FOUNDATIONS OF KINETIC THEORY

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1. Introduction

The basic equation of the kinetic theory of dilute monatomic gases is the famous nonlinear integro-differential equation of Boltzmann. In the simplest case when the molecules of the gas are hard spheres of diameter δ, which are allowed to exchange energy only through elastic collisions, the Boltzmann equation assumes the form

$$(1.1)\quad \frac{\partial}{\partial t} f(\vec{r}, \vec{v}, t) + \vec{v} \cdot \nabla_{\vec{r}} f + \vec{X}(\vec{r}) \cdot \nabla_{\vec{v}} f = \frac{\delta^2}{2} \int d\vec{w} \int d\vec{l} \cdot \{f(\vec{r}, \vec{v} + (\vec{w} - \vec{v}) \cdot \vec{l}\vec{l}, t) f(\vec{r}, \vec{w} - (\vec{w} - \vec{v}) \cdot \vec{l}\vec{l}, t) - f(\vec{r}, \vec{v}, t) f(\vec{r}, \vec{w}, t)\} \cdot |(\vec{w} - \vec{v}) \cdot \vec{l}|;$$

here $f(\vec{r}, \vec{v}, t) d\vec{r}\, d\vec{v}$ is the average number of molecues in $d\vec{r}\, d\vec{v}$ at $\vec{r}, \vec{v}$, $\nabla_{\vec{r}} f$ the gradient of f with respect to $\vec{r}$, $\nabla_{\vec{v}} f$ the gradient of f with respect to $\vec{v}$, $\vec{l}$ a unit vector and $d\vec{l}$ the surface element of the unit sphere. $\vec{X}(\vec{r})$ is an outside force (for example, gravity) acting on a particle at $\vec{r}$. If the gas is enclosed in a container of volume V and if there are no exterior forces ($X(\vec{r}) \equiv 0$) we can set

$$(1.2)\qquad f(\vec{r}, \vec{v}, t) = \frac{n}{V} f(\vec{v}, t),$$

where n is the total number of molecules, and note that it will be a solution of (1.1) if $f(\vec{v}, t)$ is a solution of the reduced Boltzmann equation

$$(1.3)\qquad \frac{\partial}{\partial t} f(\vec{v}, t) = \frac{n\delta^2}{2V} \int d\vec{w} \int d\vec{l}\, \{f(\vec{v} + (\vec{w} - \vec{v}) \cdot \vec{l}\vec{l}, t) f(\vec{w} - (\vec{w} - \vec{v}) \cdot \vec{l}\vec{l}, t) - f(\vec{v}, t) f(\vec{w}, t)\} |(\vec{w} - \vec{v}) \cdot \vec{l}|.$$

Equation (1.3) governs the temporal evolution of the velocity distribution while the spatial distribution remains uniform.

If the molecules are not hard spheres but are considered as centers of force,

$$(1.4)\qquad \frac{\delta^2}{2} |(\vec{w} - \vec{v}) \cdot \vec{l}|$$

has to be replaced by an expression depending on the nature of the force.

The most famous example is that of a Maxwell gas in which the molecules are

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assumed to repel each other with a force inversely proportional to the fifth power of their distance. In this case (1.4) has to be replaced by

$$\gamma F\left(\frac{(\vec{w}-\vec{v})\cdot\vec{l}}{|\vec{w}-\vec{v}|}\right), \tag{1.5}$$

where γ is a certain physical constant and $F(x)$ a rather complicated function related to elliptic integrals.

The reduced Boltzmann equation (1.3) is of great interest and importance especially since from it Boltzmann derived his celebrated H-theorem to the effect that

$$\frac{d}{dt}\int d\vec{v}\, f(\vec{v}, t) \log f(\vec{v}, t) \leqq 0\,, \tag{1.6}$$

the equality occurring only for the Maxwell-Boltzmann distribution

$$f_0(\vec{v}) = \frac{1}{(2\pi\sigma)^{3/2}} \exp\left[-\frac{\vec{v}\cdot\vec{v}}{2\sigma^2}\right]. \tag{1.7}$$

Mathematically, the reduced equation is probably the simplest natural example of a nonlinear integro-differential equation.

The purpose of this exposition is to reexamine critically the derivation of (1.3) and the conclusions drawn from it. The reader should be warned at the outset that more questions will be raised than answered. However, we hope to provide sharp formulations and thus perhaps pave the way toward further work on this fascinating borderline between mathematics and physics.

Several colleagues and friends helped with various parts of this exposition. Their specific contributions are acknowledged in the text. Dr. G. E. Baxter assisted me during the summer of 1954 and contributed greatly both in the large and in the small. Above all, my thanks are due to Professor G. E. Uhlenbeck for the many discussions and suggestions.

2. The "master equation"

Denote by $\vec{v}_1, \cdots, \vec{v}_n$ the velocities of the n molecules and combine them into a "master vector" $\vec{R}$(3n-dimensional)

$$\vec{R} = (\vec{v}_1, \cdots, \vec{v}_n)\,. \tag{2.1}$$

Consider now a Poisson-like process in which during time dt a "collision" can occur between the ith and jth particles ($i < j$), while the direction of the center line (that is, the line joining the centers of the ith and jth sphere, in the direction from i to j) is $\vec{l}$ within $d\vec{l}$. The probability that such a collision takes place is assumed to be of the form

$$\psi_{ij} d\vec{l}\, dt = \psi((\vec{v}_j - \vec{v}_i)\cdot\vec{l}, |\vec{v}_j - \vec{v}_i|) d\vec{l}\, dt\,. \tag{2.2}$$

For the case of hard spheres

$$\psi_{ij} = \frac{\delta^2}{V}\,\frac{|(\vec{v}_j - \vec{v}_i)\cdot\vec{l}| - (\vec{v}_j - \vec{v}_i)\cdot\vec{l}}{2} \tag{2.3}$$

which corresponds to Boltzmann's "Stosszahlansatz."[1]

If an $(i, j, \vec{l})$ collision takes place $\vec{R}$ changes into $A_{ij}(\vec{l})\vec{R}$, where

$$(2.4)\quad A_{ij}(\vec{l})\vec{R} = (\vec{v}_1, \cdots, \vec{v}_i + (\vec{v}_j - \vec{v}_i) \cdot \vec{l}\vec{l}, \cdots, \vec{v}_j - (\vec{v}_j - \vec{v}_i) \cdot \vec{l}\vec{l}, \cdots, \vec{v}_n);$$

otherwise $\vec{R}$ remains unchanged.

We can thus say that

$\vec{R} \rightarrow A_{ij}(\vec{l})\vec{R}$ with probability $\psi_{ij}\, d\vec{l}\, dt$

$\vec{R} \rightarrow \vec{R}$ with probability $1 - dt \sum_{1 \leq i \leq j \leq n} \int d\vec{l}\, \psi_{ij}$.

Since each collision preserves momentum and energy we have

$$(2.5a)\qquad \sum_1^n \vec{v}_j = \text{constant}^2$$

$$(2.5b)\qquad \sum_1^n \vec{v}_j \cdot \vec{v}_j = \text{constant} = n\sigma^2 .$$

We may as well set the constant in (2.5a) equal to 0.

Thus $\vec{R}$ is always confined to a $(3n - 3)$-dimensional sphere $S_n(\sigma)$ of radius $\sigma\sqrt{n}$. If, at time $t = 0$, we start with a distribution of points $\vec{R}$ given by the density $\phi(\vec{R}, 0)$ it is easily seen that this distribution will evolve in time according to the equation

$$(2.6)\qquad \frac{\partial\phi(\vec{R}, t)}{\partial t} = \sum_{1 \leq i < j \leq n} \int d\vec{l}\{\phi(A_{ij}(\vec{l})\vec{R}, t) - \phi(\vec{R}, t)\}\psi_{ij} .$$

This is the "master equation"[3] which is recognized as the Kolmogoroff equation for the Markoff process described above.

[1] Boltzmann's original formulation of the "Stosszahlansatz" was not framed in probabilistic terms. He simply asserted that the number of collisions in time dt between A-molecules (those whose velocities are $\vec{v}$ within $d\vec{v}$) and B-molecules (those whose velocities are $\vec{w}$ within $d\vec{w}$) which take place while the center line is in the direction $\vec{l}$ within $d\vec{l}$ is

$$n_A\, n_B\, \delta^2 \frac{|(\vec{w} - \vec{v}) \cdot \vec{l}| - (\vec{w} - \vec{v}) \cdot \vec{l}}{2}\, d\vec{l}\, dt ,$$

where n_A and n_B denote the numbers of A and B molecules, respectively.

This formulation led to the well-known paradoxes which were fully discussed in the classical article of P. and T. Ehrenfest. These writers made it clear (a) that the "Stosszahlansatz" cannot be strictly derivable from purely dynamic considerations and (b) that the "Stosszahlansatz" has to be interpreted probabilistically. The recent attempts of Born and Green, Kirkwood and Bogoliubov to derive Boltzmann's equation from Liouville's equation and hence to justify the "Stosszahlansatz" dynamically are, in our opinion, incomplete, inasmuch as they do not make it clear at what point statistical assumptions are introduced.

The "master equation" approach which we have chosen seems to us to follow closely the intentions of Boltzmann.

[2] This is incompatible with the presence of the container since collisions with the walls do not preserve momentum. This is, however, a minor point which can be circumvented by assuming that whenever a molecule collides with the wall, it is reintroduced somewhere in V without change of velocity. The origin of this slight difficulty is the fact that the reduced Boltzmann equation is not strictly valid inasmuch as the container introduces an exterior force and we cannot claim that $\vec{X}(\vec{r}) = 0$.

[3] The term "master equation" first seems to have occurred in a paper by A. Nordsieck, W. E. Lamb, Jr., and G. E. Uhlenbeck [1]. A more recent discussion of the approach to statistical equilibrium through a "master equation" was given by A. J. F. Siegert [2].

