ON PLANE SAMPLING AND RELATED GEOMETRICAL PROBLEMS

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1. Summary

We study the following problem. An isotropic plane stochastic process is observed at points making up a regular pattern. We are interested in finding patterns yielding the least limiting variance of the observed values, when the points are situated within a circle with infinitely increasing radius.

In section 4, a solution is presented for the correlation function (3.5) and some ranges of point densities. The solution is obtained by solving the related geometrical problem of covering a plane by circles in such a way that the circles mutually intersect as little as possible (see section 3). From the results obtained it follows that, in contradistinction to the linear case, no unique pattern of points is optimum for all convex correlation functions simultaneously. The efficiency of patterns in general use is, however, quite good.

In section 5, finally, we study a subclass of convex correlation functions of an isotropic plane process, consisting of functions that admit a spectral representation in terms of the simple correlation function (5.2).

2. Introduction

In this section, an expository survey of the background of the problem will be presented.

2.1. *Applications of plane sampling.* Many applications of the sampling method may be broadly described as "plane sampling." We give some examples.

Forest surveys. In the simplest case, one might want to estimate the area of a certain country or geographical district covered by forest. Similarly, one might want to estimate the proportion of a forest area covered by a certain variety of tree. In another case, one might want to estimate the number of trees or the volume of timber in a forest area. These kinds of applications are discussed in, for example, Matérn [14].

General agricultural surveys. Illustrative examples of plane sampling are furnished by large-scale surveys carried out in order to estimate the cultivated area of a certain country or geographical district or the total production of a particular crop. We refer here to, for example, Mahalanobis [12].

Analogous examples are furnished by such small-scale surveys as "field experiments" restricted to a single field on a farm. There is no need for specific references on this point.

Soil surveys. One specific example concerns the estimation of the proportion of the area of ground of a certain district that is too salty to be cultivated in the usual way; for a discussion, see Sulanke [20].

Another specific example concerns the estimation of the density of worms in soil; see Finney [5].

Geological surveys proper. The method of plane sampling has been used in order to estimate the extension, volume, and other parameters characterizing such geological deposits as black or brown coal, zinc deposits, and so on. We refer to Zubrzycki [26].

In this connection, we want to mention briefly applications of plane sampling in connection with the construction of water-power stations. In such situations, there often is need for estimating the total volume of earth to be removed (for example, as a basis for cost estimates), or the total volume of gravel available for construction purposes.

Some other examples. The examples given so far relate to sampling a geographical area of some sort. The method of plane sampling is, however, applicable to sampling other kinds of areas; some references will be given.

Drápal, Horálek, and Režny [2] discuss the problem of estimating the proportions of the surface of cast iron composed of crystals of carbon and of iron, respectively. This problem bears considerable resemblance to the "corpuscle problem," discussed by Wicksell [23], [24]. Husu [10] considers the estimation of a parameter that characterizes the "smoothness" of the surface of a metal plate. Faure [3] and Savelli [18] consider the problem of measuring the transparency of photographic film.

2.2. *The sampling theory.* The theoretical task of constructing a (probabilistic) sampling theory to cope with the problems of estimation raised by applications such as those just discussed has long been the subject of considerable research. By and large, the theory of "field experimentation" is the origin of the theory of plane sampling. However, from a rather early date, somewhat different paths of advancement have been taken.

In field experiments it is often feasible to apply randomization of the experimental units; the use of randomization may be considered as a device for getting around the need for a (realistic) model of the role played by "topographic variation." In applications of plane sampling such as those discussed above, it is often desirable, for practical reasons, to use systematic sampling procedures. As a consequence, it is necessary to account for the role played by "topographic variation" by means of a (realistic) model of this variation.

The theory of linear sampling. To a large extent the theory of plane sampling is a formal extension of the theory of "linear sampling," this term referring to sampling a one-dimensional stochastic process. Therefore we include some references to the linear case.

Early contributions include such papers as Osborne [16], Madow and Madow [11], and Cochran [1]. In the last-mentioned paper it is proved that systematic sampling is, on the average, more precise than stratified sampling, provided that the correlogram is concave upwards. Among recent contributions we may mention Hâjek [6], [7].

The theory of plane sampling. A classical contribution to the discussion of topographic variation is given by Smith [19]. In the field of plane sampling proper, the work of Mahalanobis [12] may be considered pioneer. Matérn [14] presents a most important contribution. In this work, Matérn shows how the theory of stationary stochastic processes, as developed by Khinchin and Cramér, may be used in the construction of a stochastic model of topographic variation. The paper by Quenouille [17] is another important contribution from the 1940's. Among recent contributions, we may mention Masuyama [13], Whittle [21], [22], Williams [25], and Zubrzycki [26], [27].

3. Formulation of the problem of the paper

Our problem is to find a regular pattern of points which yield the least limiting variance of values associated with an isotropic plane stochastic process observed at these points. In this section we transform this problem into an equivalent geometrical problem, the solution of which is discussed in section 4.

To begin, we briefly review some previously established results concerning the linear case. The typical problem can be found as follows.

To the points t of a real line there are assigned random variables $\eta(t)$, subject to the following assumptions.

(a) All random variables $\eta(t)$ have common expected value μ and common variance σ^2.

(b) The correlation coefficient between any two random variables $\eta(t')$ and $\eta(t'')$ depends only upon the absolute difference $|t' - t''|$; in symbols,

$$R[\eta(t'), \eta(t'')] = \rho(|t' - t''|). \tag{3.1}$$

(c) The function $\rho(t)$, with $t \geqq 0$, called the correlation function of the process, is continuous with $\rho(0) = 1$.

The assumptions (a) to (c) characterize the family of random variables $\eta(t)$ as a continuous stochastic process stationary to the second degree.

Suppose now that we want to estimate the mean $T^{-1} \int_0^T \eta(t)\, dt$ of the process by the average

$$\bar{\eta} = \frac{1}{n}[\eta(t_1) + \cdots + \eta(t_n)] \tag{3.2}$$

of its values observed at n points $t_1, \cdots, t_n$ selected in a given segment $0 \leqq t \leqq T$. Consider all probability methods of sampling n points such that the expected number of points selected from any subsegment (α, β), with $0 \leqq \alpha < \beta \leqq T$, is proportional to $\beta - \alpha$. We ask how these n points should be chosen in order to minimize the quadratic error of estimation, defined as the expected value of $[\bar{\eta} - T^{-1} \int_0^T \eta(t)\, dt]^2$. This value extends over both the sampling experiment and nature's experiment in producing the process $\eta(t)$. Using an argument of Hájek [7], involving a kind of spectral representation of the correlation function of the process with respect to a one-parameter family of properly chosen simple correlation functions, it follows that if the correlation function is convex, then the best method of choosing the n points $t_1, \cdots, t_n$ is to select them equidistantly.

We now try to generalize the discussion of the linear case to the case of a plane. Thus we consider a family of random variables $\eta(p)$ assigned to the points p of a Euclidean plane, for which the following generalizations (a′) to (c′) of the previously given assumptions (a) to (c) are fulfilled.

(a′) All random variables $\eta(p)$ have common expected value μ and common variance σ^2.

(b′) The coefficient of correlation between any two random variables $\eta(p)$ and $\eta(q)$ depends only on the vector joining the points p and q; in symbols,

$$R[\eta(p), \eta(q)] = \rho(q - p), \tag{3.3}$$

where $q - p$ is the vector difference between q and p.

(c′) The correlation function $\rho(p)$ is a continuous function of p with $\rho(0) = 1$, where 0 is the zero vector.

