PAPERS ON PROBABILITY THEORY

CONTINUITY AND HÖLDER'S CONDITIONS FOR SAMPLE FUNCTIONS OF STATIONARY GAUSSIAN PROCESSES

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1. Introduction

In the present paper we shall consider a series of questions connected with local properties of sample functions of stationary stochastically continuous separable Gaussian processes. The words stochastically continuous and separable will be omitted. The results given below were previously published in [1] without proofs. In section 2 we show that for stationary Gaussian processes the following alternatives take place: either all the sample functions are continuous, or all the sample functions are unbounded in every interval of finite length. The hypothesis about the existence of such alternatives was stated long ago by Kolmogorov (see also [2]). In section 3, conditions sufficient for the continuity of sample functions which were obtained by Hunt [3] are formulated in terms of correlation functions in formulas (44) and (45). In the same section examples are given of Gaussian processes that are unbounded with probability one in intervals of finite length. These examples show that conditions (44) and (45) cannot be strengthened materially. Conditions sufficient for almost all sample functions to satisfy a Hölder condition are discussed in section 4. The author wishes to express his deep gratitude to A. N. Kolmogorov for proposing the problems.

2. Alternatives

Let $\xi(t)$ be a stationary Gaussian process with a continuous correlation function, assuming real values. The main result of this section is

THEOREM 1. *For every Gaussian stochastically continuous stationary process $\xi(t)$ one of the following alternatives holds: either with probability one the sample functions $\xi(t)$ are continuous or with probability one they are unbounded in every finite interval.*

In order to prove this theorem we need two lemmas.

LEMMA 1. *Let $\zeta(t)$ be a random process such that in the interval $\Delta = (t', t'')$*

$$P\left\{\sup_{t\in\Delta} \zeta(t) > a\right\} > p. \tag{1}$$

Then there exists a set of points $S_n = \{t_1, \cdots, t_n\}$ *such that*

$$P\left\{\max_{t_i\in S_n} \zeta(t_i) > a\right\} > p. \tag{2}$$

LEMMA 2. *Let* $\zeta_1(t), \cdots, \zeta_m(t), \cdots, M\zeta_m(t) = 0$ *be a sequence of mutually independent stationary Gaussian processes for which in some interval* $\Delta = (t_1', t_2')$ *and for* $\epsilon > 0$, $\delta > 0$,

$$P\left\{\sup_{t\in\Delta} \zeta_k(t) > a\right\} > p, \qquad \sup_{t\in\Delta} P\left\{\sum_{i=1}^{\infty} \zeta_i(t) < -\delta\right\} < \frac{\epsilon}{2m}, \tag{3}$$

where m is defined by $(1 - p)^m < \epsilon/2$.
Then

$$P\left\{\sup_{t\in\Delta} \sum_{i=1}^{m} \zeta_i(t) > a - \delta\right\} > 1 - \epsilon. \tag{4}$$

The proof of these auxiliary lemmas is elementary and is omitted here.

As is known, the modulus of continuity of the function $f(t)$ in the interval Δ is defined as the function

$$w_f(\Delta, \delta) = \sup_{t',t''\in\Delta,|t'-t''|<\delta} |f(t') - f(t'')|. \tag{5}$$

In the case that almost all the sample functions are continuous in the interval Δ we have

$$\lim_{\delta\downarrow 0} P\{w_\xi(\Delta, \delta) > \epsilon\} = 0 \tag{6}$$

for every $\epsilon > 0$. In this notation it is sufficient to show that if

$$\lim_{\delta\downarrow 0} P\{w_\xi(\Delta', \delta) > b\} > p' \tag{7}$$

for some $b = 2a > 0$, $p' > 0$, and for some interval Δ', then for any interval Δ and for any $N > 0$

$$P\left\{\sup_{t\in\Delta} |\xi(t)| > N\right\} = 1. \tag{8}$$

If (7) is satisfied then, making use of the stationary nature of the process $\xi(t)$, it can be shown that for any interval $\Delta = (t_1, t_1')$ and for some $p = p(t_1' - t_1) > 0$,