3. Connection with the Boltzmann equation (1.3)

The derivation of the "master equation" (2.6) with ψ_{ij} given by (2.3) embodies the basic assumption ("Stosszahlansatz") of Boltzmann. Yet (2.6) is *linear* while Boltzmann's equation (1.3) is not. Clearly, to pass from (2.6) to (1.3) additional assumptions are needed.

In order to discuss these assumptions as well as to exhibit more clearly many other points we shall construct a simplified mathematical model which embodies many (if not all!) of the essential features of our problem. Let

$$\vec{R} = (x_1, \cdots, x_n) \tag{3.0}$$

be subject to the condition

$$||\vec{R}||^2 = x_1^2 + \cdots + x_n^2 = n \tag{3.1}$$

and let

$$A_{ij}(\theta)\vec{R} = (x_1, \cdots, x_i \cos\theta + x_j \sin\theta, \cdots, -x_i \sin\theta + x_j \cos\theta, \cdots, x_n) . \tag{3.2}$$

Let furthermore[4]

$$\psi_{ij} = \frac{\nu}{2\pi n} = \text{constant}. \tag{3.3}$$

The "master equation" assumes now the form

$$\frac{\partial\phi(\vec{R}, t)}{\partial t} = \frac{\nu}{n} \sum_{1 \leq i < j \leq n} \frac{1}{2\pi} \int_0^{2\pi} \{\phi(A_{ij}(\theta)\vec{R}, t) - \phi(\vec{R}, t)\} d\theta \tag{3.4}$$

and the analogue of (1.3) is

$$\frac{\partial f(x, t)}{\partial t} = \nu \int_{-\infty}^{\infty} dy \frac{1}{2\pi} \int_0^{2\pi} \{f(x\cos\theta + y\sin\theta, t) \cdot f(-x\sin\theta + y\cos\theta, t) - f(x, t)f(y, t)\} d\theta . \tag{3.5}$$

The changes we made are

(a) we dropped the conservation of momentum (2.5a)

(b) we simplified the form of ψ_{ij}

(c) we replaced the more complicated six-dimensional rotations $A_{ij}(\vec{l})$ by two-dimensional rotations $A_{ij}(\theta)$.

Let us now assume that $\phi(\vec{R}, 0)$ is symmetric in all variables $x_1, \cdots, x_n$. It then follows that $\phi(\vec{R}, t)$ is also symmetric.

Let us now introduce the following abbreviations

$$f_1^{(n)}(x, t) = \int_{x_2^2 + \cdots + x_n^2 = n - x^2} \phi(\vec{R}, t) d\sigma_1 , \tag{3.6a}$$

[4] Somewhat more generally we could set $\psi_{ij} = \nu f(\theta)/n$ where $f(-\theta) = f(\theta)$ ("microscopic reversibility") and $\int_{-\pi}^{\pi} f(\theta) d\theta = 1$, $f(\theta) \geq 0$. The theory would then go through without any serious modifications. The more general theory is analogous to the theory of the Maxwell gas.

$$f_2^{(n)}(x, y, t) = \int_{x_3^2+\cdots+x_n^2=n-x^2-y^2} \phi(\vec{R}, t)d\sigma_2 , \tag{3.6b}$$

etc.

The integrations are over spheres indicated under the integral signs, the free variables being replaced by x, y, etc. The density functions $f_1^{(n)}, f_2^{(n)}, \cdots$ will be referred to as *contractions* of ϕ, $f_k^{(n)}$ being the *k-dimensional contraction*. An easy calculation on (3.4) yields

$$\frac{\partial f_1^{(n)}(x, t)}{\partial t} = \frac{(n-1)\nu}{n} \int_{-\sqrt{n-x^2}}^{\sqrt{n-x^2}} dy \frac{1}{2\pi} \int_0^{2\pi} \{ f_2^{(n)}(x \cos\theta + y \sin\theta, -x\sin\theta + y\cos\theta, t) - f_2^{(n)}(x, y, t)\} d\theta \tag{3.7}$$

which is strongly reminiscent of (3.5). To get (3.5) one must only assume that

$$f_2^{(n)}(x, y, t) \sim f_1^{(n)}(x, t) f_1^{(n)}(y, t) \tag{3.8}$$

for all x, y in the allowable range. One is immediately faced with the difficulty that since $\phi(\vec{R}, t)$ is uniquely determined by $\phi(\vec{R}, 0)$ no additional assumptions on $\phi(\vec{R}, t)$ can be made unless they can be deduced from some postulated properties of $\phi(\vec{R}, 0)$.

A moment's reflection will convince us that in order to derive (3.5) the following theorem must first be proved.

BASIC THEOREM. *Let $\phi_n(\vec{R}, 0)$ be a sequence of probability density functions defined on spheres $||\vec{R}||^2 = x_1^2 + \cdots + x_n^2 = n$ and having the "Boltzmann property"*

$$\lim_{n\to\infty} f_k^{(n)}(x_1, \cdots, x_k, 0) = \prod_{j=1}^{k} \lim_{n\to\infty} f_1^{(n)}(x_j, 0) . \tag{3.9}$$

Then $\phi_n(\vec{R}, t)$ [that is, solutions of (3.4)] also have the "Boltzmann property":

$$\lim_{n\to\infty} f_k^{(n)}(x_1, \cdots, x_k, t) = \prod_{j=1}^{k} \lim_{n\to\infty} f_1^{(n)}(x_j, t) . \tag{3.10}$$

In other words, the Boltzmann property propagates in time!

It thus appears that the nonlinear character of Boltzmann's equation (3.5) is due solely to the extremely special assumption which the initial distribution has to satisfy.

4. Proof of the basic theorem

Consider the Hilbert space of square integrable functions $\psi_n(\vec{R})$ defined on the sphere S_n, $||\vec{R}||^2 = n$, and the linear operator Ω

$$\Omega\psi_n = \frac{1}{n} \sum_{1\leq i<j\leq n} \frac{1}{2\pi} \int_0^{2\pi} \{ \psi_n(A_{ij}(\theta)\vec{R}) - \psi_n(\vec{R})\} d\theta . \tag{4.1}$$

It is easily verified that

$$(\Omega\psi_n, \chi_n) = (\psi_n, \Omega\chi_n) \tag{4.2}$$

and that

$$(\Omega\psi_n, \psi_n) = -\frac{1}{n} \sum_{1\leq i<j\leq n} \frac{1}{2\pi} \int_0^{2\pi} \int_{S_n} \{ \psi_n(A_{ij}(\theta)\vec{R}) - \psi_n(\vec{R})\}^2 d\sigma\, d\theta . \tag{4.3}$$

The operator Ω is thus self-adjoint and clearly bounded (though the bound may, and indeed does, depend on n).

We can thus write

$$\phi_n(\vec{R}, t) = \sum_{k=0}^{\infty} \frac{\Omega^k \phi_n(\vec{R}, 0)}{k!} (\nu t)^k . \tag{4.4}$$

Let $g(\vec{R}) \equiv g(x_1)$ be a bounded function of only one variable. We have from (4.2) and (4.4)

$$\begin{aligned}(\phi_n(\vec{R}, t) g(\vec{R})) &= \int_{S_n} \phi_n(\vec{R}, t) g(x_1) d\sigma \\ &= \int_{-\sqrt{n}}^{\sqrt{n}} f_1^{(n)}(x, t) g(x) dx = \sum_{k=0}^{\infty} \frac{(\nu t)^k}{k!} (\Omega^k g, \phi_n(\vec{R}, 0)) .\end{aligned} \tag{4.5}$$

Now,

$$\Omega g = \frac{1}{n} \sum_{j=2}^{n} \frac{1}{2\pi} \int_0^{2\pi} \{g(x_1 \cos\theta + x_j \sin\theta) - g(x_1)\} d\theta \tag{4.6}$$

and setting

$$g_2(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \{g(x \cos\theta + y \sin\theta) - g(x)\} d\theta \tag{4.7}$$

we have

$$\Omega g = \frac{1}{n} \sum_{j=2}^{n} g_2(x_1, x_j) . \tag{4.8}$$

Further

$$\Omega^2 g = \frac{1}{n} \sum_{j=2}^{n} \Omega g_2(x_1, x_j) \tag{4.9}$$

and

$$\begin{aligned}\Omega g(x_1, x_2) &= \frac{1}{n} \frac{1}{2\pi} \int_0^{2\pi} \{g_2(x_1 \cos\theta + x_2 \sin\theta, -x_1 \sin\theta + x_2 \cos\theta) - g_2(x_1, x_2)\} d\theta \\ &+ \frac{1}{n} \sum_{j=3}^{n} \frac{1}{2\pi} \int_0^{2\pi} \{g_2(x_1 \cos\theta + x_j \sin\theta, x_2) - g_2(x_1, x_2)\} d\theta \\ &+ \frac{1}{n} \sum_{j=3}^{n} \frac{1}{2\pi} \int_0^{2\pi} \{g_2(x_1, x_2 \cos\theta + x_j \sin\theta) - g_2(x_1, x_2)\} d\theta .\end{aligned} \tag{4.10}$$

Since $\phi_n(\vec{R}, 0) \geqq 0$ and

$$\int_{S_n} \phi_n(\vec{R}, 0) d\sigma = 1 , \tag{4.11}$$

and since $|g(\vec{R})| < M$, we have $|g_2| < 2M$ and hence

$$(4.12) \qquad |(\Omega g, \phi_n(\vec{R}, 0))| < 2M .$$

Furthermore

$$(4.13) \qquad |\Omega g_2| < \frac{4M}{n} + \frac{2(n-2)}{n} 4M < 2!\,4M ,$$

and hence

$$(4.14) \qquad |\Omega^2 g| < 2!\,2^2 M .$$

In general,

$$(4.15) \qquad |(\Omega^k g, \phi_n(\vec{R}, 0))| < k!\,2^k M .$$

(*Added in proof:* The above proof is applicable only for $k \leqq n$. However, the formula remains valid for $k > n$, since

$$|\Omega^k g| = |\Omega^{k-n}\Omega^n g| \leqq n^{k-n} n!\,2^n M < k!\,2^k M.)$$