In the sequel we shall be concerned with processes which are, in addition, isotropic. This means that the correlation function depends only on the length $u = |p - q|$ of the vector $p - q$

$$\rho(p - q) = \rho(|p - q|) = \rho(u), \tag{3.4}$$

that is, the correlation function is a function of one real variable. In what follows we refer to this function $\rho(u)$ as the correlation function.

In the linear case it turned out that the best method of sampling n points is to select them equidistantly. It is, however, difficult to generalize this result in a straightforward manner to the case of a plane; it is not obvious which domains in the plane can replace the segment $0 \leqq t \leqq T$. Therefore we have looked for regular allocations that can be dealt with by means of limiting theorems relating to increasing domains. This approach eliminates the troublesome boundary effect from the problem. On the other hand it introduces some problems of a purely geometrical nature. An alternative device to cope with the boundary effect would be to define the stochastic process $\eta(p)$ on the surface of a sphere or a torus.

As shown in Zubrzycki [26], we can construct a two-dimensional continuous, isotropic stationary stochastic process $\eta(p)$, the correlation function of which is $\rho(u) = r(u/a)$, where

$$(3.5) \qquad r\left(\frac{u}{a}\right) = \begin{cases} \dfrac{2}{\pi}\left\{\arccos\dfrac{u}{a} - \dfrac{u}{a}\left[1 - \left(\dfrac{u}{a}\right)^2\right]^{1/2}\right\}, & 0 \leqq u \leqq a, \\ 0, & \text{otherwise,} \end{cases}$$

where a is a positive constant.

In this analytic form, the correlation function does not reveal its most important feature: the value of $r(u/a)$ for a given u can be computed as

$$(3.6) \qquad r\left(\frac{u}{a}\right) = \frac{|K_1 \cap K_2|}{\frac{\pi a^2}{4}},$$

that is, $r(u/a)$ equals the ratio of the area $|K_1 \cap K_2|$ of the common part of two circles K_1 and K_2 with radius $a/2$ and centers p_1 and p_2 at distance $u = |p_1 - p_2|$ to the area $\pi a^2/4$ of such a circle. The geometrical interpretation is illustrated in figure 1.

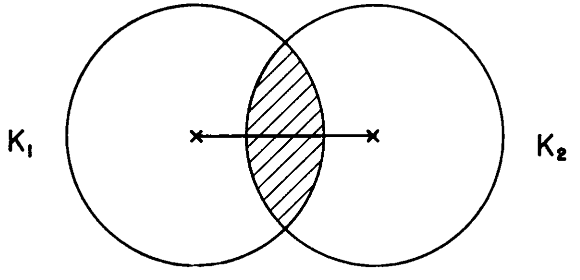

FIGURE 1

Geometrical interpretation of the value of the correlation function given by (3.6).

The mean value μ of the process may be interpreted as $|D|^{-1} \int_D \eta(p)\, dp$ for a domain D with infinitely large area $|D|$. Suppose now that we have selected n points $p_1, \cdots, p_n$ in the plane and want to estimate μ by the average

$$(3.7) \qquad \bar{\eta} = \frac{1}{n}\left[\eta(p_1) + \cdots + \eta(p_n)\right]$$

of the values $\eta(p_1), \cdots, \eta(p_n)$ at these points. This average $\bar{\eta}$ is an unbiased estimate of μ. Obviously,

$$(3.8) \qquad \operatorname{Var} \bar{\eta} = \frac{\sigma^2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho(|p_i - p_j|).$$

The double sum of the right member of (3.8) is, by virtue of the geometrical interpretation of (3.5), equal to the ratio of the sum of the areas of the common parts of all n^2 possible pairs of circles K_i with radius $a/2$ and centers p_i, with $i = 1, \cdots, n$, to the area of such a circle; in symbols,

$$\text{Var}\,\bar{\eta} = \frac{\sigma^2}{n^2} \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} |K_i \cap K_j|}{\frac{\pi a^2}{4}}. \tag{3.9}$$

We now ask which distribution of points in the plane with a given plane density yields (in the limit) the least variance. What we said in connection with the correlation function given by (3.5) shows that this problem is uniquely related to certain questions concerning distributions of circles. Let us consider a sequence of points $p_1, \cdots, p_n$ in the plane and define the density d, if it exists, as

$$d = \lim_{R\to\infty} \frac{1}{\pi R^2} \text{Card}\,[i: p_i \in K(0, R)], \tag{3.10}$$

that is, as a limit of a ratio of the number of points p_i in a circle $K(0, R)$ of radius R and center 0 to the area πR^2 of this circle, when R tends to infinity. Of course, this limit does not depend on the choice of center 0. Then, given a sequence $p_1, p_2, \cdots$ with density d, let us call a *limiting variance* the limit

$$\sigma_\infty^2 = \lim_{R\to\infty} n_R \text{Var}\,\bar{\eta}_R, \tag{3.11}$$

where $n_R = \text{Card}\,\{i: p_i \in K(0, R)\}$ and $\bar{\eta}_R$ is the average of those variables $\eta(p_i)$ for which p_i is in $K(0, R)$.

Let us introduce two more definitions. Given, in a plane, a sequence of circles $K_1, K_2, \cdots$ with radius $a/2$ and with centers $p_1, p_2, \cdots$ which have a given density d, we define the *mean covering*, for short C', as the limit

$$C' = \lim_{R\to\infty} \frac{1}{\pi R^2} \sum_i |K_i \cap K(0, R)| \tag{3.12}$$

and the *mean double covering*, for short C'', as the limit

$$C'' = \lim_{R\to\infty} \frac{1}{\pi R^2} \sum_i \sum_j |K_i \cap K_j \cap K(0, R)|. \tag{3.13}$$

In this sum the case $i = j$ is not excluded. Of course, the mean covering C' of the circles $K_1, K_2, \cdots$ and density d of their centers $p_1, p_2, \cdots$ are related by the equality $C' = |K_1|d$. As a consequence we may use C' as our measure of density of centers in comparisons where $|K_1|$ is kept constant.

Now it is clear that, for stochastic processes $\eta(p)$ with correlation functions given by (3.5), the search for a sequence of points with a prescribed density which yields the minimum limiting variance is equivalent to the search for a corresponding sequence of circles yielding the minimum mean double covering. Moreover, if there exists such a sequence of points that would realize the minimum limiting variance for all positive values of a in the correlation function given by (3.5), then it would be the best sequence also for a process with a correlation function given by

$$\rho(u) = \int_0^\infty r\left(\frac{u}{a}\right) dF(a), \tag{3.14}$$

where $F(a)$ is a distribution function with $F(0) = 0$. Unfortunately, there do not exist sequences of points that yield minimum mean covering simultaneously for all values of a in (3.5), as will be shown later in this paper.

