RANDOM SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction

In many parts of theoretical physics one has to solve a mathematical problem of the following type:

Given (a) a partial differential equation

$$\mathcal{A}(u) = 0 , \tag{1}$$

and (b) an open domain D with boundary C (in an n-dimensional Euclidean space), find a function $u(P)$ satisfying (1) in D and taking given values $f(A)$ on C,

$$u(P) \to f(A) \quad \text{if} \quad P \to A . \tag{2}$$

In each case one has to specify: (a) what conditions (continuity, boundedness, existence of derivatives, etc.) $u(P)$ must satisfy in order to be considered a regular solution; (b) what precise meaning the limit in (2) has (limit along given paths, limit in the mean, etc.); and (c) what class of functions $f(A)$ (continuous, integrable, etc.) is one allowed to use for the boundary conditions, in order to secure existence and uniqueness theorems.

When one looks at the physical facts which give support to this mathematical setting, one is readily convinced that, due to errors in measurements, to neglected fluctuations in the phenomena, etc., the boundary conditions cannot be expressed by only one well-determined function $f(A)$, but that a whole set of functions $f_\omega(A)$ must be considered. Furthermore, among these we cannot identify the one that will actually materialize in an experiment; in general, we are only able to say that some functions in the set are more likely to be observed than some others. We can translate this idea in mathematical language by saying that, in the boundary conditions corresponding to the real physical problem, the function $f(A)$ must be replaced by a random function $f_\omega(A)$, where ω represents a point in a suitable probability space $(\Omega, \mathfrak{F}, \mu)$. We have been led to this point of view by our research on the statistical theory of turbulence [2] and [3]. In our opinion the key problem in this theory is to find random solutions of the Navier-Stokes equations corresponding to a given random velocity field at time $t = 0$. But, unfortunately, since the equations of fluid dynamics are nonlinear we know almost nothing about existence and uniqueness of their solutions. Since we are attracted to this point of view and yet unable to solve this most interesting problem, we have had to satisfy ourselves by considering only linear partial differential equations. In this case it is possible to prove that the consideration of random boundary values makes sense and to build a rather complete theory of the corresponding random solutions.

We shall say that $u_\omega(P)$ is a random solution of (1) corresponding to the random boundary values $f_\omega(A)$ if

(a) for each fixed $\omega_0 \in \Omega - \Lambda$ (where $\mu(\Lambda) = 0$), the sample function $u_{\omega_0}(P)$ is a regular solution of (1) in D and takes the value $f_{\omega_0}(A)$ on C,

(b) for each fixed $P \in D$, $u_\omega(P)$ is μ-measurable.

When, in a particular problem, one has been able to prove the existence of a random solution, the next step is to look at its statistical properties and, first of all, to compute the moments. To begin with, one must give sufficient conditions on the random function $f_\omega(A)$ so that the existence of the moment

$$\gamma(A_1, \cdots, A_k) = E[f_\omega(A_1) \cdots f_\omega A_k)] \qquad A_1, \cdots, A_k \in C \tag{3}$$

implies the existence of

$$\Gamma(P_1, \cdots, P_k) = E[u_\omega(P_1) \cdots u_\omega(P_k)] \qquad P_1, \cdots, P_k \in D. \tag{4}$$

If this is the case one would expect, from heuristic considerations, that Γ is a solution of (1) with respect to each of the points P,

$$\mathcal{A}_{P_1}(\Gamma) = 0\,, \qquad \mathcal{A}_{P_k}(\Gamma) = 0 \tag{5}$$

