^{n-1}$. This L_n has the limit zero for $|z| < 1$, and elsewhere diverges except at $z = 1$, where it equals the corresponding value of f for all n. Except at $z = 1$ the limit, where it exists, seems to bear little relation to the function to which L_n interpolates. However, later work showed that if the boundary of B consists of one or more rectifiable curves, then what one should be looking for is convergence to the function

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$$F(z) = \frac{1}{2\pi i} \int \frac{f(t)}{t - z}\, dt \tag{1.1}$$

at interior points of B, where the integral is extended over the boundary of B. In such cases, if f is analytic at interior points of B and continuous in the corresponding closed region, or set of regions, then (1.1) is the familiar Cauchy integral formula, and $F(z) = f(z)$ at interior points of B. In the Méray case, a simple calculation shows that $F(z) = 0$ for $|z| < 1$.

It turns out that in the classical theory of the complex case, a key role is played by the concept of "uniformly distributed" or "equidistributed" interpolation points, where here a distribution means an asymptotic relative frequency distribution, not a probability distribution. It is the purpose of this paper to cast further light on this role through the use of the techniques of probability theory.

The concept of an equidistribution was introduced by Hermann Weyl [3] in 1916. There is an easily accessible treatment in Pólya and Szegö's *Aufgaben und Lehrsätze* ([4], pp. 67–77 and 230–242). For present purposes it is convenient to give the definition as follows: Let $\{\sigma_n : \theta_{1n}, \theta_{2n}, \cdots, \theta_{nn}\}$ be a sequence of sets of real numbers lying on the closed interval $[0, 2\pi]$ and let $N_n(\theta)$ be the number of the numbers σ_n lying on the closed interval $[0, \theta]$. If for each θ,

$$\lim_{n\to\infty} \frac{N_n(\theta)}{n} = \frac{\theta}{2\pi}, \tag{1.2}$$

then the sequence $\{\sigma_n\}$ is equidistributed on $[0, 2\pi]$ and the corresponding sequence, $\{z_{1n}, z_{2n}, \cdots, z_{nn}\}$, where $z_{kn} = \exp(i\theta_{kn})$, is equidistributed on the circle $\gamma : |z| = 1$.

The simplest example of an equidistribution on γ is given by the sequence of nth roots of unity, for which z_{kn} is $\exp(2\pi ik/n)$, for $k = 1, \cdots, n$. Another classical example is $\{\xi, \xi^2, \cdots, \xi^n\}$, where $|\xi| = 1$ but ξ is not a root of unity.

We henceforth impose the restriction that B shall be a simply connected region of the finite z-plane bounded by a Jordan curve Γ. (A Jordan curve is homeomorphic to a circle.) There are generalizations of some of the deterministic and probabilistic developments of the sequel to regions with more general boundaries, and to point sets consisting of several mutually exterior regions.

The particular extension to a general Jordan curve Γ of the concept of equidistribution on the unit circle which has usually turned out to be appropriate in the problems here under consideration is as follows. Let the analytic function

$$z = \phi(w) = cw + c_0 + \frac{c_1}{w} + \frac{c_2}{w^2} + \cdots, \qquad |w| \geqq 1, \tag{1.3}$$

map the exterior K of Γ onto the region $|w| > 1$ in the complex w-plane conformally and in a one-to-one manner so that the respective points at infinity in the two regions correspond. Then we shall say that the sequence $\{S_n : z_{1n}, z_{2n}, \cdots, z_{nn}\}$ is equidistributed on Γ if and only if the image sequence $\{w_{1n}, w_{2n}, \cdots, w_{nn}\}$ is equidistributed on $\gamma : |w| = 1$, where $z_{kn} = \phi(w_{kn})$.

The theory for the case in which f is analytic on $B + \Gamma$, and therefore on a larger region containing $B + \Gamma$, was worked out first. It is made accessible by the availability of the so-called Cauchy-Hermite formula which we shall display below in section 2 (2.1). The class of functions analytic on a given closed Jordan region $B + \Gamma$ will henceforth be designated by the symbol $(A)_\Gamma$, with Γ replaced by γ if Γ is the unit circle. This is of course a subclass of the class of all functions defined and continuous on Γ, which we denote by $(C)_\Gamma$ or $(C)_\gamma$. The principal classical result for functions in $(A)_\Gamma$ is

THEOREM 1.1. *A necessary and sufficient condition that*

$$\lim_{n\to\infty} L_n(f; z) = f(z), \qquad z \text{ on } B, \tag{1.4}$$

for every $f \in (A)_\Gamma$, *is that the sequence* $\{S_n\}$ *of sets of interpolating points be equidistributed on* Γ. *If* $\{S_n\}$ *satisfies this condition, then*

$$L_n(f; z) - f(z) = O\left(\frac{1}{R^n}\right) \tag{1.5}$$

on and inside each level curve $\Gamma_R : z = \phi[R \exp(i\theta)]$, *R fixed*, $R > 1$, $0 \leqq \theta \leqq 2\pi$, *which does not contain or pass through a singularity of* f. *If convergence for all* $f \in (A)_\Gamma$, *or indeed just for all analytic functions* f *of the type* $1/(a - z)$, *a exterior to* Γ, *is known to take place at only a single point* z_0 *of* B, *then it must take place at all points of* $B + \Gamma$ *and the points* S_n *must be equidistributed on* Γ.