4. Optimal nets of points

Consider a sequence of congruent circles $K_1, K_2, \ldots$ with centers $p_1, p_2, \ldots$, respectively. Let us denote the indicator function of K_i by $k_i(p)$, that is, let us put

$$k_i(p) = \begin{cases} 1, & p \in K_i, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Moreover, we put

$$k(p) = \sum_i k_i(p). \tag{4.2}$$

In other words $k(p)$ is equal to the number of circles covering p. In terms of these functions the definitions of the mean covering C' and the mean double covering C'' given in section 3 take on the forms

$$\begin{aligned} C' &= \lim_{R\to\infty} \frac{1}{\pi R^2} \sum_i \int_{K(O,R)} k_i(p)\, dp \\ &= \lim_{R\to\infty} \frac{1}{\pi R^2} \int_{K(O,R)} \sum_i k_i(p)\, dp \\ &= \lim_{R\to\infty} \frac{1}{\pi R^2} \int_{K(O,R)} k(p)\, dp \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} C'' &= \lim_{R\to\infty} \frac{1}{\pi R^2} \sum_i \sum_j \int_{K(O,R)} k_i(p)k_j(p)\, dp \\ &= \lim_{R\to\infty} \frac{1}{\pi R^2} \int_{K(O,R)} \sum_i \sum_j k_i(p)k_j(p)\, dp \\ &= \lim_{R\to\infty} \frac{1}{\pi R^2} \int_{K(O,R)} k^2(p)\, dp. \end{aligned} \tag{4.4}$$

Our problem is to determine sequences of congruent circles with a fixed mean covering for which the mean double covering attains its minimum. We now prove an inequality from which it follows that a sufficient condition for a sequence of circles to have this minimum property is that the set of values of the function $k(p)$ consists of two consecutive integers. It is the content of the following

LEMMA 4.1. *For any sequence of circles with mean covering C', the following inequality*

$$C'' \geqq \{2[C'] + 1\}C' - [C']\{[C'] + 1\} \tag{4.5}$$

holds, where $[C']$ is the integral part of C'; equality holds if and only if $k(p)$ takes the values $[C']$ and $[C'] + 1$ only.

PROOF. Clearly, we have

$$\begin{aligned} \frac{1}{\pi R^2} \int_{K(O,R)} k^2(p)\, dp &= \left\{\frac{1}{\pi R^2} \int_{K(O,R)} k(p)\, dp\right\}^2 - \left\{\frac{1}{\pi R^2} \int_{K(O,R)} k(p)\, dp - [C'] - \frac{1}{2}\right\}^2 \\ &\quad + \frac{1}{\pi R^2} \int_{K(O,R)} \left\{k(p) - [C'] - \frac{1}{2}\right\}^2 dp. \end{aligned} \tag{4.6}$$

Since always,

$$\left\{k(p) - [C'] - \frac{1}{2}\right\}^2 \geqq \frac{1}{4}, \tag{4.7}$$

we conclude that

$$\begin{aligned} \frac{1}{\pi R^2} \int_{K(O,R)} k^2(p)\, dp \\ \geqq \left\{\frac{1}{\pi R^2} \int_{K(O,R)} k(p)\, dp\right\}^2 - \left\{\frac{1}{\pi R^2} \int_{K(O,R)} k(p)\, dp - [C'] - \frac{1}{2}\right\}^2 + \frac{1}{4}. \end{aligned} \tag{4.8}$$

For $R \to \infty$, we get

$$C'' \geqq C'^2 - \left\{C' - [C'] - \frac{1}{2}\right\}^2 + \frac{1}{4} \tag{4.9}$$

and this is an alternative form of (4.5). Now if $[C']$ and $[C'] + 1$ are the only values of $k(p)$, then (4.7) and consequently (4.5) become equalities. This proves the lemma.

We now describe some sequences of circles minimizing the mean double covering. We confine ourselves to the case where the centers of the circles form nets composed of congruent figures such as triangles or squares. This will enable us to compute the mean covering and the mean double covering from a single mesh of a net, and we shall exploit this possibility. The minimum property will follow by our lemma, since the function $k(p)$ will take only two consecutive integers as its values in our examples. We arrange these examples by increasing values of C'.

EXAMPLE 4.1. If

$$C' \leqq \frac{\pi}{2\sqrt{3}} \fallingdotseq 0.907, \tag{4.10}$$

then the net of equilateral triangles has the optimal property. This situation is illustrated in figures 2 and 3. Clearly, the pattern considered here may as well be referred to as a net of rhombuses.

For purposes of illustration we present the details of the computation of C' and of C''. In figure 2 we put the radius of the circles equal to 1, and the side

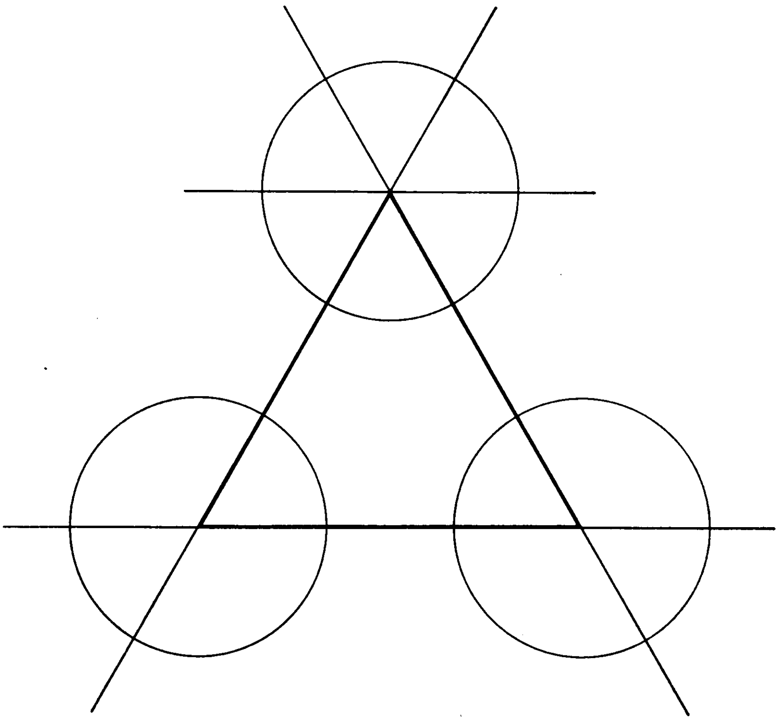

FIGURE 2

Optimum sampling pattern for $C' < \frac{\pi}{2\sqrt{3}} \doteqdot 0.907$.

of the triangle equal to $s \geqq 2$. Thus the area of the triangle is $A = (s^2\sqrt{3})/4$. The circles divide this area into four parts, $A = T_0 + 3T_1 = A_0 + A_1$. The meaning of T_0 and T_1 is shown in figure 4.

In T_0 we have $k(p) = 0$, while in T_1, we have $k(p) = 1$. Now

$$C' = \frac{1}{A} \int_{A_0} 0 \, dp + \int_{A_1} 1 \, dp = \frac{1}{A} \int_{A_1} dp. \tag{4.11}$$

Thus

$$C' = \frac{4}{s^2\sqrt{3}} \left(\frac{3\pi}{6} \right) = \frac{2\pi}{s^2\sqrt{3}}. \tag{4.12}$$

For $s > 2$, we have $C' < \pi/2\sqrt{3} \doteqdot 0.907$, while for $s = 2$, we have $C' = \pi/2\sqrt{3} \doteqdot 0.907$. Moreover, $C'' = C'$ since $k^2(p) = k(p)$.

We now compare this value of C'' with the corresponding value of C'' for a net of squares having the same value of C' and, therefore, representing the same point density. We put the side of the square equal to x. Drawing the four circles with radius equal to 1, we obtain the configuration shown in figure 5.

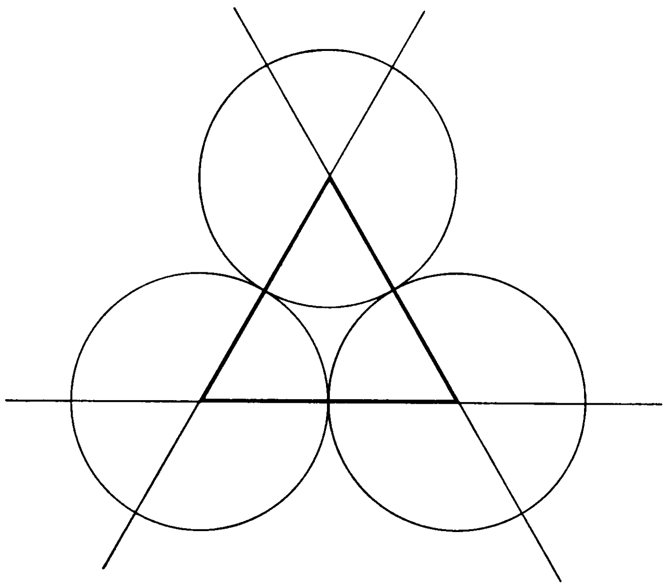

FIGURE 3

Optimum sampling pattern for $C' = \dfrac{\pi}{2\sqrt{3}} \doteqdot 0.907$.