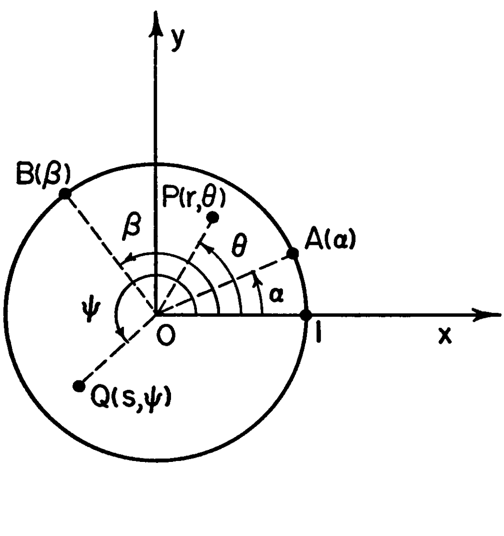
and that

$$\Gamma(P_1, \cdots, P_k) \to \gamma(A_1, \cdots, A_k) \qquad P_1 \to A_1, \cdots, P_k \to A_k. \tag{6}$$

Rather than confine ourselves to this rather general plane we find it more illuminating to look in some detail at two particular cases. We have chosen them to be as different as possible: one elliptic equation in a finite domain, one parabolic equation in an infinite domain.

2. Random harmonic functions in the unit circle

A function $u(P)$ is harmonic in D if (a) $u(P)$ is continuous in D, (b) $u_{x_1}(P) \cdots u_{x_n}(P)$ exist and are continuous in D, (c) $u_{x_1x_1}(P), \cdots, u_{x_nx_n}(P)$ exist and are finite in D, and (d) $u_{x_1x_1} + \cdots + u_{x_nx_n} = 0$ in D. Here we shall consider only the case in which D is the interior of the unit circle. We shall say that $u(P) \in Ha$ if it is the difference



of two nonnegative harmonic functions in the unit circle. Let us introduce the kernel

$$k(r,\ \theta) = \frac{1}{2\pi}\,\frac{1 - r^2}{1 - 2r\cos\theta + r^2}\,. \tag{7}$$

There is a one-to-one correspondence between the functions of class Ha and the signed Borel measures on C, given by the Poisson-Stieltjes integral

$$u(P) = \int_0^{2\pi} k(r,\ \theta - \alpha)d\nu\ . \tag{8}$$

The signed Borel measure ν is the difference $\nu = \nu^+ - \nu^-$ of two Borel measures; a Borel measure is itself a set function defined for every set $E \in \mathcal{B}$ (where $\mathcal{B}$ is the Borel field generated by the intervals $0 \leqq \alpha_1 \leqq \alpha < \alpha_2 \leqq 2\pi$), completely additive and finite

$$0 \leqq \nu(E) \leqq \nu(C) < +\infty\ . \tag{9}$$

Let us consider an arbitrary probability space $(\Omega, \mathfrak{F}, \mu)$.

DEFINITION 1. *$\nu_\omega(E)$ is a random measure on C if*

(a) *$\nu_\omega(E)$ is a real-valued function on $\mathcal{B} \times \Omega$,*

(b) *for each $\omega \in \Omega$, the sample ν_ω is a signed Borel measure,*

(c) *for each $E \in \mathcal{B}$, $\nu_\omega(E)$ is μ-measurable.*

THEOREM 1. *Given a random measure ν_ω on C, there is one and only one random function $u_\omega(P)$ in D such that, for each sample,*

(A) $u_\omega(P) \in Ha$,

(B) $\lim_{r\uparrow 1} \int_0^{\phi} u_\omega(r,\ \theta)d\theta = \nu_\omega(J_\phi)\ , \qquad J_\phi = \{\alpha\colon 0 \leqq \alpha < \phi \leqq 2\pi\}\ .$