$$\lim_{\delta\downarrow 0} P\{w_\xi(\Delta, \delta) > b\} > p. \tag{7'}$$

We now note that if for some interval $\Delta = (t_1, t_1')$ we have $b = 2a > 0$, $p > 0$, $P\{w_\xi(\Delta, \delta) > 2a\} > p$, then

$$P\left\{\sup_{t\in\Delta} |\xi(t) - \xi(t_1)| > a\right\} > p. \tag{9}$$

Taking into account the symmetry of Gaussian distributions it follows that

$$P\left\{\sup_{t\in\Delta} [\xi(t) - \xi(t_1)] > a\right\} > \frac{p}{2}. \tag{10}$$

It follows from (7′) that the spectrum of the process $\xi(t)$ is unbounded, since in the opposite case almost all the sample functions would be entire and analytic [4]. Therefore (7′) holds for any $L > 0$ for the process

$$\xi_L(t) = \int_{|\lambda| \geq L} e^{it\lambda}\Phi(d\lambda). \tag{11}$$

Here $\Phi(d\lambda)$ is the spectral probability measure corresponding to the process $\xi(t)$. Therefore, for the process $\xi_L(t)$ we have

$$P\left\{\sup_{t\in\Delta} [\xi_L(t) - \xi_L(t_1)] > a\right\} > \frac{p}{2}. \tag{12}$$

Suppose that we are given $\epsilon > 0$, $\delta > 0$. First we select an interval $\Delta = (t_1, t_1')$ sufficiently small and then L sufficiently large that

$$\begin{gathered}\inf_{t\in\Delta} P\{\xi(t) - \xi(t_1) > -\delta\} > 1 - \epsilon,\\ \inf_{t\in\Delta} P\{\xi_L(t) - \xi_L(t_1) > -\delta\} > 1 - \frac{\epsilon}{2m},\end{gathered} \tag{13}$$

where m is an integer such that

$$\left(1 - \frac{1}{2}p\right)^m < \frac{\epsilon}{2}. \tag{14}$$

It follows from lemma 1 and from (12) that for some finite set $T_1 \subset \Delta$

$$P\left\{\max_{t_i\in T_1} [\xi_L(t_i) - \xi_L(t_1)] > a\right\} > \frac{p}{2}. \tag{15}$$

But

$$\xi_L(t) = \operatorname*{l.i.m.}_{N_1\to\infty} \int_{N_1 > |\lambda| \geq L} e^{it\lambda}\Phi(d\lambda), \tag{16}$$

where l.i.m. means convergence in the mean square.

Therefore, a sufficiently large N_1 can be found such that for

$$\xi_1(t) = \int_{N_1 > |\lambda| \geq L} e^{it\lambda}\Phi(d\lambda) \tag{17}$$

we have

$$P\left\{\max_{t_i\in T_1} [\xi_1(t_i) - \xi_1(t_1)] > a\right\} > \frac{p}{2}. \tag{18}$$

Since for the process

$$\zeta_1(t) = \int_{|\lambda| \geq N_1} e^{it\lambda}\Phi(d\lambda) \tag{19}$$

we have

$$\lim_{\delta\downarrow 0} P\{w_{\zeta_1}(\Delta, \delta) > 2a\} > p, \tag{20}$$

we can find by the same method an $N_2 > N_1$ and in fact a finite set $T_2 \subset \Delta$ such that for

$$\xi_2(t) = \int_{N_2 > |\lambda| \geqq N_1} e^{it\lambda}\Phi(d\lambda) \tag{21}$$

we have

$$P\left\{\max_{t_i \in T_2} [\xi_2(t_i) - \xi_2(t_1)] > a\right\} > \frac{p}{2}. \tag{22}$$

Proceeding further in the same way we can construct a sequence of mutually independent stationary Gaussian processes

$$\xi_k(t) = \int_{N_k > |\lambda| \geqq N_{k-1}} e^{it\lambda}\Phi(d\lambda), \qquad k = 3, 4, \cdots, \tag{23}$$

for which

$$P\left\{\sup_{t \in \Delta} [\xi_k(t) - \xi_k(t_1)] > a\right\} > \frac{p}{2}. \tag{24}$$