Moreover,

$$(4.16) \qquad \lim_{n\to\infty} (\Omega g, \phi_n(\vec{R}, 0)) = \iint_{-\infty}^{\infty} f_2(x, y) g_2(x, y) dx dy ,$$

where

$$(4.17) \qquad f_2(x, y) = \lim_{n\to\infty} f_2^{(n)}(x, y, 0)$$

and in general

$$(4.18) \qquad \lim_{n\to\infty} (\Omega^k g, \phi_n(\vec{R}, 0)) = \int \cdots_{-\infty}^{\infty} \cdots \int f_{k+1}(x_1, \cdots, x_{k+1}) \cdot g_{k+1}(x_1, \cdots, x_{k+1}) dx_1 \cdots dx_{k+1}$$

where the g_{k+1} are defined inductively as follows:

$$g_1(x) \equiv g(x)$$

$$(4.19) \qquad g_{k+1}(x_1, \cdots, x_{k+1}) = \sum_{j=1}^{k} \frac{1}{2\pi} \int_0^{2\pi} \{g_k(x_1, \cdots, x_j \cos\theta + x_{k+1} \sin\theta, \cdots, x_k) - g_k(x_1, \cdots, x_k)\} d\theta .$$

Since we have assumed that $\phi_n(\vec{R}, 0)$ has the Boltzmann property we have

$$(4.20) \qquad \lim_{n\to\infty} (\Omega^k g, \phi_n(\vec{R}, 0)) = \int \cdots_{-\infty}^{\infty} \cdots \int f(x_1) \cdots f(x_{k+1}) \cdot g_{k+1}(x_1, \cdots, x_{k+1}) dx_1 \cdots dx_{k+1}$$

where

$$(4.21) \qquad f(x) = f_1(x, 0) = \lim_{n\to\infty} f_1^{(n)}(x, 0) .$$

From (4.5), (4.15) and (4.20) it follows that for $0 \leqq t < 1/2\nu$

$$\text{(4.22)} \qquad \int_{-\infty}^{\infty} f(x, t)g(x)dx = \sum_{k=0}^{\infty} \frac{(\nu t)^k}{k!} \int \cdot \overset{\infty}{\underset{-\infty}{\vdots}} \cdot \int f(x_1) \cdots f(x_{k+1}) \cdot g_{k+1}(x_1, \cdots, x_{k+1})dx_1 \cdots dx_{k+1}$$

where[5] $f(x, t) \equiv f_1(x, t) = \lim_{n\to\infty} f_1^{(n)}(x, t)$.

Starting now from a function $\gamma_2(x_1, x_2) = g(x_1)h(x_2)$ and defining $\gamma_k(x_1, \cdots, x_k)$ inductively by formula (4.19) we obtain again for $0 \leqq t < 1/2\nu$

$$\text{(4.23)} \qquad \iint_{-\infty}^{\infty} f_2(x_1, x_2, t)g(x_1)h(x_2)dx_1dx_2 = \sum_{k=0}^{\infty} \frac{(\nu t)^k}{k!} \int \cdot \overset{\infty}{\underset{-\infty}{\vdots}} \cdot \int f(x_1) \cdots f(x_{k+2})\gamma_{k+2}(x_1, \cdots, x_{k+2})dx_1 \cdots dx_{k+2} .$$

It is easily checked that

$$\text{(4.24)} \qquad \gamma_3(x_1, x_2, x_3) = h_1(x_2)g_2(x_1, x_3) + g_1(x_1)h_2(x_2, x_3) ,$$

$$\text{(4.25)} \qquad \gamma_4(x_1, x_2, x_3, x_4) = h_2(x_2, x_4)g_2(x_1, x_3) + h_1(x_2)g_3(x_1, x_3, x_4) + g_2(x_1, x_4)h_2(x_2, x_3) + g_1(x_1)h_3(x_2, x_3, x_4) ,$$

etc.

Thus, for instance,

$$\text{(4.26)} \qquad \iiiint_{-\infty}^{\infty} f(x_1) \cdots f(x_4)\gamma_4(x_1, x_2, x_3, x_4)dx_1dx_2dx_3dx_4$$
$$= \left(\int_{-\infty}^{\infty} f(x)h(x)dx\right)\left(\iiint_{-\infty}^{\infty} f(x_1)f(x_2)f(x_3)g_3(x_1, x_2, x_3)dx_1dx_2dx_3\right)$$
$$+ 2\left(\iint_{-\infty}^{\infty} f(x_1)f(x_2)g_2(x_1, x_2)dx_1dx_2\right)\left(\iint_{-\infty}^{\infty} f(x_1)f(x_2)h_2(x_1, x_2)dx_1dx_2\right)$$
$$+ \left(\int_{-\infty}^{\infty} f(x)g(x)dx\right)\left(\iiint_{-\infty}^{\infty} f(x_1)f(x_2)f(x_3)h_3(x_1, x_2, x_3)dx_1dx_2dx_3\right) ,$$

and since similar formulas hold it is seen that for $0 \leqq t < 1/2\nu$,

[5] It should be clear that the limit is to be understood in the weak sense in the space $L(-\infty, \infty)$. Thus once the propagation of the Boltzmann property is established, we have a proof that the Boltzmann equation (3.5) does have a solution. Uniqueness follows from considerations of section 7 where, starting from (3.5), we show how to determine uniquely the Fourier-Hermite coefficients of $f(x, t)$; the needed fact that the Hermite functions are complete in $L(-\infty, \infty)$ is, of course, well known. Since the master equation is truly descriptive of the physical situation, and since existence and uniqueness of the solution of the master equation are almost trivial, the preoccupation with existence and uniqueness theorems for the Boltzmann equation appears to be unjustified on grounds of physical interest and importance.

$$(4.27)\quad \iint_{-\infty}^{\infty} f_2(x_1, x_2, t)g(x_1)h(x_2)dx_1dx_2 = \int_{-\infty}^{\infty} f(x_1, t)g(x_1)dx_1 \int_{-\infty}^{\infty} f(x_2, t)h(x_2)dx_2 .$$

Since g and h are arbitrary we have for $0 \leqq t < 1/2\nu$,

$$(4.28)\quad f_2(x_1, x_2, t) = f(x_1, t)f(x_2, t) .$$

By a similar, but more tedious, argument we also get

$$(4.29)\quad f_k(x_1, \cdots, x_k, t) = f(x_1, t) \cdots f(x_k, t) .$$

The restriction on t can now be removed by observing that it *does not depend on the initial distribution.*

In fact, we can start with some t_o, $0 < t_o < (2\nu)^{-1}$, and repeating the argument extend the proof of Boltzmann's property to the range $t_o \leqq t < t_o + (2\nu)^{-1}$. Proceeding this way we can clearly cover the whole time range $0 \leqq t < \infty$.

The above proof suffers from the defect that it works only if the restriction on time is independent of the initial distribution. It is therefore inapplicable to the physically significant case of hard spheres because in this case our simple estimates yield a time restriction which depends on the initial distribution. A general proof that Boltzmann's property propagates in time is still lacking.

5. Distributions having Boltzmann's property

We shall now show that

$$(5.1)\quad \phi_n(\vec{R}) = \frac{\prod_{j=1}^{n} c(x_j)}{\int_{S_n} \prod_{j=1}^{n} c(x_j)d\sigma} ,$$

where $c(x) \geqq 0$, and subject to some mild restrictions, has the Boltzmann property.

First we determine the asymptotic behavior of

$$(5.2)\quad \int_{S_n} \prod_{j=1}^{n} c(x_j)d\sigma .$$

This is done by a method frequently used in statistical physics (see, for example, [1]) and we shall restrict ourselves to going through the formal steps without rigorous justification.

Set

$$(5.3)\quad F_n(r) = \int_{\Sigma x_j^2 = r^2} \prod_{1}^{n} c(x_j)d\sigma ,$$

and note that for $\Re(s) > 0$,

$$(5.4)\quad \frac{1}{2}\int_0^{\infty} e^{-s\rho} \frac{F_n(\sqrt{\rho})}{\rho^{1/2}} d\rho = \int_0^{\infty} e^{-sr^2} F_n(r)dr = \left(\int_{-\infty}^{\infty} e^{-sx^2} c(x)dx\right)^n .$$

Using the complex inversion formula we get

$$\frac{F_n(\sqrt{\rho})}{2\rho^{1/2}} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{z\rho}\left(\int_{-\infty}^{\infty} e^{-zx^2}c(x)dx\right)^n dz\,, \tag{5.5}$$

where $\gamma > 0$ and otherwise arbitrary.[6] Thus

$$F_n(\sqrt{n}) = \frac{2\sqrt{n}}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(e^z\int_{-\infty}^{\infty} e^{-zx^2}c(x)dx\right)^n dz\,, \tag{5.6}$$

and we are in position to use the method of steepest descent.