Runge [5] published in 1904 a proof that convergence takes place inside the unit circle for all $f \in (A)_\gamma$ when S_n is the set of nth roots of unity. Fejér [6] announced the general sufficiency condition as given above in 1918; Kalmar [7] the necessity in 1926. The statement in the last sentence was proved by Curtiss [8] in 1941. Walsh and Szegö have established various generalizations. (The standard reference work on complex polynomial approximation in general, and complex interpolation in particular, is Walsh's Colloquium volume [9], which has a bibliography complete to 1956.)

The convergence problem for functions known only to be in class $(C)_\Gamma$ is more delicate, and a number of questions still remain open, some of which motivated the present investigation. The following statement partially summarizes the facts which have so far been established.

THEOREM 1.2. *If* Γ *is rectifiable, then a necessary and sufficient condition that* $\lim_{n\to\infty} L_n(f; z) = F(z)$ *at a single point* z_0 *of* B, *for every* $f \in (C)_\Gamma$, *where* $F(z)$ *is given by* (1.1), *is that both of the following relations shall hold true:*

$$\lim_{n\to\infty} L_n[(z - a)^{-k}; z_0] = 0, \qquad k = 1, 2, \cdots, \tag{1.6}$$

where a is an arbitrary fixed point on B; and there exists a number $M > 0$ *independent of n such that*

$$\sum_1^n \left| \frac{\omega_n(z_0)}{(z_0 - z_{kn})\omega_n'(z_{kn})} \right| < M, \qquad n = 1, 2, \cdots, \tag{1.7}$$

where $\omega_n(z) = (z - z_{1n})(z - z_{2n}) \cdots (z - z_{kn})$. *Convergence is uniform on any subset of B on which these conditions hold uniformly.*

If Γ is rectifiable and S_n is the transform under (1.3) *of the nth roots of unity, then* (1.6) *holds true uniformly for z_0 on any closed subset of B, and if $\phi''(w)$ is nonvanishing and of bounded variation for $|w| = 1$, then both* (1.6) *and* (1.7) *hold uniformly for z_0 on any closed subset of B.*

The first part of the theorem was only recently announced [10]. The basic theory for the sufficient conditions involving the roots of unity was published in 1935 [11] with improvements in 1941 [12]. Earlier, Walsh had proved convergence for the case in which Γ is the unit circle and S_n consists of the nth roots of unity (see [9], p. 179).

A class of functions contained in $(C)_\Gamma$ but containing $(A)_\Gamma$ is that of the functions which are continuous on $\Gamma + B$ and analytic on B. We shall call this the class $(CA)_\Gamma$, or $(CA)_\gamma$ when Γ is the unit circle. *For these functions it turns out* ([10], theorem 4) *that* (1.7) *above is a sufficient condition for convergence of $\{L_n\}$ on B.* Fejér [6] in 1918 established the convergence inside the unit circle when S_n consists of the nth roots of unity.

The implication of the italicized statement in the preceding paragraph is that if (1.7) holds at just one point z_0 of B, the sequence of interpolation points $\{S_n\}$ must be equidistributed on Γ, because $(CA)_\Gamma$ contains $(A)_\Gamma$ and the last sentence of theorem 1.1 becomes applicable. Yet in a broad sense just what is the role of equidistribution in the convergence process? In the $(A)_\Gamma$ case, it is clear from theorem (1.1) that the role is a crucial one, but the relevance (if any) of equidistribution is not evident in interpolation to less heavily restricted classes of functions. It is possible (see [13]) to construct sequences of interpolation points equidistributed on the unit circle γ for which neither (1.6) nor (1.7) is valid, and yet such that for functions f of class $(CA)_\gamma$ satisfying a light restriction on their boundary values the convergence of $\{L_n\}$ to $f(z)$ takes place everywhere inside γ. One such construction involves merely adjoining a single point ξ, with $|\xi| = 1$ but not a root of unity, to the nth roots of unity for each n. The failure of formula (1.6) to hold true means in particular that $\{L_n(1/z; z_0)\}$ does not converge to the "right" value. It seems strange that the convergence process should be so delicate, even for a function like $1/z$ which is analytic on γ, that this process can be upset by just adding another interpolation point at each stage. Some of the questions now open relate to whether equidistribution is always sufficient for convergence if $f \in (CA)_\Gamma$, and to what restriction on an equidistribution will make (1.3) and (1.4) hold true.

The $(CA)_\Gamma$ question was posed to the author by the late Professor Aurel Wintner some years ago, and now seems to be of considerable interest because of its bearing on parallel and unresolved convergence problems concerning the convergence of harmonic interpolating polynomials.

There is of course a close relationship between numerical equidistributions of the type we have been discussing here and probability distributions. The study on which we shall now report was motivated primarily by the hope that the methods of probability theory might cast some light on a few of the unresolved questions concerning the role of equidistribution in the deterministic theory. It was undertaken also with the idea that it might perhaps be helpful in studying other function-theoretic interpolation problems. A final motivation, of course, was just curiosity.

