ON THE PROBABILISTIC THEORY OF COMPLEX STRUCTURES

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1. Introduction

The advent of exceedingly complex electrical and mechanical systems, such as high-speed electronic computers, guidance systems for missiles, or the mechanical structure of a contemporary airplane, has made it a matter of great importance to follow up the design of such a system by some evaluation of the probability that it will perform its task without failure. If the probabilities of successful performance are known for each component of a structure, then it is usually theoretically possible to compute the probability that the entire structure will perform by carefully tracing the design step by step. This procedure may become prohibitive when the structure becomes very complex, that is, consists of a very large number of components.

It is the purpose of this report to present some results obtained by J. D. Esary, S. C. Saunders, and the writer, which deal with certain properties of complex structures and their reliabilities, and either do not depend on the number of components, or else display an asymptotic behavior as the number of components increases. In section 2, a number of properties of structures will be discussed which may be described as combinatorial and which, while interesting in themselves, are preliminary to a probabilistic discussion of structure reliability. In section 3 the probability of successful performance will be studied for the kinds of structures analyzed in section 2. A number of our results are generalizations ideas of originated by von Neumann [1] and systematically developed by Moore and Shannon [2]. The present report is meant to give a survey of the main findings and does not contain the mathematical derivations of the theory. A detailed mathematical presentation is contained in a manuscript submitted for publication [3].

2. Structures and their combinatorial properties

2.1. *Dichotomic structures.* We will limit our presentation to devices (components as well as systems built of components) which can be in only one of two states: they either perform or fail. The state will be described by the value of an indicator variable which will be given the value 1 when the device performs and

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the value 0 when it fails. In studies of this kind, methods of Boolean algebra could be applied, but we will not make use of this mathematical tool.

It should be mentioned that the limitation of the possible states of a device to performance or failure only is in many instances an oversimplification of the real situation. To emphasize this simplifying assumption, we shall refer to structures capable only of these two states as "dichotomic structures." In reality one frequently encounters devices which are capable of a continuous range of partial performance, from perfect performance to total failure. A large part of the statements reported in the following can be generalized to such structures with partial performance, but a detailed theory of this kind has not yet been fully developed.

2.2. *Structure functions.* By a *structure function* of order n we will understand a function $\phi(x_1, x_2, \cdots, x_n)$ which ascribes to each of the 2^n vertices $(0, 0, \cdots, 0, 0)$, $(0, 0, \cdots, 0, 1)$, $(0, 0, \cdots, 1, 0)$, $\cdots$, $(1, 0, \cdots, 0, 0)$, $\cdots$, $(1, 1, \cdots, 1, 1)$ of the n-dimensional unit cube one of the values 0 or 1.

The intuitive interpretation of such a structure function is that it describes a design such that $x_1, x_2, \cdots, x_n$ are the indicator variables corresponding to the first, second, and nth component, and ϕ ascribes to each state of these n components a state of the entire structure. Instead of "structure function ϕ" we shall often say in short "structure ϕ."

To each *state vector* $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ we ascribe a nonnegative integer defined by

$$S(\mathbf{x}) = S(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} x_i \tag{2.2.1}$$

called the *size* of that state vector. Clearly $S(\mathbf{x})$ is the number of components which perform at the particular state $\mathbf{x}$ of the components.

A state vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is called a *path* for the structure ϕ if $\phi(\mathbf{x}) = \phi(x_1, x_2, \cdots, x_n) = 1$, and it will be called a *cut* if $\phi(\mathbf{x}) = \phi(x_1, x_2, \cdots, x_n) = 0$.

We furthermore define *path numbers* and *cut numbers* for a given structure ϕ as follows:

$$A_j = j\text{th path number} = \text{number of paths of size } j \text{ for } \phi, \tag{2.2.2}$$

$$A_j^* = j\text{th cut number} = \text{number of cuts of size } j \text{ for } \phi. \tag{2.2.3}$$

One verifies easily that

$$A_j + A_j^* = \binom{n}{j}. \tag{2.2.4}$$

2.3. *Coherent structures.* Not all of the possible 2^{2^n} structures of order n are of equal practical importance. In particular, practically all designs occurring in practice have the property that, vaguely speaking, the better the components perform the better the performance of the structure. To give a precise formulation of this property of structures we state the following definitions.

We say that $(x_1, \cdots, x_n) \geqq (y_1, \cdots, y_n)$ when $x_i \geqq y_i$ for $i = 1, \cdots, n$.