This random function $u_\omega(P)$ is defined by the Poisson-Stieltjes integral

$$u_\omega(P) = \int_0^{2\pi} k(r,\ \theta - \alpha)d\nu_\omega\ . \tag{10}$$

The fact that the asserted properties are true for any sample is an obvious consequence of well-known theorems of the classical theory of harmonic functions (see, for instance, pp. 46–54 in [1]); the proof that, for any fixed $P \in D$, $u_\omega(P)$ is μ-measurable is almost trivial here. Since $(r,\ \theta)$ is fixed the kernel $k(r,\ \theta - \alpha)$ is a continuous function of α in $[0, 2\pi]$ and one has

$$m = \frac{1}{2\pi}\,\frac{1 - r}{1 + r} \leqq k(r,\ \theta - \alpha) \leqq \frac{1}{2\pi}\,\frac{1 + r}{1 - r} = M\ . \tag{11}$$

Divide the interval $[m, M]$ into N parts and consider the approximating sum

$$\sum_1^N a_k\nu_\omega(E_k)\ . \tag{12}$$

Each term is μ-measurable by hypothesis and thus the sum is measurable and so is the integral (10), as limit of such sums.

Let us turn now to the case where the boundary conditions on C are expressed by a random function $f_\omega(\alpha)$. Let us call m the Lebesgue measure defined on C by giving the length $\alpha_2-\alpha_1$ to the interval $0\leqq\alpha_1\leqq\alpha<\alpha_2\leqq 2\pi$. We shall say that $g(\alpha)\in L(C)$ if the Lebesgue integral $\int_0^{2\pi}|g(\alpha)|d\alpha$ exists.

DEFINITION 2. *A random function* $f_\omega(\alpha)$ *belongs to the class* K *if*

(K_1) $f_\omega(\alpha)$ *is measurable with respect to the product measure* $m\times\mu$

(K_2) *for each* $\alpha\in C$, $f_\omega(\alpha)\in L^2(\Omega)$.

Introduce the covariance

$$E[f_\omega(\alpha)f_\omega(\beta)]=\gamma(\alpha,\beta)\,, \tag{13}$$

(K_3) $\gamma(\alpha,\alpha)\in L(C)$.

Let us remark that since Ω is of finite measure, $(K_2)\Rightarrow f_\omega(\alpha)\in L(\Omega)$, and thus $E[f_\omega(\alpha)]=F(\alpha)$ exists for all $\alpha\in C$.

LEMMA 1. $f_\omega(\alpha)\in K\Rightarrow f_\omega(\alpha)\in L^2(C)$ *with probability* 1 (*that is, if* $\omega\in\Omega-\Lambda$, $\mu(\Lambda)=0$).

PROOF. $f_\omega(\alpha)$ is $m\times\mu$-measurable and we have by (K_3) that

$$\int_0^{2\pi}\left[\int_\Omega|f_\omega(\alpha)|^2d\mu\right]d\alpha<+\infty\,. \tag{14}$$

Thus, by Tonelli's theorem this implies that $f_\omega(\alpha)\in L^2(C\times\Omega)$. Now the result follows by Fubini's theorem.

Note that the measure of C being finite, lemma $1\Rightarrow f_\omega(\alpha)\in L(C)$ for $\omega\in\Omega-\Lambda$. Thus if we put

$$\nu_\omega(E)=\int_E f_\omega(\alpha)d\alpha,\qquad E\in\mathcal{B}\,,\qquad \omega\in\Omega-\Lambda\,, \tag{15}$$

we define a random Borel-measure, to which we can apply theorem 1. Thus

$$u_\omega(P)=\int_0^{2\pi}k(r,\,\theta-\alpha)f_\omega(\alpha)d\alpha \tag{16}$$

is the unique random function satisfying, for all $\omega\in\Omega-\Lambda$,

(A) $u_\omega(P)\in Ha$

(B′) $\lim_{r\uparrow 1}u_\omega(P)=f_\omega(\theta)$, except at most for a set of θ of m-measure zero.