2. The stochastic formulation and the principal new results

As in section 1, we shall be dealing with a function f given on a Jordan curve Γ of the complex z-plane. The interior of Γ will be denoted by B and the exterior of Γ will be denoted by K. We let $L_n(f: z)$ again denote the polynomial in z of degree at most $n - 1$ which interpolates to f in n designated points S_n lying on Γ. But now we shall postulate that the interpolation points are random variables defined on a certain probability space, and this means that $L_n(f; z)$ will be a random variable too. It seems to serve no useful purpose to continue to make a distinction between the interpolation points in the set S_n and the first n interpolation points in any successive set S_{n+k}, so our basic set of interpolation points will be taken to be the infinite family of random variables $z_1, z_2, \cdots$, with S_n designating the first n of them.

We assign the probability measure as follows: Let $\theta_1, \theta_2, \cdots$, be an infinite family of mutually independent random variables, and let θ_k have a uniform (that is, rectangular) marginal probability distribution on the real interval $[0, 2\pi]$. If Γ is a circle of radius R, we define z_k by $z_k = R \exp (i\theta_k)$, for $k = 1, 2, \cdots$. More generally, if Γ is any Jordan curve, we let z_k be given by $z_k = \phi\,[\exp (i\theta_k)]$, for $k = 1, 2, \cdots$, where ϕ is the mapping function (1.3) discussed above.

The interpolation points z_k will thus be mutually independent. This is a more restrictive condition than will be needed for some of the developments below, but in this first treatment it does not seem inappropriate.

The question of possible coincidences in a given sample sequence $z_1^*, z_2^*, \cdots$ of the process $\{z_n\}$ should receive passing attention. First we shall review briefly some of the formulas for $L_n(f; z)$.

If $f \in (A)_\Gamma$ the following formula, known as the Cauchy-Hermite formula, is available:

$$L_n(f; z) = \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(t)}{t - z} \left[1 - \frac{\omega_n(z)}{\omega_n(t)}\right] dt, \tag{2.1}$$

where

$$\omega_n(z) = (z - z_1)(z - z_2) \cdots (z - z_n) \tag{2.2}$$

and Γ' is a suitably chosen rectifiable Jordan curve containing $B + \Gamma$ in its interior. If f is analytic on Γ but not throughout B then (2.1) is still valid when the path of integration consists not only of the exterior curve Γ' but also of another suitably chosen curve Γ'' lying in B. If f is analytic on $\Gamma + K$, where K is the region exterior to Γ, then we have recourse to a slightly unfamiliar form of the Cauchy integral formula (Osgood, in [14], Vol. 1, p. 344, ascribes the theorem to D. R. Curtiss):

$$f(z) = f(\infty) + \frac{1}{2\pi i} \int_{\Gamma''} \frac{f(t)}{t - z}\, dt, \qquad z \text{ exterior to } \Gamma'', \tag{2.3}$$

where Γ'' is suitably chosen on B. This formula implies that the following representation for $L_n(f; z)$ is valid for all z when f is analytic on $\Gamma + K$:

$$L_n(f; z) = \frac{1}{2\pi i} \int_{\Gamma''} \frac{f(t)}{t - z}\left[1 - \frac{\omega_n(z)}{\omega_n(t)}\right] dt + f(\infty). \tag{2.4}$$

A special case of these integral formulas in which the integrals can be evaluated by residues, is

$$L_n\left[\frac{1}{a - z}; z\right] = \frac{1}{a - z}\left[1 - \frac{\omega_n(z)}{\omega_n(a)}\right], \qquad z \neq a. \tag{2.5}$$

The value of L_n at $z = a$ is $\omega_n'(a)/\omega_n(a)$. Finally for an f unrestricted except for finiteness at $z_1, z_2, \cdots$, we have the so-called Lagrange interpolation polynomial

$$L_n(f; z) = \sum_{k=1}^{n} f(z_k) \frac{\omega_n(z)}{(z - z_k)\omega_n'(z_k)}. \tag{2.6}$$

There are many equivalents of (2.6) which can be written down by using divided differences.

Now the Cauchy-Hermite formula (2.1) and its variations (2.4) and (2.5) automatically make provision for coincidences among the points of interpolation. The polynomial L_n as so represented takes on the value of f and of its first $(k - 1)$ derivatives at a coincidence of k points. The same is not true of the Lagrange interpolation formula (2.6), which is formally meaningless in case of a coincidence because of the presence of $\omega_n'(z_k)$ in the denominator of the summand. But in our stochastic model, the probability of a coincidence of any multiplicity is always zero. Thus, for any function in $(C)_\Gamma$, we can say that (2.6) defines the interpolation polynomial L_n with probability one, and with these remarks the question of coincidences will receive no further attention.

We shall now summarize the principal results concerning the stochastic convergence of the process $\{L_n(f; z)\}$.

THEOREM 2.1. *If $f \in (A)_\Gamma$, then*

$$\lim_{n \to \infty} L_n(f; z) = f(z), \qquad z \text{ on } B, \tag{2.7}$$

with probability one. In fact, with probability one a sample sequence of interpolation points will cause uniform convergence of $\{L_n\}$ to f for z on and inside each level curve Γ_R: $z = \phi[R \exp(i\theta)]$, $0 \leqq \theta \leqq 2\pi$, R fixed, $R > 1$, which does not contain or pass through a singularity of f.