A structure ϕ is called *coherent* when

(2.3.1) $\phi(x_1, \cdots, x_n) \geqq \phi(y_1, \cdots, y_n)$ for all $(x_1, \cdots, x_n) \geqq (y_1, \cdots, y_n)$

and

(2.3.2) $$\phi(0, 0, \cdots, 0) = 0, \qquad \phi(1, 1, \cdots, 1) = 1.$$

The assumption that a structure is coherent imposes considerable restrictions on its path numbers and cut numbers. For example, one can prove that the path numbers for a coherent structure must satisfy the inequalities

(2.3.3) $$\frac{A_j}{\binom{n}{j}} \leqq \frac{A_{j+1}}{\binom{n}{j+1}} \qquad \text{for} \quad j = 0, 1, \cdots, n-1.$$

The class of coherent structures has a number of useful properties. Most of them will be discussed in the next section but it should be mentioned here that the combinatorial theory of reduction of structures, that is, of replacing a given structure by an equivalent structure with the smallest possible number of components, takes a particularly simple form for coherent structures.

In the practice of designing circuits, or mechanical structures, one frequently proceeds by first designing a structure of fairly low order and later on replacing each component by a structure which itself consists of a number of components. This is for example the case when "moduls" are put into the place of single components in circuit design. This practical procedure has its counterpart in a formal operation applied to structure functions which is called "combination" of structures. The class of coherent structures is closed under the operation of combination, that is, when all structures used are coherent then the structure obtained by combination is coherent.

3. The reliability function

3.1. *Assumptions and definitions.* From now on it will be assumed that all components of a given structure may perform or fail in a random manner and independently from each other. More specifically, for a structure ϕ of order n we assume that its n components are independent random variables $\mathbf{X} = (X_1, \cdots, X_n)$ each with the same probability distribution

(3.1.1) $$P\{X_i = 1\} = p, \qquad P\{X_i = 0\} = 1 - p, \qquad i = 1, 2, \cdots, n,$$

so that $\phi(\mathbf{X}) = \phi(X_1, \cdots, X_n)$ is also a random variable capable of the values 1 and 0.

The *reliability function* $h(p)$ is defined by

(3.1.2) $$h_\phi(p) = E[\phi(\mathbf{X}); p] = P\{\phi(\mathbf{X}) = 1; p\}.$$

One verifies easily that the reliability function for a given structure ϕ can be expressed in terms of the path numbers of that structure by the formula

(3.1.3) $$h_\phi(p) = \sum_{j=0}^{n} A_j p^j (1-p)^{n-j}.$$

For ϕ coherent one clearly has $A_0 = 0$, $A_n = 1$.

For any structure ϕ, the size $S(\mathbf{X})$ of the random vector is a random variable with a binomial distribution, $\phi(\mathbf{X})$ a random variable with the probabilities $P\{\phi(\mathbf{X}) = 1\} = h(p)$ and $P\{\phi(\mathbf{X}) = 0\} = 1 - h(p)$, and $S(\mathbf{X})$ and $\phi(\mathbf{X})$ are dependent random variables. It is natural to consider the conditional expectations

(3.1.4) $$E[S(\mathbf{X})|\phi(\mathbf{X}) = 1] = L(p) = \text{mean path,}$$

(3.1.5) $$E[n - S(\mathbf{X})|\phi(\mathbf{X}) = 0] = W(p) = \text{mean cut for the structure } \phi.$$

Using the same pair of random variables $\phi(\mathbf{X})$, $S(\mathbf{X})$ we introduce the definition:

A structure ϕ is *coherent in probability* when

(3.1.6) $$P\{\phi(\mathbf{X}) = 1|S(\mathbf{X}) = k\} \leqq P\{\phi(\mathbf{X}) = 1|S(\mathbf{X}) = k+1\} \quad \text{for } k = 0, 1, \cdots, n-1$$

and

(3.1.7) $$P\{\phi(\mathbf{X}) = 1|S(\mathbf{X}) = 0\} = 0, \qquad P\{\phi(\mathbf{X}) = 1|S(\mathbf{X}) = n\} = 1.$$

3.2. *Propeities of the reliability function.* If ϕ is coherent in probability, then $h(p)$ is nondecreasing for $0 \leqq p \leqq 1$. Since a coherent structure is always coherent in probability, a coherent structure must always have a nondecreasing reliability function.

For any structure ϕ of order n (without assumption of coherence) the inequality

(3.2.1) $$L(p) + W(p) > n$$

is necessary and sufficient for the reliability function $h(p)$ being strictly increasing for $0 < p < 1$. The inequality

(3.2.2) $$L(p) + W(p) > n + 1$$

is necessary and sufficient in order that the function

(3.2.3) $$\sigma_h(p) = \frac{h(p)}{1 - h(p)} \frac{1-p}{p}$$

be strictly increasing.

