THE HERTZSPRUNG-RUSSELL DIAGRAM

BENGT STRÖMGREN
YERKES AND McDONALD OBSERVATORIES

1. Introduction

Around 1900 it was becoming clear to astrophysicists that stars differ not only with respect to atmospheric temperature but also, for one and the same atmospheric temperature, with respect to luminosity, in other words, that two parameters are necessary to characterize a star.

In 1911 and the following years E. Hertzsprung and H. N. Russell, basing their researches on observational material pertaining to spectral types, colors and luminosities of stars, discussed the distribution of stars in color-luminosity, or color-spectral class diagrams, and described the most essential features of the distribution. A diagram in which stars are represented by points according to their spectral class and luminosity, or their color index and luminosity, is generally referred to as a Hertzsprung-Russell diagram (H-R diagram).

The relation between spectral class and effective temperature is now fairly well established. It depends to a certain degree on the luminosity. The same is true with respect to the relation between color index (which measures the intensity ratio between two suitably chosen wave-length regions) and effective temperature. Thus to each point in an H-R diagram (abscissa, spectral type or color index; ordinate, luminosity) we can assign an effective temperature. Since the effective temperature T_e determines the flux of energy F per unit area of the surface of the star, the latter quantity is a known function of position in the H-R diagram.

The luminosity L, which gives the total energy radiation per second, is usually measured by the absolute bolometric magnitude $M_{\text{bol}} = \text{constant} - 2.5 \log_{10} L$. The photometric observations yield quantities such as the visual magnitude and the photographic magnitude which measure the light intensity in wave-length regions around 5600 A and 4300 A, respectively, while the bolometric magnitude by definition measures the total energy of the spectrum. However, the reductions of visual or photographic magnitudes to bolometric magnitudes, the so-called bolometric corrections, are known as functions of the spectral class. Hence a visual or photographic absolute magnitude-color index diagram can easily be converted into a bolometric absolute magnitude-effective temperature diagram.

The luminosity of a star is equal to the surface times the flux F,

$$L = 4\pi R^2 F \tag{1}$$

where R is the radius of the star. Hence the radius can be computed from the luminosity and the effective temperature. In fact, since by the definition of the effective temperature T_e the flux F is proportional to T_e^4, we have

$$\frac{L}{L_\odot} = \left(\frac{T_e}{T_\odot}\right)^4 \left(\frac{R}{R_\odot}\right)^2 \tag{2}$$

where $L_{\odot}$, $T_{\odot}$ and $R_{\odot}$ denote the known values of the luminosity, the effective temperature and the radius of the sun (see figure 1).

Any two stars represented by coinciding points in the H-R diagram will thus be identical with respect to effective temperature, luminosity, and radius. The important question arises whether or not stars represented in the same point in the H-R diagram are identical in every respect. If they are indeed identical, then we have ideal two-dimensional classification. If not, at least one additional parameter is necessary for the complete characterization of a star.

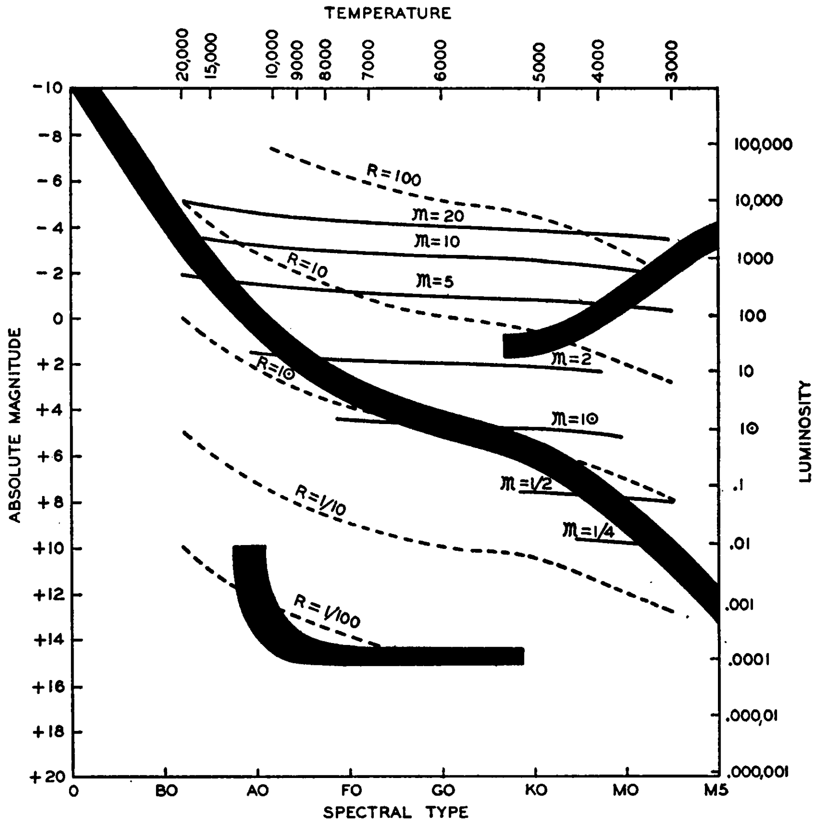

FIGURE 1

H-R diagram with lines of equal mass and equal radius (after Otto Struve, *Stellar Evolution*, Princeton University Press, 1950, p. 46, figure 8). The giant sequence is to the right of the main sequence, the white-dwarf sequence to the left.

This question may be investigated in two ways. First, the spectra of stars represented in the same point (or small area) in the H-R diagram may be compared in detail and investigated for differences. Second, the masses of stars from the same small area of the H-R diagram may be compared. The observational material of masses (visual or spectroscopic binaries) is very important, but it is small. It is therefore significant that masses can be derived in certain cases from a quantitative discussion of the properties of the spectrum. We shall return to the latter question in section 5.