THEOREM 2.2. *For the function $f(z) = 1/(a - z)$, with a on B, the following relations hold at each z on B, where $z \neq a$.*

(a) *For any $\epsilon > 0$*

$$\lim_{n \to \infty} P\{|L_n(f; z) - f(z)| < \epsilon\} = \frac{1}{2}. \tag{2.8}$$

(b) *If* $g(z) \neq f(z)$ *and if* $\eta < |g(z) - f(z)|$, *then*

$$\lim_{n\to\infty} P\{|L_n(f; z) - g(z)| < \eta\} = 0. \tag{2.9}$$

(c) *Given any* $M > 0$, *however large,*

$$\lim_{n\to\infty} P\{|L_n(f; z) - f(z)| > M\} = \frac{1}{2}. \tag{2.10}$$

(d) *Given any* $M > 0$, *however large,*

$$\lim_{n\to\infty} P\{|L_n(f; a)| > M\} = 1. \tag{2.11}$$

Part (a) of theorem 2.2 seems surprising, partly because of the following fact. Let Γ' be any rectifiable Jordan curve lying inside B and containing the point a in its interior. Let λ be the length of Γ'. Then with any sample sequence of interpolation points z_1^*, z_2^*, $\cdots$, it is impossible to have the inequality $|L_n^*(f; z) - f(z)| < \epsilon$ holding everywhere on Γ' if $\epsilon < 1/\lambda$. Here $f(z)$ is still $1/(a - z)$ and $\{L_n^*\}$ is the sample sequence of interpolating polynomials corresponding to $\{z_n^*\}$. The reason is that $\int_{\Gamma'} L_n^*(f; z)\, dz = 0$ by the Cauchy integral theorem; also $\int_{\Gamma'} [1/(a - z)]\, dz = -2\pi i$, and if the inequality were to hold everywhere on Γ' we would have the contradiction:

$$\begin{aligned} 2\pi &= \left| \int_{\Gamma'} [L_n^*(f; z) - f(z)]\, dz \right| \\ &\leqq \int_{\Gamma'} |L_n^*(f; z) - f(z)|\, |dz| < \epsilon\lambda < 1. \end{aligned} \tag{2.12}$$

But still the theorem says that for a fixed large n and a fixed z, approximately fifty per cent of the sample sequences of interpolation points will cause the value of L_n^* to fall arbitrarily close to that of f.

Part (b) merely provides further evidence on how sensitive the convergence of our interpolation procedure is to the spacing of points, even for functions analytic on Γ. It will be recalled that theorem 1.2 states that with $g(z) = 0$, $\{L_n[1/(a - z); z]\}$ *does* converge to $g(z)$ when L_n is found by interpolation to $1/(a - z)$ in the nth roots of unity and Γ is rectifiable. Our present stochastic theory is easily modified so as to admit the nth roots of unity as a possible sample sequence. It would now appear that there are "almost no other" equidistributed sequences for which $\{L_n\}$ will behave properly. Analogous results can be established for more general functions f for which the representation (2.3) is valid.

In the following theorems, E denotes the expected value operator.

THEOREM 2.3. *If Γ is a circle with center at z_0, and if f belongs to $(C)_\Gamma$ and its* $(n - 1)$st *derivative is bounded in absolute value on Γ, then for all z*

$$E[L_n(f; z)] = a_0 + a_1(z - z_0) + \cdots + a_{n-1}(z - z_0)^{n-1}, \tag{2.13}$$

where $\sum_0^\infty a_k(z - z_0)^k$ is the Taylor expansion of the analytic function

$$(2.14) \qquad F(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t - z}\, dt, \qquad z \text{ on } B,$$

around the point z_0. *Thus* $\lim_{n\to\infty} E[L_n(f; z)]$ *exists and equals* $F(z)$ *for all* z *on* B, *and the limit also exists on* Γ *itself. If* $f \in (A)_\Gamma$, *then the limit exists and equals* $f(z)$ *on each closed disk with center at* z_0 *which does not contain a singularity of* f.

THEOREM 2.4. *If* f *is analytic on* $\Gamma + K$, *where* Γ *is an arbitrary Jordan curve and* K *is the region exterior to* Γ, *then* $E[L_n(f; z] = f(\infty)$ *for all* z.

THEOREM 2.5. *For any* Γ *there exists a smallest number* $\overline{R} = \overline{R}(\Gamma)$, *where* $1 \leqq \overline{R} < \infty$,*with the following property. For all* f *analytic on and interior to the level curve* $\Gamma_{\overline{R}}$: $z = \phi[\overline{R} \exp(i\theta)]$, $0 \leqq \theta \leqq 2\pi$, $\lim_{n\to\infty} E[L_n(f; z)] = f(z)$ *for every* z *on* $\Gamma + B$, *uniformly for* z *on any closed subset of* B. *If* $\overline{R} > 1$, *there exist functions* $f \in (A)_\Gamma$ *such that* $\{E[L_n(f; z)]\}$ *diverges to infinity on a subset of* B.

