A SPECIAL PROBLEM OF BROWNIAN MOTION, AND A GENERAL THEORY OF GAUSSIAN RANDOM FUNCTIONS

PAUL LÉVY
ÉCOLE POLYTECHNIQUE

1. Introduction

1.1. The subject of this paper is twofold: a special problem and a general theory. The reader may wonder why the general theory is not stated in part 2 and then applied to the special problem. The answer is that the general theory appeared as a necessary generalization of theorems stated in part 2, after the two first parts had been written, and the author thought that he would not have enough time before this Symposium to reorganize the paper. Moreover, part 2 will be a good introduction to the general theory. In the introduction the author will begin with the general theory, and the reader who wishes to do so may begin with part 3.

1.2. The problem considered in this theory is to find an explicit representation of a Gaussian r.f.[1] of a real variable t that may be considered as the canonical form of this function. By subtraction of a known function, it may be reduced to a Gaussian r.f. $\phi(t)$ with identically zero expectation. Such a r.f. is generally defined by its covariance $\Gamma(t_1, t_2)$, or by a stochastic differential equation with a Cauchy condition. None of these methods gives an explicit representation of $\phi(t)$.

In his previous papers [5] and [6], the author has considered the relation between these two classical methods, and solved the problem of deducing one of these representations from the other. He has also called attention to an explicit representation, which may be written in the form

$$\phi(t) = \int_0^t F(t, u)\, \xi_u \sqrt{du}\,, \tag{1.2.1}$$

where the r.v. ξ_u are independent reduced Gaussian r.v. Here $\xi_u\sqrt{du}$ may be replaced by $dX(u)$, where $X(u)$ is the Wiener r.f. (see, for instance, formula (3.2.15) in [6]). We shall call $\mu + \sigma\xi$ the *canonical form* of a Gaussian X, where ξ is a reduced Gaussian r.v., μ is the expectation of X, and σ is its standard deviation. This leads naturally to the following definition: if the conditional canonical form of $\phi(t')$ is $\mu(t' \mid t) + \xi\sigma(t' \mid t)$ when $\phi(u)$ is given in $(0, t)$ (with $0 < t < t'$), then formula (1.2.1) gives the *canonical form* of $\phi(t)$ if

$$\mu(t' \mid t) = \int_0^t F(t', u)\, \xi_u \sqrt{du}\,. \tag{1.2.2}$$

This paper was supported (in part) by funds provided under Contract AF-18(600)-958 with the Air Research and Development Command.

[1] We shall use the following abbreviations: r.v., random variable (or variables); r.f., random function (or functions); a.s., almost sure (or surely).

Hence

$$(1.2.3)\quad \xi\sigma(t'\mid t)=\int_t^{t'}F(t',u)\,\xi_u\sqrt{du}\,,\qquad \sigma^2(t'\mid t)=\int_t^{t'}F^2(t',u)\,du\,.$$

The kernel $F(t, u)$ will be called the *canonical kernel* of $\phi(t)$, and, *roughly speaking*, the theory may be summarized by the following *existence and uniqueness theorem: every Gaussian r.f.* $\phi(t)$ *with identically zero expectation may be represented in the form* (1.2.1); *there exists only one canonical representation, and, in exceptional cases,* $\phi(t)$ *has representations of the form* (1.2.1) *that are not canonical.*[2]

1.3. Now, to give a precise statement of this theorem, a few restrictions are necessary. For the *existence theorem*, the Schwartz theory of distributions has to be introduced, and further it is necessary to assume that, for every fixed t, $F^2(t, u)$ is the Schwartz derivative of a nondecreasing function of u. Under these conditions every Gaussian r.f. with identically zero expectation and *satisfying a minor restrictive continuity condition* may be represented by formula (1.2.1).

In section 4.5 the author will explain, as well as he is able, the nature of this condition. It is his considered opinion that it is not possible to give a simple necessary and sufficient condition. The reader will not be surprised by this fact if he recalls, for instance, that the absolute convergence of a complex Fourier series does not correspond to a simple property of the represented function.

As for the *uniqueness theorem*, let us notice that all the functions $F(t, u) = \epsilon(u)F_1(t, u)$, with the same $F_1(t, u)$ and $\epsilon(u) = \pm 1$, are possible kernels for the same r.f. Let us call the class of these functions the *class of equivalent kernels*. The uniqueness theorem is then true, not for the functions $F(t, u)$, but for the classes of equivalent kernels, and, hence, to a well-defined $\phi(t)$ corresponds only one canonical class.