If we denote by $\zeta_k(t) = \xi_k(t) - \xi_k(t_1)$, for $k = 1, \cdots$, then, making use of (13) and (14) and lemma 2, it can be seen easily that

$$P\left\{\sup_{t \in \Delta} [\xi(t) - \xi(t_1)] > a - 2\delta\right\} > 1 - 2\epsilon. \tag{25}$$

Since $\epsilon > 0$, $\delta > 0$, and the interval Δ can be chosen arbitrarily small, it follows that for any interval $\Delta = (t_1, t_1')$ we have

$$P\left\{\sup_{t \in \Delta} [\xi(t) - \xi(t_1)] > a\right\} = 1. \tag{26}$$

This completes the first step in the proof of the theorem.

We now show that it follows from (26) that

$$P\left\{\sup_{t \in \Delta} [\xi(t) - \xi(t_1)] > na\right\} = 1, \qquad n = 2, 3, \cdots. \tag{27}$$

We present the argument for $n = 2$. Suppose that (26) holds. Again assign arbitrarily small numbers $\epsilon > 0$, $\delta > 0$. It follows from (26) and lemma 1 that there exists a finite set of points $S \subset \Delta$ for which

$$P\left\{\max_{t_i \in S} [\xi(t_i) - \xi(t_1)] > a\right\} > 1 - \frac{\epsilon}{4}. \tag{28}$$

Since

$$\xi(t) = \underset{M \to \infty}{\text{l.i.m.}} \int_{|\lambda| < M} e^{it\lambda}\Phi(d\lambda), \tag{29}$$

we can select a sufficiently large $M > 0$, such that for

$$\eta_1(t) = \int_{|\lambda| < M} e^{it\lambda}\Phi(d\lambda) \tag{30}$$

we have

$$P\left\{\max_{t_i \in S} [\eta_1(t_i) - \eta_1(t_1)] > a\right\} > 1 - \frac{\epsilon}{4}, \tag{31}$$

and such that for

$$\eta_2(t) = \int_{|\lambda| \geqq M} e^{it\lambda}\Phi(d\lambda) \tag{32}$$

we have

$$\inf_{t \in \Delta} P\{\eta_2(t) - \eta_2(t_1) > -\delta\} > 1 - \frac{\epsilon}{2}. \tag{33}$$

The process $\eta_1(t)$ has a bounded spectrum. It follows that almost all the sample functions of this process are continuous. Therefore, near every point $t_k \in S = \{t_2, \cdots, t_N\}$ with $t_N < t_1'$, it is possible to find a sufficiently small interval $\Delta_k = (t_k - \delta', t_k + \delta') \subset \Delta$, such that in case $\eta_1(t_k) - \eta_1(t_1) > a$, then $\eta_1(t) - \eta_1(t_1) > a$, everywhere in the interval Δ_k for all k simultaneously with probability greater than $1 - \epsilon/4$. Taking (31) into account, this gives

$$P\left\{ \bigcup_{k=2}^{N} \left(\inf_{t \in \Delta_k} [\eta_1(t) - \eta_1(t_1)] > a \right) \right\} > 1 - \frac{\epsilon}{2}. \tag{34}$$

We now consider the process $\eta_2(t)$. Repeating the argument in the first part of the theorem, it can be shown that for the interval Δ_k, for $k = 2, \cdots, N$, we have

$$P\left\{ \sup_{t \in \Delta_k} [\eta_2(t) - \eta_2(t_k - \delta')] > a \right\} = 1. \tag{35}$$

From this equation and from (33) it follows that for the event A_k determined by

$$\sup_{t \in \Delta_k} [\eta_2(t) - \eta_2(t_1)] > a - \delta, \tag{36}$$

we have

$$P\{A_k\} \geqq P\left\{ \sup_{t \in \Delta_k} [\eta_2(t) - \eta_2(t_k - \delta')] > a \right\} - P\{\eta_2(t_k - \delta') - \eta_2(t_1) < -\delta\} \geqq 1 - \frac{\epsilon}{2}. \tag{37}$$