Let us put

$$U(P)=E[u_\omega(P)] \tag{17}$$

$$\Gamma(P,Q)=E[u_\omega(P)u_\omega(Q)]\,. \tag{18}$$

Then, by an immediate application of Fubini's theorem we have

THEOREM 2. *If* $f_\omega(\alpha)\in K$, *then*

$$U(P)=\int_0^{2\pi}k(r,\,\theta-\alpha)F(\alpha)d\alpha \tag{19}$$

$$\Gamma(P,\,Q)=\int_0^{2\pi}\int_0^{2\pi}k(r,\,\theta-\alpha)k(s,\,\psi-\beta)\gamma(\alpha,\,\beta)d\alpha d\beta\,, \tag{20}$$

$$P\in D\,,\qquad Q\in D\,.$$

Thus $U(P)$ is a harmonic function, and $\Gamma(P, Q)$ is harmonic in P (for fixed Q) and in Q (for fixed P). More precisely, $U(P)$ is the unique function satisfying

$$U(P) \in Ha \quad \text{and} \quad \lim_{r \uparrow 1} U(P) = F(\theta) , \qquad \text{a.a.}\,\theta , \tag{21}$$

and $\Gamma(P, Q)$ is the unique function satisfying

$$\Gamma(P, Q) \in Ha \text{ in } P \text{ for fixed } Q , \tag{22}$$

$$\Gamma(P, Q) \in Ha \text{ in } Q \text{ for fixed } P , \tag{23}$$

$$\lim_{r \uparrow 1,\, s \uparrow 1} \Gamma(P, Q) = \gamma(\theta, \psi) , \qquad \text{a.a.}\,\theta \quad \text{and} \quad \psi . \tag{24}$$

One can raise an interesting question: Suppose that we know the exact value of the boundary condition at a given point $A_1(= \alpha_1)$; let us say $f_\omega(\alpha_1) = \eta$. From this knowledge, what kind of information do we get about the random function $u_\omega(P)$? Let us assume, to take the simpler cases, that the boundary condition is expressed by a normal (Gaussian) function such that

$$E[f_\omega(\alpha)] = 0 . \tag{25}$$

Then, obviously, $u_\omega(P)$ is also a normal random function, and

$$E[u_\omega(P)] = 0 . \tag{26}$$

Put

$$z_\omega = u_\omega(P) - \frac{\Gamma(P, A_1)}{\gamma(\alpha_1, \alpha_1)} f_\omega(\alpha_1) , \tag{27}$$

where

$$\Gamma(P, A_1) = E[u_\omega(P) f_\omega(\alpha_1)] = \int_0^{2\pi} k(r, \theta - \alpha)\gamma(\alpha, \alpha_1) d\alpha . \tag{28}$$

The random variable z_ω is also normal, has expectation 0 and satisfies

$$E[z_\omega f_\omega(\alpha_1)] = 0 . \tag{29}$$

Thus z_ω is independent of $f_\omega(\alpha_1)$. Consequently, we can easily compute the conditional expectation and variance of $u_\omega(P)$,

$$E[u_\omega(P) | f_\omega(\alpha_1) = \eta] = \frac{\Gamma(P, A_1)}{\gamma(\alpha_1, \alpha_1)} \eta = \eta_1 , \tag{30}$$

$$E[(u_\omega(P) - \eta_1)^2 | f_\omega(\alpha_1) = \eta] = \Gamma(P, P) \left[1 - \frac{\Gamma(P, A_1)^2}{\Gamma(P,P)\gamma(\alpha_1, \alpha_1)} \right] . \tag{31}$$

The conditional expectation is always smaller than the expectation and the ratio

$$0 \leqq \frac{\Gamma(P, A_1)^2}{\Gamma(P, P)\gamma(\alpha_1, \alpha_1)} \leqq 1 \tag{32}$$

can be conveniently taken as a measure of the information given by the knowledge of the value of $f_\omega(\alpha)$ at the point α_1.