The reader may be surprised by the third part of the fundamental theorem. For this reason, the author thought that a theory of Gaussian sequences, which is summarized in section 4.1, would be a good introduction to the theory of Gaussian r.f. For sequences, the existence of representations that are not canonical seems quite natural, and it is easy to recognize whether a given sequence has such representations and to determine the canonical one. Returning to a Gaussian r.f. of t, one may expect that a representation of form (1.2.1) need not be canonical. The problem arises as to how to recognize whether a given kernel is canonical; this problem is easily reduced to a Volterra equation. In the regular case, this equation has only one solution and the kernel is canonical. On the contrary, in the singular case, the problem may be difficult. It was solved in [11] in a very special case only, and, in this paper, theorem 4.8 gives for the Goursat kernels only a necessary condition that is not sufficient.

1.4. The theory that we have summarized may be applied to many classical problems. For instance, to determine whether or not $\phi(t)$ is a stationary r.f., it is necessary to assume that the process begins at time $-\infty$ instead of 0, and to replace 0 in formula (1.2.1) by $-\infty$. Then *the necessary and sufficient condition for the stationary character is that the canonical kernel has the form* $\epsilon(u)f(t-u)$, *with* $\epsilon = \pm 1$.

In sections 4.6–4.8, the necessary and sufficient conditions for the existence of the derivatives, and also for the Markovian character of given order, in the wide sense and in the restricted sense (see definitions in section 4.8) are given. In this summary, we only

[2] Attention was directed in footnote 17 in [6] to the importance of condition (1.2.2). However, formula (3.2.15) in [6] and the present formula (1.2.1) are not exactly the same. In (3.2.15), the kernel is always canonical, and the author did not know that other kernels can exist.

mention the condition for the case of wide sense Markovian character: the condition is that the canonical kernel be a Goursat kernel of the same order; then, if other kernels exist, they have the same or larger order. Examples of these two cases are $\Psi_2(t)$ and $M_5(t)$ of section 3.6.

Another problem is to find the relations between $F(t, u)$ and the other definitions of $\phi(t)$. The covariance is obviously

$$(1.4.1) \qquad \Gamma(t_1, t_2) = \int_0^t F(t_1, u) F(t_2, u)\, du, \qquad t = \min(t_1, t_2),$$

and in order to deduce $F(t, u)$ from $\Gamma(t_1, t_2)$ it is necessary to solve an integral equation of degree 2. However, in a very general case that has been considered in [5] and [6], if $F(t, u)$ is the canonical kernel, the problem is easier. This case is the one when $\sigma^2(t + dt \mid t)$ has the form $\sigma^2(t)dt$, with $\sigma(t) \neq 0$. Then it is possible to deduce a stochastic differential equation from $\Gamma(t_1, t_2)$, and to determine $F(t, u)$ by integrating this equation.[3] Aside from obtaining $\sigma(t) = F(t, t)$ from $\sigma^2(t)$, which is given by

$$(1.4.2) \qquad \sigma^2(t) = \lim_{\tau \to 0} \frac{1}{\tau} [\Gamma(t, t) + \Gamma(t + \tau, t + \tau) - 2\Gamma(t, t + \tau)],$$

one has only to solve linear integral equations with kernels depending on $\Gamma(t_1, t_2)$.

1.5. Let us consider now the special problem concerning the Brownian motion depending on a point A of the Euclidean space E_n or of the Hilbert space E_ω. The Brownian function $X(A)$, the existence of which was proved in [2], is the Gaussian r.f. defined, up to an additive constant, by the formula

$$(1.5.1) \qquad X(A) - X(B) = \xi \sqrt{r(A, B)},$$

where ξ is a reduced Gaussian r.v., and r is the distance between A and B. We shall let $\bar{M}_n(t)$ denote the average of $X(A)$ on the sphere Ω_t with center O and radius t, and $M_n(t) = \bar{M}_n(t) - X(O)$. Whenever no confusion is possible, $\bar{M}_n(t)$ and $M_n(t)$ will be replaced by $\bar{M}(t)$ and $M(t)$.