Let B_k be the event that for $2 \leqq i < k$,

$$\inf_{t \in \Delta_i} [\eta_1(t) - \eta_1(t_1)] \leqq a, \qquad \inf_{t \in \Delta_k} [\eta_1(t) - \eta_1(t_1)] > a. \tag{38}$$

In this notation (34) can be written in the form

$$\sum_{k=2}^{N} P\{B_k\} > 1 - \frac{\epsilon}{2}. \tag{34'}$$

Taking (34′) into account, we have

$$P\left\{ \sup_{t \in \Delta} [\xi(t) - \xi(t_1)] > 2a - \delta \right\} \geqq P\left\{ \bigcup_{k=1}^{N} B_k A_k \right\} = \sum_{k=1}^{N} P\{B_k A_k\} = \sum_{k=1}^{N} P\{B_k\} P\{A_k\} \geqq 1 - \epsilon. \tag{39}$$

Since $\epsilon > 0$ and $\delta > 0$ can be selected arbitrarily small, (27) follows for $n = 2$. Repeating the argument it is possible, beginning with (27) for $n = 2$, to establish (27) for $n = 4$, and so on. Therefore, for any interval and for any $N > 0$, we have

$$P\left\{\sup_{t\in\Delta}\xi(t) > N\right\} = 1. \tag{40}$$

The theorem is proved.

3. Sufficient conditions for continuity. Examples of everywhere unbounded processes

We shall now consider conditions sufficient for the continuity of sample functions of stationary Gaussian processes. The strongest result known to the writer at present is due to Hunt [3]. The sufficient conditions obtained by Hunt are formulated in terms of spectral functions. In order to reformulate these conditions in terms of correlation functions the following lemma is useful.

LEMMA 3. *If $F(\lambda)$ is a nondecreasing function of bounded variation and if*

$$\varphi(h) = \int_0^\infty (1 - \cos\lambda h)\, dF(\lambda), \tag{41}$$

and if for some $C > 0$, $a > 0$ and for all sufficiently small h we have

$$\varphi(h) \leqq \frac{C}{|\log|h||^a} \tag{42}$$

then it follows that for all $b < a$

$$\int_0^\infty [\log(1+\lambda)]^b\, dF(\lambda) < \infty. \tag{43}$$

Conversely, if (43) *holds for some $b > 0$, then* (42) *holds for all $a \leqq b$, $C > 0$, and for all sufficiently small h, where $|h| < \delta = \delta(C, a)$.*

Hunt showed that if the spectral function $F(\lambda)$ of the stationary Gaussian process $\xi(t)$ satisfies condition (43) for some $b > 1$, then almost all sample functions of the process $\xi(t)$ are continuous. On the other hand, by lemma 3 above and by (42) with $a > 1$, it follows that (43) holds for all b, with $1 < b < a$. Therefore, condition (42) for $a > 1$ is also sufficient for the continuity of sample functions. Hence we have

THEOREM 2. *In order that almost all sample functions of the stationary Gaussian process $\xi(t)$ be continuous it is sufficient that one of the following equivalent conditions be satisfied either for some $b > 1$*

$$\int_0^\infty [\log(1+\lambda)]^b\, dF(\lambda) < \infty \tag{44}$$

or for some $a > 1$, $C > 0$, and for all sufficiently small h

$$M|\xi(t+h) - \xi(t)|^2 \leqq \frac{C}{|\log|h||^a}. \tag{45}$$

We now derive some conditions sufficient for almost all sample functions of a stationary Gaussian process to be unbounded in every finite interval.

THEOREM 3. *If for a stationary Gaussian process $\xi(t)$ the spectral density $f(\lambda)$ exists and is such that for some $C > 0$, $\lambda_0 > 0$ and for all $\lambda \geqq \lambda_0$*

$$f(\lambda) \geqq \frac{C}{\lambda(\log \lambda)^2}, \tag{46}$$

then almost all sample functions are unbounded in every interval of finite length.