The saddle point is determined from the equation

$$\int_{-\infty}^{\infty} e^{-z_0x^2}c(x)dx = \int_{-\infty}^{\infty} x^2e^{-z_0x^2}c(x)dx \tag{5.7}$$

and we shall assume that a real solution exists. It then follows easily that it is unique and we can write (setting $\gamma = z_0$, $z = z_0 + i\xi$)

$$F_n(\sqrt{n}) = \frac{2\sqrt{n}e^{nz_0}}{2\pi}\int_{-\infty}^{\infty} d\xi\left(\int_{-\infty}^{\infty} e^{-z_0x^2}e^{i\xi(1-x^2)}c(x)dx\right)^n. \tag{5.8}$$

Under mild additional assumptions on $c(x)$ we get

$$F_n(\sqrt{n}) \sim \frac{2ne^{nz_0}}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} e^{-z_0x^2}c(x)dx\right)^n. \tag{5.9}$$

Let now $g(x)$ and $h(x)$ be bounded continuous functions defined in $(-\infty, \infty)$. Then, by the same process we obtain

$$\int_{S_n} g(x_1)h(x_2)\prod_1^n c(x_j)d\sigma \tag{5.10}$$

$$\sim \frac{\left(\int_{-\infty}^{\infty} g(x)e^{-z_0x^2}c(x)dx\right)\left(\int_{-\infty}^{\infty} h(x)e^{-z_0x^2}c(x)dx\right)}{\left(\int_{-\infty}^{\infty} e^{-z_0x^2}c(x)dx\right)^2}\frac{2ne^{nz_0}}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} e^{-z_0x^2}c(x)dx\right)^n,$$

and it follows that

$$f_1(x) = \frac{e^{-z_0x^2}c(x)}{\int_{-\infty}^{\infty} e^{-z_0x^2}c(x)dx}\,, \tag{5.11}$$

$$f_2(x_1, x_2) = f_1(x_1)f_1(x_2)\,, \tag{5.12}$$

etc.

The most severe restriction placed upon $c(x)$ is that for some z

$$e^{-zx^2}c(x) \tag{5.13}$$

be integrable in $(-\infty, \infty)$. If this is violated, our method fails and it is far from clear what happens.

[6] Note that we assume that $c(x)$ is such that for some z, $\Re z > 0$, the integral $\int_{-\infty}^{\infty} e^{-zx^2}c(x)dx$ exists. If $c(x)$ vanishes at infinity so rapidly that for some z with $\Re z < 0$ the above integral exists, we can, of course, take $\gamma = 0$.

Consider now probability densities $\psi_n(\vec{R})$ which have the Boltzmann property but which are not necessarily of the form (5.1). Suppose furthermore that the limiting one-dimensional contraction of $\psi_n(\vec{R})$ is $f(x)$. Since

$$n = n\int \psi_n(\vec{R})d\sigma = \int (x_1^2 + \cdots + x_n^2)\psi_n(\vec{R})d\sigma = n\int x_1^2\psi_n(\vec{R})d\sigma = n\int_{-\sqrt{n}}^{\sqrt{n}} x^2 f_1^{(n)}(x)dx \tag{5.14}$$

we have (provided $\int_{-\infty}^{\infty} x^2 f(x)dx < \infty$)

$$\int_{-\infty}^{\infty} x^2 f(x)dx = 1\ . \tag{5.15}$$

If we set

$$c(x) = f(x)\ , \tag{5.16}$$

we see that

$$\phi_n(\vec{R}) = \frac{\prod_1^n f(x_j)}{\int_{S_n} \prod_1^n f(x_j)d\sigma} \tag{5.17}$$

contracts (in the limit $n \to \infty$) to $f(x)(z_0 = 0$ in view of (5.15)) and has the Boltzmann property.

It seems plausible that, in some sense, $\phi_n(\vec{R})$ is an approximation to $\psi_n(\vec{R})$ but we have been unable to state this precisely. It would already be of interest to prove or disprove the conjecture that for every $\alpha > 1$

$$\lim_{n\to\infty} \left(\frac{\int \phi_n^\alpha(\vec{R})d\sigma}{\int \psi_n^\alpha(\vec{R})d\sigma} \right)^{\frac{1}{n}} = 1\ . \tag{5.18}$$

6. The H-theorem

Starting with the master equation (3.4)

$$\frac{\partial \phi}{\partial t} = \frac{\nu}{n} \sum_{1\leq i<j\leq n} \frac{1}{2\pi} \int_0^{2\pi} \phi\{(A_{ij}(\theta)\vec{R},\ t) - \phi(\vec{R},\ t)\}d\theta \tag{6.1}$$

we obtain [see (4.3)]

$$\frac{d\int \phi^2(\vec{R},\ t)d\sigma}{dt} = -\frac{\nu}{n} \sum_{1\leq i<j\leq n} \frac{1}{2\pi} \int_0^{2\pi} d\theta \int \{\phi(A_{ij}(\theta)\vec{R},\ t) - \phi(\vec{R},\ t)\}^2 d\sigma\ , \tag{6.2}$$

and hence

$$\frac{d}{dt}\int \phi^2(\vec{R},\ t)d\sigma \leq 0\ . \tag{6.3}$$

Furthermore, if $\alpha > 1$, we have by Hölder's inequality

$$(6.4)\quad \int \phi^{\alpha-1}(\vec{R})\phi(A_{ij}(\theta)\vec{R})d\sigma \leqq \left(\int \phi^{\alpha}(\vec{R})d\sigma\right)^{(\alpha-1)/\alpha}\left(\int \phi^{\alpha}(A_{ij}(\theta)\vec{R})d\sigma\right)^{1/\alpha} = \int \phi^{\alpha}(\vec{R})d\sigma,$$

and hence

$$(6.5)\qquad \frac{d}{dt}\int \phi^{\alpha}(\vec{R},\ t)d\sigma \leqq 0\ .$$

Since

$$(6.6)\qquad \log \phi = \lim_{\epsilon\to 0} \frac{\phi^{\epsilon}-1}{\epsilon}\ ,$$

we obtain as a corollary of (6.5) that

$$(6.7)\qquad \frac{d}{dt}\int \phi(\vec{R},\ t)\log \phi(\vec{R},\ t)d\sigma \leqq 0\ .$$

The equality in (6.3), (6.5) and (6.7) occurs only if

$$(6.8)\qquad \phi(\vec{R},\ t) = \text{constant} = \frac{1}{S_n(\sqrt{n})}\ ,$$

where $S_n(\sqrt{n})$ denotes the surface of the sphere $||R||^2 = n$.

The one-dimensional contraction of (6.8) is readily seen to be

$$(6.9)\qquad \frac{\left(1-\dfrac{x^2}{n}\right)^{\frac{n-3}{2}}}{\displaystyle\int_{-\sqrt{n}}^{\sqrt{n}}\left(1-\frac{x^2}{n}\right)^{\frac{n-3}{2}}dx}$$

which in the limit $n \to \infty$ becomes the Maxwell-Boltzmann density[7]

$$(6.10)\qquad \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\ .$$

We must now prove that[8]

$$(6.11)\qquad \phi(\vec{R},\ t) \to \frac{1}{S_n(\sqrt{n})}\ ,$$

as $t \to \infty$, at least in the weak sense, that is, for every $\chi(\vec{R}) \in L^2(S_n)$

$$(6.12)\qquad \lim_{t\to\infty}\int \phi(\vec{R},\ t)\chi(\vec{R})d\sigma = \frac{\int \chi(\vec{R})d\sigma}{S_n(\sqrt{n})}\ .$$

Since the master equation is of the form

$$(6.13)\qquad \frac{\partial\phi}{\partial t} = \Omega\phi\ ,$$

[7] The observation that the one-dimensional contraction of the uniform distribution on the sphere $||\vec{R}||^2 = n$ leads, in the limit, to the Maxwell-Boltzmann distribution is due to Maxwell but is often attributed to Borel [3].

[8] This is simply the ergodic property of the Markoff process under consideration. Rather than to appeal to general theorems we prefer to keep the exposition self-contained and provide a proof which, in this case, is very simple.

where Ω is a bounded, self-adjoint, negative operator we have

$$(6.14)\qquad (\phi,\ \chi) = \int \phi(\vec{R},\ t)\chi(\vec{R})d\sigma = \int_{-\infty}^{0} e^{\lambda t}d_{\lambda}(E(\lambda)\phi(\vec{R},\ 0),\ \chi(\vec{R}))\ ,$$

where $E(\lambda)$ are the projection operators involved in the resolution of the identity of the operator Ω.

The function

$$(6.15)\qquad r(\lambda) = (E(\lambda)\phi(\vec{R},\ 0),\ \chi(\vec{R}))$$

is of bounded variation and since Ω is bounded $r(\lambda)$ is constant for sufficiently large negative λ.

Thus

$$(6.16)\qquad \frac{d}{dt}\int \phi(\vec{R},\ t)\chi(\vec{R})d\sigma = \int_{-\infty}^{0} \lambda e^{\lambda t}dr(\lambda)\ ,$$

and consequently

$$(6.17)\qquad \lim_{t\to\infty}\frac{d}{dt}\int \phi(\vec{R},\ t)\chi(\vec{R})d\sigma = 0\ .$$

From (6.3) it follows that a sequence $t_s \to \infty$ exists such that $\phi(\vec{R},\ t_s)$ converges weakly to a function $\phi_0(\vec{R})$, that is,

$$(6.18)\qquad \lim_{s\to\infty}\int \phi(\vec{R},\ t_s)\chi(\vec{R})d\sigma = \int \phi_0(\vec{R})\chi(\vec{R})d\sigma$$

and

$$(6.19)\qquad \lim_{s\to\infty}\int \phi(\vec{R},\ t_s)\chi(A_{ij}(\theta)\vec{R})d\sigma = \int \phi_0(\vec{R})\chi(A_{ij}(\theta)\vec{R})d\sigma\ .$$

