ON THE SOLUTION OF TWO-STAGE LINEAR PROGRAMS UNDER UNCERTAINTY

GEORGE B. DANTZIG
AND
ALBERT MADANSKY
THE RAND CORPORATION

1. Introduction

In this paper we shall study the program whose constraints are given by

$$Ax + By = b, \qquad x \geqq 0, \qquad y \geqq 0 \tag{1}$$

where A and B are known $m \times n_1$ and $m \times n_2$ matrices, x and y are n_1 and n_2-dimensional vectors, and b is a random m-dimensional vector with known distribution. We wish to minimize with respect to x

$$E \min_y (c'x + f'y), \tag{2}$$

where c and f are known n_1 and n_2-dimensional vectors, E denotes expectation taken with respect to the distribution of b, and prime denotes transpose.

As an example of a situation giving rise to such a program, consider the set of possible polyhedra given by $Ax = b$, where $x \geqq 0$, when b is random. Here, in contrast to the usual case [6] where one minimizes $c'x$ subject to x lying in the intersection over b of these polyhedra, one is instead allowed, after selecting an x and subsequently observing b, to compensate with a vector $y \geqq 0$ for infeasibility of the selected x at a penalty cost $f'y$, where $f \geqq 0$. In this case By would be $y^+ - y^-$ and the vector y which yields the smallest penalty cost for each b and x would be composed of two parts, $y^+ = b - Ax$, $y^- = 0$ if $b \geqq Ax$ or $y^- = Ax - b$, $y^+ = 0$ if $b < Ax$. As choice of y depends on b as well as x, we alter the objective from minimizing $c'x$ to minimizing $c'x$ plus the expected smallest penalty cost.

Many short range inventory problems can be expressed mathematically as such a program. The vector x may represent an inventory which is to be bought at cost $c'x$ before the random demand b is observed. Once b is observed, one must compensate by a vector y, at cost $f'y$, for imbalances $(b - Ax)$ between the original inventory and the demand so as to satisfy (1). For example, coordinates of this vector y may represent the amount of additional inventory to be bought immediately to meet the excess of demand over supply or the amount of inventory to discard in case of an excess of inventory over demand.

As the structure of the problem involves a decision x to be made first, after which the random vector b is observed and a second decision y is made, we term this a *two-stage* problem. It may be that some coordinates of b are not random and that the corresponding equations do not involve y. We shall call these equations "fixed constraints" on x.

The structure of the matrices A and B may impose further constraints on x. For example, B may be a positive matrix, in which case $Ax + By = b$ and $y \geqq 0$ imply that $Ax \leqq b$. We shall call such constraints "induced constraints." These constraints of course may depend on the value of the random vector b.

If some equations do not involve y but the corresponding subvector of b is random, we wish to restrict ourselves to the modified problem where we take as fixed restraints on x that x satisfies some specific one of all these equations. We further assume that, for each $x \geqq 0$ and satisfying all existing fixed and all possible induced constraints and for each b, there exists a y such that (x, y) is feasible, that is, satisfies (1).

This last assumption may seem very restrictive, as it says that, given any feasible x and possible b, the set of linear equations $By = b - Ax$ must have a nonnegative solution. This assumption is motivated by the desire to solve a class of problems which can be expressed as the program given in (1) and where this assumption is necessary for the problem to have a solution. The constraints of this class of problems have the structure

$$\begin{aligned} A_{11}x \qquad\quad &= b_1, \\ A_{21}x + A_{22}y &= b_2, \end{aligned} \qquad x \geqq 0, \qquad y \geqq 0 \tag{3}$$

where b_1 is a known vector and b_2 is a random vector with known distribution. It is clear that these constraints can be written as those of the program in (1) and that in this format, if the objective function for (3) is of the form of (2), we have a two-stage program with fixed constraints on x. A further description of the underlying problem which gives rise to the structure in (3) and an example of such a problem are given in [2].

As an alternative to this assumption, we can define K as the convex set of the x such that each $x \in K$ is nonnegative and has an associated y for each b such that (x, y) is feasible. The problem is, then, to find $x \in K$ which minimizes $c'x + E \min_y f'y$. These x certainly satisfy all fixed and induced constraints.

2. Optimality conditions

Let us first study the program

$$By = b - Ax, \qquad f'y = \min, \qquad y \geqq 0 \tag{4}$$

and its dual for each b, x. We assume the existence of a π which is feasible for the dual of the program given by (4) for each b, x. By the existence theorem for linear programming, this assumption plus the feasibility property of the convex set K for the program given by (1) and (2) guarantees the existence of at least one optimal dual vector $\pi = \bar{\pi}(b, x)$ which maximizes $\pi'(b - Ax)$ subject to $\pi' B \leqq f'$.