Returning to the general case, let us now assume that $f_\omega(\alpha)$ is a stationary (wide sense) random function

$$\gamma(\alpha, \beta) = g(\alpha - \beta) . \tag{33}$$

Then, obviously $g(\alpha) \in L(C)$. Let $G(P)$ be the unique function satisfying

$$G(P) \in Ha, \qquad \lim_{r \uparrow 1} G(P) = g(\theta), \qquad \text{a.a. } \theta . \tag{34}$$

We have

$$G(P) = G(r, \theta) = \int_0^{2\pi} k(r, \theta - \alpha) g(\alpha) d\alpha . \tag{35}$$

From the known property of the kernel k

$$k(rs, \theta - \psi) = \int_0^{2\pi} k(r, \theta - \alpha) k(s, \psi - \alpha) d\alpha , \tag{36}$$

we deduce the following theorem from (20).

THEOREM 3. *If $f_\omega(\alpha) \in K$ and $f_\omega(\alpha)$ is stationary (wide sense) then*

$$\Gamma(P, Q) = G(rs, \theta - \psi) . \tag{37}$$

Two interesting facts appear here. For r fixed, *as function of θ, the random function $u_\omega(P)$ is stationary (wide sense)*. Moreover, the covariance depends only on the combination rs. Let us put

$$\rho = \log \frac{1}{r} , \qquad \sigma = \log \frac{1}{s} . \tag{38}$$

Now the covariance depends only on $\rho + \sigma$. According to the terminology of Loève [7] $u_\omega(P)$ is thus an exponentially convex function of ρ. Returning to (37) the fact that the covariance depends only on the combination rs reflects the semigroup property of the transformation

$$T_r f = \int_0^{2\pi} k(r, \theta - \alpha) f(\alpha) d\alpha , \tag{39}$$

which, due to (36), is expressed by the equation

$$T_{rs} = T_r T_s = T_s T_r , \qquad 0 \leqq r < 1 , \qquad 0 \leqq s < 1 . \tag{40}$$

3. Random solution of the heat equation on an infinite rod

Because of the well-known interpretation of the heat equation

$$u_t = u_{xx} \tag{41}$$

as giving the velocity profile in the so-called shear-flow of an incompressible viscous fluid, this problem was the first to attract our attention. We have given in [4] the solution in the case when the random initial temperature is stationary (wide sense); we shall present here only brief remarks on the general case in order to attract attention to an interpretation which seems to be most useful in problems of this type. This is the case when each sample of $f_\omega(A)$ and $u_\omega(P)$ belongs to a Banach space. Then the theory turns out to be quite similar to the classical presentation of statistical mechanics.

Let us introduce the kernel (point-source)

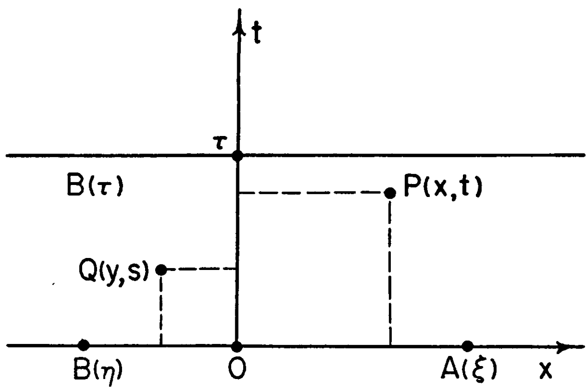
(42) $$k(x,\ t) = (4\pi t)^{-1/2}\, e^{-x^2/4t}\,.$$

As for the harmonic functions, one could start from solution of the heat equation given by

(43) $$u(x,\ t) = \int_{-\infty}^{+\infty} k(x - \xi,\ t) d\nu\ ,$$

where ν is a signed Borel measure on the infinite rod $-\infty < \xi < +\infty$, and $\nu(E)$ represents the heat content of E, where E is any Borel set on the rod. Let us call $B(\tau)$ the strip

(44) $$B(\tau) = \{(x, t)\colon -\infty < x < +\infty, 0 < t < \tau \leqq +\infty\}\,.$$



From the results of Widder [11] it is known that every solution of the heat equation, which is the difference of two nonnegative solutions of the heat equation in the strip $B(1:4a)$, is given by (43) where ν is such that

(45) $$\int_{-\infty}^{+\infty} e^{-a\xi^2}\, d|\nu| < +\infty\ .$$

Thus the analog of theorem 1 could be given here. However, overlooking these generalities, we shall immediately consider the particular case which is of interest to us.