The method of proof of this theorem is as follows. Let $\eta_n(t)$ be mutually independent stationary Gaussian processes with

$$M\eta_n(t) = 0, \qquad n = 1, 2, \cdots, \tag{47}$$

that have spectral densities $f_n(\lambda) = 1/2^{n+1}$ for $0 \leqq \lambda < 1/2^n$, and $f_n(\lambda) = 0$ for $\lambda \geqq 1/2^n$. Consider the random process

$$\eta(t) = \sum_{n=1}^{\infty} C_n \eta_n(t), \qquad \sum_{n=1}^{\infty} C_n^2 = 1, \qquad \sum_{k=m}^{\infty} C_k^2 \geqq \frac{C_1'}{m} \tag{48}$$

for all $m \geqq m_0$ and for some $C_1' > 0$. It can be verified that, for this process,

$$\lim_{\delta \downarrow 0} P\{w_\eta(\Delta, \delta) > \sqrt{C_1'}\} \geqq \frac{1}{2}. \tag{49}$$

It follows from the alternatives established in section 2 that almost all sample functions of the process $\eta(t)$ are unbounded in every finite interval. The spectral density $f_\eta(\lambda)$ of the random process $\eta(t)$ satisfies the inequalities

$$\frac{k_1}{\lambda(\log \lambda)^2} \leqq f_\eta(\lambda) \leqq \frac{k_2}{\lambda(\log \lambda)^2} \tag{50}$$

for some $0 < k_1 < k_2$. The stationary Gaussian process $\eta_w(t)$, with spectral density $g_w(\lambda) = 0$ for $\lambda < w$, and $g_w(\lambda) = f_\eta(\lambda)$ for $\lambda > w$, possesses the same property of unboundedness. Noting that the process $\xi(t)$ can be represented as the sum of two mutually independent stationary Gaussian processes, one of which has spectral density of the form $\alpha g_w(\lambda)$, where $\alpha > 0$ and $w > 0$, we obtain the theorem.

It is possible to construct examples of unbounded Gaussian processes from the properties of correlation functions. The following lemma, proved by A. D. Ventcel, is very useful.

LEMMA 4. *Let $\xi_1, \cdots, \xi_n$ be Gaussian random variables*

$$M\xi_i = 0, \qquad M\xi_i^2 = \sigma_i^2, \qquad M\xi_i\xi_j = r_{ij} < 0, \qquad i \neq j;\ i, j = 1, \cdots, n. \tag{51}$$

Then

$$P\left\{\max_{1 \leqq i \leqq n} \xi_i \leqq a\right\} \leqq \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma_i} \int_{-\infty}^{a} e^{-x^2/2\sigma_i^2}\,dx. \tag{52}$$

If the correlation function $B(h) = M\xi(t+h)\xi(t)$ of the stationary random process $\xi(t)$ is convex for $0 \leqq h \leqq \delta$, where $\delta > 0$, then

$$M[\xi(t_1 + h) - \xi(t_1)][\xi(t_2 + h) - \xi(t_2)] \leqq 0, \qquad |t_2 - t_1| < \delta. \tag{53}$$

Using lemma 4 we can show that if for a stationary Gaussian process $\xi(t)$,

$$M|\xi(t+h) - \xi(t)|^2 \geqq \frac{C}{|\log|h||}, \qquad C > 0, \tag{54}$$

for all sufficiently small h, and if $B(h)$ is a convex function, then

$$\lim_{\delta \downarrow 0} P\{w_\xi(\Delta, \delta) > \sqrt{C}\} = 1. \tag{55}$$

Again applying the alternatives of section 2, we obtain

THEOREM 4. *If the correlation function $B(h)$ of the stationary Gaussian process $\xi(t)$ is convex for $h \in (0, \delta)$, where $\delta > 0$, and if (54) holds, then almost all sample functions of the process $\xi(t)$ are unbounded in all intervals of finite length.*

4. Hölder's conditions

In the case of stationary Gaussian processes that are continuous with probability one, there arises the problem of studying the modulus of continuity of sample functions. From general results we can note the following. If $f(h) \geqq 0$ and $h/f(h) \to 0$ as $h \to 0$, then the probability of the event

$$\left\{\overline{\lim_{h\to 0}} \frac{|\xi(t+h) - \xi(t)|}{f(h)} > 1\right\} \tag{56}$$

can be either zero or one. Another theorem of the same nature is a generalization of a result of Baxter [5] in the case of stationary Gaussian processes.