The significance of theorem 2.5 is that the process $\{L_n\}$ will be asymptotically unbiased only for a subclass of $(A)_\Gamma$ characterized by having singularities sufficiently distant from Γ. The required distance is never infinite. Theorem 2.3 shows that if Γ is a circle, then $\overline{R} = 1$ and there is no restriction on the position of the singularities of a function $f \in (A)_\gamma$. But if Γ is an ellipse, it turns out that $\overline{R} > 1$, and this in fact is the general situation. A somewhat more explicit statement of the result in the last sentence of the theorem is that if $\overline{R} > 1$, there exist functions $f \in (A)_\Gamma$ analytic on and interior to any level curve given by $z = \phi[R \exp(i\theta)]$, $0 \leqq \theta \leqq 2\pi$, R fixed, $1 \leqq R < \overline{R}$ such that the sequence of mean values of L_n diverges to infinity on a subset of B.

THEOREM 2.6. *If* f *is analytic on some closed disk of radius* R *containing* Γ, *and with center at a point* z_0 *on* B, *and if the random interpolation points on* Γ *are the points* $\chi[\exp(i\theta_k)]$, *for* $k = 1, 2, \cdots$, *where* $z = \chi(w)$ *maps* $B + \Gamma$ *conformally onto* $|w| \leqq 1$ *so that the point* $w = 0$ *corresponds to* $z = z_0$, *then*

$$(2.15) \qquad E[L_n(f; z)] = \sum_{0}^{n-1} a_k(z - z_0)^{k-1},$$

where the numbers a_k *are the coefficients of the Taylor expansion of* f *about* z_0. *Thus*

$$(2.16) \qquad \lim_{n\to\infty} E[L_n(f; z)] = f(z), \qquad |z - z_0| \leqq R.$$

THEOREM 2.7. *If* Γ *is a circle of radius* R *and* f *is analytic on and inside a concentric circle of radius* $2R$, *then*

$$(2.17) \qquad \lim_{n\to\infty} E[|L_n(f; z) - f(z)|^2] = 0$$

uniformly for $|z| \leqq R$. *There are functions* $f \in (A)_\Gamma$ *for which this expected value diverges to infinity for* z *on some subset of* Γ. *For all functions of the type* $1/(a - z)$, a *on* B, *this expected value diverges to infinity everywhere.*

Theorem 2.7 seems to indicate that stochastic convergence in the mean is not a useful type of stochastic convergence to study in connection with the process $\{L_n(f; z)\}$, and so no attempt was made to generalize to noncircular regions.

3. Indications of proofs, and further results

Theorem 2.1 is a translation into probability language of the sufficient condition in theorem 1.1, and thus really gives us no new information. If a sample sequence of interpolation points is equidistributed on Γ, then the classical theory indicates that the corresponding sequence $\{L_n^*\}$ has the convergence property stated in theorem 1.1. But by the Glivenko-Cantelli theorem ([15], pp. 20–21), the sample sequences of our sequence $z_1, z_2, \cdots$ of random interpolation points have the equidistribution property with probability one, so the indicated convergence of the stochastic process $\{L_n\}$ takes place with probability one.

Theorem 2.2, parts (a), (b), and (c), depend on the following result, in which as usual $\omega_n(z) = \prod_1^n (z - z_k)$.

LEMMA 3.1. *Given any two real numbers m_1 and m_2, with $0 < m_1 < m_2$, if z and a lie on B,*

$$\text{(a)} \qquad \lim_{n\to\infty} P\left\{\left|\frac{\omega_n(z)}{\omega_n(a)}\right| < m_1\right\} = \frac{1}{2},$$

$$\text{(b)} \qquad \lim_{n\to\infty} P\left\{m_1 \leqq \left|\frac{\omega_n(z)}{\omega_n(a)}\right| \leqq m_2\right\} = 0,$$

$$\text{(c)} \qquad \lim_{n\to\infty} P\left\{\left|\frac{\omega_n(z)}{\omega_n(a)}\right| > m_2\right\} = \frac{1}{2},$$

$$\text{(d)} \qquad \lim_{n\to\infty} P\left\{\left|\frac{\omega'(a)}{\omega(a)}\right| > m_2\right\} = 1.$$

For the proof of parts (a) to (c), we first observe that if θ is a random variable with a uniform distribution on $[0, 2\pi]$, then

$$\text{(3.1)} \qquad \begin{aligned} E[\log|z - \phi(e^{i\theta})|] &= \frac{1}{2\pi}\int_0^{2\pi} \log|z - \phi(e^{i\theta})|\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{z - \phi(e^{i\theta})}{ce^{i\theta}}\right| d\theta + \log|c|, \end{aligned}$$

where of course ϕ is given by (1.3). Now if z lies on B, then the function

$$\text{(3.2)} \qquad \frac{z - \phi(w)}{cw} = \frac{z}{cw} - 1 - \frac{c_0}{w} - \frac{c_1}{w^2} -$$

is continuous for $|w| \geqq 1$ and analytic for $|w| > 1$ including at $w = \infty$, where it equals one, and there is no w, with $|w| \geqq 1$, such that $\phi(w) = z$. Therefore the logarithm of the absolute value of this function is harmonic for $|w| > 1$ and continuous for $|w| \geqq 1$, and equals zero at $w = \infty$; so by the Gauss mean value theorem the integral in the third member of (3.1) has the value zero. So we have shown that

$$\text{(3.3)} \qquad \begin{aligned} E[\log|z - \phi(e^{i\theta})|] &= \log|c|, \\ E\left[\log\left|\frac{z - \phi(e^{i\theta})}{a - \phi(e^{i\theta})}\right|\right] &= 0. \end{aligned}$$

It should also be noted that the variance of $\log \{|z - \phi[\exp(i\theta)]|/|a - \phi[\exp(i\theta)]|\}$ exists for z and a on B. We denote it by σ^2.