If ϕ is a coherent structure then either (a) $h(p) = p$ or (b) $\sigma_h(p)$ is strictly increasing and $h(0) = 0$, $h(1) = 1$.

If ϕ is a coherent structure then the conditions

(3.2.4) $$A_1 = 0 \quad \text{and} \quad A_{n-1} = n$$

are necessary and sufficient in order that $\sigma_h(p)$ assume the value 1 in $0 < p < 1$.

The last statement has an important intuitive interpretation. It is easily seen that $h(0) = 0$, $h(1) = 1$, $\sigma_h(p)$ strictly increasing, and $\sigma_h(\rho) = 1$ for some

$0 < \rho < 1$, imply that $h(p) < p$ for $0 < p < \rho$ and $h(p) > p$ for $\rho < p < 1$, so that $h(p)$ is an S-shaped function, that is, has the form indicated in figure 1. We see therefore that any coherent structure for which $A_1 = 0$ and $A_{n-1} = n$ has an S-shaped reliability function.

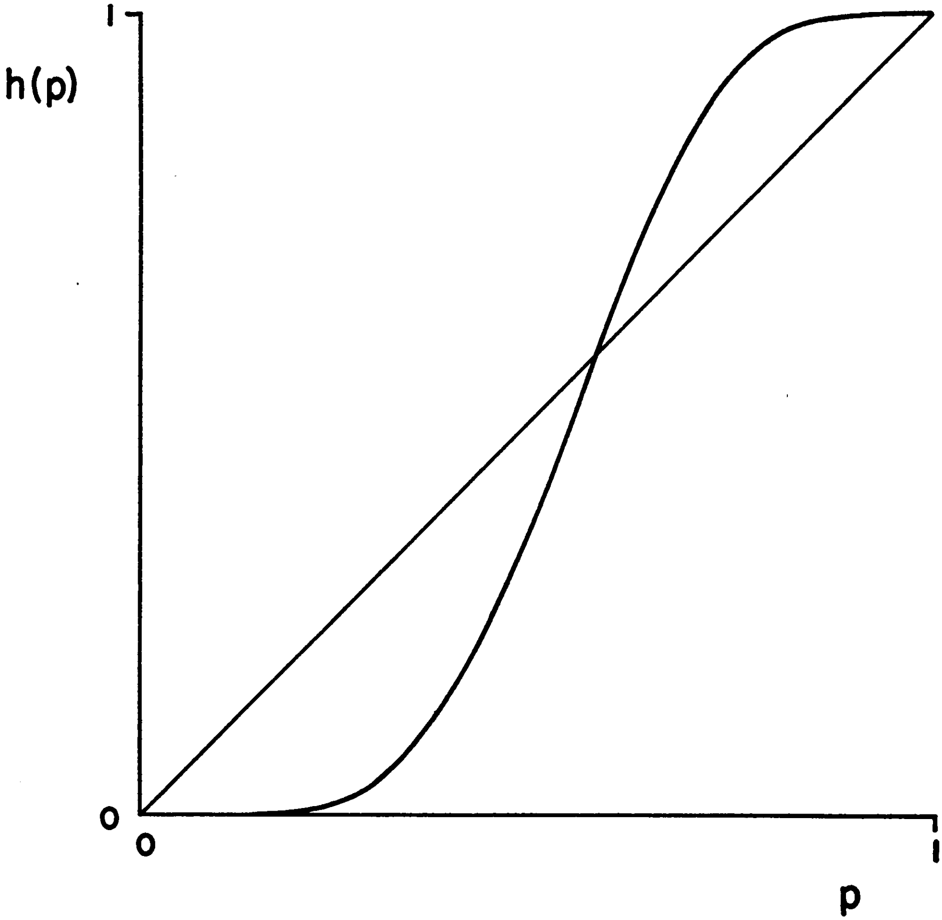

FIGURE 1

S-shaped reliability function

This establishes the fact that the very wide class of coherent structures, for which there are no paths of size 1 and all vectors of size $n - 1$ are paths, have S-shaped reliability functions, and therefore exhibit two practically important properties:

(1) There exists a value ρ such that for $p < \rho$ the structure is less reliable and for $p > \rho$ more reliable than a single component.