Quite generally it can be said that classification of stars according to two parameters (effective temperature and luminosity) gives a very good description of the properties of stars. As a rule, any group of stars with representative points all within a small area of the H-R diagram is remarkably homogeneous. There are exceptions, however, of which the following are the most important: (1) Certain peculiar stars have spectra that differ quite markedly from those of the normal stars corresponding to their locations in the H-R diagram. Such stars are quite important from the point of view of the astrophysicist, but they are relatively very few in number. (2) It has become clear during the last few years that there is at least one region in the H-R diagram, namely that of main-sequence F stars, where one finds a mixture of stars with pronounced spectral differences and presumably of markedly different chemical composition. They are stars of Baade's population I and extreme population II, respectively. However, in any sample of F stars from our galactic neighborhood the population I stars dominate to the extent that the phenomenon has been difficult to detect and establish. In fact, the conclusion has been reached in an indirect way through comparative studies of stars in globular cluster and in our neighborhood. It should be emphasized that these striking differences pertain to two kinds of stars that are both occurring in very large numbers in our galaxy as a whole. (3) The region of the K giants contains a mixture of stars of somewhat different chemical composition, the observations of the spectra indicating the necessity of the introduction of a third population parameter. There is also some indication that in this region stars at the same point in the H-R diagram may differ appreciably in mass. We shall return to these questions in section 6.

2. Methods of determining the distribution of the stars in the H-R diagram

The determination of the spectral class or the color index of a star for the purpose of its representation in an H-R diagram generally presents no problem. Difficulties arise, however, in connection with the determination of absolute magnitudes. Photometry yields apparent magnitudes m, and knowledge of the distance p (in parsecs) of the star is required for the reduction to absolute magnitude: $M = m + 5 - 5 \log p$. A relative accuracy of approximately 5 per cent in the distance determination is necessary in order to obtain a precision of one-tenth of a magnitude in M.

For nearby stars the trigonometric parallax gives a good determination of the distance. For about 6,000 stars the trigonometric parallax has been determined, with probable errors generally in the range 0″.005 to 0″.012. For stars with trigonometric parallaxes greater than 0″.1, that is, with distances smaller than 10 parsecs, the accuracy of the absolute magnitudes derived from the trigonometric parallax is very satisfactory, and for a parallax equal to 0″.05 the procedure is still usable for individual stars. However, for parallaxes considerably smaller than this the relative uncertainty becomes too great, and the trigonometric parallaxes are then in general useful only for such purposes as the determination of the average parallax for groups of stars.

Studies of the stars within, say, 10 parsecs considered as a typical sample of the galactic neighborhood have proved very useful, particularly in determining the distribution of the stars in the H-R diagram. The sample of stars nearer than 5 parsecs is believed to be fairly complete for stars brighter than absolute magnitude 15. For the stars nearer than 10 parsecs, the limit of completeness is at a higher luminosity, and for the sample of stars within 20 parsecs it is at still higher absolute brightness. Presumably less than 20 per cent of the stars within 20 parsecs have known trigonometric parallaxes.

There are only 36 known stars within 5 parsecs (47 when components of double and multiple stars are counted separately). This sample is small, and even when one goes to the distance limit 20 parsecs, the sample is too small to give a sufficient representation of the stars brighter than visual absolute magnitude $2^m - 3^m$. For the upper part of the H-R diagram, methods other than the use of trigonometric parallaxes must therefore be depended upon for a determination of the distribution of stars in the diagram.

The procedures that have been used include the following: (1) Determination of the distribution of absolute magnitudes from observed distribution of proper motions, separately for different intervals of spectral class. (2) Utilization of observed spectral classes or color indices, and apparent magnitudes for stars belonging to clusters. The stars in a cluster are practically at the same distance from the observer so that the difference between apparent and absolute magnitude is very nearly constant for stars in the same cluster. (3) The use of absolute magnitudes determined from stellar spectra.

Applications of the method of trigonometric parallaxes and of the cluster method are described in the contributions by O. J. Eggen, H. L. Johnson, and G. E. Kron to this Symposium. We shall consider here the method of spectroscopic absolute magnitudes, and particularly in sections 3 and 4 the method based on photoelectric photometry in narrow wave-length regions.

A method for determining absolute magnitudes from slit spectra, using estimated stellar absorption line intensities, was developed at Mount Wilson Observatory by Adams and Kohlschütter about forty years ago. Adams, Joy, Humason and Brayton [1] in 1935 published a catalogue of spectroscopic absolute magnitudes for 4,179 stars, mostly of spectral classes A, F, G, K and M, and brighter than apparent visual magnitude 7, and discussed the distribution of the stars in the H-R diagram.

Morgan, Keenan and Kellman [2] have described a system of two-dimensional spectral classification from slit spectra of a dispersion of about 120 A per min., which was developed at Yerkes Observatory.

Lindblad ([3] through [7]) developed methods for two-dimensional classification from low-dispersion spectra photographed with an objective prism about thirty years ago. The classification was made according to the intensities of relatively strong features in the spectra the strength of which could be estimated, or preferably measured through photographic photometry, even in the case of short unwidened spectra.

The success of the Lindblad method indicated that it should be possible to achieve accurate two-dimensional spectral classification through photoelectric photometry in wave-length bands of widths of the order of 50–100A, isolated for instance through the use of interference filters. The author ([8] through [10]) has investigated some of the possibilities. Results obtained for F stars are described in [11], and for G and K stars by Strömgren and Gyldenkerne, [12], [13]. In section 3 the results obtained are summarized, and in particular the application of the method to stars in the range A3–F0 is discussed.

For the spectral classes B, A and F the classification through photoelectric photometry with interference filters is based on an evaluation of hydrogen criteria, namely the strength of the absorption line Hβ and the magnitude of the Balmer discontinuity. Lindblad and his associates through their photographic-photometric work showed that the strength of the hydrogen lines is a very good luminosity criterion for early-type stars. The great usefulness of the Balmer discontinuity as an indicator of spectral class (effective temperature) for early-type stars was established through photographic-photometric work by Barbier, Chalonge, and Vassy [14], and by Öhman [15]. Barbier and Chalonge [16], [17] developed a method of two-dimensional classification through photographic spectrophotometry of the spectral region around the wave-length of the Balmer discontinuity. Extensive work according to this method has recently been carried out by Chalonge and Divan [18]. Öhman [19] utilized photographic-photometric determinations of the strength of hydrogen lines and the K-line for classification.