Now

$$(4.13) \qquad C' = \frac{1}{A} \int_{A_0} 0\, dp + \int_{A_1} 1\, dp + \int_{A_2} 2\, dp = \frac{\pi}{x^2}.$$

From $C' = \pi/2\sqrt{3}$ we get

$$(4.14) \qquad x^2 = \sqrt{12}; \qquad\qquad x = \sqrt[4]{12} \doteqdot 1.86,$$

that is, the circles intersect as shown in figure 5.

If A_2 stands for the area of that portion of the square where $k(p) = 2$, we get

$$(4.15) \qquad A_2 = 8\left\{\frac{v\pi}{360} - \frac{y}{2}\frac{\sqrt[4]{12}}{2}\right\} = 8\left\{\frac{v\pi}{360} - \frac{\sqrt[4]{12}}{4}\sin v\right\},$$

with v and y having the meaning indicated in figure 6. Carrying out the computations gives $A_2 = 0.14$. Clearly $A_1 = \pi - 2A_2 = 2.86$. Thus

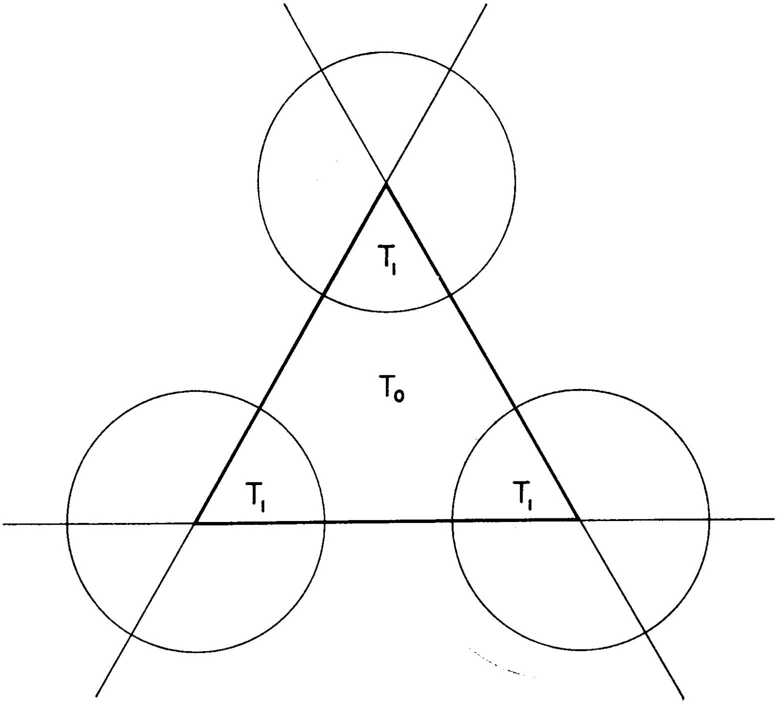

FIGURE 4

Illustration of the meaning of T_0 and T_1.

$$C' = \frac{1}{\sqrt{12}} \{2.86 + 2(0.14)\} = 0.907 \tag{4.16}$$

as it should be, and

$$C'' = \frac{1}{\sqrt{12}} \{2.86 + 2^2(0.14)\} = 0.988 > 0.907. \tag{4.17}$$

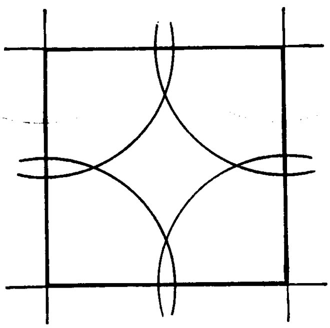

FIGURE 5

Computation of C'' for a net of squares.

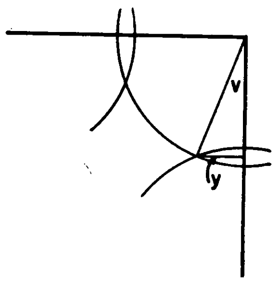

FIGURE 6

Meaning of v and y.

EXAMPLE 4.2. If

$$0.907 \doteqdot \frac{\pi}{2\sqrt{3}} \leqq C' \leqq \frac{2\pi}{3\sqrt{3}} \doteqdot 1.209, \tag{4.18}$$

then the net of equilateral triangles is still optimal; see figure 7. However, in this case $k(p)$ has three values: 0, 1, and 2, so that our lemma 4.1 does not apply. The optimality of the net in question is a consequence of a known inequality

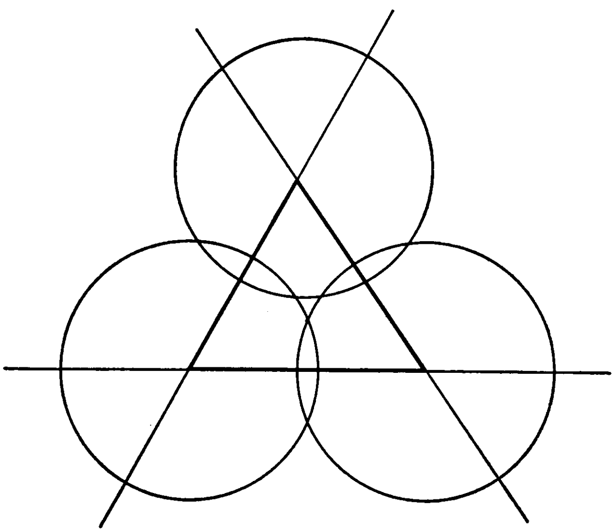

FIGURE 7

Optimum sampling pattern for $0.907 \doteqdot \frac{\pi}{2\sqrt{3}} < C' < \frac{2\pi}{3\sqrt{3}} \doteqdot 1.209$.

(Fejes Tóth [4], inequality (3), p. 80), from which it follows that among all convex hexagons of a given area and all circles of a given area, the maximum possible area of a common part of a hexagon and circle is reached when the hexagon is equilateral and the circle is concentric with it.

The statement of example 4.2 follows if we apply the quoted inequality to the cells that are formed by attaching each point of a plane to the nearest circle center. The above-mentioned inequality is of its greatest interest when the mean covering is in the range indicated in example 4.2. Let us note, however, that it also implies the statement in example 4.1 to the effect that the circles should be disjoint.