By the duality theorem,

$$\min_y f'y = \bar{\pi}'(b, x)(b - Ax), \tag{5}$$

so that

$$\min_y c'x + f'y = c'x + \bar{\pi}'(b, x)(b - Ax) = C(b, x), \tag{6}$$

say, and

$$EC(b, x) = c'x + E\bar{\pi}'(b, x)(b - Ax). \tag{7}$$

The optimal x is then the x that minimizes $EC(b, x)$ subject to $x \in K$. When for a given b and x there are many optimal π, we shall mean by $\bar{\pi}(b, x)$ any vector chosen from these optima, unless some explicit statement to the contrary is made.

Following is an immediate necessary condition for some vector $x = \bar{x}$ to be optimal.

THEOREM 1. *Let $\bar{x}$ be optimal and let $\bar{\pi}(b, x_1)$ be any vector which optimizes the dual of* (4) *for given b and $x = x_1$, where $x_1 \in K$. Then*

$$[c' - E\bar{\pi}'(b, x_1)A]\bar{x} \leqq [c' - E\bar{\pi}'(b, x_1)A]x_1. \tag{8}$$

PROOF. Since $\bar{x}$ is optimal and x_1 is feasible

$$c'\bar{x} + E\bar{\pi}'(b, \bar{x})(b - A\bar{x}) \leqq c'x_1 + E\bar{\pi}'(b, x_1)(b - Ax_1). \tag{9}$$

Also

$$E\bar{\pi}'(b, \bar{x})(b - A\bar{x}) \geqq E\bar{\pi}'(b, x_1)(b - A\bar{x}) \tag{10}$$

since $\bar{\pi}'(b, \bar{x})$ optimizes the dual of (4) when $x = \bar{x}$.

Hence

$$c'\bar{x} + E\bar{\pi}'(b, x_1)(b - A\bar{x}) \leqq c'x_1 + E\bar{\pi}'(b, x_1)(b - Ax_1). \tag{11}$$

The following lemmas enable us to obtain another necessary condition for some vector $x = x_0$ to be optimal.

LEMMA 1. *$EC(b, x)$ is a convex function of x.*

PROOF. It is easy to check directly from the definition of convexity that $C(b, x)$ is convex in x for each b [1]. Then $EC(b, x)$ is a convex function of x.

LEMMA 2. *$[c' - E\bar{\pi}'(b, x_1)A]x + E\bar{\pi}'(b, x_1)b$ is a support plane to $EC(b, x)$ at $x = x_1$.*

PROOF. Since at $x = x_1$ this plane intersects $EC(b, x)$, all we need show is that, for $x \neq x_1$,

$$[c' - E\bar{\pi}'(b, x_1)A]x + E\bar{\pi}'(b, x_1)b \leqq [c' - E\bar{\pi}'(b, x)A]x + E\bar{\pi}'(b, x)b. \tag{12}$$

But this is true if and only if

$$E\bar{\pi}'(b, x_1)(b - Ax) \leqq E\bar{\pi}'(b, x)(b - Ax) \tag{13}$$

and this is so since $\bar{\pi}(b, x)$ is optimal for the dual of (4).

A consequence of these lemmas is the following theorem.

THEOREM 2. *Let $\bar{x}$ be a relative interior point of K and let $EC(b, x)$ be differentiable in the neighborhood of $\bar{x}$. Then there exists a $\bar{\pi}(b, \bar{x})$ such that $c' - E\bar{\pi}'(b, \bar{x})A = 0$ if and only if $\bar{x}$ is optimal.*

PROOF. Since the convex function $EC(b, x)$ is differentiable at $\bar{x}$, the supporting hyperplane

$$z = [c' - E\bar{\pi}'(b, \bar{x})A]x + E\bar{\pi}'(b, \bar{x})b \tag{14}$$

is tangent to $EC(b, x)$ at $x = \bar{x}$. Hence

$$\frac{\partial EC(b, x)}{\partial x}\bigg|_{x=\bar{x}} = \frac{\partial z}{\partial x}\bigg|_{x=\bar{x}} = c' - E\bar{\pi}'(b, \bar{x}) = 0 \tag{15}$$

is a necessary condition for $\bar{x}$ to be optimal. As $\bar{x}$ is a relative interior point of K, it is also sufficient.