Since

$$(6.20)\qquad \frac{d}{dt}\int \phi(\vec{R},\ t)\chi(\vec{R})d\sigma = \frac{\nu}{n}\sum_{1\leqq i<j\leqq n}\frac{1}{2\pi}\int_0^{2\pi} d\theta \int d\sigma\phi(\vec{R},\ t)\{\chi(A_{ij}(\theta)\vec{R}) - \chi(\vec{R})\}\ ,$$

it follows from (6.17) (by letting $t \to \infty$ through the sequence t_s) that

$$(6.21)\qquad \begin{aligned} 0 &= \sum_{1\leqq i<j\leqq n}\frac{1}{2\pi}\int_0^{2\pi} d\theta \int d\sigma\phi_0(\vec{R})\{\chi(A_{ij}(\theta)\vec{R}) - \chi(\vec{R})\} \\ &= \int d\sigma\chi(\vec{R})\sum_{1\leqq i<j\leqq n}\frac{1}{2\pi}\int_0^{2\pi} \{\phi_0(A_{ij}(\theta)\vec{R}) - \phi(\vec{R})\}d\theta\ , \end{aligned}$$

and since $\chi(\vec{R})$ is arbitrary we have

$$(6.22)\qquad \sum_{1\leqq i<j\leqq n}\frac{1}{2\pi}\int_0^{2\pi} \{\phi_0(A_{ij}(\theta)\vec{R}) - \phi_0(\vec{R})\}d\theta = 0\ .$$

Multiplying both sides of (6.22) by $\phi_0(\vec{R})$ and integrating over S_n we obtain

$$(6.23)\qquad \sum_{1\leqq i<j\leqq n}\int d\sigma\,\frac{1}{2\pi}\int_0^{2\pi} \{\phi_0(A_{ij}(\theta)\vec{R}) - \phi_0(\vec{R})\}^2 d\theta = 0$$

and hence

$$\phi_0(A_{ij}(\theta)\vec{R}) = \phi_0(\vec{R}) \tag{6.24}$$

for almost every θ and almost every $\vec{R}$. To prove that (6.24) implies that $\phi_0(\vec{R}) \equiv$ constant almost everywhere, we need the fact that the $A_{ij}(\theta)$ generate a transitive subgroup of the full n-dimensional rotation group. This is almost trivial because starting with

$$\vec{T} = (\xi_1, \cdots, \xi_n) \in S_n \tag{6.25}$$

we can by an appropriate $A_{12}(\theta)$ turn (6.25) into

$$(\sqrt{\xi_1^2 + \xi_2^2}, 0, \xi_3, \cdots, \xi_n) . \tag{6.26}$$

An appropriate A_{13} will turn (6.26) into

$$(\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}, 0, 0, \cdots, \xi_n) , \tag{6.27}$$

and proceeding this way we see that an appropriate product

$$A_{1n}(\theta_{n-1}) \cdots A_{12}(\theta_1) \tag{6.28}$$

will turn (6.25) into $(\sqrt{n}, 0, \cdots, 0) = \vec{R}_0$. Assuming that $\phi_0(\vec{R}_0)$ is defined we see that

$$\phi_0(\vec{T}) = \phi_0(\vec{R}_0) \tag{6.29}$$

provided, of course, $\vec{T}$ is such that $\phi_0(\vec{T})$ is defined and the angles $\theta_1, \cdots, \theta_{n-1}$ (which clearly depend on $\vec{T}$) do not belong to the exceptional sets of measure 0. It is clear that for almost every $\vec{T}$ the angles $\theta_1, \cdots, \theta_{n-1}$ will not lie in the exceptional set and hence

$$\phi_0(\vec{R}) = \text{constant} = \frac{1}{S_n(\sqrt{n})} \tag{6.30}$$

almost everywhere.

Since $\phi_0(\vec{R})$ is unique it follows that

$$\lim_{t\to\infty} \phi(\vec{R}, t) = \phi_0(\vec{R}) , \tag{6.31}$$

where the limit is taken in the weak sense. The above analysis goes through without any modifications for the general master equation (2.6) except that the proof that the $A_{ij}(\vec{l})$ generate a transitive subgroup of the orthogonal group in $3n - 3$ dimensions is more tricky. A proof was communicated to me by my colleagues Drs. Feit and Hunt but we shall not reproduce it here.

We have thus shown that the master density $\phi(\vec{R}, t)$ approaches, as $t \to \infty$, the equilibrium density

$$\phi_0(\vec{R}) = \frac{1}{S_n(\sqrt{n})} \tag{6.32}$$

and that the approach is "irreversible" as implied by (6.3), (6.5) or (6.7). From the fact that the master density approaches (6.32) and from the fact that the one-dimensional contraction of (6.32) is (6.9), it follows (again in the weak sense) that

$$\lim_{t\to\infty} f_1^{(n)}(x,\,t) = \frac{\left(1-\frac{x^2}{n}\right)^{\frac{n-3}{2}}}{\int_{-\sqrt{n}}^{\sqrt{n}}\left(1-\frac{x^2}{n}\right)^{\frac{n-3}{2}}dx} \sim \frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}\,. \tag{6.33}$$

We shall now discuss the relation of Boltzmann's famous H-theorem to our development.

The H-theorem asserts that

$$\frac{d}{dt}\int_{-\infty}^{\infty} f(x,\,t)\log f(x,\,t)dx \leqq 0 \tag{6.34}$$

and is easily derivable from (3.5) by following Boltzmann's original derivation. However, in contrast with the statements (6.3), (6.5) and (6.7) (which can be generalized further by replacing ϕ^α or $\phi\log\phi$ by $M(\phi)$ provided M is concave upward) the functional

$$H(f) = \int f\log f\,dx \tag{6.35}$$

is the only one, discovered so far, which exhibits the monotonic behavior.

To elucidate this situation we must recall that (3.5) is applicable only to distributions having Boltzmann's property. If, in some sense, we could say that

$$\phi_n(\vec{R},\,t) \sim \frac{\prod_1^n f(x_j,\,t)}{\int\prod_1^n f(x_j,\,t)d\sigma}\,, \tag{6.36}$$

we would have

$$\begin{aligned}\int\phi_n\log\phi_n d\sigma &\sim \frac{1}{C_n}\int\prod_1^n f(x_j,\,t)\left(\sum_1^n \log f(x_j,\,t) - \log C_n\right)d\sigma \\ &= -\log C_n + \frac{n}{C_n}\int d\sigma\log f(x_1,\,t)\prod_1^n f(x_j,\,t)\,,\end{aligned} \tag{6.37}$$

where

$$C_n = \int\prod_1^n f(x_j,\,t)d\sigma \tag{6.38}$$

and $f(x,\,t)$ is the limiting one-dimensional contraction of $\phi_n(\vec{R},\,t)$.

Asymptotically, for large n, we would have

$$\int\phi_n\log\phi_n d\sigma \sim -\log C_n + n\int_{-\infty}^{\infty} f(x,\,t)\log f(x,\,t)dx \tag{6.39}$$

and from the fact that

$$\int\phi_n\log\phi_n d\sigma \tag{6.40}$$

decreases in time it would follow that so does

$$H(f) = \int_{-\infty}^{\infty} f \log f \, dx \,. \tag{6.41}$$

If the above steps could be made rigorous we would have a thoroughly satisfactory derivation of Boltzmann's H-theorem.

Along the same lines we could then construct other "entropy-like" functions. For instance, starting with (6.3) we could write, in virtue of (6.36),

$$\int \phi_n^2 \, d\sigma \sim \frac{\int \prod_1^n f^2(x_j, t) d\sigma}{C_n^2} \tag{6.42}$$

and applying the argument of section 5 we would have

$$\frac{1}{C_n^n} \sim 1 \tag{6.43}$$

(since by (5.10) $z_0 = 0$) and

$$\left(\int \phi_n^2 \, d\sigma\right)^{\frac{1}{n}} \sim \int_{-\infty}^{\infty} e^{z_0(f^2)(1-x^2)} f^2(x, t) dx \,, \tag{6.44}$$

where $z_0(f^2)$ denotes the real root (if it exists) of the equation

$$\int_{-\infty}^{\infty} (1 - x^2) e^{-z_0 x^2} f^2(x, t) dx = 0 \,. \tag{6.45}$$

Since

$$\left(\int \phi_n^2 d\sigma\right)^{\frac{1}{n}}$$

decreases in time one might expect that so does the functional

$$K(f) = \int_{-\infty}^{\infty} e^{z_0(f^2)(1-x^2)} f^2(x, t) dx \,. \tag{6.46}$$

It is an interesting open problem to prove or disprove the "K-theorem" to the effect that

$$\frac{dK(f)}{dt} \leqq 0 \,. \tag{6.47}$$

7. Relaxation times

In the preceding section we have shown that the master density approaches the equilibrium density

$$\phi_0(\vec{R}) = \frac{1}{S_n(\sqrt{n})}$$

and that the approach exhibits the important feature of irreversibility. We shall now study the approach to equilibrium in a more detailed manner.

We have already seen that we can write

$$\phi(\vec{R}, t) = \int_{-\infty}^{0} e^{\lambda t} d\{E(\lambda)\phi(\vec{R}, 0)\} \,, \tag{7.1}$$

and it would thus appear that $\phi(\vec{R}, t) - \phi_0(\vec{R})$ might decay exponentially.

For this to be true one has to prove that 0 which is a simple eigenvalue of Ω (with $\phi_0(R)$ as the corresponding eigenfunction) is *isolated*. In other words one has to prove that

$$\text{l.u.b.}\ \frac{(\Omega\psi,\ \psi)}{(\psi,\ \psi)} < 0\ , \tag{7.2}$$

where the l.u.b. is taken over all functions orthogonal to $\phi_0(\vec{R})$, that is,

$$\int \psi d\sigma = 0\ . \tag{7.3}$$

Actually one needs more than (7.2), namely,

$$\lim_{n\to\infty} \text{l.u.b.}\ \frac{(\Omega\psi,\ \psi)}{(\psi,\ \psi)} < 0\ . \tag{7.4}$$

Surprisingly enough this seems quite difficult and we have not succeeded in finding a proof. Even for the simplified model we have been considering, the question remains unsettled although we are able to give a reasonably explicit solution of the master equation.