EXAMPLE 4.3. If

$$1.209 \fallingdotseq \frac{2\pi}{3\sqrt{3}} \leqq C' \leqq \frac{2\pi}{2+\sqrt{3}} \fallingdotseq 1.684, \tag{4.19}$$

then the net of isosceles triangles is optimal. This is seen as follows. We start with the situation shown in figure 8. We then increase the mean covering without

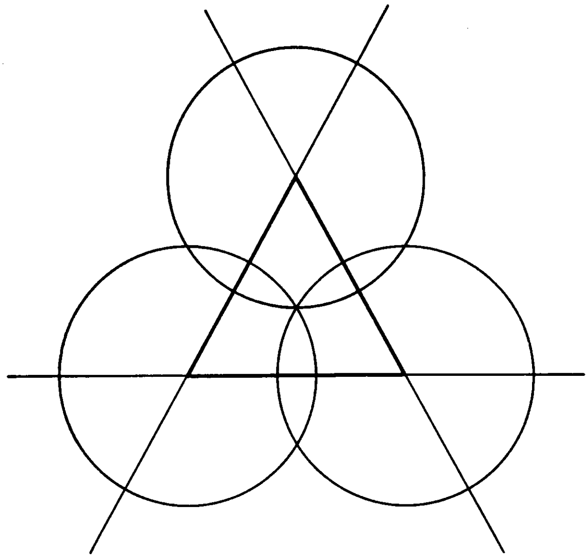

FIGURE 8

Optimum sampling pattern for $C' = \dfrac{2\pi}{3\sqrt{3}} \fallingdotseq 1.209.$

spoiling the property that $k(p)$ has as values only two consecutive integers, letting the base of the triangle diminish and its height increase, so that the three circles still intersect in one point. We can continue this procedure until the length of the base becomes equal to the radius of our circles, as shown in figure 9.

EXAMPLE 4.4. If

$$1.571 \fallingdotseq \frac{\pi}{2} \leqq C' \leqq \frac{\pi}{\sqrt{3}} \fallingdotseq 1.814, \tag{4.20}$$

a net of rectangles has the optimal property. We start with a net of squares and circles intersecting in the centers of the squares as indicated in figure 10. This

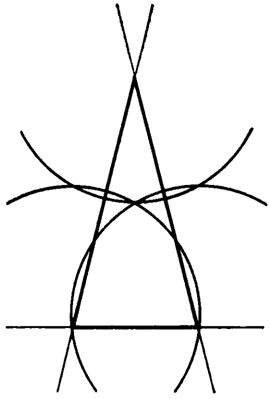

FIGURE 9

Optimum sampling pattern for $C' = \dfrac{2\pi}{2+\sqrt{3}} \doteqdot 1.684.$

corresponds to $C' = \pi/2 \doteqdot 1.571$. We then enlarge the mean covering by lengthening two sides of the square and shortening the other two, while the circles still intersect in the middle; the radius of the circles is then equal to the shorter side of the rectangle. This corresponds to $C' = \pi/\sqrt{3} \doteqdot 1.814$.

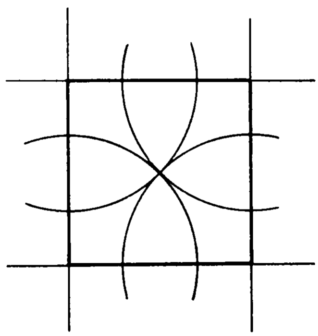

FIGURE 10

Optimum sampling pattern for $C' = \dfrac{\pi}{2} \doteqdot 1.571.$

We note that the intervals for C' corresponding to examples 4.3 and 4.4 respectively overlap. The nature of this situation will be elucidated somewhat. Instead of considering the pattern with which we start in example 4.3 as made up by a net of equilateral triangles, we think of this pattern in terms of rhombuses, with the base angle $v = 60°$, corresponding to $C' = 2\pi/3\sqrt{3} \doteqdot 1.209$. If we increase v to $v = 90°$, that is, if we change the rhombuses into squares, C' will increase to $C' = \pi/2 \fallingdotseq 1.571$. Thereafter, by "stretching" the squares into rectangles, we may further increase C' to $C' = \pi/\sqrt{3} \fallingdotseq 1.814$.

EXAMPLE 4.5. If

$$2.418 \fallingdotseq \frac{4\pi}{3\sqrt{3}} \leqq C' \leqq \frac{16\pi}{7\sqrt{7}} \fallingdotseq 2.714, \tag{4.21}$$

the optimal property is possessed by a net of hexagons which have two perpendicular axes of symmetry and can be inscribed in a circle; in general, they are not equilateral.

We start with a net of equilateral, congruent hexagons and place the centers of circles at the vertices, the radius of the circles being equal to the side of the hexagons. In this case $C' = 4\pi/3\sqrt{3} \fallingdotseq 2.418$. This case is shown in figure 11.

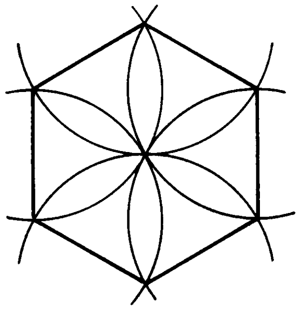

FIGURE 11

Optimum sampling pattern for $C' = \dfrac{4\pi}{3\sqrt{3}} \fallingdotseq 2.418$.

We let C' increase without spoiling the property that $k(p)$ has as values only two consecutive integers, by suitably narrowing our hexagons. We can continue this procedure until the circles corresponding to the vertices of neighboring hexagons touch. Figure 12 shows the extreme situation, which corresponds to $C' = 16\pi/7\sqrt{7} \fallingdotseq 2.714$.

EXAMPLE 4.6. If

$$C' = \frac{2\pi}{\sqrt{3}} \fallingdotseq 3.628, \tag{4.22}$$

the net of equilateral triangles has the optimal property. This pattern is represented in figure 13.

We may summarize the previous findings. By means of lemma 4.1, optimal regular nets of sample points were found for the following values of C' shown in table I.

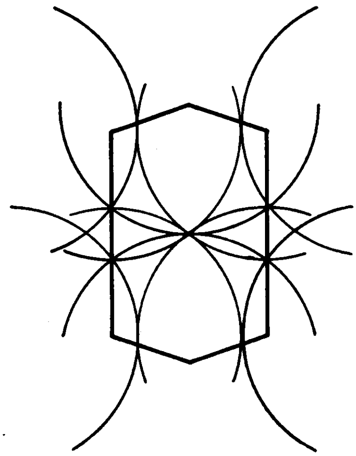

FIGURE 12

Optimum sampling pattern for $C' = \frac{16\pi}{7\sqrt{7}} \doteqdot 2.714$.

We observe that lemma 4.1 does not provide solutions for values of C' close to 1, 2, and 3 respectively. We conjecture that the same holds true for sufficiently large values of C'. This leads us to formulate the following

PROBLEM 4.1. *Determine the range of values of C' for which*

$$C'' = \{2[C'] + 1\}C' - [C']\{[C'] + 1\}. \tag{4.23}$$

In the solution of this problem the paper by Heppes [9] should prove valuable.

It is of some interest to compare, for given values of C', the corresponding

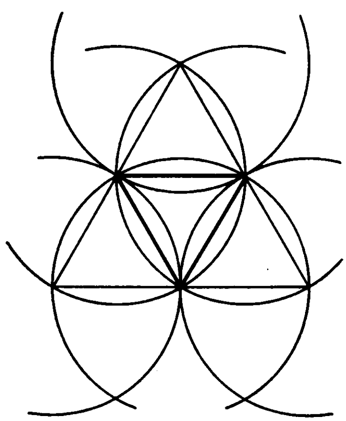

FIGURE 13

Optimum sampling pattern for $C' = \frac{2\pi}{\sqrt{3}} \doteqdot 3.628$.

TABLE I

SUMMARY OF VALUES OF C'

Example	
4.1	$C' \leqq \frac{\pi}{2\sqrt{3}} \risingdotseq 0.907$
4.2	$0.907 \risingdotseq \frac{\pi}{2\sqrt{3}} \leqq C' \leqq \frac{2\pi}{3\sqrt{3}} \risingdotseq 1.209$
4.3 and 4.4	$1.209 \risingdotseq \frac{2\pi}{3\sqrt{3}} \leqq C' \leqq \frac{\pi}{\sqrt{3}} \risingdotseq 1.814$
4.5	$2.418 \risingdotseq \frac{4\pi}{3\sqrt{3}} \leqq C' \leqq \frac{16\pi}{7\sqrt{7}} \risingdotseq 2.714$
4.6	$C' = \frac{2\pi}{\sqrt{3}} \risingdotseq 3.628$

values of C'' for different nets of sample points. In table II we present such a comparison.