Returning once again to the program given in (4), let $\bar{y}(b, x)$ denote the solution of this program for given b, x. Then $\bar{\pi}(b, x)$ and $\bar{y}(b, x)$ are saddle points of the function

$$\Psi(y, \pi|x) = f'y + \pi'(b - Ax - By). \tag{16}$$

Let $\phi(x, y, \pi) = c'x + E\Psi(y, \pi|x)$. We then have the following results.

THEOREM 3. *Let $\bar{x}$ be optimal for the two-stage problem and $\bar{y}(b, x)$, $\bar{\pi}(b, x)$ be optimal for (4) and its dual. Then*

$$\phi(\bar{x}, \bar{y}, \pi) \leqq \phi(\bar{x}, \bar{y}, \bar{\pi}) \leqq \phi(x, \bar{y}, \bar{\pi}). \tag{17}$$

Conversely, if there exist vectors $\bar{x} \in K$ and $\bar{\pi}$ feasible for the dual of (4) which satisfy (17) then $\bar{x}$ is optimal for the two-stage problem and $\bar{\pi}$ is optimal for the dual of (4) when $x = \bar{x}$.

PROOF. Since $\Psi(\bar{y}, \pi|\bar{x}) \leqq \Psi(\bar{y}, \bar{\pi}|\bar{x})$,

$$\phi(\bar{x}, \bar{y}, \pi) = c'\bar{x} + E\Psi(\bar{y}, \pi|\bar{x}) \leqq c'\bar{x} + E\Psi(\bar{y}, \bar{\pi}|\bar{x}) = \phi(\bar{x}, \bar{y}, \bar{\pi}). \tag{18}$$

Now write $\phi(x, \bar{y}, \bar{\pi})$ as

$$\phi(x, \bar{y}, \bar{\pi}) = c'x + E\bar{\pi}'(b, x)(b - Ax) + E[f' - \bar{\pi}'(b, x)B]\bar{y}(b, x). \tag{19}$$

Since $\bar{\pi}(b, x)$ is optimal for the dual of (4), by the duality theorem $f' - \bar{\pi}'(b, x)B$ has the property that, if the ith coordinate of $\bar{y}(b, x)$ is positive, its ith coordinate is zero. This and the nonnegativity of $\bar{y}(b, x)$ imply that

$$\phi(x, \bar{y}, \bar{\pi}) = c'x + E\bar{\pi}'(b, x)(b - Ax) = EC(b, x) \tag{20}$$

and, by optimality of $\bar{x}$,

$$\phi(\bar{x}, \bar{y}, \bar{\pi}) \leqq \phi(x, \bar{y}, \bar{\pi}). \tag{21}$$

To prove the converse, we must first show that $\bar{\pi}(b, \bar{x})$ is optimal for the dual of (4). Then, since $\phi(x, \bar{y}, \bar{\pi}) = EC(b, x)$, it is immediately clear that $\bar{x}$ is optimal for the two-stage stochastic program.

Let $\pi^*(b, x)$ be optimal for the dual of (4). Then

$$E\pi^{*\prime}(b, \bar{x})(b - A\bar{x}) \geqq E\bar{\pi}'(b, \bar{x})(b - A\bar{x}), \tag{22}$$

and also

$$(23) \qquad [f' - \pi^{*\prime}(b, \bar{x})B]\bar{y}(b, \bar{x}) = 0.$$

But $\bar{\pi}(b, x)$ satisfies $\phi(\bar{x}, \bar{y}, \pi^*) \leqq \phi(\bar{x}, \bar{y}, \bar{\pi})$, or

$$(24) \qquad E\pi^{*\prime}(b, \bar{x})(b - A\bar{x}) \leqq E\bar{\pi}'(b, \bar{x})(b - A\bar{x}) + E[f' - \bar{\pi}'(b, x)B]\bar{y}(b, \bar{x}).$$

If $\bar{\pi}(b, \bar{x})$ is not optimal, then $[f' - \bar{\pi}'(b, x)B]\bar{y}(b, \bar{x}) > 0$, and hence (24) contradicts (22).

This theorem is the analogue in the two-stage problem dealt with here of the duality theorem for the one-stage linear team decision problem under uncertainty given in [6]. Based on this theorem we can prove in general the following sufficient condition for optimality of $x = \bar{x}$.