(2) We consider *iterated combinations* $\phi_1, \phi_2, \cdots, \phi_{j+1}, \cdots$ of a given structure ϕ, defined as follows:

$$\begin{aligned}
&\phi_1(X_1, X_2, \cdots, X_n) = \phi(X_1, X_2, \cdots, X_n),\\
&\phi_2(X_1, X_2, \cdots, X_{n^2})\\
&\quad = \phi_1[\phi(X_1, \cdots, X_n), \phi(X_{n+1}, \cdots, X_{2n}), \cdots, \phi(X_{n(n-1)+1}, \cdots, X_{n^2})],\\
&\cdots,\\
&\phi_{j+1}(X_1, X_2, \cdots, X_{n^j})\\
&\quad = \phi_j[\phi(X_1, \cdots, X_n), \phi(X_{n+1}, \cdots, X_{2n}), \cdots, \phi(X_{n(n^j-1)+1}, \cdots, X_{n^{j+1}})].\\
&\cdots.
\end{aligned} \tag{3.2.5}$$

If $h(p)$ is the reliability function for ϕ, then the reliability functions for $\phi_1, \phi_2, \cdots, \phi_{j+1}, \cdots$ are

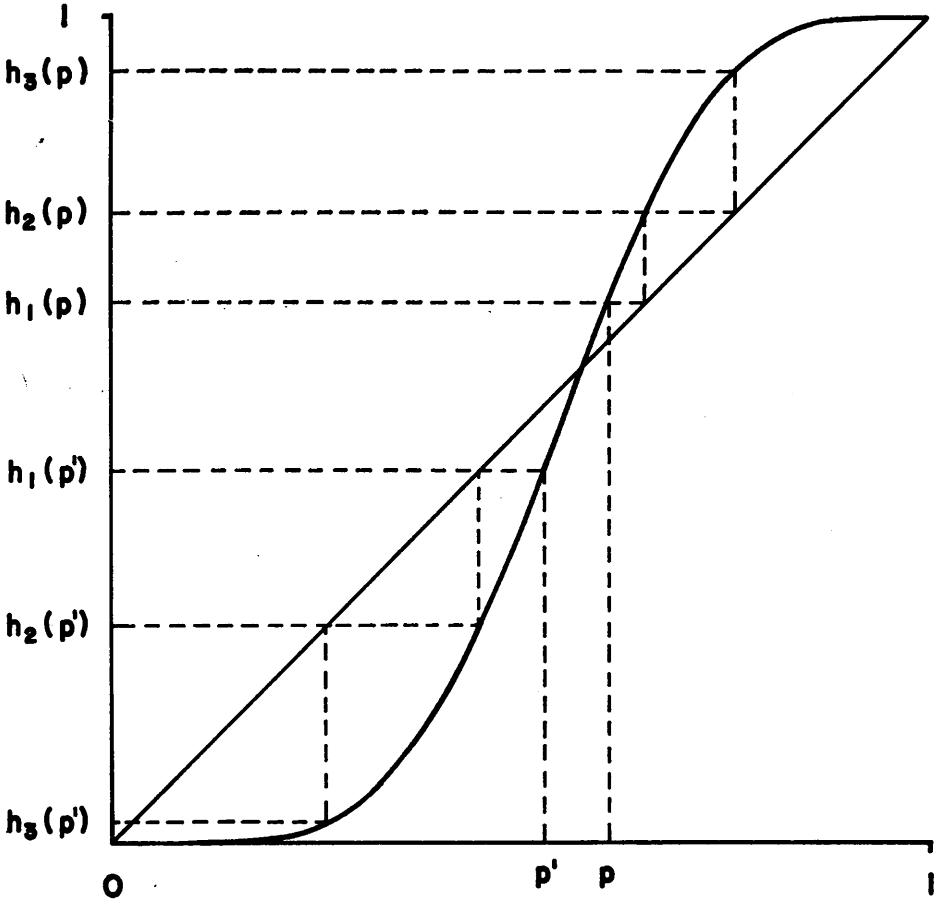

FIGURE 2

Ordering of $h_j(p)$ for an S-shaped reliability function.

$$
\begin{aligned}
h_1(p) &= h(p), \\
h_2(p) &= h_1[h(p)], \\
&\cdots, \\
h_{j+1}(p) &= h_j[h(p)], \\
&\cdots .
\end{aligned}
\tag{3.2.6}
$$

For a structure ϕ whose reliability function $h(p)$ is S-shaped, an argument indicated in figure 2 shows that

$$h_1(p) > h_2(p) > \cdots > h_{j+1}(p) \underset{j\to\infty}{\longrightarrow} 0, \qquad p < \rho, \tag{3.2.7}$$

and

$$h_1(p) < h_2(p) < \cdots < h_{j+1}(p) \underset{j\to\infty}{\longrightarrow} 1, \qquad p > \rho, \tag{3.2.8}$$

so that by iterated compositions of ϕ one can obtain structures whose reliability functions are as close to 1 as desired if the component reliability is $p > \rho$, but tend to 0 if $p < \rho$.

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