Roughly speaking, the problem is to consider the determinism of $X(A)$. When $X(A)$ is given either on Ω_t, or in the whole region $r(O, A) > t$, then $X(O)$ has a conditional standard deviation, which, for finite n, is $\sigma'_n = k'_n\sqrt{t}$ in the first case, and $\sigma_n = k_n\sqrt{t}$ in the second case. Since the standard deviation is a nonincreasing function of the information, it follows that $k_n \leqq k'_n$ (and even $k_n < k'_n$), and that k_n and k'_n are decreasing functions of n. Consequently they tend to limits k_ω and k'_ω. It is easy to find k'_ω, which is positive. However, it is much more difficult to find k_ω, and this paper is the result of research that was first undertaken to solve this problem. The answer is $k_\omega = 0$. Hence, in E_ω, $X(A)$ has a deterministic character. This function may be known at O even if nothing is assumed concerning its values in a neighborhood of O.

In the first problem, where $X(A)$ is given on Ω_t, the conditional expectation of $X(O)$ is $\bar{M}_n(t)$. In the second problem it is necessarily a linear function of the values of $\bar{M}_n(u)$ in (t, ∞). Hence, we are led to consider the r.f. $M_n(t)$. In the first part its covariance $\Gamma_n(t_1, t_2)$ is found when $n = 2p + 1$. The case $n = 2p$, in which Γ_n depends on an elliptic integral, is more difficult. In this case, $M_n(t)$ has no Markovian character. However, it is not necessary to know what happens for even n; knowledge of the character of $M_{2p+1}(t)$ enables us to pass to the limit and find the character of $M_\omega(t)$.

In part 2, the canonical form of $M_{2p+1}(t)$ is given. If $p > 1$, this function has another

[3] See, for instance, part (4) of section 3.5 and section 3.2 of [6].

representation of form (1.2.1). More exactly, this representation is given for $M_5(t)$ and $M_7(t)$, and it seems likely that it exists for every $p > 1$. $M_{2p+1}(t)$ is a Markovian r.f. of order $p + 1$, in the restricted sense. Hence, its canonical kernel is a Goursat kernel of order $p + 1$; the other kernel (at least if $p = 2$ or 3) is of order $p + 2$.

At the end of part 2 and in part 4, the problem of the continuation of $M_{2p+1}(t)$ is considered. The continuation to the right is immediately given by the canonical representation. It results from the Markovian character of this function that it may be defined as the solution of a stochastic differential equation of order $p + 1$, and that the problem is exactly the same as if only the Cauchy conditions [$M_{2p+1}(t)$ and its p derivatives] were given at a point t. The continuation to the left is not so simple. However, by application of the Legendre polynomials, it is possible to find k_{2p+1} and to prove that $k_\omega = 0$.

A more precise result is also proved. The r.f. $M_\omega(t)$ is a.s. analytic. If $t = re^{i\theta}$, it is regular for $|\theta| < \pi/6$ (and $r > 0$), and all the points of the lines $\theta = \pm\pi/6$ are singular points.

The last part contains remarks and theorems concerning what may be termed the Markovian character of $X(A)$ (in a slightly modified sense). In section 6.1, we consider the following problem: in the space E_{2p+1}, has $X(A)$ a Markovian character of order $p + 1$? The answer seems to be yes, but even if the considered surface is a sphere, the author is unable to prove it. In the other sections, the case of the Hilbert space is considered. Since another paper [12] on the same subject was presented by the author at the I.M.S. meeting, some preliminary results are summarized without proof. The first of the important results, which is an immediate consequence of the properties of $M_\omega(t)$, is that if $X(A)$ is given in a neighborhood of a sphere Ω of the Hilbert space, then $X(A)$ is known in the inside of Ω, and its average is known on every sphere with center inside of Ω.

This result may be extended to every closed surface in the Hilbert space. Other results about open surfaces or curves are also stated. These problems are connected with theorems on the Hilbert space given by the author in his book [3], and these theorems lead to very curious consequences. Let us give here only one of these consequences: let V denote a volume, S its boundary, and C a curve that divides S into two surfaces. It may happen, for suitable curves C, that, if $X(A)$ is given in an arbitrarily small neighborhood of C, then it is known in the whole volume V.