The rest of the proof now follows quickly from standard probability theory. The inequality $|\omega_n(z)|/|\omega_n(a)| < m_1$ is equivalent to

$$\frac{\sum_1^n \log \left|\frac{z - \phi(e^{i\theta_k})}{a - \phi(e^{i\theta_k})}\right|}{\sigma\sqrt{n}} < \frac{\log m_1}{\sigma\sqrt{n}}. \tag{3.4}$$

By the central limit theorem the expression on the left converges in distribution to the normal distribution with zero mean and unit variance. The uniformity of the convergence insures that the limiting value of the probability of (3.4) is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-u^2/2}\, du = \frac{1}{2}, \tag{3.5}$$

as indicated in the statement of part (a) of the lemma. Parts (b) and (c) are also now immediate consequences. For part (d) we observe that

$$\frac{\omega'(a)}{\omega(a)} = \sum_1^n \frac{1}{a - \phi(e^{i\theta_k})}. \tag{3.6}$$

The function $1/[a - \phi(w)]$ is analytic for $|w| > 1$ and continuous for $|w| \geqq 1$. It vanishes at $w = \infty$. Therefore

$$E\left[\frac{1}{a - \phi(e^{i\theta_k})}\right] = \frac{1}{2\pi i} \int_{|w|=1} \left[\frac{1}{a - \phi(w)}\right] \frac{dw}{w} = 0 \tag{3.7}$$

by the standard calculus of residues for regions containing the point at infinity ([16], pp. 110–112). The variance of the function is finite. Reference to any of various well-known results concerning sums of random variables, for example, again the central limit theorem, now completes the proof of part (d).

We now return to theorem 2.2. Part (a) of that theorem is, by (2.5), equivalent to stating that the limiting probability of the inequality

$$\left|\frac{1}{a - z}\left[1 - \frac{\omega_n(z)}{\omega_n(a)}\right] - \frac{1}{a - z}\right| = \left|\frac{1}{a - z}\right| \left|\frac{\omega_n(z)}{\omega_n(a)}\right| < \epsilon \tag{3.8}$$

is 1/2, and with $m_1 = \epsilon|a - z|$, that is just what part (a) of the lemma does state. In part (b) of theorem 2.2 we observe that

$$\begin{aligned} P\left\{\left|\frac{1}{a - z}\left[1 - \frac{\omega_n(z)}{\omega_n(a)}\right] - g(z)\right| < \eta\right\} \\ = P\left\{\left|\frac{\omega_n(z)}{\omega_n(a)} - [1 - (a - z)g(z)]\right| < \eta|a - z|\right\}. \end{aligned} \tag{3.9}$$

This probability can be no greater than

(3.10)

$$P\left\{|1 - (a - z)g(z)| - \eta|a - z| < \left|\frac{\omega_n(z)}{\omega_n(a)}\right| < |1 - (a - z)g(z)| + \eta|a - z|\right\},$$

because whenever the event in braces in (3.9) occurs, so does the event in braces in (3.10), but the one in (3.10) can occur without the occurrence of that in (3.9). If η is small enough so that the left member of the inequality in (3.10) is positive, then part (b) of the lemma becomes applicable, which establishes part (b) of theorem 2.2. The proofs of parts (c) and (d) proceed similarly.

Our proof of theorem 2.3 makes use of the recursion relation

$$L_n(f; z) = (z - z_n)L_{n-1}\left[\frac{f - f(z_n)}{z - z_n}; z\right] + f(z_n), \tag{3.11}$$

and proceeds by induction in parallel applied to f and its first difference quotient. The computations are too long to reproduce here and will be published elsewhere [13]. A consequence of the theorem worth mentioning is the following, which is obtained by combining equation (2.13) with a result due to Walsh ([9], pp. 153–154).

THEOREM 3.1. *Let Γ be the unit circle, and let f be analytic for $|z| < \rho > 1$ but have a singularity on $|z| = \rho$. Let $P_n(z)$ be the polynomial of degree at most $n - 1$ found by interpolation to f in the* nth *roots of unity. Then*

$$\lim_{n\to\infty} \{P_n(z) - E[L_n(f; z)]\} = 0 \tag{3.12}$$

for $|z| < \rho^2$, uniformly for $|z| \leqq r < \rho^2$.

We turn now to theorems 2.4 and 2.5. For the proofs of these, we need the following result.

LEMMA 3.2.

$$E\left[\frac{\omega_n(z)}{\omega_n(t)}\right] = \begin{cases} 1, & \text{all } z;\, t \text{ on } B \\ \left[1 + \dfrac{z - t}{w_t\phi'(w_t)}\right]^n, & \text{all } z;\, t \text{ on } K. \end{cases} \tag{3.13}$$

where $t = \phi(w_t)$.