As domain D we shall take here a strip $B(\tau)$ (which possibly could be the half-plane if $\tau = +\infty$).

Definition 3. $u(P) = u(x, t) \in He$ *if*

(a) $u(x, t)$ *is continuous in* D

(b) u_t, u_x, u_{xx} *exist and are continuous in* D

(c) $u_t = u_{xx}$ *in* D.

Let us call m the Lebesgue measure on the real line X, $-\infty < x < +\infty$, and consider a probability space $(\Omega, \mathfrak{F}, \mu)$. The initial temperature at the point x of the rod will be the random function $f_\omega(x)$.

DEFINITION 4. *If* $p \geqq 1$; $\alpha(x) \geqq 0$ *and* $\alpha(x) \in L[a, b]$ *for every finite interval on* X, *we say that the random function* $f_\omega(x) \in \mathfrak{L}^p(\alpha)$ *if*

(H_1) $f_\omega(x)$ *is measurable with respect to the product measure* $m \times \mu$,

(H_2) $f_\omega(x) \in L^p(\Omega)$ *for all* $x \in X$; *put* $M_p(x) = \int_\Omega |f_\omega(x)|^p d\mu$,

(H_3) $\alpha(x)\, M_p(x) \in L(X)$.

Obviously if $0 \leqq \alpha_2(x) \leqq \alpha_1(x)$, then $f_\omega \in \mathfrak{L}^p(\alpha_1) \Rightarrow f_\omega \in \mathfrak{L}^p(\alpha_2)$. By the same combination of Tonelli's and Fubini's theorems as in lemma 1 we prove the following lemma.

LEMMA 2. $f_\omega(x) \in \mathfrak{L}^p(\alpha) \Rightarrow \alpha(x)|f_\omega(x)|^p \in L(X)$ *with probability one, that is, if* $\omega \in \Omega - \Lambda$, $\mu(\Lambda) = 0$.

THEOREM 4. *If there exists* $a \geqq 0$ *such that* $f_\omega(x) \in \mathfrak{L}^p[\exp(-ax^2)]$ *then the Poisson-Fourier integral*

$$u_\omega(x, t) = \int_{-\infty}^{+\infty} k(x - \xi, t) f_\omega(\xi) d\xi \tag{46}$$

is a random function. Each sample corresponding to an $\omega \in \Omega - \Lambda$ *has the following property:*

(A) $u_\omega(x, t) \in He$ *in* $B(1:4a)$

(B) $\lim_{t \to 0+} u_\omega(x, t) = f_\omega(x)$ *except possibly for a set* $E(\omega)$ *of points* x *of* m*-measure zero.*

PROOF. (See [6].) The proof of (A) is based on a theorem of Tychonoff [10], asserting that, given a function $g(x)$ on X, if for one $\tau > 0$ we have $k(x, \tau)g(x) \in L(X)$, then

$$\int_{-\infty}^{+\infty} k(x - \xi, t) g(\xi) d\xi \in He \quad \text{in the strip } B(\tau) . \tag{47}$$

Suppose first that $a > 0$. Then, due to lemma 2, for $\omega \in \Omega - \Lambda$, $k(x, 1:4a)|f_\omega(x)|^p \in L(X)$. But, using the Hölder inequality,

$$\int_{-\infty}^{+\infty} k\left(x, \frac{1}{4a}\right) |f_\omega(x)| dx \leqq \left[\sqrt{\frac{\pi}{a}} \int_{-\infty}^{+\infty} k\left(x, \frac{1}{4a}\right) |f_\omega(x)|^p dx\right]^{1/p} . \tag{48}$$

With $k(x, 1:4a) f_\omega(x) \in L(X)$ for $\omega \in \Omega - \Lambda$, Tychonoff's theorem is equivalent to (A).