2. The covariance $\Gamma_n(t_1, t_2)$

2.1. In this part, we shall suppose $X(O) = 0$. Hence $M(t) = \bar{M}(t)$ is the average $\mathcal{M}_t[X(A)]$ of $X(A)$ on the sphere Ω_t, with center O and radius t. If t_1, t_2 and r are respectively the distances OA, OB and AB, one has

$$(2.1.1)\quad \begin{aligned} E\{X(A)X(B)\} &= \tfrac{1}{2}E\{X^2(A) + X^2(B) - [X(A) - X(B)]^2\} \\ &= \tfrac{1}{2}(t_1 + t_2 - r) \end{aligned}$$

and the covariance $\Gamma_n(t_1, t_2)$ of $M(t)$ is

$$(2.1.2)\quad \begin{aligned} \Gamma_n(t_1, t_2) &= \mathcal{M}\{E[X(A)X(B)]\}, &\quad \mathcal{M} &= \mathcal{M}_{t_1}\mathcal{M}_{t_2}, \\ &= \tfrac{1}{2}(t_1 + t_2 - \rho_n), &\quad \rho_n &= \mathcal{M}(r) = \mathcal{M}_{t_1}(r). \end{aligned}$$

2.2. Let us first calculate $\Gamma_n(t, t)$. If $t_1 = t_2 = t$, if θ is the angle AOB, and if we set

$$(2.2.1)\quad I_h = \int_0^{\pi/2} \sin^h \theta\, d\theta, \qquad J_h = \int_0^{\pi} \sin^h \theta \sin\frac{\theta}{2}\, d\theta,$$

we have

(2.2.2) $$r = 2t\sin\frac{\theta}{2}, \qquad \rho_n = t\frac{J_{n-2}}{I_{n-2}}.$$

The value of I_{n-2} is

(2.2.3) $$I_{n-2} = \tfrac{1}{2}B\left(\frac{n-1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma(\frac{1}{2})}{2\Gamma\left(\frac{n}{2}\right)} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)},$$

where B and Γ are the Eulerian functions. By the same method, we obtain

(2.2.4) $$J_{n-2} = 2^{n-2}\int_0^{\pi}\sin^{n-1}\frac{\theta}{2}\cos^{n-2}\frac{\theta}{2}\,d\theta = 2^{n-1}\int_0^{\pi/2}\sin^{n-1}\alpha\cos^{n-2}\alpha\,d\alpha$$

$$= 2^{n-2}B\left(\frac{n-1}{2}, \frac{n}{2}\right) = 2^{n-2}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\Gamma(n-\frac{1}{2})}.$$

Hence

(2.2.5) $$\rho_n = \frac{2^{n-1}}{\sqrt{\pi}}\frac{\Gamma^2\left(\frac{n}{2}\right)}{\Gamma(n-\frac{1}{2})}t$$

and

(2.2.6) $$\lim_{n\to\infty}\rho_n = t\sqrt{2}.$$

This formula (2.2.6) was almost obvious. As $N \to \infty$, the value of r for $\theta = \pi/2$ becomes dominant in the calculation of the average ρ_n, the total weight of the other values tending to 0.

Finally, we deduce from (2.1.1)

(2.2.7) $$\Gamma_n(t, t) = t - \frac{2^{n-2}}{\sqrt{\pi}}\frac{\Gamma^2\left(\frac{n}{2}\right)}{\Gamma(n-\frac{1}{2})}t,$$

(2.2.8) $$\lim_{n\to\infty}\Gamma_n(t, t) = \left(1 - \frac{1}{\sqrt{2}}\right)t.$$

2.3. If $t_1 \neq t_2$, the first formula of (2.2.2) must be replaced by

(2.3.1) $$r^2 = t_1^2 + t_2^2 - 2t_1t_2\cos\theta, \qquad r > 0.$$

Let us notice at once that, as $n \to \infty$, the value of r for $\theta = \pi/2$ is again dominant, and we get

(2.3.2) $$\lim_{n\to\infty}\Gamma_n(t_1, t_2) = \tfrac{1}{2}(t_1 + t_2 - \sqrt{t_1^2 + t_2^2}).$$

This is, for every positive t_1 and t_2, an analytic function, and it follows from a well-known theorem of Loève that, if $n = \infty$ (case of the Hilbert space), $M(t)$ is infinitely differentiable. In this case

(2.3.3) $$e^{-u}M(e^{2u})$$

has the covariance

(2.3.4) $$\cosh u - \sqrt{\frac{\cosh 2u}{2}}, \qquad u = u_1 - u_2.$$

Hence it is a stationary Gaussian random function and is infinitely differentiable.