In a photographic-spectrophotometric investigation by Hack [20] the measures of the Balmer discontinuity and the strength of Hδ are used for accurate two-dimensional classification of B, A, and F stars. Westerlund [21] following Lindblad has developed and investigated a scheme of two-dimensional classification through photographic photometry of short, widened objective prism spectra, using classification criteria that are practically independent of interstellar reddening. For B-A7 stars his classification is based on measures of the hydrogen lines and the K-line. Hossack [22] has obtained a high degree of accuracy in two-dimensional classification of late-type stars through the use of an oscilloscope microphotometer in the evaluation of slit spectra with a dispersion of 33A per min.

Classification from short objective prism spectra is of the greatest importance in survey work in extended fields, while classification with spectra obtained one at a time is more accurate. The classification procedures based on photoelectric photometry in narrow wave-length region give high accuracy and freedom from systematic errors to a relatively high degree. Simultaneous photoelectric photometry in several suitably selected narrow wave-length regions in comparison with photographic spectrophotometry has the great advantage of reaching much fainter stars: with the same telescope and exposure time, and for equal classification precision, the limiting magnitude is about 4 magnitudes fainter for the photoelectric procedure. The great importance of systematic classification studies based on photographic spectra to precede the working-out of photoelectric procedures should however be emphasized.

3. Two-dimensional classification through photoelectric photometry in narrow wave-length regions

In an investigation referred to above [11] it was shown that satisfactory two-dimensional classification for F stars can be obtained through photoelectric photometry with interference filters that isolate wave-length regions suitable for defining an Hβ strength index and a Balmer discontinuity index.

A dielectric Hβ filter (Bausch and Lomb) having a half-width of 35A was used together with two comparison interference filters with transmission peaks at 4700A and 5000A to measure the strength of the Hβ absorption line. Three narrow-band filters peaked at 3550A, 4030A, and 4500A gave an index sensitive to the magnitude of the Balmer discontinuity. Photoelectric intensity measures through these filters yielded two classification indices l and c defined as follows:

$$l = 2.5\{\tfrac{1}{2}[\log I(4700) + \log I(5000)] - \log I(4861)\} + \text{constant} \tag{3}$$

$$c = 2.5\{2 \log I(4030) - \log I(3550) - \log I(4500)\} + \text{constant} \tag{4}$$

where the I's denote the intensities through the filters with the transmission peaks indicated. With this choice of wave-length l is practically unaffected by interstellar reddening, while c is practically or very nearly unaffected.

The technique of photoelectric photometry that yields l and c is briefly described in [11]. The observations were made with the 82-inch reflector of the McDonald Observatory. The probable error of one observation is $\pm 0\overset{m}{.}003$ for l, and $\pm 0\overset{m}{.}008$ for c.

Comparison with spectral classes and luminosity classes on the MK system, determined by Morgan, showed that the indices l (Hβ index) and c (Balmer discontinuity index) give good two-dimensional classification for the F stars [11]. The l–c classification procedure was further tested for accuracy and calibrated in the following way: (1) For a number of stars for which the distances were so small that interstellar reddening could safely be assumed to be negligible photoelectric color indices in H. L. Johnson's (B-V)-system were measured. A linear relation between B-V, and l and c was derived, and it was then tested how accurately the color index can be predicted from l and c. (2) Visual absolute magnitudes M_v were derived for a number of stars from trigonometric parallaxes, and from spectral classes and luminosity classes determined spectroscopically by Morgan. Again for the limited range in spectral class considered, a linear expression in l and c proved adequate, and the accuracy of the reproduction of M_v was tested. The results of the investigation [11] are summarized in table III (p. 59).

For G and K stars the Hβ absorption line and the Balmer discontinuity are no longer strong features of the spectrum, and other classification criteria were used for the two-dimensional classification through photoelectric photometry with interference filters [12], [13]. Three of the criteria used by Lindblad and his collaborators in the photographic work with short objective prism spectra proved to be adequate for the purpose, namely, the strength of the K-line, the discontinuity at the G-band, and the strength of the cyanogen absorption near 4200A. Comparisons were made with spectral classes and luminosity classes on the MK system, and with B-V color indices for nearby stars not appreciably reddened by interstellar absorption. For a summary of the results see table III (p. 59).

Going from the F stars in the direction of higher effective temperature one finds that the strength of Hβ and the Balmer discontinuity increase and reach maxima for the A stars. The occurrence of the maxima complicates the problem of classification of the A stars somewhat. However a recent investigation has shown that the l-c procedure gives very satisfactory results for the spectral range beyond F0 up to about A3. The observations in question were made by the author with the 82-inch reflector of the McDonald Observatory in November, 1954, and a preliminary report is given below.

The classification indices l and c, measuring the strength of Hβ and the Balmer discontinuity, respectively, were determined for a number of A stars brighter than $5^{m}.5$ and between declinations 0° and +60°. So far the program was limited to the right ascensions accessible during the month of November, and about one-half of all A stars brighter than $5^{m}.5$ and in the declination zone mentioned were measured. Photoelectric color indices B-V were also determined, and since the stars, with the exception of a few stars of luminosity class I had distances less than 100 parsecs (mostly less than 50 parsecs), the effect of interstellar reddening upon the color indices is presumably quite small.

For the A stars with B-V $\geqq 0^{m}.10$ (later than about A3) a linear expression for B-V in terms of l and c was assumed and the coefficients determined by a least-squares solution.

The calibration of the classification of indices l and c in terms of absolute visual magnitudes M_v presents greater difficulty as in the range considered; the number of A stars with sufficiently accurate trigonometric parallaxes is small. However, accurate values of M_v are available for a number of stars belonging to the Hyades moving cluster. The values determined by Van Bueren [23] were used for seven Hyades stars, and one star with a good trigonometric parallax and a normal point corresponding to the average of three stars with somewhat less accurate trigonometric parallaxes were added.

The following linear expressions for the color index B-V, and for M_v, in terms of l and c were obtained,

$$\text{B-V} = 0.626 - 0.564\,l - 0.347\,c \tag{5}$$

$$M_v = 2.84 + 26.4\,l - 7.2\,c \tag{6}$$

A3–A8 stars, B-V = $0^{m}.10 - 0^{m}.20$
Luminosity classes III, IV and V

Table I gives the residuals O-C for the color index B-V, and for M_v.