THEOREM 4. *Let $\bar{\pi}(b, \bar{x})$ be optimal for the dual of* (4) *when $x = \bar{x}$. If for all $x \in K$, $[c' - E\bar{\pi}'(b, \bar{x})A]\bar{x} \leqq [c' - E\bar{\pi}'(b, \bar{x})A]x$, then $\bar{x}$ is optimal for the two-stage problem.*

PROOF. Optimality of $\bar{\pi}(b, \bar{x})$ for the dual of (4) when $x = \bar{x}$ immediately yields the inequality

$$(25) \qquad \phi(\bar{x}, \bar{y}, \pi) \leqq \phi(\bar{x}, \bar{y}, \bar{\pi}).$$

Now by definition, hypothesis, and noting the facts that $f' - \bar{\pi}'(b, x)B \geqq 0$, $[f' - \bar{\pi}'(b, x)B]\bar{y}(b, x) = 0$ for all x, and $b - Ax - B\bar{y}(b, x) = 0$, we see that

$$
\begin{aligned}
(26) \qquad \phi(\bar{x}, \bar{y}, \bar{\pi}) &= [c' - E\bar{\pi}'(b, \bar{x})A]\bar{x} + E\bar{\pi}'(b, \bar{x})b \\
&\leqq [c' - E\bar{\pi}'(b, \bar{x})A]x + E\bar{\pi}'(b, \bar{x})b + E[f' - \bar{\pi}'(b, \bar{x})B]\bar{y}(b, x) \\
&= c'x + Ef'\bar{y}(b, x) + E\bar{\pi}'(b, \bar{x})[b - Ax - B\bar{y}(b, x)] \\
&= c'x + Ef'\bar{y}(b, x) + E\bar{\pi}'(b, x)[b - Ax - B\bar{y}(b, x)] \\
&= \phi(x, \bar{y}, \bar{\pi}).
\end{aligned}
$$

Hence, by theorem 3, $\bar{x}$ is optimal.

This indicates that if $\bar{x} \in K$ minimizes the linear form $[c' - E\bar{\pi}'(b, \bar{x})A]x$, it is optimal. It would be worthwhile to know whether the converse of theorem 4 is true. We have established this converse if $EC(b, x)$ is differentiable and $\bar{x}$ exists and is an interior point of K, see theorem 2. We shall consider later the finite case. A kind of converse of theorem 4 is the following.

THEOREM 5. *Let $\bar{x}$ and $\bar{y}(b, \bar{x})$ be optimal for the two-stage problem for given b, and let $\bar{y}(b, x)$ and $\bar{\pi}(b, x)$ be optimal for* (4) *and its dual for given x. Then*

$$
\begin{aligned}
(27) \qquad & E[(c', f') - \bar{\pi}'(b, \bar{x})(A, B)][\bar{x}, \bar{y}(b, \bar{x})] \\
& \qquad \leqq E[(c', f') - \bar{\pi}'(b, \bar{x})(A, B)][x, \bar{y}(b, x)].
\end{aligned}
$$

PROOF. It is seen from the definition of ϕ and the fact that $b - Ax - B\bar{y}(b, x) = 0$ that

$$(28) \qquad \phi[x, \bar{y}(b, x), \bar{\pi}(b, x)] = \phi[x, \bar{y}(b, x), \bar{\pi}(b, \bar{x})].$$

Now the righthand member of (28) is the right member of (27), plus $E\bar{\pi}'(b, x)b$, and the left member of (28) is $\phi(x, \bar{y}, \bar{\pi})$, which, by theorem 4, is greater than or equal to $\phi(\bar{x}, \bar{y}, \bar{\pi})$, that is, the left member of (27) plus $E\bar{\pi}'(b, x)b$.

When there are only a finite number of possible $b: b_1, \cdots, b_N$, with associated probabilities $p_1, \cdots, p_N$, where $\sum_{i=1}^{N} p_1 = 1$, then the two-stage program can be written as follows. Find $x, y_1, \cdots, y_N$ and min z satisfying

$$
\begin{array}{lllll}
Ax + By_1 & & & & = b_1 \\
Ax + & By_2 & & & = b_2 \\
\cdot & & \cdot & & \cdot \\
\cdot & & & \cdot & \cdot \\
\cdot & & & \cdot & \cdot \\
Ax + & & & By_N & = b_N
\end{array}
\tag{29}
$$

$$c'x + p_1 f'y_1 + p_2 f'y_2 + \cdots + p_N f'y_N = z$$

$$x \geqq 0,\ y_1 \geqq 0,\ y_2 \geqq 0,\ \cdots,\ y_N \geqq 0$$

As an application of the duality theorem and the optimality test of the simplex method in this special case, we obtain the following theorem.