Since $\Gamma_1(t_1, t_2) = \min(t_1, t_2)$ has discontinuous derivatives (along the line $t_1 = t_2$), and $M_1(t) = X(t)/\sqrt{2}$ where $X(t)$ is the Wiener r.f., we may foresee that $\Gamma_n(t_1, t_2)$ and also, a.s., $M_n(t)$ both have a finite number of continuous derivatives, and this number increases indefinitely with n. This property will be proved in section 2.4 for Γ_n, and in part 3 for M_n.

2.4. Let us consider a surface Ω in E_n. We shall set

$$U(A) = \int_\Omega r\, d\Omega, \tag{2.4.1}$$

where r is the distance between A and a point B that runs over the surface Ω, and $d\Omega$ is the element of area. The distance between A and Ω will be designated by δ. Obviously, $U(A)$ is an analytic function and Ω is the locus of its singular points.

THEOREM 2.4. *If A crosses Ω at a regular point,*[4] *$U(A)$ and its derivatives of orders $<n$ are continuous. The nth normal derivative is discontinuous if $n = 2p + 1$; the derivatives of order n of*

$$U(A) - (-1)^p \frac{\omega_n}{2n} \delta^n \tag{2.4.2}$$

(where ω_n is the area of a sphere with radius 1) are continuous.

PROOF. It is known that

$$\Delta r^a = a(n + a - 2) r^{a-2} \tag{2.4.3}$$

(Δ is the Laplacian operator). If $n = 2p + 1$, it follows that

$$\Delta^p r = (-1)^{p-1} \frac{(2p)!}{(2p-1) r^{2p-1}} = (-1)^{p-1} \frac{(n-1)!}{(n-2) r^{n-2}} \tag{2.4.4}$$

and

$$\Delta^p U(A) = (-1)^{p-1} \frac{(n-1)!}{n-2} \int_\Omega \frac{d\Omega}{r^{n-2}}. \tag{2.4.5}$$

The integral is a Newtonian potential. It is continuous, and the sum

$$\int_\Omega \frac{d\Omega}{r^{n-2}} + \frac{(n-2)\omega_n}{2} \delta \tag{2.4.6}$$

has continuous derivatives of the first order. Thus the differences

$$\Delta^p U(A) - (-1)^p \frac{(n-1)!\,\omega_n}{2} \delta, \tag{2.4.7}$$

$$\frac{d^{2p} U(A)}{d\nu^{2p}} - (-1)^p \frac{(n-1)!\,\omega_n}{2} \delta, \tag{2.4.8}$$

$d/d\nu$ being a normal derivative, also have continuous derivatives of the first order, and the difference (2.4.2) has, at every regular point of Ω, continuous derivatives of order n. Q.E.D.

Let us now apply the result to the sphere Ω_{t_2}. We have

$$\rho_n = \frac{1}{\omega_n t_2^{n-1}} U(A), \qquad I_n(t_1, t_2) = \tfrac{1}{2}(t_1 + t_2 - \rho_n). \tag{2.4.9}$$

[4] A point is here considered as regular if, in a neighborhood of this point, the tangent plan is well defined and varies continuously.

Hence,

FIRST CONTINUITY THEOREM. *If $n = 2p + 1$, the differences*

$$(2.4.10) \qquad \rho_n - (-1)^p \frac{|t_2 - t_1|^n}{2n t_1^p t_2^p}, \qquad \Gamma_n(t_1, t_2) - (-1)^{p-1} \frac{|t_2 - t_1|^n}{4n t_1^p t_2^p},$$

are continuous and have continuous derivatives (even if $t_1 = t_2 > 0$) *of all orders* $\leqq n$ (and of all orders if t_1 and t_2 are different and positive).

If $n = 2p$, the nth normal derivative of $\Gamma_n(t_1, t_2)$ has a logarithmic discontinuity. Since we shall not use this theorem, we do not need to prove it.