Observations of F stars obtained previously with the 82-inch reflector of the McDonald Observatory [11] were supplemented in November, 1954, by observations of F0 stars. A program consisting of all F0 stars brighter than $5^{m}.5$ and between declination 0° and +60° was thus two-thirds completed.

Photoelectric color indices B-V were available for a number of practically unreddened stars, and a linear expression for B-V in terms of l and c was again derived.

For the purpose of the M_v-calibration accurate distances were available for 6 Hyades stars, 2 Ursa Major nucleus stars, and 2 stars with good trigonometric parallaxes. In addition, 5 normal points were formed by taking averages for stars with nearly the same values of l and c and having moderately accurate trigonometric parallaxes.

The linear expressions for B-V and M_v derived by least-squares solutions are given below,

$$\text{B-V} = 0.576 - 1.102\,l - 0.147\,c \tag{7}$$

$$M_v = 6.36 + 24.7\,l - 10.3\,c \tag{8}$$

A9-F1 stars, B-V 0.25–0.35
Luminosity classes III, IV and V;
$l \geqq 0.122$

The residuals O-C for B-V, and for M_v, are given in table II.

For stars of type earlier than A2 or A3, corresponding to B-V $< 0^{m}.10$, the l-c

procedure of classification becomes less accurate. Figure 2 illustrates the situation. It shows an *l-c* diagram, with the curve corresponding to main-sequence stars, luminosity class V. The shaded area according to the observations contains both main-sequence A0-1 stars, and A2-A3 stars of luminosity somewhat higher than for the corresponding main-sequence stars. Consequently the color index B-V of the stars within even a very small area of the *l-c* diagram varies within a range of about 0^m06. Photoelectric measures of a third classification index, for example, an index measuring the strength of the K-line, would presumably make it possible to avoid the ambiguity.

TABLE I

SPECTRAL RANGE A3–A8

HR		l	c	B-V observed	O-C for B-V	M_v from (4)	O-C for M_v
812	38 Ari	0^m198	0^m890	0^m20	-0^m01		
1905	122 Tau	0.198	0.898	0.20	0.00		
1414	79 Tau	0.215	0.918	0.20	+0.02	1^m9	$+0^m2$
3569	ι UMa	0.225	0.882	0.19	0.00	2.4	−0.2
8984	λ Psc	0.220	0.900	0.19	0.00		
184	π Cas	0.242	0.930	0.18	+0.01		
1412	θ^2 Tau	0.208	0.995	0.18	+0.01	1.1	−0.1
8880	τ Peg	0.212	0.980	0.17	0.00		
1427		0.228	0.995	0.16	+0.01	1.7	+0.2
9039	82 Peg	0.220	0.975	0.16	0.00		
2375		0.212	0.990	0.16	−0.01		
3555	σ^2 Cnc	0.235	0.978	0.15	−0.01		
1380	64 Tau	0.235	0.990	0.15	0.00	1.9	0.0
1620	ι Tau	0.225	1.032	0.15	+0.01	1.3	+0.1
553	β Ari	0.250	0.995	0.14	0.00	2.2	
613	κ Ari	0.242	0.992	0.14	−0.01		
3572	α Cnc	0.232	1.040	0.14	0.00		
1387	65 Tau	0.228	1.050	0.14	0.00	1.3	−0.2
403	δ Cas	0.222	1.080	0.14	+0.01		
8463		0.245	0.992	0.13	−0.01		
269	37 And	0.240	1.048	0.13	+0.01	1.6	
622	β Tri	0.228	1.058	0.13	0.00		
324	41 And	0.255	1.052	0.12	0.00	2.0	
620	58 And	0.230	1.075	0.11	−0.01		
2763	λ Gem	0.232	1.062	0.10	−0.02		
1389	68 Tau	0.240	1.090	0.06		1.3	0.0
β Ari, 37 And, 41 And, normal point						1.9	0.0

Figure 2 also shows the curve in the *l-c* diagram for stars one magnitude brighter than the main-sequence stars, according to the relations (5), (6), and (7), (8). The curve, however, is only drawn for B-V $\geqq 0^m10$.

The very luminous A and F stars (luminosity classes Ia, Ib and II) according to the measures are very well segregated from the rest of the stars in the *l-c* diagram, having quite small values of *l* for their *c*.

The curve in the *l-c* diagram (figure 2) to the right of the main-sequence curve represents the metallic line stars. It is seen that these are also segregated from the other stars.

From spectral class B9 or B8 and earlier, the *l*-*c* procedure again gives accurate spectral classification [8], [9], [10]. However, rough spectral types must be available in order to separate the B stars from stars of later type. For instance, a B5V star occupies about the same position in the *l*-*c* diagram as an F2V star. Alternatively, a third classification index must be measured photoelectrically.