The examples discussed above and given in table II show that the net of equilateral triangles is not, in general, the optimal net. In other words, this net does not yield the minimum limiting variance for an isotropic stochastic process in a plane with correlation function given by (3.5). The differences are, however, rather small and their sign alternates as C' increases. Therefore we state the following

PROBLEM 4.2. *Is it true that the net of equilateral triangles yields the minimum*

TABLE II

THE RELATION BETWEEN C'' AND THE PATTERN OF THE NET OF POINTS FOR SOME VALUES OF C'

C'		$[C']$	C'' when the Pattern of the Net of Points is		
Exact value	Num. value		Equilateral triangles	Squares	Equilateral Hexagons
$\frac{\pi}{2\sqrt{3}}$	0.907	0	0.907	0.988	1.165
$\frac{2\pi}{3\sqrt{3}}$	1.209	1	1.627	1.695	1.870
$\frac{\pi}{2}$	1.571	1	2.861	2.712	2.814
$\frac{4\pi}{3\sqrt{3}}$	2.418	2	6.395	6.389	6.110
$\frac{16\pi}{7\sqrt{7}}$	2.714	2	7.711	7.936	7.736

limiting variance for a stationary isotropic stochastic process with exponential correlation function?

An affirmative answer to this problem would imply the optimality of the net of equilateral triangles for all isotropic processes with any completely monotone correlation function. Numerical examples contained in Matérn [15] give some support to such an affirmative answer. Matérn compares the limiting variances for an isotropic process with correlation function $\exp(\mu)$, if triangular, square, and hexagonal nets of some chosen densities are used. The effect is that in all cases the triangular net proved to be somewhat better than the other ones.

Finally, we mention without formal proof the almost obvious relation concerning the limiting behavior of the mean double covering. Let us write it down as

LEMMA 4.2. *For any regular net of points the ratio of the mean double covering C'' and the square of the mean covering C'^2 tends to unity when the mean covering increases over all bounds; in symbols,*

$$\lim_{C' \to \infty} \frac{C''}{C'^2} = 1. \tag{4.24}$$

5. A class of planar isotropic correlation functions

In this section, which is due to Hájek and Zubrzycki, we discuss the class of convex planar correlation functions admitting representation

$$\rho(u) = \int_0^\infty r\left(\frac{u}{a}\right) dF(a), \tag{5.1}$$

where $F(a)$ is a distribution function with $F(0+) = 0$, and

$$r(u) = \begin{cases} \dfrac{2}{\pi}\left[\arccos u - u(1 - u^2)^{1/2}\right], & 0 \leqq u \leqq 1, \\ 0, & u > 1. \end{cases} \tag{5.2}$$

For some related results, see Hammersley and Nelder [8].

Clearly,

$$r'(u) = \begin{cases} -\dfrac{4}{\pi}(1 - u^2)^{1/2}, & 0 \leqq u < 1, \\ 0, & u > 1, \end{cases} \tag{5.3}$$

and

$$r''(u) = \begin{cases} \dfrac{4}{\pi}\dfrac{u}{(1 - u^2)^{1/2}}, & 0 \leqq u < 1, \\ 0, & u > 1. \end{cases} \tag{5.4}$$

LEMMA 5.1.

$$\int_0^\infty \frac{1}{u^2} r''\left(\frac{a}{u}\right) r''\left(\frac{u}{\mu}\right) du = \begin{cases} \dfrac{8}{\pi\mu}, & \mu > a > 0, \\ 0, & a > \mu. \end{cases} \tag{5.5}$$

PROOF. The case $a > \mu$ is clear. If $\mu > a$, we have, in view of (5.4),

$$\int_0^\infty \frac{1}{u^2} r''\left(\frac{a}{u}\right) r''\left(\frac{u}{\mu}\right) du = \left(\frac{4}{\pi}\right)^2 \int_a^\mu \frac{a\,du}{u[(u^2 - a^2)(\mu^2 - u^2)]^{1/2}} = \frac{8}{\pi\mu}. \tag{5.6}$$

THEOREM 5.1. *The correlation functions $\rho(u)$ of the type given by (5.1) are characterized by the following properties*

(a) *$\rho(u)$ is continuous, convex, and with $\rho(\infty) = 0$.*

(b) *$\rho'(u)$ is absolutely continuous.*

(c) $\int_0^\infty (1/u^2) r''(a/u)\rho''(u)\,du$ *is a* ~~*nondecreasing*~~ *function of a.*

The functions $F(a)$ and $\rho''(u)$ are linked by the inversion formulas

$$dF(a) = -\frac{\pi}{8} a^3\, d \int_0^\infty \frac{1}{u^2} r''\left(\frac{a}{u}\right) \rho''(u)\,du, \tag{5.7}$$

and

$$\rho''(u) = \int_0^\infty \frac{1}{a^2} r''\left(\frac{u}{a}\right) dF(a). \tag{5.8}$$

Condition (c) is fulfilled, for example, if $\rho''(u)/u$ is a nondecreasing function of u, which means, provided that $\rho''(u)$ is absolutely continuous, that

$$\rho''(u) - u\rho'''(u) \geqq 0, \qquad u \geqq 0. \tag{5.9}$$

If (5.9) holds, then $F(a)$ is absolutely continuous and therefore

$$\frac{dF(a)}{da} = -\frac{1}{2} a^2 \int_0^{\pi/2} \rho'''\left(\frac{a}{\sin\theta}\right) \frac{d\theta}{\sin^2\theta}. \tag{5.10}$$

PROOF. Property (a) follows easily from the corresponding property of correlation functions $r(u/a)$. Property (b), and simultaneously the relation given by (5.8), will be proved if we show that the indefinite integral of the right side of (5.8) equals $\rho'(u)$. Now

$$\begin{aligned}\int_u^\infty \int_0^\infty \frac{1}{a^2} r''\left(\frac{u}{a}\right) dF(a)\,du &= \int_0^\infty \frac{1}{a}\left[\int_u^\infty \frac{1}{a} r''\left(\frac{u}{a}\right) du\right] dF(a) \\ &= -\int_0^\infty \frac{1}{a} r'\left(\frac{u}{a}\right) dF(a) = -\rho'(u).\end{aligned} \tag{5.11}$$

The change of integration order is justified by Fubini's theorem, since $r''(u) \geqq 0$. The last identity follows from (5.1) by differentiation under the integral sign, which is justified, since $(1/a)r'(u/a)$ is uniformly bounded for $u > \epsilon > 0$.

In view of (5.8) we have, by using Fubini's theorem again,

$$\begin{aligned}\int_0^\infty \frac{1}{u^2} r''\left(\frac{a}{u}\right) \rho''(u)\,du &= \int_0^\infty \frac{1}{u^2} r''\left(\frac{a}{u}\right) \int_0^\infty \frac{1}{\mu^2} r''\left(\frac{u}{\mu}\right) dF(\mu)\,du \\ &= \int_0^\infty \frac{1}{\mu^2}\left[\int_0^\infty \frac{1}{u^2} r''\left(\frac{a}{u}\right) r''\left(\frac{u}{\mu}\right) du\right] dF(\mu),\end{aligned} \tag{5.12}$$

which gives, in accordance with (5.5),

$$\int_0^\infty \frac{1}{u^2} r''\left(\frac{a}{u}\right) \rho''(u)\, du = \frac{8}{\pi} \int_a^\infty \frac{1}{\mu^3}\, dF(\mu). \tag{5.13}$$

The last relation, however, is equivalent to (5.7).