THEOREM 5. *Let $\theta_i(\bar{x})$ be the ith subvector $(i = 1, \cdots, N)$ in the vector of prices associated with a basic solution $x = \bar{x}$, $y_i = \bar{y}_i$, $i = 1, \cdots, N$, for* (29). *Then $\bar{x}$, $\{\bar{y}_i\}$, is optimal if*

$$
\begin{aligned}
&c' - \sum_{i=1}^{N} \theta_i'(\bar{x})A \geqq 0, \qquad [c' - \sum_{i=1}^{N} \theta_i'(\bar{x})A]\bar{x} = 0 \\
&p_i f' - \theta_i'(\bar{x})B \geqq 0, \qquad [p_i f' - \theta_i'(\bar{x})B]\bar{y}_i = 0, \quad i = 1, \cdots, N.
\end{aligned}
\tag{30}
$$

Further, if $\bar{x}$, $\{\bar{y}_i\}$ is optimal, then there exist prices $\theta_i(\bar{x})$ satisfying (30).

We can easily prove theorem 4 in the finite case.

THEOREM 4′. *Let $x = \bar{x}$ and suppose that there exist optimal prices $\bar{\pi}(b, \bar{x})$ for the program in* (4) *such that*

$$c' - E\bar{\pi}'(b, \bar{x})A \geqq 0, \qquad [c' - E\bar{\pi}'(b, \bar{x})A]\bar{x} = 0. \tag{31}$$

Then $\bar{x}$ is optimal.

PROOF. Take

$$\theta_i(\bar{x}) = p_i \bar{\pi}(b_i, \bar{x}). \tag{32}$$

Then

$$E\bar{\pi}'(b, x) = \sum_{i=1}^{N} p_i \bar{\pi}(b_i, \bar{x}) = \sum_{i=1}^{N} \theta_i(\bar{x}). \tag{33}$$

Also, since $\bar{\pi}'(b_i, \bar{x})$ is optimal for the dual of (4), it satisfies

$$f' - \bar{\pi}'(b_i, \bar{x})B \geqq 0, \qquad [f' - \bar{\pi}'(b_i, \bar{x})B]\bar{y}(b_i, \bar{x}) = 0. \tag{34}$$

Hence for $\bar{y}_i = \bar{y}(b_i, \bar{x})$, $\theta_i(\bar{x})$ so defined satisfies (30) and, by theorem 5, $\bar{x}$ is optimal.

The converse of this theorem is also easy to prove in the finite case.

Theorem 6. *Let $\bar{x}$ be optimal. Then there exist optimal prices $\bar{\pi}(b, \bar{x})$ for the program in* (4) *such that*

$$c' - E\bar{\pi}'(b, \bar{x})A \geqq 0, \qquad [c' - E\bar{\pi}'(b, \bar{x})A]\bar{x} = 0. \tag{35}$$

Proof. Let $\pi(b_i, \bar{x}) = \theta_i(\bar{x})/p_i$, where the $\theta_i(\bar{x})$ satisfy (30). Then we have $[f' - \pi'(b_i, \bar{x})B]\bar{y}_i = 0$, so that $\bar{y}_i = \bar{y}(b_i, \bar{x})$, and $\pi(b_i, \bar{x})$ is a set of optimal prices for (4).

It is interesting to contrast this necessary condition for the finite case with that of theorem 2. In this case, not only is $c' - E\bar{\pi}'(b, \bar{x})A \geqq 0$, but further if the ith coordinate of $\bar{x}$ is positive then the ith coordinate of $c' - E\bar{\pi}'(b, \bar{x})A$ is zero. In the case dealt with in theorem 2, all coordinates of $c' - E\bar{\pi}'(b, \bar{x})A$ are zero. Of course, in the finite case $EC(b, x)$ does not satisfy the conditions of theorem 2 because here $\bar{x}$ is an extreme point of the convex region of interest and $EC(b, x)$ is not differentiable in the neighborhood of the extreme points of this convex region.

A simple example of a two-stage problem satisfying the conditions of theorem 2 is the problem solved in [2]. There x' was a two-dimensional nonnegative vector (x_1, x_2) with fixed constraint $x_1 + x_2 = 100$ so that K was a line segment in the first quadrant, and b was distributed uniformly between 70 and 80, $EC(b, x)$ was differentiable at the optimum $\bar{x} = (75, 25)$, a relative interior point of K, and, though

$$\bar{\pi}(b, x) = \begin{cases} (0, \pi_2), & b \leqq x_1, \\ (2, \pi_2), & b > x_1, \end{cases} \tag{36}$$

was optimal for the dual of (4), where π_2 could take on any value, the particular $\bar{\pi}(b, \bar{x})$ for which theorem 2 held had $\pi_2 = 0$.

3. Computational procedures

As the determination of an optimal y, given x and b, is a straightforward application of linear programming techniques to the program defined in (4), our objective is to find methods of determining an optimal x other than solving the large program (29). As will be seen shortly, an application of the decomposition principle [4] to the dual of (29) will reduce the size of the program greatly and will directly obtain for us only the optimal x and not the optimal set of y.