TABLE II

Spectral Range A9–F1

HR		*l*	*c*	B-V observed	O-C for B-V	M_v from (5)	O-C for M_v
4931	78 UMA	$0^{\mathrm{m}}.130$	$0^{\mathrm{m}}.600$	$0^{\mathrm{m}}.36$	$+0^{\mathrm{m}}.02$	$3^{\mathrm{m}}.4$	$-0^{\mathrm{m}}.3$
840	16 Per	0.125	0.748	0.35	+0.02	1.8	
1201		0.132	0.630	0.35	+0.01	3.2	−0.1
1292	45 Tau	0.125	0.625	0.34	−0.01	3.0	+0.2
1269	ψ Tau	0.125	0.615	0.33	0.00		
4310	χ Leo	0.140	0.705	0.33	+0.01		
2777	δ Gem	0.130	0.705	0.33	0.00	2.3	+0.1
3757	23 UMa	0.130	0.742	0.33	+0.01	1.9	
5936	λ CrB	0.135	0.688	0.33	0.00	2.6	
4141	37 UMa	0.135	0.608	0.33	−0.01	3.4	−0.1
623	14 Ari	0.138	0.820	0.33	+0.02		
2085	η Leo	0.138	0.635	0.33	0.00	3.2	
1432	85 Tau	0.138	0.660	0.33	0.00	3.0	0.0
1287	44 Tau	0.125	0.735	0.32	−0.01		
2852	ρ Gem	0.132	0.652	0.32	−0.01	2.9	+0.1
3015	4 Pup	0.140	0.680	0.32	+0.01	2.1	
5570	16 Lib	0.145	0.730	0.31	0.00	2.4	
1408	76 Tau	0.142	0.712	0.30	−0.01	2.6	+0.5
2707	21 Mon	0.145	0.810	0.30	0.00		
813	μ Ceti	0.160	0.775	0.30	+0.01	2.4	
3888	υ UMa	0.135	0.810	0.29	−0.02	1.4	
4031	ζ Leo	0.138	0.892	0.29	0.00		
4916	8 Dra	0.150	0.700	0.29	−0.02	2.9	
8830	7 And	0.158	0.738	0.29	0.00	2.7	
717	12 Tri	0.165	0.775	0.29	+0.01		
569	λ Ari	0.160	0.805	0.28	0.00	2.0	
8494	ε Cep	0.170	0.792	0.28	+0.01	2.4	
1331	51 Tau	0.172	0.792	0.27	0.00	2.5	−0.3
114	28 And	0.160	0.875	0.25	−0.02		
4090	30 LMi	0.165	0.920	0.25	−0.01		
23 UMa, 16 Per, 4 Pup		normal point				2.0	−0.6
υ UMa, λ Ari		normal point				1.7	0.0
η Lep, λ CrB		normal point				3.0	+0.1
7 And, 16 Lib, 8 Dra		normal point				2.7	0.0
ε Cep, μ Ceti		normal point				2.4	−0.1

For the B stars *l* is a measure of the visual absolute magnitude which is nearly independent of *c*, or the spectral class [8], [9], [10]. This corresponds to the correlation between the strength of the hydrogen lines measured in low-dispersion objective prism spectra and the absolute magnitude which was established by Lindblad and utilized by him for the luminosity classification of early-type stars. Reference is also made to the investigations of Petrie [24] on the determination of absolute magnitudes from the strength of Hγ measured through photographic photometry of slit spectra.

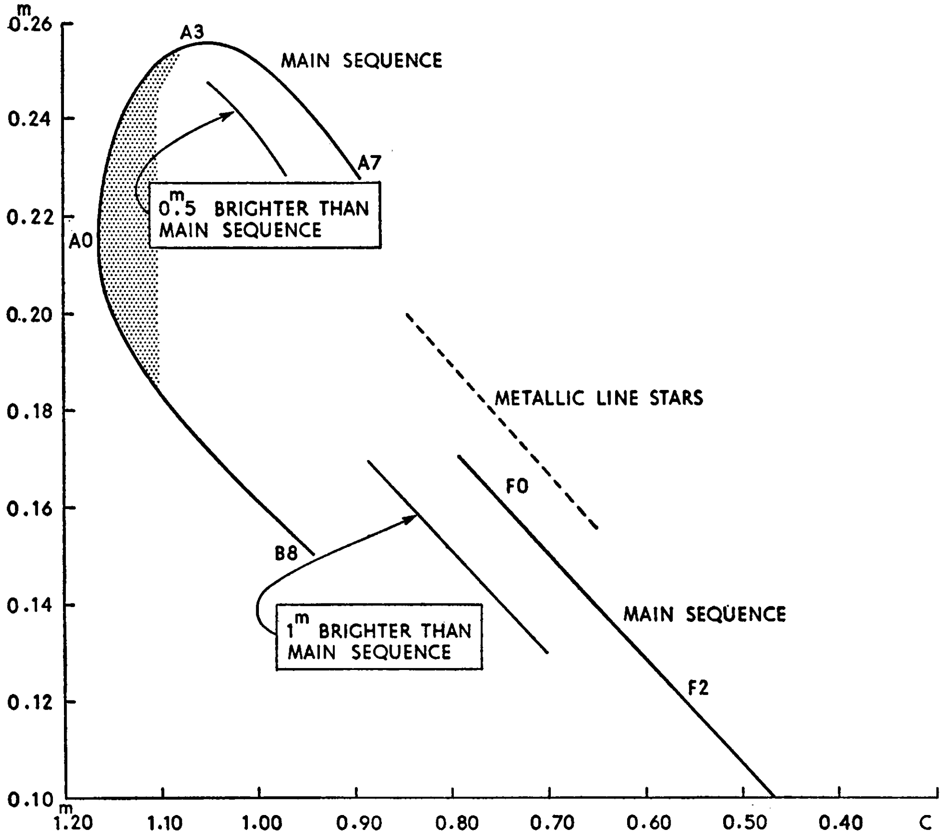

FIGURE 2

The representative point of a star in this diagram is according to the Balmer discontinuity index c (abscissa) and the Hβ index l (ordinate).

The index c is a good indicator of the spectral class for the B stars. It corresponds to the color index ratio of W. Becker [25], [26], and the quantity Q defined by Johnson and Morgan. The use of interference filters makes it possible to choose the wave-lengths that define c rather close together, which gives the advantage of an index that is not very sensitive to changes in the law of wave-length dependence of interstellar reddening.

Table III summarizes the results obtained concerning the accuracy of spectral classification through photoelectric photometry with interference filters.

The photometric probable errors for B-V and M_v are derived from the linear expressions in terms of l and c, the probable errors for one observation of these quantities being known.

The differences O-C between the observed values and those given by the linear expressions for B-V and M_v yield certain probable errors of one difference, O-C. These can be analyzed in terms of (1) The photometric probable error of the B-V or M_v predicted from the linear expressions; (2) The probable error of the observed B-V or M_v; (3) The effect of cosmical scatter, due to causes such as variation of chemical composition from star to star, and the influence of duplicity of some of the stars. The analysis (see [11], [12] and [13]) leads to estimates of the effect of cosmical scatter. These estimates may be too high in some cases, as it is difficult to take the effects of any existing interstellar reddening into account, and also because the values of the probable errors of the photometry derived from internal agreement may be a little too low. Deviations from linearity in the relation between l and c, and B-V and M_v, respectively, will also tend to increase the average differences O-C. However, we may assume that the true probable error will be somewhere in between the photometric value and the estimated value (see table III), probably closer to the latter.