Suppose now $a = 0$, that is, $f_\omega(x) \in \mathfrak{L}^p(1)$ and take a decreasing sequence $a_n \downarrow 0$. For every n, $f_\omega \in \mathfrak{L}^p(1) \Rightarrow f_\omega \in \mathfrak{L}^p[\exp(-a_n x^2)]$. Thus by the preceding result we know that for $\omega \in \Omega - \Lambda_n$, $u_\omega(x, t) \in He$ in the strip $B(1:4a_n)$. Obviously $\Lambda_n \subset \Lambda_{n+1}$. Let Λ be the limit of the Λ_n so that $\mu(\Lambda_n) = 0 \Rightarrow \mu(\Lambda) = 0$. Thus if $\omega \in \Omega - \Lambda$, $u_\omega(x, t) \in He$ in every strip $B(1:4a_n)$, that is, in the half-plane $B(+\infty)$, which proves (A).

Moreover, for each sample (B) is simply a known result of Titchmarsh [9], p. 31.

Finally the fact that $u_\omega(x, t)$ is μ-measurable at every point (x, t) where the Poisson-Fourier integral exists follows in exactly the same way as for harmonic functions, by using the approximating sums.

Consider now the moments (mean and covariance) of the random function $u_\omega(x, t)$ and put

$$E[f_\omega(\xi)] = F(\xi) , \qquad E[f_\omega(\xi) f_\omega(\eta)] = \gamma(\xi, \eta) , \tag{49}$$

$$E[u_\omega(x, t)] = U(x, t) , \qquad E[u_\omega(x, t) u_\omega(y, s)] = \Gamma(x, y, t, s) . \tag{50}$$

The same use of Fubini's theorem as for theorem 2, gives the next theorem.

THEOREM 5. *If* $f_\omega(x) \in \mathcal{L}^p(e^{-ax^2})$, $p \geqq 1$, *then*

$$U(x, t) = \int_{-\infty}^{+\infty} k(x - \xi, t)F(\xi)d\xi , \tag{51}$$

$U(x, t) \in He$ *in* $B(1:4a)$ *and* $\lim_{t \to 0+} U(x, t) = F(x)$ a.a.x. *If* $f_\omega(x) \in \mathcal{L}^p(e^{-ax^2})$, $p \geqq 2$, *then*

$$\Gamma(x, y, t, s) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} k(x - \xi, t)k(y - \eta, s)\gamma(\xi, \eta)d\xi d\eta ; \tag{52}$$

for a fixed (y, s), *as function of* (x, t), $\Gamma \in He$ *in* $B(1:4a)$; *and for a fixed* (x, t), $\Gamma \in He$ *in* $B(1:4a)$. *Moreover* $\lim_{t \to 0+, s \to 0+} \Gamma(x, y, t, s) = \gamma(x, y)$ *for* a.a. *x and y.*

In [4] we proved that if $f_\omega(x)$ is a stationary (wide sense) random function,

$$\gamma(\xi, \eta) = g(\xi - \eta) \tag{53}$$

which implies that $f_\omega \in \mathcal{L}^2(1)$, the random function $u_\omega(x, t) \in He$ on the whole half-plane $B(+\infty)$ and that for every (x, t) and (y, s) on the half-plane

$$\Gamma(x, y, t, s) = G(x - y, t + s) , \tag{54}$$

where

$$G(x, t) = \int_{-\infty}^{+\infty} k(x - \xi, t)g(\xi)d\xi . \tag{55}$$

This result means that, as a function of x, the random function $u_\omega(x, t)$ is stationary (wide sense), and as a function of t is exponentially convex, according to the Loève terminology. The remarkable fact that the covariance depends only on $t + s$ reflects the semigroup property of the transformation

$$T_t f(x) = \int_{-\infty}^{+\infty} k(x - \xi, t)f(\xi)d\xi \tag{56}$$

$$T_{t+s} = T_t T_s = T_s T_t , \qquad 0 < s < +\infty, \qquad 0 < t < +\infty. \tag{57}$$