By virtue of theorems 4′ and 6, the problem dual to (29) can be expressed as

$$\begin{aligned} p_1A'\pi_1 + p_2A'\pi_2 + \cdots + p_NA'\pi_N &\leqq c \\ B'\pi_1 \qquad\qquad\qquad\qquad &\leqq f \\ B'\pi_2 \qquad\qquad\quad &\leqq f \\ \ddots \qquad\qquad &\vdots \\ B'\pi_N &\leqq f \\ p_1b_1'\pi_1 + p_2b_2'\pi_2 + \cdots + p_Nb_N'\pi_N &= \max \end{aligned} \tag{37}$$

We now note that the dual problem is in the standard form for application of the decomposition algorithm [4] to solve the program. To utilize the algorithm, it is convenient to call the last N sets of inequalities in (37) a single subprogram and the first set of inequalities the master program. This will reduce the program from one with $n_1 + Nn_2$ to one with $n_1 + 1$ constraints.

For notational convenience in describing the algorithm, let $\tilde{B}$ be a $Nm \times Nn_2$ block diagonal matrix of the form

$$\tilde{B} = \begin{bmatrix} B & & 0 \\ & \ddots & \\ 0 & & B \end{bmatrix}, \tag{38}$$

$\tilde{f}$ be a Nn_2-dimensional vector of the form $\tilde{f}' = [f' \cdots f']$, $\tilde{\pi}' = [\pi_1' \cdots \pi_N']$, $\tilde{A}' = [p_1A' \cdots p_NA']$, and $\tilde{b}' = [p_1b_1' \cdots p_Nb_N']$. Then (37) can be rewritten as,

$$\tilde{A}'\tilde{\pi} \leqq c, \qquad \tilde{B}'\tilde{\pi} \leqq \tilde{f}, \qquad \tilde{b}'\tilde{\pi} = \max. \tag{39}$$

Let $S = \{\tilde{\pi} | \tilde{B}'\, \tilde{\pi} \leqq \tilde{f}\}$ and let $W = \{\tilde{\pi}_1, \cdots, \tilde{\pi}_k\}$ be the set of extreme points of the convex set S. We assume here that S is a bounded set. The slight modification in the algorithm for unbounded S is given in [4]. Also let $P_j = \tilde{A}'\tilde{\pi}_j$ and $r_j = \tilde{b}'\tilde{\pi}_j$ for $j = 1, \cdots, k$. The extremal problem corresponding to (39) is to find numbers $\lambda_1, \cdots, \lambda_k$ such that

$$\begin{aligned} P_1\lambda_1 + P_2\lambda_2 + \cdots + P_k\lambda_k &\leqq c \\ \lambda_1 + \lambda_2 + \cdots + \lambda_k &= 1 \\ r_1\lambda_1 + r_2\lambda_2 + \cdots + r_k\lambda_k &= \max \\ \lambda_1 \geqq 0,\ \lambda_2 \geqq 0,\ \cdots,\ \lambda_k &\geqq 0. \end{aligned} \tag{40}$$

Then, as is shown in [4], $\tilde{\pi} = \sum_{j=1}^{k} \lambda_j\tilde{\pi}_j$ solves (39).

Now this program has even more variables than (39), namely kNm; moreover k can be very large and hence not practical to determine W explicitly. However, (40) only has $n_1 + 1$ constraints and one need never carry more than $n_1 + 1$ variables in solving (40) when using the decomposition algorithm.

The algorithm is initiated once one has a feasible basis for (40), that is, $(n_1 + 1)$ vectors π_j, to determine the necessary $(n_1 + 1)$ vectors P_j and $(n_1 + 1)$ vectors λ_j which are positive and satisfy the constraints of (40). This may be obtained using phase one of the simplex method (see [4]). The prices for this basis are then determined and are used to generate those extreme points of S that appear promising and to suppress the others. This is done with each new basis formed during the process. In this case x will be the price vector for the first r_1 constraints and we will call z the price for the last constraint. When (40) is solved, the resulting price x is optimal for the original program (29).