TABLE III

Range of Spectral Class and Luminosity Class	Photometric Probable Error, One Observation			Estimated Probable Error, One Observation including Effect of Cosmical Scatter		
	B-V	Spectral Class	M_v	B-V	Spectral Class	M_v
B0–B8 Ib–V	$\pm 0^{m}.003$	±0.008	$\pm 0^{m}.15$			$\pm 0^{m}.3$
A3–A8 III–V	±0.003	±0.010	±0.10	$\pm 0^{m}.005$	±0.017	±0.12
A9–F1 III–V	±0.003	±0.010	±0..11	±0.006	±0.020	±0.17
F2–F9 III–V	±0.006	±0.020	±0.18	±0.008	±0.024	±0.2
G7–K5 I–V	±0.009	±0.014	±0.35	±0.012	±0.018	±0.5

The method of spectral classification through photoelectric photometry with interference filters was developed with a view to applications that include the determination of color excesses through the comparison of color indices predicted from classification indices such as l and c, and measured color indices. It is clear from table III that very accurate color excesses can be obtained, and that the distribution of the interstellar reddening medium may be studied in detail with the help of color excesses thus determined.

4. On the determination of the distribution of stars in the H-R diagram with the help of spectral classification through photoelectric photometry in narrow wave-length regions

The method described in the previous section has not yet been systematically applied to the problem of the determination of the distribution of stars in the H-R diagram. However, we shall consider quite briefly the successive steps of the procedure by which this would be done, and also discuss one specific question pertaining to the distribution of the stars in the spectral range A3–F2 in the H-R diagram.

The steps in question are the following:

(a) The observation of classification indices, say l and c, for stars brighter than a given apparent magnitude, say 6^{m}, for the spectral range under investigation.

(b) Calibration of the indices to obtain the relation with B-V and M_v. Trigonometric parallaxes, cluster parallaxes, and proper motions and radial velocities would be used for the calibration.

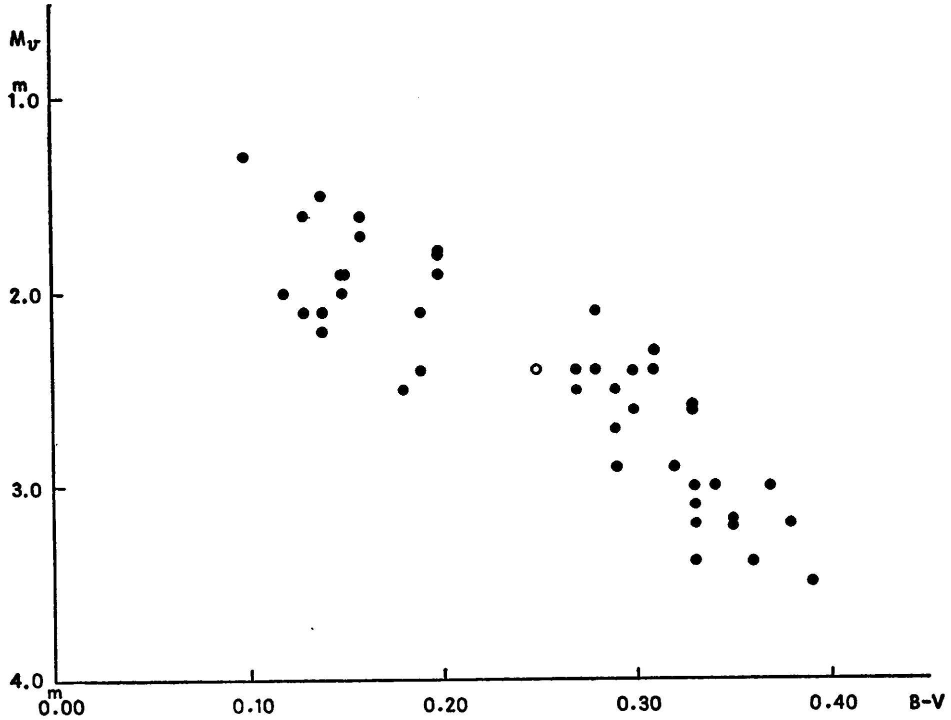

FIGURE 3

H-R diagram for stars within $0^{m}.7$ of the main sequence for the range of color index B-V from $0^{m}.10$ to $0^{m}.40$ (spectral type A3 to F2). The open circle refers to 67 Tau, a member of the Hyades cluster.

(c) Determination of the distribution of the stars in the l-c diagram, and correction for accidental errors. Translation of the two-dimensional distribution to the (B-V) $-$ M_v distribution with the help of the calibration just established.

(d) Application of the well-known correction for the fact that the representative sample volume depends upon the absolute magnitude. Correction for the effect of change of stellar density with distance from the sun.

As was mentioned in section 2, the procedure is of importance primarily in the upper part of the H-R diagram, for stars brighter than absolute magnitude 2-3.

We shall now turn to a specific question concerning the distribution of the stars in the H-R diagram which can be discussed up to a point with the aid of the observational material already available.

In a discussion of color indices B-V (blue minus visual) and U-B (ultraviolet minus blue) for bright stars, Johnson and Morgan [27] constructed a diagram U-B versus B-V. In this diagram there is a discontinuity at a point about B-V = $0^{\mathrm{m}}.3$, corresponding to a spectral class about F0. The discontinuity is shown more clearly in a similar diagram for main-sequence stars by Morgan, Harris and Johnson [28].

Harris (unpublished) in an investigation of color indices of bright stars has found a statistical deficiency of main-sequence stars of colors B-V between the limits $0^{\mathrm{m}}.20$ and $0^{\mathrm{m}}.30$.

The gap in the main sequence, if it exists, is not a very pronounced one, and accurate measures are necessary in order to establish the nature of the statistical distribution of the stars in this particular section of the H-R diagram. As emphasized by Harris certain galactic clusters, particularly Praesepe, contain main-sequence stars in the color index interval in question. Among the field stars in our galactic neighborhood α Aquilae is known to have main-sequence absolute magnitude, and a color index in the interval.