Let $H_k^{(r)}(\vec{R})$, $1 \leqq r \leqq \gamma(k)$,[9] be the linearly independent n-dimensional spherical harmonics of order k.

Since $H_k^{(r)}(A_{ij}(\theta)\vec{R})$ is a spherical harmonic of order k it must be a linear combination of the $H_k^{(r)}(\vec{R})$. Thus

$$H_k^{(r)}(A_{ij}(\theta)\vec{R}) = \sum_{s=1}^{\gamma(k)} C_k^{(r,s)}(\theta,\ i,\ j) H_k^{(s)}(\vec{R})\ . \tag{7.5}$$

Thus

$$\Omega H_k^{(r)} = \frac{\nu}{n} \sum_{s=1}^{\gamma(k)} d_k^{(r,\ s)}\ H_k^{(s)}(\vec{R})\ , \tag{7.6}$$

where

$$d_k^{(r,\ s)} = \sum_{1\leqq i<j\leqq n} \left\{\frac{1}{2\pi}\int_0^{2\pi} C_k^{(r,\ s)}\ (\theta;\ i,\ j) d\theta - 1\right\}. \tag{7.7}$$

The $\gamma(k) \times \gamma(k)$ matrix

$$D_k = ((d_k^{(r,\ s)})) \tag{7.8}$$

is easily seen to be symmetric and, if one writes

$$\phi(\vec{R},\ 0) \sim \sum_{k=0}^{\infty} \sum_{s=1}^{\gamma(k)} a_k^{(s)}\ H_k^{(s)}\ (\vec{R})\ , \tag{7.9}$$

one obtains

$$\phi(\vec{R},\ t) \sim \sum_{k=0}^{\infty} (a_k) e^{(\nu/n)D_k t}\ (H_k(\vec{R}))\ . \tag{7.10}$$

[9] Actually $\gamma(k)$ depends also on n but we suppress it to simplify the formulas.

Here (a_k) is the row vector $(a_k^{(1)}, \cdots, a_k^{\gamma(k)})$ and $(H_k(\vec{R}))$ the column vector

$$\begin{pmatrix} H_k^{(1)}(\vec{R}) \\ H_k^{(2)}(\vec{R}) \\ \cdot \\ \cdot \\ \cdot \\ H_k^{\gamma(k)}(\vec{R}) \end{pmatrix} . \tag{7.11}$$

The "time constants" in the expansion (7.10) are

$$-\frac{\nu}{n} \times \text{eigenvalues of } D_k , \tag{7.12}$$

and it is not clear that the set of numbers (7.12) does not have 0 as a limit point (especially if one allows n to approach ∞).

Consider now the one-dimensional contraction $f_1^{(n)}(x_1, t)$ of $\phi(\vec{R}, t)$. For a function $g(x_1)$ such that $\int g^2(x_1)d\sigma < \infty$ we have

$$\int_{-\sqrt{n}}^{\sqrt{n}} g(x)f_1^{(n)}(x, t)dx = \sum_{k=0}^{\infty} (a_k)e^{(\nu t/n)D_k} \left(\int_{-\sqrt{n}}^{\sqrt{n}} g(x)\omega_k(x)dx\right) , \tag{7.13}$$

where $\omega_k^{(s)}(x)$ is the one-dimensional contraction of $H_k^{(s)}(\vec{R})$.

The functions $H_k^{(s)}(R)$ can be so chosen that $\omega_k^{(s)}(x) = 0$, $s = 2, 3, \cdots, \gamma(k)$, in which case $\omega_k^{(1)}(x)$ is the well known Gegenbauer function.

Denoting by $-\lambda_k^{(s)}(n)$ the eigenvalues of the matrix $n^{-1}D_k$ we can rewrite (7.13) in the form

$$\int_{-\sqrt{n}}^{\sqrt{n}} g(x)f_1^{(n)}(x, t)dx = \sum_{k=0}^{\infty} \left(\sum_{s=1}^{\gamma(k)} b_k^{(s)}e^{-\gamma\lambda_k^{(s)}(n)t}\right)\int_{-\sqrt{n}}^{\sqrt{n}} \omega_k^{(1)}(x)g(x)dx , \tag{7.14}$$

where the $b_k^{(s)}$ are certain linear combinations of the $a_k^{(s)}$ with coefficients involving the eigenvectors of D_k. It should be borne in mind that $\gamma(k)$, $b_k^{(s)}$ and $\omega_k^{(1)}(x)$ depend also on n. It seems very difficult to go beyond (7.14) but it suggests what happens in the limit $n \to \infty$.

It is known that as $n \to \infty$ $\omega_k^{(1)}(x)$, when properly normalized, approaches

$$e^{-x^2/2}h_k(x) , \tag{7.15}$$

where $h_k(x)$ is the Hermite polynomial

$$h_k(x) = e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2} . \tag{7.16}$$

It thus appears natural to expect that

$$f(x, t) \sim \sum_{k=0}^{\infty} a_k(t)h_k(x)e^{-x^2/2} , \tag{7.17}$$

where moreover $a_k(t)$ might be expected to be of the form

$$a_k(t) = \int_{-\infty}^{0} e^{\nu\lambda t}d\beta_k(\lambda) . \tag{7.18}$$

If $\phi(\vec{R}, 0)$ has the Boltzmann property we know that $f(x, t)$ satisfies the nonlinear equation (3.5) and using this information we shall be able to determine the $a_k(t)$ explicitly.

From (3.5) we deduce almost immediately that

$$(7.19)\quad \frac{d}{dt}\int_{-\infty}^{\infty} f(x, t)h_k(x)dx = \frac{\nu}{2}\iint_{-\infty}^{\infty} f(x, t)f(y, t)\frac{1}{2\pi}$$

$$\cdot \int_0^{2\pi} \{h_k(x\cos\theta + y\sin\theta) + h_k(-x\sin\theta + y\cos\theta) - h_k(x) - h_k(y)\}d\theta dx dy$$

and using the known relation

$$(7.20)\qquad h_k(x\cos\theta + y\sin\theta) = \sum_0^k \binom{k}{l}\cos^l\theta\sin^{k-l}\theta\, h_l(x)h_{k-l}(y)$$

we obtain

$$(7.21)\qquad \frac{db_k}{dt} = -\nu b_k \text{ if } k \text{ is odd}$$

and

$$(7.22)\qquad \frac{db_{2m}}{dt} = \nu\sum_{l=0}^m \binom{2m}{2l}\beta_{2l,\,2m-2l}\, b_{2l}\, b_{2m-2l} - \nu b_{2m}$$

where

$$(7.23)\qquad b_k(t) = \int_{-\infty}^{\infty} f(x, t)h_k(x)dx$$

and

$$(7.24)\qquad \beta_{2l,\,2m-2l} = \frac{1}{2\pi}\int_0^{2\pi}\cos^{2l}\theta\sin^{2m-2l}\theta\, d\theta\, .$$

Formula (7.22) can be rewritten in the more convenient form

$$(7.25)\qquad \frac{db_{2m}}{dt} = \nu(2\alpha_{2m} - 1)b_{2m} + \nu\sum_{l=1}^{m-1}\binom{2m}{2l}\beta_{2l,\,2m-2l}\, b_{2l}\, b_{2m-2l}\, ,$$

where

$$(7.26)\qquad \alpha_{2m} = \beta_{2m,\,0} = \frac{1}{2\pi}\int_0^{2\pi}\cos^{2m}\theta\, d\theta = \frac{1}{2^{2m}}\binom{2m}{m}\, .$$

It is also clear that b_k differs from a_k by a numerical factor only.

Since

$$(7.27)\qquad \int_{-\infty}^{\infty} f(x, t)dx = 1\, , \quad \int_{-\infty}^{\infty} x^2 f(x, t)dx = 1$$

we have $b_0 = 1 (a_0 = 1/\sqrt{2\pi})$ and it follows from (7.25) (since $\alpha_2 = \frac{1}{2}$) that

$$(7.28)\qquad \frac{db_2}{dt} = 0$$

or

$$(7.29)\qquad b_2(t) \equiv 0\, .$$

The odd b's are easily determined from (7.21)

$$b_k(t) = b_k(0)e^{-\nu t} . \tag{7.30}$$

The early even b's are also easily determined and one gets

$$b_4(t) = b_4(0)e^{\nu(2\alpha_4-1)t} , \tag{7.31}$$

$$b_6(t) = b_6(0)e^{\nu(2\alpha_6-1)t} , \tag{7.32}$$

where in deriving these formulas use has been made of the fact that $b_2(t) = 0$. To determine $b_8(t)$ we have the equation

$$\frac{db_8}{dt} = \nu(2\alpha_8 - 1)b_8 + \nu \binom{8}{4} \beta_{4,4}\, b_4^2 , \tag{7.33}$$

and since we know b_4 [see (7.31)] we can solve for b_8 and it is clear that it is a linear combination of

$$e^{\nu(2\alpha_8-1)t} \quad \text{and} \quad e^{2\nu(2\alpha_4-1)t} . \tag{7.34}$$

Similarly,

$$\frac{db_{10}}{dt} = \nu(2\alpha_{10} - 1)b_{10} + 2\nu \binom{10}{4} \beta_{4,6}\, b_4 b_6 \tag{7.35}$$

and b_{10} is a linear combination of the exponentials

$$e^{\nu(2\alpha_{10}-1)t} \quad \text{and} \quad e^{\nu[(2\alpha_4-1)+(2\alpha_6-1)]t} . \tag{7.36}$$

It is clear that in this way all the b's can be determined but formulas will get progressively more and more complex.