The set of all functions $f(x)$ such that $\exp(-ax^2)\,|f(x)|^p \in L(X)$ is a Banach space, but, in general, the solution $u(x, t)$ of the heat equation corresponding to the initial temperature $f(x)$ does not belong to the same Banach space for any $t > 0$. If we want $f(x)$ and $u(x, t)$ (for each t in $0 < t < 1:4a$) to belong to the same Banach space, we have to consider another Banach space. One of the simplest and most useful (because it applies to the case of stationary random functions) seems to be the Banach space of all functions such that

$$\frac{|f(x)|^p}{1 + x^2} \in L(X) . \tag{58}$$

This Banach space, which we will call $\mathcal{A}^p$, seems convenient for our purposes because $f(x) \in \mathcal{A}^p \Rightarrow u(x, t) \in \mathcal{A}^p$ for all $t > 0$.

Let us now suppose that the random function $f_\omega(x) \in \mathcal{L}^p(1:1 + x^2)$. Due to lemma 2, almost all samples belong to $\mathcal{A}^p$. There is thus a correspondence $\omega \to \phi$ from the point $\omega \in \Omega$ to the point $\phi = f_\omega(x) \in \mathcal{A}^p$. The probability measure μ in Ω induces a measure λ in $\mathcal{A}^p$; a set $\Phi \subset \mathcal{A}^p$ is λ-measurable if its inverse image $\phi^{-1}(\Phi) \subset \Omega$ is

μ-measurable and $\lambda(\Phi) = \mu[\phi^{-1}(\Phi)]$. We can now consider $\mathfrak{A}^p$ as a probability space and suppose that the points $\phi = f_\omega(x)$ is chosen at random in $\mathfrak{A}^p$ according to the probability law defined by

$$\Pr\{\phi \in \Phi\} = \lambda(\Phi). \tag{59}$$

Because $f_\omega(x) \in \mathcal{L}^p(1: 1 + x^2) \Rightarrow f_\omega(x) \in \mathcal{L}^p \exp(-ax^2)$ for any $a > 0$, the random function $u_\omega(x, t) \in He$ in the half-plane $B(+\infty)$ and, moreover, almost all samples belong to $\mathfrak{A}^p$. For a sample $\omega \in \Omega - \Lambda$ we can consider the Fourier-Stieltjes integral

$$u_\omega(x, t) = \int_{-\infty}^{+\infty} k(x - \xi, t) f_\omega(\xi) d\xi \tag{60}$$

as a transformation of $\mathfrak{A}^p$ into itself, transforming the point $\phi \to f_\omega(x)$ into the point $\phi_t \to u_\omega(x, t)$, let us say

$$\phi_t = T_t\phi. \tag{61}$$

For $0 < t < +\infty$, the transformations T_t define a strongly continuous semigroup of transformations of the Banach space into itself.

In this way our problem takes the same shape as the classical problem of statistical mechanics, with the Banach space $\mathfrak{A}^p$ playing the role of the phase-space. A state of the system (that is, the temperature distribution in the rod) is represented by a point ϕ in $\mathfrak{A}^p$. At the initial time, we choose at random (according to a given probability law) one point ϕ; then all the states of the system are represented by the trajectory consisting of the set of points ϕ_t deduced from ϕ by the transformation $\phi_t = T_t\phi$. This interpretation seems to be very fruitful in many cases; it allows the use of the interesting theory, due to E. Mourier [8], of random elements in Banach spaces.

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