The observations of stars in the spectral range A3–F1 discussed in the previous section give further indications concerning the nature of the discontinuity. The *l-c* classification was used to eliminate all stars more than $0^{\mathrm{m}}.7$ above the lower limit of the main sequence, and metallic line stars were also eliminated. The remaining main-sequence stars in the color-index interval B-V from $0^{\mathrm{m}}.10$ to $0^{\mathrm{m}}.40$ are plotted in figure 3. It is seen that there is a relative deficiency of main-sequence stars in the interval $0^{\mathrm{m}}.205$-$0^{\mathrm{m}}.265$. In fact, only one star (67 Tauri) is found here, while there are 8, 10, 12 and 12 main-sequence stars in the color-index intervals, $0^{\mathrm{m}}.085$–$0^{\mathrm{m}}.145$, $0^{\mathrm{m}}.145$–$0^{\mathrm{m}}.205$, $0^{\mathrm{m}}.265$–$0^{\mathrm{m}}.325$, and $0^{\mathrm{m}}.325$–$0^{\mathrm{m}}.385$, respectively.

As mentioned in section 4, the observations A and F0 stars in the declination zone 0° to +60°, and brighter than $5^{\mathrm{m}}.5$, are only one-half and two-thirds complete, respectively, but it is very unlikely that there would be any selection effect that would appreciably affect the color distribution of the main-sequence stars.

However, considerably more observational material is necessary for a satisfactory determination of the statistical distribution for the region of the H-R diagram in question.

5. Conversion of the absolute magnitude-color index diagram into a gravity-effective temperature diagram

The properties of the emitted spectrum of a star are a function of the following parameters that define the structure of the stellar atmosphere: (1) The effective temperature T_e which gives the flux of energy F through the atmosphere; (2) The gravitational acceleration g toward the center of the star; (3) The relative abundance of the elements.

In the theory of stellar atmospheres (Milne, Pannekoek, Russell, Unsöld and others), methods have been developed for the quantitative determination of the physical parameters T_e and g as well as the chemical parameters, that is, the abundances. As yet the theory is incomplete in certain respects, and it has been necessary to introduce simplifying assumptions and approximations so that the accuracy in the determination of the parameters is not very high, but ultimately the approach will presumably yield results of great value.

Here we shall consider only very briefly one application of the general method, namely, the determination of T_e and g from two intensity ratios in the continuous spectrum, one of which pertains to intensities on either side of the Balmer discontinuity.

Model stellar atmosphere calculations give the intensity distribution in the emitted continuous spectrum as a function of T_e and g (compare, for example, [29], [30]). Recent calculations of this kind by Osawa (unpublished) have yielded the atmospheric structures and the emitted continuous spectrum for atmospheres in radiative equilibrium. The continuous absorption coefficient, which is an important determining factor, was computed including the effect of the negative hydrogen ion according to the quantum-mechanical calculation by Chandrasekhar and Breen [31].

TABLE IV (a)

2.5 log [I(5050)/I(6250)] AS A FUNCTION OF T_e AND log g
(This quantity is most sensitive to changes in T_e)

T_e	log g =		
	3.5	4.0	4.5
9000°	0.18	0.15	0.14
7500°	0.00	−0.01	−0.03

TABLE IV (b)

2.5 log [I(6250)/I(3647s)] AS A FUNCTION OF T_e AND log g
(This quantity is most sensitive to changes in log g)

T_e	log g =		
	3.5	4.0	4.5
9000°	1.15	1.06	0.99
7500°	1.18	1.08	0.96

As an illustration of the principle, tables IV (a) and (b) give the intensity ratios for the wave-lengths 3647s A (shortward of the Balmer discontinuity), 5050A, and 6250A, as a function of T_e and g for a range of temperatures corresponding to spectral class A2–F2. An inspection of the table shows that observed intensity ratios, 6250/5050 and 6250/3647s, determine corresponding values of T_e and g.

When the effective temperature T_e and the surface gravity g have been determined for a star the values can be combined with the absolute magnitude to yield the mass of the star. It should be emphasized that a determination of the absolute magnitude independent of the spectroscopic data is essential. The method may be applied where trigonometric or cluster parallaxes are available.

The absolute magnitude gives the luminosity $L = 4\pi R^2 F$, and since the effective temperature determines F, the radius R can be computed. The surface gravity is given by $g = GMR^{-2}$ where G is the constant of gravitation and M is the mass. It follows that the mass M can be found from T_e, g, and R. This is the method for determining the mass that we have referred to above. Its application to stars in clusters may become particularly important.

6. Some problems concerning the interpretation of the H-R diagram

According to the Russell-Vogt theorem the radius, the luminosity and the whole structure of a star in secular equilibrium is uniquely determined by the mass and the chemical composition of the stars. That a star is in secular equilibrium means that the total production of energy through nuclear processes in the star, per second, equals the luminosity.

The chemical composition of a star depends upon its initial chemical composition, and the changes of composition that have taken place as a consequence of nuclear transmutations during the lifetime of the star.

During the last twenty years it has become increasingly clear that the nuclear transmutations in stellar interiors, in particular the conversion of hydrogen into helium, give rise to differences in chemical composition between different zones of the star. The chemical composition varies as a function of distance from the center, and the variations become more and more pronounced with the increasing age of the star. Sweet [32] and Öpik [33] have shown that the mixing of matter in stellar interiors is generally not effective outside convection zones, and that therefore the differences in composition caused by nuclear transformation are not smoothed out by mixing currents. Only in stars with exceptionally high rotational velocities is mixing of importance.

Variations of chemical composition with distance from the center of the star have a great influence on the internal structure, and on the radius and luminosity for a given mass of the star. Quite generally, the inhomogeneities of chemical composition will give rise to an increase of the radius. Very pronounced evolutionary changes of stars are thus produced by the nuclear transmutations. We can distinguish three cases:

(a) If there are no convection zones in the central regions of the star where the nuclear transformations take place, then the evolutionary changes will produce a continuous variation of the mean molecular weight, from a maximum at the center to a constant value in the outer parts of the star. The variation will become more and more pronounced as the star becomes older. The structure of the star will gradually change, and its radius will increase [34]. From a certain point on there will be a central zone in which the hydrogen has been exhausted.