A moment's reflection will convince us that in the formula for $b_{2m}(t)$ we shall have only exponentials with time constants

$$\nu \sum_{s=2}^{m} l_s(2\alpha_{2s} - 1) , \tag{7.37}$$

where the l_s are nonnegative integers subject to the condition

$$\sum_{s=2}^{m} l_s s = m . \tag{7.38}$$

This observation is closely related to an interesting property of Hermite polynomials. Let us define the "Boltzmann bracket $[g, h]$" as follows

$$\begin{aligned} &[g, h] \\ &= \int_{-\infty}^{\infty} e^{-y^2/2} \frac{1}{2\pi} \int_0^{2\pi} \{g(x \cos \theta + y \sin \theta)h(-x \sin \theta + y \cos \theta) - g(x)h(y)\} d\theta dy \end{aligned} \tag{7.39}$$

and note that $[g, h]$ is a function of x. If g and h are polynomials of even degree $[g, h]$ is a polynomial whose degree is the sum of degrees of g and h.

Substituting (7.17) in (3.5) we obtain formally

$$\frac{\partial f}{\partial t} = \nu \sum_{k,l=0}^{\infty} a_k(t)a_l(t)[h_k, h_l]\, e^{-x^2/2} = \nu \sum_{m=0}^{\infty} \sum_{k+l=m} a_k(t)a_l(t)[h_k, h_l]\, e^{-x^2/2} . \tag{7.40}$$

Consider now $[h_k, h_l]$ with $k + l = 2m$. If both k and l are odd the bracket is identically 0 and it is sufficient to consider $[h_{2k}, h_{2l}]$ with $2k + 2l = 2m$. The polynomial $[h_{2k}, h_{2l}]$ is of degree $2m$ and hence a linear combination of $h_0, h_2, \cdots, h_{2m}$. The interesting fact is that it is actually only a multiple of h_{2m}.

To see this, note that if $[h_{2k}, h_{2l}]$ did have a lower component, say, h_{2r} $(r < m)$, $a_{2k}a_{2l}$ would have to be part of the coefficient of $h_{2r} \exp(-x^2/2)$ in the expansion of $\partial f/\partial t$ (and hence of f) and this is impossible in view of (7.37) and (7.38).

Thus

$$[h_{2k}, h_{2l}] = C_{k,l}h_{2k+2l} \tag{7.41}$$

and the multiplier $C_{k,l}$ can be easily determined. In fact

$$C_{k,l} = C_{2k}C_{2l}, \tag{7.42}$$

where

$$a_{2k}(t) = C_{2k}b_{2k}(t), \tag{7.43}$$

provided neither k nor l are 0, in which case the formula has to be slightly modified.

A direct proof of the bracket relation (7.41) was communicated to us by Dr. G. E. Baxter.

Comparing our solution of (3.5) with the solution (7.14) of the master equation we notice that, whereas in (7.14) the coefficient of $\omega_k^{(1)}(x)$ is composed of a large number of exponentials (in fact $\gamma(k)$ of them), the corresponding coefficient of $h_k(x) \exp(-x^2/2)$ contains only a finite number (remember that $\gamma(k) \to \infty$ as $n \to \infty$).

There may be several causes for this enormous reduction.

(a) Since we are starting with distributions having Boltzmann's property most temporal modes are absent.[10]

(b) Some eigenvalues of D_k are degenerate.

(c) Some eigenvalues of D_k approach ∞ as $n \to \infty$.

(d) Some $b_k^{(s)}$ approach 0 as $n \to \infty$.

A particularly interesting possibility is that there are $\lambda_k^{(s)}$ which approach 0, but that the corresponding coefficients $b_k^{(s)}$ also approach 0. If this were the case there would be temporal modes which decay extremely slowly but which are unobservable because of the smallness of their amplitude.

8. The linear Boltzmann equation

So far we have primarily concentrated on the evolution of $\phi(\vec{R}, t)$ in case $\phi(\vec{R}, t)$ had the Boltzmann property. There is another case which is of great interest, namely,

$$\phi(\vec{R}, 0) = \frac{c(x_1)}{\int c(x_1)d\sigma}. \tag{8.1}$$

[10] Since functions having Boltzmann's property are, by definition, symmetric, only symmetric spherical harmonics are involved in (7.10). The number of linearly independent symmetric spherical harmonics of degree k remains bounded as $n \to \infty$ but their number is still considerably larger than the number of distinct exponentials which form the coefficient of $h_k(x)$. Part of the excess can be explained by the fact that some of the spherical harmonics contract to 0. Whether this accounts for the whole discrepancy is not yet settled.

The contractions of (8.1) are

$$
(8.2)\qquad
\begin{aligned}
f_1^{(n)}(x_1, 0) &= \frac{c(x_1)\left(1 - \frac{x_1^2}{n}\right)^{\frac{n-3}{2}}}{\int_{-\sqrt{n}}^{\sqrt{n}} c(x_1)\left(1 - \frac{x_1^2}{n}\right)^{\frac{n-3}{2}} dx_1} \\
f_2^{(n)}(x_1, x_2, 0) &= \frac{c(x_1)\left(1 - \frac{x_1^2 + x_2^2}{n}\right)^{\frac{n-4}{2}}}{\iint\limits_{x_1^2+x_2^2 \leq n} c(x_1)\left(1 - \frac{x_1^2 + x_2^2}{n}\right)^{\frac{n-4}{2}} dx_1 dx_2},
\end{aligned}
$$

etc.

In the limit $n \to \infty$ these become

$$
(8.3)\qquad f_1(x_1, 0) \equiv f(x_1) = \frac{c(x_1)\exp\left[-\frac{x_1^2}{2}\right]}{\iint_{-\infty}^{\infty} c(x_1)\exp\left[-\frac{x_1^2 + x_2^2}{2}\right] dx_1 dx_2},
$$

etc.

By the argument of section 4 we have

$$
\begin{aligned}
(8.4)\qquad & \int_{-\infty}^{\infty} f(x, t) g(x) dx \\
&= \sum_{k=0}^{\infty} \frac{(\nu t)^k}{k!} \int_{-\infty}^{\infty}\cdots\int f_{k+1}(x_1, \cdots, x_{k+1}, 0)\, g_{k+1}(x_1, \cdots, x_{k+1})\, dx_1 \cdots dx_{k+1} \\
&= \sum_{k=0}^{\infty} \frac{(\nu t)^k}{k!} \frac{1}{(\sqrt{2\pi})^k} \int_{-\infty}^{\infty}\cdots\int f(x_1, 0) \exp\left[-\frac{x_2^2 + \cdots + x_k^2}{2}\right] g_{k+1}(x_1, \cdots, x_{k+1}) dx_1 \cdots dx_{k+1}.
\end{aligned}
$$

Recall now that

$$
\begin{aligned}
(8.5)\qquad & g_{k+1}(x_1, \cdots, x_{k+1}) \\
&= \sum_{j=1}^{k} \frac{1}{2\pi} \int_0^{2\pi} \{g_k(x_1, \cdots, x_j \cos\theta + x_{k+1}\sin\theta, \cdots, x_k) - g_k(x_1, \cdots, x_k)\} d\theta
\end{aligned}
$$

and note that in the corresponding integral all terms except those which correspond to $j = 1$ vanish.

We thus have

$$
\begin{aligned}
(8.6)\qquad & \frac{1}{(\sqrt{2\pi})^k} \int_{-\infty}^{\infty}\cdots\int f(x_1) \exp\left[-\frac{x_2^2 + \cdots + x_{k+1}^2}{2}\right] g_{k+1}(x_1, \cdots, x_{k+1}) dx_1 \cdots dx_{k+1} \\
&= \frac{1}{(\sqrt{2\pi})^{k-1}} \int_{-\infty}^{\infty}\cdots\int L(f) \exp\left[-\frac{x_2^2 + \cdots + x_k^2}{2}\right] g_k(x_1, \cdots, x_k) dx_1 \cdots dx_k,
\end{aligned}
$$

where

(8.7) $L(f)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} \{ f(x \cos\theta + y \sin\theta) e^{-(-x\sin\theta + y\cos\theta)^2/2} - f(x) e^{-y^2/2} \} \, d\theta dy \; .$$

Repeated applications of (8.6) gives

$$\int_{-\infty}^{\infty} f(x, t) g(x) dx = \sum_{k=0}^{\infty} \frac{(\nu t)^k}{k!} \int_{-\infty}^{\infty} L^k(f(x)) g(x) dx \tag{8.8}$$

or

$$f(x, t) = e^{\nu t L} f(x) \; . \tag{8.9}$$

Thus $f(x, t)$ satisfies the *linear Boltzmann equation*

(8.10) $\dfrac{\partial f(x, t)}{\partial t}$

$$= \frac{\nu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} \{ f(x \cos\theta + y \sin\theta, t) e^{-(-x\sin\theta + y\cos\theta)^2/2} - f(x, t) e^{-y^2/2} \} \, d\theta dy \; .$$

In contrast with section 4 convergence difficulties are of a trivial nature and the reasoning extends easily to more realistic cases. The reason why we now have a much simpler situation is clear on physical grounds.

In fact, the assumption

$$\phi(\vec{R}, 0) = \frac{c(x_1)}{\int c(x_1) d\sigma} \tag{8.11}$$

means that all particles except the first are already in statistical equilibrium [as can be seen from (8.2)] and we have here a situation in which a particle moves in a medium which, statistically speaking, remains stationary.

This is analogous to the theory of Brownian motion except that a collision may now produce large changes in velocity and hence instead of a simple operator like $D(\partial^2 f/\partial x^2)$ we have the more complicated integral operator of (8.10).

The general solution (7.14) of the master equation suggests again that the solution of (8.10) is to be sought in the form

$$\sum_{k=0}^{\infty} a_k(t) h_k(x) e^{-x^2/2} \; . \tag{8.12}$$

It is easily verifiable that $h_k(x) \exp(-x^2/2)$ is an eigenfunction of the operator L, the corresponding eigenvalue being $\lambda_{2m+1} = -\nu$ or $\lambda_{2m} = \nu(\alpha_{2m} - 1)$, and consequently $a_k(t) = c_k e^{\lambda_k t}$. The time constants λ_{2m} are not among those encountered in the solution of the nonlinear equation.