As the algorithm is iterative, generating new prices with each iteration, we shall append the superscript t to x and z to denote the appropriate iteration. Given the initial feasible basis, we obtain comcomitantly x^0 and z^0. To test optimality of x^0 and, if not optimal, to generate a new basis, one must first find $\tilde{\pi} = \tilde{\pi}^0$ which maximimizes $\tilde{\pi}'(\tilde{b} - \tilde{A}x^0) - z^0$ subject to $\tilde{B}'\tilde{\pi} \leqq \tilde{f}$). If $\tilde{\pi}^{0\prime}(\tilde{b} - \tilde{A}x^0) - z^0 \geqq 0$, the algorithm is terminated, and x^0 solves (29). If not, a new column is added to the $n_1 + 1$ columns of the initial basis, namely,

$$\begin{bmatrix} \tilde{A}'\tilde{\pi}^0 \\ 1 \end{bmatrix} \tag{41}$$

and its associated cost $\tilde{b}'\tilde{\pi}^0$ is added to the objective function. Using the simplex method, a new basis and new prices x^1, z^1 are determined, $\tilde{\pi} = \tilde{\pi}^1$ which maximizes $\tilde{\pi}'(\tilde{b} - \tilde{A}x^1) - z^1$ subject to $\tilde{B}'\tilde{\pi} \leqq \tilde{f}$ is determined, and once again we test $\tilde{\pi}^1$ for optimality. The process terminates when for some iteration t and prices x^t, z^t, we have

$$\max_{\tilde{B}'\tilde{\pi} \leqq \tilde{f}} \tilde{\pi}'(\tilde{b} - \tilde{A}x^t) - z^t \geqq 0. \tag{42}$$

Notice that all one needs to carry along in the computation are the columns of the current basis, $n_1 + 1$ in all. At iteration t, if (42) is not satisfied, the column

$$\begin{bmatrix} \tilde{A}'\tilde{\pi}^t \\ 1 \end{bmatrix} \tag{43}$$

is added to the basis and the simplex method determines the appropriate column to remove from the basis and hence from consideration in subsequent calculation.

Since

$$\tilde{b}'\tilde{\pi} = \sum_{i=1}^{N} p_i \pi'(b_i, x) b_i = E\pi'(b, x)b \tag{44}$$

and

$$\tilde{\pi}'\tilde{A}x = \sum_{i=1}^{N} p_i \pi'(b_i, x)Ax = E\pi'(b, x)Ax, \tag{45}$$

we see that the test for optimality can be rewritten as

$$\max_{\tilde{B}'\tilde{\pi} \leqq \tilde{f}} E\pi'(b, x^t)b - E\pi'(b, x^t)Ax^t - z^t \geqq 0. \tag{46}$$

Also, it can be shown that, at the optimal $\tilde{\pi}$

$$z^t = \min_{\{y_i | x^t\}} \sum_{i=1}^{N} f'y_i \tag{47}$$

where $y_i \geqq 0$ satisfies $By_i = b_i - Ax^t$. Since, for any feasible y_i and π_i of the subprogram,

$$\min_{\substack{y_i \geqq 0 \\ By_i = b_i - Ax}} f'y_i \geqq \max_{B'\pi(b_i, x) \leqq f} p_i \pi'(b_i, x)(b - Ax) \tag{48}$$

summing (48) for $i = 1, \cdots, N$ yields (46) with the inequality reversed. Hence the test for optimality can be interpreted as checking whether equality holds in (46) or, substituting the value of z^t given in (47), whether $\{\pi(b_i, x^t), i = 1, \cdots, N\}$ satisfies

$$E \min_y f'y = E\pi'(b, x^t)(b - Ax^t), \tag{49}$$

that is, is optimal for the dual of (4), and

$$E\pi'(b, x^t)A \leqq c. \tag{50}$$

In other words, the test for optimality of the decomposition algorithm is an implementation of the sufficient condition for optimality given in theorem 4′.

As $EC(b, x)$ is a convex function of x defined over a convex set, the problem of minimizing $EC(b, x)$ is a problem in convex programming. The following procedure for solving this problem is an application of lemma 2 and a technique due to Kelley [5] (see also [7]).

We assume here that aside from the condition $x \geqq 0$, there are enough fixed and induced constraints on x so that x lies in a bounded convex polyhedron defined by, say, $\bar{A}x = \bar{b}$. The problem is then the following.

$$\begin{aligned} \bar{A}x &= \bar{b}, \\ EC(b, x) &\leqq z, \\ x &\geqq 0, \\ z &= \min. \end{aligned} \tag{51}$$

To initiate the algorithm, let x^0 be feasible. Consider the linear program

$$\begin{aligned} \bar{A}x &= \bar{b}, \\ c'x + E\bar{\pi}'(b, x^0)(b - Ax) &\leqq z, \\ x &\geqq 0, \\ z &= \min. \end{aligned} \tag{52}$$

Let x^1 solve this program, and now consider the program

$$\begin{aligned} \bar{A}x &= \bar{b}, \\ c'x + E\bar{\pi}'(b, x^0)(b - Ax) &\leqq z, \\ c'x + E\bar{\pi}'(b, x^1)(b - Ax) &\leqq z, \\ x &\geqq 0, \\ z &= \min. \end{aligned} \tag{53}$$

Let x^2 solve this program.