(b) If the star has a convective core in which practically all of the nuclear transmutations take place, the core will have a mean molecular weight, constant within the core, but higher than for the surroundings. The corresponding evolutionary effects have been studied by Schönberg and Chandrasekhar [35] (see also Ledoux [36]). Again the radius of the star will increase with age. In this case, when the evolution has progressed to a point where hydrogen is nearly exhausted in the convective core, the increase in radius becomes very pronounced. The resulting structures and evolutionary changes have been studied through detailed investigations by Sandage and Schwarzschild [37] (see also Sandage [38]).

(c) In a rapidly rotating star internal mixture may be so strong that the star remains nearly homogeneous, and the change with age is then caused by the gradual decrease of the hydrogen content in the whole star. These changes are much smaller.

The position of a secularly stable star in the H-R diagram is determined by three properties: the mass, the initial chemical composition, and the age. The evolutionary path in the H-R diagram along which the star moves as its age increases will depend on whether the internal structure is as in case (a), (b), or (c).

Since we observe the distribution of the stars in the H-R diagram at a definite moment, the age gives the epoch of formation of the star. Therefore, if a definite change with time of the composition of the matter out of which the stars have been formed has taken place, a correlation between age and initial chemical composition will result.

There can be little doubt that the population I main sequence consists of stars in secular equilibrium that have very similar chemical compositions and that have not yet developed inhomogeneity of chemical composition in the interior to a degree that would very appreciably affect the structure, the radius, and the luminosity. Essentially then the stars form a one-parameter series in the H-R diagram, the mass being the parameter.

The fainter stars on the main sequence do not have a convective core. Going to brighter stars along the main sequence one reaches a point, presumably in the range around the sun, where the stars have a convective core. For the sun it has not yet been possible to decide whether or not there is a convective core (see [39] and the references there given). Ledoux [40] has made the important point that the variations of molecular weight outside convective zones, caused by transformation of hydrogen into helium, tend to stabilize the whole internal structure against convective currents. In view of this it is rather probable that the sun and perhaps even main-sequence stars with masses up to 1.2–1.3 solar masses have structures without convective cores. However, for the A stars where energy production is mostly through the carbon cycle, it is practically certain that there is a convective core.

Presumably stars of population II form a main sequence which lies below the population I main sequence. For some globular clusters the evidence is fairly conclusive (compare Arp, Baum and Sandage [41], and H. L. Johnson [42]). Reference is also made to the observational evidence pertaining to the group of subdwarfs in our galactic neighborhood that are characterized by very low heavy-element content and extreme kinematical properties [43].

Conclusions concerning the reason why the main sequences of population I and II differ are as yet uncertain, but the evidence from the spectra of the brighter globular cluster stars, and particularly the analysis of subdwarf spectra (Chamberlain and Aller [44]), indicates that the explanation might well be in terms of low heavy-element content for the population II stars. Investigations by Reiz [45] on the internal structure of stars of very low heavy-element content lend support to this view. These investigations further indicate that stars now remaining on the population II main sequence probably do not have convective cores.

The stars belonging to one galactic cluster, or one globular cluster, are presumably all of nearly the same age and chemical composition. Essentially they differ in mass only, and should form one-dimensional sequences in the H-R diagram, each characterized by a certain age and chemical composition.

Referring to the remarks made above concerning the existence or nonexistence of convective cores, and the evolution patterns (a) and (b), we shall consider four observationally well-established cluster sequences in the H-R diagram [46]. These are represented in figures 4 and 6 in [46].

(1) It is generally assumed that the Pleiades sequence consists of population I

stars which are so relatively young that evolutionary effects are small up to absolute magnitude 0, and not very pronounced even at $M_v = -2$.

(2) The Praesepe stars are presumably population I stars which are older than the Pleiades so that stars with masses in the range 1.5–2.5 solar masses deviate from the main sequence because of the effect of an increased mean molecular weight in the convective core, according to Schönberg and Chandrasekhar. It is quite plausible to assume that the Praesepe K giants have masses just above and continuous with this range, and it may be conjectured that their evolution according to the evolution pattern (b) has caused a very rapid increase of radius at the particular mass and age.

(3) H. L. Johnson [46] has shown that the M 67 sequence probably consists of population I stars still older than the Praesepe stars so that deviation from the main sequence already begins for stars of approximately 1.0–1.2 solar masses. We might then expect that evolution follows pattern (a) (corresponding to no convective core). This might explain why the corresponding sequence in the H-R diagram is continuous through the subgiants up to the K giants. If this conjecture is correct, then the subgiants would have masses just a little higher than the mass of the sun, and the K giants in M 67 would have masses of the order of 1.5 solar masses.

(4) The situation concerning the globular cluster sequences is as yet not very clear, but it appears possible that these sequences might correspond to evolution according to pattern (a), and that the shift to the left in the H-R diagram relative to a sequence, such as that for M 67, is due to lower heavy-element content.

Although the discussion of cluster sequences has not yet reached a point where very certain conclusions can be drawn, it would seem that definite results might be obtained through further observations, and particularly through further calculations such as those carried out by Schwarzschild and Sandage concerning the evolution pattern (b). Computations aimed at clarifying the nature of evolution according to pattern (a) have been planned by several investigators.

We shall not here discuss questions of finer details of the distribution of stars in the H-R diagram such as the possible narrow gap in the main sequence around B-V $= 0^{m}.25$. Reference is made, however, to the suggestion of Rudkjöbing [47] that consideration of the role of the outer hydrogen convection zone might be of importance in discussions of certain features of the distribution of stars in the H-R diagram. It is quite probable that the depth of the outer convection zone is considerable for the sun and main-sequence stars below the sun [48], while the zone is narrow and deviations from radiative equilibrium small in the case of A stars. It would appear important to study the effect of the transition with regard both to internal constitution and atmospheric structure of main-sequence stars.

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