2.5. To obtain the exact value of ρ_n, θ being again the angle AOB, we have to start from the formulas

$$(2.5.1) \qquad r^2 = t_1^2 + t_2^2 - 2t_1 t_2 \cos\theta ,$$

$$(2.5.2) \qquad 2I_{n-2}\rho_n = \int_0^\pi r \sin^{n-2}\theta \, d\theta .$$

Considering $\cos\theta$ as parameter in the integral, we see at once that, if $n = 2p$, the right side is an elliptic integral which reduces to an elementary integral only if $t_1 = t_2$; if $n = 2p + 1$, it is an elementary integral. For this reason, we shall only consider the second case, in which we shall obtain simple expressions for $\Gamma_n(t_1, t_2)$ and $M(t)$. *Therefore, we shall always suppose*

$$(2.5.3) \qquad n = 2p + 1 .$$

Instead of $\cos\theta$, it is preferable to choose r as parameter. Since it varies from $|t_2 - t_1|$ to $t_1 + t_2$, we shall set

$$(2.5.4) \qquad t = \min(t_1, t_2) , \qquad t' = \max(t_1, t_2) .$$

Then

$$(2.5.5) \qquad t \leqq t', \qquad \Gamma(t_1, t_2) = \Gamma(t, t') ,$$

and the formula (2.5.2) becomes

$$(2.5.6) \qquad I_{n-2}\rho_n = \frac{1}{(2tt')^{n-2}} \int_{t'-t}^{t'+t} [t^2 t'^2 - (t^2 + t'^2 - r^2)^2]^{p-1} r^2 \, dr .$$

Obviously, this integral is a homogeneous polynomial in t and t', of degree $4p - 1$, and is an odd function of t. This enables us to write

$$(2.5.7) \qquad \rho_n = \frac{P_n^*(t^2, t'^2)}{t^{n-3} t'^{n-2}},$$

where P_n^* is a homogeneous polynomial of degree $n - 2$. Further, ρ_n, which is a weighted average of r, is a number in the interval $(t' - t, t' + t)$, and, for indefinitely increasing t', $\rho_n = t' + O(t)$. Hence we have

$$(2.5.8) \qquad \rho_n = t' + \frac{t^2}{t'^{n-2}} P_n(t^2, t'^2) ,$$

where $P_n(\cdot\,\cdot)$ is a homogeneous polynomial of degree $p - 1$, that is,

$$(2.5.9) \qquad P_n(t^2, t'^2) = a_1 t'^{2p-2} + a_2 t^2 t'^{2p-4} + \cdots + a_p t^{2p-2}$$

Thus

$$\Gamma_n(t_1, t_2) = \frac{t}{2} - \frac{1}{2}\sum_1^p a_h \frac{t^{2h}}{t'^{2h-1}}. \tag{2.5.10}$$

2.6. To find the numbers a_h, we shall use the same transformation as in 2.3. The covariance of the r.f. $e^{-u}M(e^{2u})$ is here

$$\gamma_n(u) = \tfrac{1}{2} e^{-|u|} - \frac{1}{2}\sum_1^p a_h e^{-(4h-1)|u|}, \qquad u = u_1 - u_2, \tag{2.6.1}$$

and, by the continuity theorem, $\gamma_n(u)$ and its $n-1$ first derivatives are continuous. This happens if and only if the derivatives of odd orders $1, 3, 5, \cdots, 2p-1$ vanish for $u = 0$. Setting

$$a'_h = (4h-1)\, a_h, \tag{2.6.2}$$

we have

$$2\left[\frac{d^{\nu+1}\gamma_n(u)}{du^{\nu+1}}\right]_{u=+0} = -1 + \sum_1^p (4h-1)\, a'_h, \tag{2.6.3}$$

and we obtain the equations

$$\sum_{h=1}^p (4h-1)^{2\nu'} a'_h = 1, \qquad \nu' = 0, 1, 2, \cdots, p-1, \tag{2.6.4}$$

which define $a'_1, a'_2, \cdots, a'_h$.

Their determinant is the Vandermonde determinant

$$\| (4h-1)^{2\nu'} \| = \prod_{h,k} [(4h-1)^2 - (4k-1)^2] = \prod_{h,k} 8(k-h)(2h+2k-1), \tag{2.6.5}$$

$$h = 1, 2, \cdots, p; \qquad k = h+1, h+2, \cdots, p.$$

It is $\neq 0$, and the system (2.6.3) has one and only one solution.