Because of the independence and common distribution of the random variables $z_1, z_2, \cdots$, we have the equation

$$E\left[\frac{\omega_n(z)}{\omega_n(t)}\right] = \left\{E\left[\frac{z - \phi(e^{i\theta_1})}{t - \phi(e^{i\theta_1})}\right]\right\}^n. \tag{3.14}$$

By again using the theory of residues for regions containing the point at infinity ([16], pp. 110–112) it is easy to show that for all z and for t on K,

$$I(z, t) = E\left[\frac{z - \phi(e^{i\theta_1})}{t - \phi(e^{i\theta_1})}\right] = \frac{1}{2\pi}\int_0^{2\pi} \frac{z - \phi(e^{i\theta})}{\phi(w_t) - \phi(e^{i\theta})}\, d\theta = 1 + \frac{z - \phi(w_t)}{w_t\phi'(w_t)}, \tag{3.15}$$

and for t on B, $I(z, t) = 1$. The proof of the lemma is complete.

Theorem 2.4 now follows at once from formula (2.4) and the lemma; the interchange of the order of application of the E operator and the integration over Γ'' is of course permissible.

The proof of theorem 2.5 is not quite so immediate. For any $f \in (A)_\Gamma$ there will exist an appropriate path of integration Γ' on K such that (2.1) is valid. It is permissible to change the order of integration, so we obtain, using the lemma,

$$E[L_n(f;z)] = \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(t)}{t-z} \{1 - [I(z,t)]^n\}\, dt, \tag{3.16}$$

and the problem is to determine the conditions under which $|I(z,t)| < 1$. Let $M(R) = M(z,R) = \max_{|w_t|=R} |I(z,t)|$. Since $z = \phi(w)$ gives a schlicht mapping of $|w| > 1$ onto K, it follows that $\phi'(w)$ cannot vanish for $|w| > 1$. Therefore each of the two functions

$$1 - \frac{\phi(w)}{w\phi'(w)}, \qquad \frac{1}{w\phi'(w)} \tag{3.17}$$

is analytic for $|w| > 1$ and by inspection of (1.2) it can be seen that each of them vanishes at infinity. It follows from the standard maximum modulus theory ([17], pp. 165–168) that for each z, $M(R)$ is a continuous monotonically decreasing function of R, where $1 < R < \infty$, and also $M(\infty) = 0$. We now define for each fixed z on $\Gamma + B$ a number R_z as follows:

Case (a) If $M(R) < 1$ for $1 < R < \infty$, then $R_z = 1$.

Case (b) If $M(R) = 1$ for some value of R, $1 < R < \infty$, then R_z is taken as this value of R.

Because of the continuity and monotonicity of $M(R)$, these cases exhaust the possibilities insofar as the behavior of $M(R)$ on the interval $1 < R < \infty$ is concerned, and they uniquely define R_z for each z. We note that in either case, $M(R) < 1$ for $R_z < R < \infty$, and in case (b), $M(R) > 1$ for $1 < R < R_z$.

Now for all z on $\Gamma + B$,

$$M(z,R) \leqq \max_{|w_t|=R} \left| 1 - \frac{\phi(w_t)}{w_t\phi'(w_t)} \right| + (\max_{z \text{ on } \Gamma+B} |z|) \left[\max_{|w_t|=R} \left| \frac{1}{w_t\phi'(w_t)} \right| \right]. \tag{3.18}$$

Since $\max |z|$ for z on $\Gamma + B$ is surely finite, and since each of the other maximum values decreases monotonically with R to zero, there certainly must exist some value of R, say R', $1 \leqq R' < \infty$, such that the right member of this inequality is less than one. Therefore for all z on $\Gamma + B$, $1 \leqq R_z \leqq R'$. So the least upper bound of the numbers R_z as z ranges over $\Gamma + B$ is finite and not greater than R'. *We take this least upper bound to be the number $\bar{R}$ referred to in theorem* 2.5. It has the property that for all z on $B + \Gamma$ and all $R > \bar{R}$, $M(R) < 1$, and if $\bar{R} > 1$, then every interval $\bar{R} - \epsilon < R \leqq \bar{R}$, $\epsilon > 0$, contains at least one of the numbers R_z. The implication of the latter statement is that every interval $\bar{R} - \epsilon < R < \bar{R}$ contains values of R such that for at least one z on $\Gamma + B$, $M(R,z) > 1$.

Now if f is analytic on and inside the level curve $\Gamma_{\bar{R}}$, it will be analytic on and inside some level curve $\Gamma_{R''}$, $R'' > \bar{R}$. Thus taking $\Gamma_{R''}$ to be Γ' in (3.16) we obtain

$$|E[L_n(f;z)] - f(z)| \leqq \frac{[M(z,R'')]^n}{2\pi d} m\lambda, \qquad z \text{ on } \Gamma + B, \tag{3.19}$$

where d is the distance from $\Gamma_{R''}$ to Γ, and m is the maximum of $|f|$ on $\Gamma_{R''}$, and λ is the length of $\Gamma_{R''}$. But for each z on $\Gamma + B$ we have $M(z,R'') < 1$, so the above inequality establishes the convergence of $\{E[L_n(f;z)]\}$ to $f(z)$ pointwise on $\Gamma + B$. However, it also shows that $\{|E(L_n)|\}$ is a uniformly bounded sequence for z on $\Gamma + B$, and uniform convergence on any closed subset of B then follows from Vitali's theorem ([17], pp. 168–169).