THEOREM 5. *Let $\xi_i(t)$, for $i = 1, 2$, be two stationary Gaussian processes with $M|\xi_i(t+h) - \xi_i(t)|^2 = \varphi_i(h)$ such that $h^2/\varphi_i(h) \to 0$, as $h \to 0$, and $\overline{\lim}_{h\to 0} [\varphi_1(h)/\varphi_2(h)] = \alpha > 1$. Then a sequence of numbers $h_n \downarrow 0$, as $n \to \infty$, can be found such that with probability one*

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \frac{[\xi_1(t+h_n) - \xi_1(t)]^2}{\varphi_1(h_n)} = 1, \tag{57}$$

and

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \frac{[\xi_2(t+h_n) - \xi_2(t)]^2}{\varphi_1(h_n)} = \frac{1}{\alpha}. \tag{58}$$

The proof of this theorem can be obtained by using the fact that for a stationary Gaussian process $\zeta(t)$, which is not differentiable in the mean square, the values

$$\frac{\zeta(t+h_1) - \zeta(t)}{[M|\zeta(t+h_1) - \zeta(t)|^2]^{1/2}} \tag{59}$$

and

$$\frac{\zeta(t+h_2) - \zeta(t)}{[M|\zeta(t+h_2) - \zeta(t)|^2]^{1/2}} \tag{60}$$

asymptotically become mutually independent random variables when $h_1 =$ constant and $h_2 \downarrow 0$.

We shall say that a random process $\xi(t) \in H(\alpha, C)$, with $0 < \alpha < 1$, $C > 0$, if for every $C' > C$, with probability one uniformly over all t in every interval Δ of finite length, there exists a $\delta(\Delta, C')$ such that for all $h < \delta(\Delta, C')$

$$|\xi(t+h) - \xi(t)| \leqq C'|h|^\alpha. \tag{61}$$

The following theorem holds for processes of this general type.

THEOREM 6. *In order that a stochastically continuous random process* $\xi(t) \in H(\alpha, C)$ *it is sufficient that*

$$\sum_{n=1}^{\infty} \sum_{k=0}^{2^n-1} P\left\{\left|\xi\left(\frac{k+1}{2^n}\right) - \xi\left(\frac{k}{2^n}\right)\right| > K_\alpha C \frac{1}{2^{n\alpha}}\right\} < \infty, \tag{62}$$

where $K_\alpha = (2^\alpha - 1)/2^{1+2\alpha}$. *If for some* $a > 0$, *we have* $M|\xi(t+h) - \xi(t)|^a < \varphi(h) \downarrow 0$ for $h \to 0$, *then from*

$$\sum_{n=1}^{\infty} n^{\alpha a} \varphi\left(\frac{1}{n}\right) < \infty \tag{63}$$

it follows that $\xi(t) \in H(\alpha, C)$ *for every* $C > 0$.

The proof of this theorem is analogous to the proof of a well-known theorem of Kolmogorov [6] about the continuity of sample functions (see also the Russian edition of Doob [7], p. 576).

THEOREM 7. *If the correlation function* $B(h)$ *of a stationary Gaussian process is such that for all sufficiently small* h

$$M|\xi(t+h) - \xi(t)|^2 \leqq C_1 \frac{|h|^{2\alpha}}{|\log|h||} \tag{64}$$

then $\xi(t) \in H(\alpha, \sqrt{2C_1}/K_\alpha)$. *If, however, for* $h \in (0, \delta)$, $\delta > 0$, $B(h)$ *is a convex function such that for all sufficiently small* h *and for* $0 < C_2 < C_1$ *we have*

$$C_2 \frac{|h|^{2\alpha}}{|\log|h||} \leqq M|\xi(t+h) - \xi(t)|^2 \leqq C_1 \frac{|h|^{2\alpha}}{|\log|h||} \tag{65}$$

then $\xi(t) \in H(\alpha, \sqrt{2C}/K_\alpha)$, *but* $\xi(t) \notin H(\alpha, C')$ *with probability one, if* $C' < \sqrt{2C_2}$.