One sees that on the kth iteration of this procedure one solves the program

$$\begin{aligned} \bar{A}x &= \bar{b}, \\ c'x + E\bar{\pi}'(b, x^i)(b - Ax) &\leqq z, \qquad i = 0, \cdots, k - 1 \\ x &\geqq 0, \\ z &= \min. \end{aligned} \tag{54}$$

Kelley has shown that $\lim_{k\to\infty} EC(b, x^k)$ is the minimum of $EC(b, x)$ though $\lim_{k\to\infty} x^k$ does not necessarily solve the convex program.

An alternative procedure [3] rewrites (51) as

$$\begin{aligned} \bar{A}x &= \bar{b}, \\ x &\geqq 0, \\ w + EC(b, x) &= 0, \\ w &= \max, \end{aligned} \tag{55}$$

and uses the decomposition principle with $\bar{A}x = \bar{b}$, $x \geqq 0$ as the master program and $w + EC(b, x) = 0$ as the single subprogram.

The extremal problem which is equivalent to the master program is

$$\begin{aligned} \lambda_1 + \cdots + \lambda_k &= 1, \\ (\bar{A}x^1)\lambda_1 + \cdots + (\bar{A}x^k)\lambda_k &= \bar{b}, \\ EC(b, x^1)\lambda_1 + \cdots + EC(b, x^k)\lambda_k &= \min, \end{aligned} \tag{56}$$

where $x^1, \cdots, x^k$ need not be feasible, provided some convex combination of them is. Once this program is solved and prices ρ_0^k, ρ^k are generated, one must solve the subprogram

$$\begin{aligned} w + EC(b, x) &= 0, \\ w + \rho^{k\prime}\bar{A}x + \rho_0^k &= \max \end{aligned} \tag{57}$$

or equivalently, one must find $x = x^{k+1}$ such that

$$\delta_k(x) = EC(b, x) - \rho^{k\prime}\bar{A}x - \rho_0^k \tag{58}$$

is a minimum. If for all x

$$EC(b, x) \geqq \rho^{k\prime}\bar{A}x + \rho_0^k \tag{59}$$

then $EC(b, x^{k+1})$ is minimal. If not, another column

$$\begin{bmatrix} 1 \\ \bar{A}x^{k+1} \end{bmatrix} \tag{60}$$

is added to (56), $EC(b, x^{k+1})\lambda_{k+1}$ is added to the cost form in (56), and the algorithm is iterated. In [3] it is shown that $\lim_{k\to\infty} EC(b, x^k)$ is the minimum of $EC(b, x)$.

4. Discussion

It is well known (see, for example, [2]) that replacing the random b by its expected value is of little help in solving the stochastic linear program. However, we have seen from the above discussion that the expected value of the prices, for a given x, for the program given in (4) play a critical role in solving the stochastic linear program. In fact roughly speaking, the vector $[c' - E\bar{\pi}'(b, \bar{x})A]'$ acts as the gradient of the function $EC(b, x)$. An interesting area for future consideration is the effect of sampling from the distribution of b, estimating $E\bar{\pi}(b, x)$ and $E\bar{\pi}'(b, x)b$ for each value of x generated by the iterative procedures given above, and using these estimates as $E\bar{\pi}(b, x)$ and $E\bar{\pi}'(b, x)b$ in those procedures.

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REFERENCES

[1] E. M. L. Beale, "On mimimizing a convex function subject to linear inequalities," *J. Roy. Statist. Soc., Ser. B*, Vol. 17 (1955), pp. 173–184.
[2] G. B. Dantzig, "Linear programming under uncertainty," *Management Sci.*, Vol. 1 (1955), pp. 157–206.
[3] ———, "General convex objective forms," The RAND Corporation Paper, P-1664, 1959.
[4] G. B. Dantzig and P. Wolfe, "Decomposition principle for linear programs," *Operations Res.*, Vol. 8 (1960), pp. 101–111.
[5] J. E. Kelley, Jr., "The 'cutting plane' method for solving convex programs," to appear in *J. Soc. Indust. Appl. Math.*
[6] R. Radner, "The linear team: an example of linear programming under uncertainty," *Proceedings of the Second Symposium in Linear Programming*, Washington, D.C., 1955, pp. 381–396.
[7] P. Wolfe, "Accelerating the cutting plane method for nonlinear programming," The RAND Corporation Paper, P-2010, 1960.