Finally consider the case $\bar{R} > 1$. *We shall show that for any level curve $\Gamma_{\bar{R}-\epsilon}$, where $\epsilon > 0$ is arbitrarily small, there exists a function analytic on and inside $\Gamma_{\bar{R}-\epsilon}$, for which $\{E(L_n)\}$ diverges to infinity on a subset of B.* To do this we use the fact, mentioned above, that given any $\epsilon > 0$, the Cartesian product of the z-set $\Gamma + B$ with the R-interval $\bar{R} - \epsilon < R < \bar{R}$ contains a point (z_0, R_0) such that $M(z_0, R_0) > 1$. Let t_0 be a point on Γ_{R_0} such that $|I(z_0, t_0)| = M(z_0, R_0)$. Consider the function $f(z) = 1/(t_0 - z)$. This function is analytic on $\Gamma_{\bar{R}-\epsilon}$. By (2.5) and lemma 3.2,

$$E\left[L_n\left(\frac{1}{t_0 - z}; z\right)\right] = \frac{1}{t_0 - z}(1 - [I(z, t_0)]^n). \tag{3.20}$$

We know that $|I(z_0, t_0)| > 1$. It is also clear from continuity that this inequality holds for all z in some neighborhood of z_0, and such a neighborhood will surely contain points of B. Wherever the inequality holds, the sequence $\{E(L_n)\}$ obviously diverges to infinity.

This completes the proof of theorem 2.5.

The actual value of $\bar{R}$, which is a characteristic constant of Γ like the so-called "conformal radii" and "transfinite diameter," may be hard to determine in given cases. It is slightly tiresome to compute even for an ellipse. If the foci of the ellipse are at $z = \pm 1$ and the major axis is of length $\rho + (1/\rho)$, then

$$I(z, t) = \frac{z - \dfrac{1}{\rho w_t}}{t - \dfrac{1}{\rho w_t}}, \qquad t \text{ on } K, \tag{3.21}$$

from which at least the fact that $\bar{R} > 1$ can easily be deduced.

The proof of theorem 2.6 is similar to that of theorem 2.5 but is simpler, since there is no problem of finding the location of a critical level curve. We omit the proof.

Finally, we shall look briefly at the proof of theorem 2.7. It is no restriction to take the center of Γ at zero, which we shall do. Suppose that f is analytic on $|z| \leqq 2R$, and therefore on $|z| \leqq R'$, where R' is a suitably chosen number greater than $2R$. Formula (2.1) now states that for $|z| < R'$,

$$|L_n(f; z) - f(z)|^2 = \left|\frac{1}{2\pi i}\int_{|t|=R'} \frac{f(t)}{t - z}\frac{\omega_n(z)}{\omega_n(t)}\,dt\right|^2, \tag{3.22}$$

and using the Schwarz inequality, we find that

$$|L_n(f; z) - f(z)|^2 \leqq \frac{m^2R'}{2\pi}\int_{|t|=R'} \left|\frac{\omega_n(z)}{\omega_n(t)}\right|^2 |dt|, \tag{3.23}$$

where m is the maximum of $|f(t)|/|t - z|$ for $|z| \leqq R$, and $|t| = R'$. Therefore

$$E|L_n(f; z) - f(z)|^2 \leqq \frac{m^2R'}{2\pi} \max E\left|\frac{\omega_n(z)}{\omega_n(t)}\right|^2, \tag{3.24}$$

the maximum being restricted by $|z| \leqq R$, $|t| = R'$. We must now evaluate the expectation on the right side of the inequality. The result is

LEMMA 3.3.

$$E\left|\frac{\omega_n(z)}{\omega_n(t)}\right|^2 = \begin{cases} \left[1 + \dfrac{|z - t|^2}{R^2 - |t|^2}\right]^n, & \text{all } z;\ |t| < R \\ \left[\dfrac{\left|\dfrac{R}{t}\right|^2 + \left|z - \dfrac{R^2}{t}\right|^2}{|t|^2 - R^2}\right]^n, & \text{all } z;\ |t| > R. \end{cases} \tag{3.25}$$

The lemma is easily proved by the calculus of residues. By examining the geometry of the situation it can be shown that for all $|z| \leqq R$, the expression in square brackets in the lemma for the case $|t| > R$ is less than one in absolute value for $|t| > 2R$, but if $|t| < 2R$, there are values of z on $|z| < R$ for which the absolute value of the expression is greater than one. The convergence property stated in theorem 2.7 then follows from (3.24), and the divergence indicated in the second sentence of the theorem is established by considering (2.5) for a suitably chosen function $1/(a - z)$. The third sentence of the theorem follows mainly from the fact that the absolute value of the expression in square brackets in the lemma for the case $|t| < R$ is greater than one if $z \neq t$. Therefore divergence in the mean will take place for all functions of the type $1/(a - z)$, with $|a| < R$, at least for all z not equal to a. The case in which $z = a$ is not meaningful insofar as the expression $|L_n - f|^2$ is concerned, but it can be shown that

$$E\left|L_n\left(\frac{1}{a - z}; a\right)\right|^2 = O(n). \tag{3.26}$$

We omit the details.

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