PROOF. If (64) holds then

$$P\left\{\left|\xi\left(\frac{k+1}{2^n}\right) - \xi\left(\frac{k}{2^n}\right)\right| > CK_\alpha \frac{1}{2^{n\alpha}}\right\} \leqq \exp\left\{-\frac{C^2 n K_\alpha^2 \log 2}{2C_1}\right\}. \tag{66}$$

Therefore, the series (62) converges when $C > \sqrt{2C_1}/K_\alpha$. In order to prove that $\xi(t) \notin H(\alpha, C')$, when $C' < \sqrt{2C_2}$, we make use of lemma 4. By this lemma and by (65) we have that if

$$\log n - \frac{1}{2}\left[\frac{C'(\log n)^{1/2}}{\sqrt{C_2}} + 1\right]^2 \to +\infty \text{ as } n \to \infty, \tag{67}$$

then

$$\begin{aligned} P\left\{\max_{1 \leqq k \leqq n}\left[\xi\left(\frac{k+1}{n}\right) - \xi\left(\frac{k}{n}\right)\right] \leqq \frac{C'}{n^\alpha}\right\} \\ \leqq \left[P\left\{\left[\xi\left(\frac{k+1}{n}\right) - \xi\left(\frac{k}{n}\right)\right] \leqq \frac{C'}{n^\alpha}\right\}\right]^n \\ \leqq \left[1 - \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[\frac{C'(\log n)^{1/2}}{\sqrt{C_2}} + 1\right]^2\right\}\right] \to 0, \end{aligned} \tag{68}$$

as $n \to \infty$. This will be the case if $C' < \sqrt{2C_2}$. Hence $\xi(t) \notin H(\alpha, C')$ for $C' < \sqrt{2C_2}$. This proves the theorem.

In the following theorem the conditions for $\xi(t) \in H(\alpha, C)$ are formulated in terms of spectral functions.

THEOREM 8. *If the spectral function $F(\lambda)$ of a stationary Gaussian process $\xi(t)$ is such that*

$$\int_0^\infty \lambda^{2\alpha} \log(1 + \lambda)\, dF(\lambda) < \infty, \tag{69}$$

then $\xi(t) \in H(\alpha, C)$ for every $C > 0$.

PROOF. It can be shown that if (69) holds, then for every $C_1 > 0$ and for sufficiently small h, with $|h| \leqq \delta(C_1)$, (64) holds and the statement of theorem 8 is a consequence of theorem 7.

Analogously we prove the following result which was first obtained by another method by Hunt [3].

THEOREM 9. *If the spectral function $F(\lambda)$ of a stationary Gaussian process $\xi(t)$ is such that*

$$\int_0^\infty \lambda^{2\alpha}\, dF(\lambda) < \infty, \tag{70}$$

then almost all sample functions satisfy a generalized Hölder condition of the form

$$|\xi(t + h) - \xi(t)| < C|h|^\alpha \left(\log \frac{1}{|h|}\right)^{1/2} \tag{71}$$

uniformly over all t in any interval of finite length for every $C > 0$ and all sufficiently small h.

In the case of stationary Gaussian processes continuous differentiability with probability one is equivalent to a Lipshitz condition. In fact, if for a sufficiently small h the inequality

$$|\xi(t + h) - \xi(t)| < C_w|h|, \tag{72}$$

is satisfied uniformly in t in some interval of finite length with probability one, where $C_w > 0$ is a random variable, then almost all sample functions are absolutely continuous. Moreover, the derivative, which is also a stationary Gaussian process, is bounded with probability one. The continuity of the derivative is a consequence of theorem 1.

In conclusion we note that for Gaussian processes whose correlation functions are analytic, almost all sample functions are also analytic and possess many properties (see [1] and [4]).

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