9. Linearized equation. Method of successive approximations

We shall now discuss a method of solving the nonlinear Boltzmann equation, which in its essence is a perturbation method.

We start by postulating $f(x, t)$ in the form

$$f(x, t) = f_0(x)(1 + \theta p_1(x, t) + \theta^2 p_2(x, t) + \cdots) , \tag{9.1}$$

where θ is an auxiliary artificial parameter. Substituting (9.1) into Boltzmann's equation (3.5) we get by comparing coefficients of powers of θ

$$\frac{dp_1}{dt} = \nu\{[1, p_1] + [p_1, 1]\}, \tag{9.2}$$

$$\frac{dp_2}{dt} = \nu\{[1, p_2] + [p_2, 1] + [p_1, p_1]\},$$

etc.

Here

$$f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{9.3}$$

and the brackets [,] are the Boltzmann brackets defined by (7.39).

Equations (9.2) can be solved successively and the solution $f(x, t)$ is then given by (9.1) with $\theta = 1$.

The operator

$$\Lambda p \equiv [1, p] + [p, 1] = \int_{-\infty}^{\infty} f_0(y) \frac{1}{2\pi} \cdot \int_0^{2\pi} \{p(x \cos \theta + y \sin \theta) + p(-x \sin \theta + y \cos \theta) - p(x) - p(y)\} d\theta dy \tag{9.4}$$

is linear and the equation

$$\frac{dp_1}{dt} = \nu \Lambda p_1 \tag{9.5}$$

will be referred to as the *linearized Boltzmann equation.*

In order to solve equations (9.2) we must know $p_1(x, 0)$, $p_2(x, 0)$, $\cdots$. It is simplest to set $p_2(x) \equiv 0$, $p_3(x, 0) \equiv 0$, $\cdots$ and

$$p(x) \equiv p_1(x) = \frac{f(x, 0)}{f_0(x)} - 1 \tag{9.6}$$

so that

$$f(x, 0) = f_0(x)(1 + p(x)) . \tag{9.7}$$

The background of the successive scheme (9.2) will now emerge from (4.22). In fact, if we substitute (9.7) into (4.22) we obtain

$$\int_{-\infty}^{\infty} f(x, t) g(x) dx = \int_{-\infty}^{\infty} f_0(x) g(x) dx + \sum_{k=0}^{\infty} \frac{(\nu t)^k}{k!} \cdot \int \cdots \int_{-\infty}^{\infty} f_0(x_1) \cdots f_0(x_{k+1}) \left(\sum_1^{k+1} p(x_j)\right) g_{k+1}(x_1, \cdots, x_{k+1}) dx_1 \cdots dx_{k+1}$$
$$+ \sum_{k=1}^{\infty} \frac{(\nu t)^k}{k!} \int \cdots \int_{-\infty}^{\infty} f_0(x_1) \cdots f_0(x_{k+1}) \cdot \left(\sum_{1 \leq i < j \leq k+1} p(x_i) p(x_j)\right) g_{k+1}(x_1, \cdots, x_{k+1}) dx_1 \cdots dx_{k+1} + \cdots . \tag{9.8}$$

We shall now show that the successive infinite sums in this rearranged series are

$$(9.9)\qquad \int_{-\infty}^{\infty} f_0(x)p_1(x,\, t)g(x)dx\ ,\qquad \int_{-\infty}^{\infty} f_0(x)p_2(x,\, t)g(x)dx\ ,\ \cdots .$$

To see this we note that from the definition (4.19) of g_k we have

$$(9.10)\qquad \int\cdots\int_{-\infty}^{\infty} f_0(x_1)\cdots f_0(x_{k+1}) \left(\sum_{1}^{k+1} p(x_j)\right) g_{k+1}(x_1,\cdots,x_{k+1})dx_1\cdots dx_{k+1}$$

$$= \int\cdots\int_{-\infty}^{\infty} f_0(x_1)\cdots f_0(x_k) \left(\sum_{1}^{k} \Lambda p(x_j)\right) g_k(x_1,\cdots,x_k)dx_1\cdots dx_k$$

$$= \int_{-\infty}^{\infty} f_0(x)\Lambda^k p(x)g(x)dx$$

and hence formally the first infinite sum of (9.8) is equal to

$$(9.11)\qquad \int_{-\infty}^{\infty} f_0(x)g(x)(e^{\nu t\Lambda}\, p(x))dx$$

which is clearly

$$(9.12)\qquad \int_{-\infty}^{\infty} f_0(x)g(x)p_1(x)dx\ .$$

The computations with the second infinite sum are more involved and we shall just give the result leaving the detailed verification to the reader.

For a symmetric function $p(x_1, x_2)$ of two variables we define the operators $\Lambda_{(2)}$ and B as follows:

$$(9.13)\qquad \Lambda_2 p(x_1,\, x_2) = \Lambda_{(x_2)}p(x_1,\, x_2) + \Lambda_{(x_1)}p(x_1,\, x_2)\ ,$$

where $\Lambda_{(x_2)}(\Lambda_{(x_1)})$ is the operator Λ defined by (9.4) applied to $p(x_1, x_2)$ while keeping $x_2(x_1)$ constant, and

$$(9.14)\qquad Bp(x_1,\, x_2) = \int_{-\infty}^{\infty} dx_2 f_0(x_2)\,\frac{1}{2\pi}$$

$$\cdot \int_0^{2\pi} \{p(x_1\cos\theta + x_2\sin\theta - x_1\sin\theta + x_2\cos\theta) - p(x_1,\, x_2)\}\, d\theta\ .$$

From the definition of g_k one gets

$$(9.15)\qquad \int\cdots\int_{-\infty}^{\infty} f_0(x_1)\cdots f_0(x_{k+1}) \left(\sum_{1\leqq i<j\leqq n} p(x_i,\, x_j)\right) g_{k+1}(x_1,\cdots,x_{k+1})dx_1\cdots dx_{k+1}$$

$$= \int_{-\infty}^{\infty} f_0(x)g(x)\ \{\Lambda^{k-1}Bp + \Lambda^{k-2}B\Lambda_{(2)}p + \cdots + B\Lambda_{(2)}^{k-1}p\}\ dx$$

and setting

$$(9.16)\qquad P(x,\, t) = \sum_{k=1}^{\infty} \frac{(\nu t)^k}{k\,!}\ \{\Lambda^{k-1}\, Bp + \Lambda^{k-2}\, B\Lambda_{(2)}\, p + \cdots + B\Lambda_{(2)}^{k-1}p\}$$

we see that

$$\Lambda P = \frac{1}{\nu}\frac{\partial P}{\partial t} - B\{e^{\nu t \Lambda^{(2)}} p\} . \tag{9.17}$$

It can now be verified that if

$$p(x_1, x_2) = p(x_1)p(x_2) , \tag{9.18}$$

one has

$$B\{e^{\nu t \Lambda^{(2)}} p\} = [p_1(x, t), p_1(x, t)] , \tag{9.19}$$

and thus

$$P(x, t) = p_2(x, t) . \tag{9.20}$$

The eigenfunctions of the operator Λ are again the Hermite polynomials $h_k(x)$, the corresponding eigenvalues being[11]

$$\mu_{2m+1} = -\nu , \qquad \mu_{2m} = \nu(2\alpha_{2m} - 1) . \tag{9.21}$$

Consequently

$$p_1(x, t) \sim \sum_{1}^{\infty} \delta_k e^{\mu_k t} h_k(x) e^{-x^2/2} . \tag{9.22}$$

Using the bracket relation (7.41) and the fact that $[h_k, h_l] = 0$ if either k or l is odd we get

$$[p_1, p_1] \sim \sum_{1}^{\infty} \left(\sum_{k+l=m} \delta_{2k}\delta_{2l} C_{k,l} e^{(\mu_{2k}+\mu_{2l}) t} \right) h_{2m}(x) e^{-x^2/2} . \tag{9.23}$$

Hence $p_2(x, t)$ will be linear combination of exponentials with time constants

$$-(\mu_{2k} + \mu_{2l}). \tag{9.24}$$

Similarly $p_3(x, t)$ will involve exponentials with time constants

$$-(\mu_{2k} + \mu_{2l} + \mu_{2m}) , \tag{9.25}$$

etc.

All this agrees, of course, with what we have found in section 7.

The *linearized Boltzmann equation* is used to discuss systems which are so close to equilibrium that $p_1(x, t)p_1(y, t)$ and other quadratic terms can be neglected. If we compare $p_1(x, t)$ with the exact solution of the nonlinear Boltzmann equation it appears that it is *actually exact* as far as the first few (slowest) exponentials are concerned. However, denoting by λ_0 the smallest time constant of Λ which is different from 0, we see that the part of $p_1(x, t)$ which involves exponentials with time constants larger than $2\lambda_0$ is *meaningless* because $p_2(x, t)$ introduces larger terms.

This observation holds in general (that is, for the physically significant cases of

[11] Note that $\mu_2 = \mu_0 = 0$ so that 0 is a doubly degenerate eigenvalue. In an actual physical case 0 is fivefold degenerate corresponding to given conservation laws (particles, energy and three components of momentum). In our case, only the number of particles and the energy are conserved, hence only double degeneracy. For the linear equation (8.10), 0 is a simple eigenvalue because energy is no longer conserved.

hard spheres and Maxwell molecules) even though the spectrum of Λ may not even be discrete.

The scheme (9.2) of solving Boltzmann's equation is not really a successive approximation scheme because higher "approximations" involving $p_2(x, t)$, $p_3(x, t)$, etc. introduce terms which are larger than terms kept in in $p_1(x, t)$.

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