Now assume that the correlation function $\rho(u)$ fulfills conditions (a), (b), and (c), and consider the function $F(a)$ given by (5.7). In view of property (c), $F(a)$ will be nondecreasing.

The fact that the total variation of $F(a)$ equals 1 follows from the subsequent lemmas 5.2 and 5.3 and from the following relations,

$$\begin{aligned} \int_0^\infty dF(a) &= -\frac{\pi}{8} \int_0^\infty a^3\, d \int_0^\infty \frac{1}{u^2} r''\left(\frac{a}{u}\right) \rho''(u)\, du\, da \\ &= \left[-\frac{\pi}{8} a^3 \int_0^\infty \frac{1}{u^2} r''\left(\frac{a}{u}\right) \rho''(u)\, du \right]_0^\infty \\ &\quad + \frac{3\pi}{8} \int_0^\infty \left[\int_0^\infty \frac{a^2}{u^2} r\left(\frac{a}{u}\right) da \right] \rho''(u)\, du \\ &= \int_0^\infty u\rho''(u)\, du = -\int_0^\infty \rho'(u)\, du \\ &= \rho(0) - \rho(\infty) = 1. \end{aligned} \tag{5.14}$$

It remains to show that $\rho'(u)$ is uniquely determined by relation (5.7), that is, that $\rho(u)$ coincides with the correlation function obtained from (5.1). However, (5.7) is equivalent to (5.13), if rewritten in the form

$$\frac{8}{\pi} \int_0^\infty \frac{1}{\mu^3}\, dF(\mu) = \int_0^\infty \frac{1}{u^2} r''\left(\frac{a}{u}\right) d\rho'(u). \tag{5.15}$$

Comparing (5.15) with (5.8) we can see that the $\rho'(u)$ may be determined from $(8/\pi) \int_a^\infty (1/\mu^3)\, dF(\mu)$ in the same way as $F(a)$ has been determined from $\rho''(u)$. The fact that $\rho'(u)$ may not have finite variation is irrelevant.

Before proceeding to the rest of the proof, we observe that by substituting $u = (a/\sin\theta)$ into (5.7) we get

$$dF(a) = -\frac{1}{2} a^3\, d \int_0^{\pi/2} \frac{\rho''\left(\dfrac{a}{\sin\theta}\right)}{\dfrac{a}{\sin\theta}}\, d\theta. \tag{5.16}$$

From this form it is easily seen that condition (c) is fulfilled if $\rho''(u)/u$ is a nondecreasing function of u, or, more especially, if (5.9) holds; notice that $\{[\rho''(u)]/u\}' = -1/u^2[\rho''(u) - u\rho'''(u)]$. Now if (5.9) holds, we can differentiate (5.16) under the integral sign (Fubini's theorem), which gives

$$(5.17) \qquad \frac{dF(a)}{da} = -\frac{1}{2} a^3 \int_0^{\pi/2} \left[\rho'''\left(\frac{a}{\sin\theta}\right)\frac{1}{a} - \rho''\left(\frac{a}{\sin\theta}\right)\frac{\sin\theta}{a^2}\right] d\theta$$

$$= -\frac{1}{2} a^2 \int_0^{\pi/2} \rho'''\left(\frac{a}{\sin\theta}\right) d\theta + \left[-\frac{1}{2} a\rho''\left(\frac{a}{\sin\theta}\right)\cos\theta\right]_0^{\pi/2}$$

$$-\frac{1}{2}\int a^2\rho'''\left(\frac{a}{\sin\theta}\right)\frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= -\frac{1}{2} a^2 \int_0^{\pi/2} \rho'''\left(\frac{a}{\sin\theta}\right)\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= -\frac{1}{2} a^2 \int_0^{\pi/2} \rho'''\left(\frac{a}{\sin\theta}\right)\frac{d\theta}{\sin^2\theta}.$$

The relation $\rho''(\infty) = 0$ which we have used follows from the subsequent lemma 5.3. Our theorem is thus completely proved.

LEMMA 5.2.

$$(5.18) \qquad \int_0^\infty u^2 r''(u)\, du = \frac{8}{3\pi}.$$

PROOF. Integrating by parts, we have

$$(5.19) \qquad \int_0^\infty u^2 r''(u)\, du = -\int_0^\infty 2ur'(u)\, du$$

$$= \frac{8}{\pi}\int_0^1 u(1 - u^2)^{1/2} = \frac{8}{3\pi}.$$

LEMMA 5.3. *Any correlation function $\rho(u)$, which fulfills conditions* (a), (b), *and* (c) *of theorem* 5.1, *has the properties*

$$(5.20) \qquad \lim_{a\to 0} a^3 \int_0^\infty \frac{1}{u^2} r''\left(\frac{a}{u}\right)\rho''(u)\, du = \lim_{a\to\infty} a^3 \int_0^\infty \frac{1}{u^2} r''\left(\frac{a}{u}\right)\rho''(u)\, du = 0$$

and

$$(5.21) \qquad \lim_{a\to 0} a\rho'(a) = \lim_{a\to\infty} a\rho'(a) = 0.$$

If, moreover, $\rho''(u)/u$ is non~~decreasing~~, then

$$(5.22) \qquad \lim_{a\to 0} a^2\rho''(a) = \lim_{a\to\infty} a^2\rho''(a) = 0.$$

PROOF. Since $\rho(u)$ is convex, $-\rho'$ is nonnegative and nonincreasing, and we have

$$(5.23) \qquad 0 \leqq -\frac{1}{2} a\rho'(a) \leqq -\int_{a/}^{a} \rho'(u)\, du = \rho\left(\frac{a}{2}\right) - \rho(a).$$

Since $\rho(u)$ is continuous at the points 0 and ∞, (5.21) is clear.

Now, if (5.9) holds true, the function $\rho''(u)/u$ is nonnegative and nonincreasing, so that

$$
\begin{aligned}
(5.24)\qquad 0 \leqq \frac{1}{3}\, a^2\rho''(a) &= \frac{1}{3}\, a^3\, \frac{\rho''(a)}{a} \leqq a^2 \int_{a/2}^{a} \frac{\rho''(u)}{u}\, du \\
&\leqq 2a \int_{a/2}^{a} \rho''(u)\, du = 2a \left[\rho'(a) - \rho'\left(\frac{1}{2}\, a\right) \right],
\end{aligned}
$$

where the expression converges to 0 if $a \to 0$ or $a \to \infty$.

The same consideration will be used in proving (5.20). In view of condition (c) we have

$$
\begin{aligned}
(5.25)\qquad 0 &\leqq \frac{1}{3}\, a^3 \int_0^\infty \frac{1}{u^2}\, r''\left(\frac{a}{u}\right) \rho''(u)\, du \\
&\leqq \int_{a/2}^{a} \int_0^\infty \frac{\mu^2}{u^2}\, r''\left(\frac{\mu}{u}\right) \rho''(u)\, du\, d\mu \\
&= \int_0^\infty \left[\int_{a/2}^{a} \frac{\mu^2}{u^2}\, r''\left(\frac{\mu}{u}\right) d\mu \right] \rho''(u)\, du \\
&= \int_0^\infty \left[\int_{a/2u}^{a/u} u^2 r''(u)\, du \right] u\rho''(u)\, du,
\end{aligned}
$$

where, using (5.18)

$$
(5.26)\qquad \int_{a/2u}^{a/u} u^2 r''(u)\, du \begin{cases} \leqq \dfrac{8}{3\pi}, & \dfrac{a}{u} > 0, \\[2ex] < \dfrac{8}{3\pi}\, \epsilon, & \dfrac{a}{u} \leqq \epsilon, \\[2ex] = 0, & \dfrac{a}{u} > 1. \end{cases}
$$

Consequently, on the one hand,

$$
(5.27)\qquad \int_0^\infty \left[\int_{a/2u}^{a/u} u^2 r''(u)\, du \right] u\rho''(u)\, du \leqq \frac{8}{3\pi} \left[\epsilon + \int_0^{a/\epsilon} u\rho''(u)\, du \right]
$$

and, on the other hand,

$$
(5.28)\qquad \int_0^\infty \left[\int_{a/2u}^{a/u} u^2 r''(u)\, du \right] u\rho''(u)\, du \leqq \frac{8}{3\pi} \int \quad u\rho''(u)\, du.
$$

The inequalities (5.25) together with the inequality (5.27) or (5.28) prove the relation (5.20) for $a \to 0$ or $a \to \infty$, respectively. Notice that $u\rho''(u)$ is integrable with $\int_0^\infty u\rho''(u) = 1$.

EXAMPLE 5.1. The convex correlation function $\exp(-cu)$ has a negative third derivative and therefore fulfills condition (5.9). Hence it admits the representation (5.1), where the spectral density is given by (5.10).

EXAMPLE 5.2. The convex correlation function

$$\rho(u) = \frac{2}{\sqrt{2\pi}} \int_u^\infty e^{-u^2/2}\, du \tag{5.29}$$

also admits the representation (5.1), since $\rho''(u)/u = (1/\sqrt{2\pi}) \exp(-u^2/2)$ is a nonincreasing function of u.

EXAMPLE 5.3. The linear convex correlation function

$$\rho(u) = \begin{cases} 1 - u, & u \leqq 1, \\ 0, & u \geqq 1, \end{cases} \tag{5.30}$$

has a discontinuous first derivative, and therefore does not admit the representation (5.1). It may be shown, however, that $\rho(u)$ is not a planar isotropic correlation function. Actually, considering a square net of points with coordinates

$$p_{ij} = \left(\frac{i}{\sqrt{2}}, \frac{j}{\sqrt{2}}\right), \qquad 0 \leqq i, j \leqq n - 1, \tag{5.31}$$

we have

$$\sum_{i,j=0}^{n} \sum_{i',j'=0}^{n} \rho(|p_{ij} - p_{i'j'}|)(-1)^{i+j+i'+j'} = n^2 - 4\left(1 - \frac{1}{\sqrt{2}}\right) n(n - 1), \tag{5.32}$$

which is negative for a sufficiently large n.

Example 5.3 shows that the class of planar isotropic convex correlation functions is smaller than the class of linear convex correlation functions. One might suspect that all planar isotropic correlation functions are expressible in the form (5.1). This is, however, disproved by

THEOREM 5.2. *There exist isotropic stationary stochastic processes in a plane with correlation function $g(x, y) = f[(x^2 + y^2)^{1/2}]$ such that $f(u)$, where $0 \leqq u < \infty$, is a convex function which cannot be represented in the form*

$$f(u) = \int_0^\infty r\left(\frac{u}{a}\right) dF(a), \tag{5.33}$$

where

$$r(u) = \begin{cases} \dfrac{2}{\pi} \{\arccos u - u(1 - u^2)^{1/2}\}, & 0 \leqq u \leqq 1, \\ 0, & 1 \leqq u < \infty, \end{cases} \tag{5.34}$$

and $F(a)$ is a distribution function with $F(0+) = 0$.

This theorem follows from the following two lemmas.

LEMMA 5.4. *The function $g(x, y) = f[(x^2 + y^2)^{1/2}]$, where*

$$f(u) = \begin{cases} \dfrac{2}{\pi}\left(\dfrac{\pi}{2} - u\right), & 0 \leqq u \leqq 1, \\ \dfrac{2}{\pi}\left\{\left(\arcsin \dfrac{1}{u}\right) - [u - (u^2 - 1)^{1/2}]\right\}, & 1 \leqq u \leqq \infty, \end{cases} \tag{5.35}$$

is a correlation function of a stationary isotropic stochastic process (*see figure* 14).

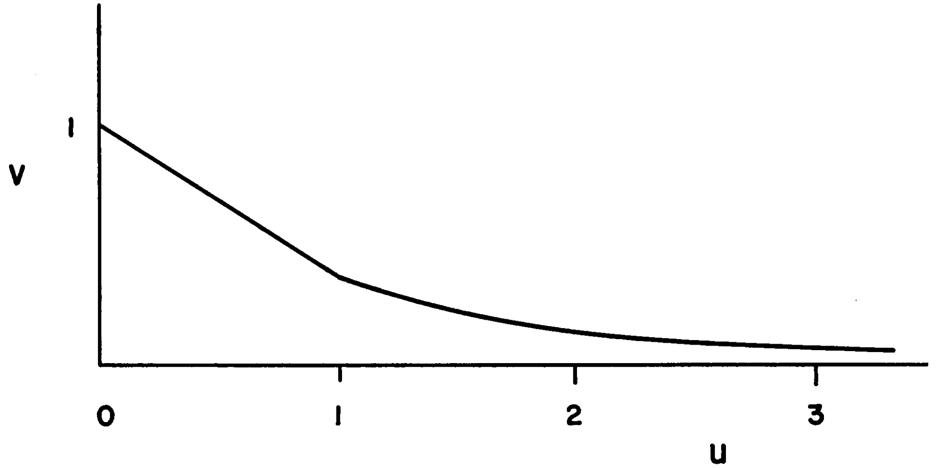

FIGURE 14

$v = f(u)$, with $f(u)$ given by (5.35).

LEMMA 5.5. *If the function* $f(u)$ *is representable in the form* (5.33), *then* $f''(u) > 0$ *for all* $u > 0$ *with* $u < a$, *where* $F(a) < 1$.

To prove lemma 5.4 we consider a linear stationary stochastic process $\eta(t)$ with correlation function $h(t)$ given by

$$h(t) = \begin{cases} 1 - |t|, & |t| \leqq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{5.36}$$

Define then a plane stochastic process $\xi(x, y)$ by putting

$$\xi(x, y) = \eta(x \cos \alpha, y \sin \alpha), \tag{5.37}$$

where α is a random variable independent of the process $\eta(t)$ with

$$P\{\alpha < a\} = \begin{cases} 0, & a \leqq 0, \\ \dfrac{a}{2\pi}, & 0 \leqq a \leqq 2\pi, \\ 1, & 2\pi \leqq a. \end{cases} \tag{5.38}$$

In other words we first define a plane stochastic process which depends only upon one coordinate and has with respect to it correlation function $h(t)$, and then we randomize the direction. It is seen that $\xi(x, y)$ is an isotropic stationary stochastic process with correlation function $g(x, y) = f[(x^2 + y^2)^{1/2}]$, where

$$f(u) = \frac{1}{2\pi} \int_0^{2\pi} [1 - h(u \sin \psi)]\, d\psi, \tag{5.39}$$

which leads to (5.35).

To prove lemma 5.5 we note that the second derivative of a function $f(u)$ given by (5.33) is given by

$$f''(u) = \int_0^\infty \frac{1}{a^2} r''\left(\frac{u}{a}\right) dF(a), \tag{5.40}$$

where

$$r''(u) = \begin{cases} \dfrac{4}{\pi} \dfrac{u}{(1-u^2)^{1/2}} > 0, & 0 < u < 1, \\ 0, & 1 < u < \infty. \end{cases} \tag{5.41}$$

This proves lemma 5.5.

Now the function $f(u)$ given by (5.35) has a second derivative which vanishes for $0 < u < 1$. This contradicts lemma 5.5 and thus proves our theorem 5.2.

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