Each a'_h is the ratio of two Vandermonde determinants, and, to find its value, it is sufficient to write the factors which are not the same in both. Therefore, we have

$$a'_h = \prod_{k \neq h} \frac{k(2k-1)}{(k-h)(2h+2k-1)}, \tag{2.6.6}$$

$$a_h = \frac{1}{4h-1} \prod_{k \neq h} \frac{k(2k-1)}{(k-h)(2h+2k-1)} \tag{2.6.7}$$

(the values of k are here $1, 2, \cdots, h-1, h+1, h+2, \cdots, p$), and $\gamma_n(u)$ and $\Gamma_n(t_1, t_2)$ are given by (2.6.1) and (2.5.10). As we have $h-1$ negative differences $k-h$, the sign of a_h is $(-1)^{h-1}$.

The reader may notice that, for any fixed $n = 2p+1$, we have two easy verifications. From (2.2.7) we deduce

$$1 - \frac{2^{n-2}\,\Gamma^2(p+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(2p+\frac{1}{2})} = \frac{\Gamma_n(t, t)}{t} = \frac{1}{2}\left(1 - \sum_1^p a_h\right), \tag{2.6.8}$$

and, from the deepest part of the continuity theorem (which was not used yet), that is, from the continuity of the nth derivative of the difference (2.4.2), we may deduce the value of the right side of (2.6.2) for $\nu = 2p$.

2.7. *Particular values of n.* One has

$$\Gamma_1(t_1, t_2) = \frac{t}{2}, \tag{2.7.1}$$

$$\Gamma_3(t_1, t_2) = \frac{t}{2} - \frac{t^2}{6t'}, \tag{2.7.2}$$

$$\Gamma_5(t_1, t_2) = \frac{t}{2} - \frac{t^2}{5t'} + \frac{t^4}{70t'^3}, \tag{2.7.3}$$

$$\Gamma_7(t_1, t_2) = \frac{t}{2} - \frac{3t^2}{14t'} + \frac{t^4}{42t'^3} - \frac{t^6}{462t'^5}, \tag{2.7.4}$$

$$\Gamma_9(t_1, t_2) = \frac{t}{2} - \frac{2t^2}{9t'} + \frac{t^4}{33t'^3} - \frac{2t^6}{429t'^5} + \frac{t^8}{2574t'^7}. \tag{2.7.5}$$

The first of these formulas is obvious. The values of Γ_3 and Γ_5 may be easily deduced from the formula (2.5.6), without using the continuity theorem; the author found them first by this elementary method. But, as p increases, the computation becomes more and more complicated, and it is easier to deduce Γ_7 and Γ_9 from the general formulas (2.5.10) and (2.6.7).

3. Analytic expressions of $M(t)$

3.1. Considering only the case $n = 2p + 1$, where $\Gamma_n(t_1, t_2)$ is a Goursat kernel, we shall obtain for $M_n(t)$ analytic expressions of the form

$$\phi(t) = \int_0^t F(t, u)\, \xi_u \sqrt{du}, \qquad t > 0, \tag{3.1.1}$$

where the variables ξ_u are independent reduced Gaussian random variables. Then, if $X(t)$ is the ordinary Brownian (or Bachelier-Wiener) function, if $X(0) = 0$, and if $F(t, u)$ is continuous and differentiable in u,

$$\phi(t) = \int_0^t F(t, u)\, dX(u) = X(t)\, F(t, t) - \int_0^t \frac{\partial F(t, u)}{\partial u} X(u)\, du, \tag{3.1.2}$$

and $\phi(t)$ may be defined by an ordinary Riemann integral. It may also be defined as the Gaussian r.f. having expectation zero and covariance

$$\Gamma(t_1, t_2) = \int_0^t F(t, u)\, F(t', u)\, du, \tag{3.1.3}$$

where t and t' are defined by formulas (2.5.4).

We shall present in the next part a general theory of the functions $\phi(t)$. In this part we need only three simple lemmas.

3.2. LEMMA 1. *If $F(t, u)$ is a homogeneous function of degree α, then $\Gamma(t_1, t_2)$ is a homogeneous function of degree $2\alpha + 1$, and $t^{-\alpha-1/2}\phi(t)$ is a stationary r.f. of* $\log t$.

LEMMA 2. *If*

$$F(t, u) = \sum_0^p f_h(t)\, \varphi_h(u), \tag{3.2.1}$$

then

$$(3.2.2)\qquad \Gamma(t_1, t_2) = \sum_0^p f_h(t')\, g_h(t)\,, \qquad g_h(t) = \int_0^t F(t, u)\, \varphi_h(u)\, du\,.$$

These lemmas are both obvious consequences of formula (3.1.3). Let us now consider the integral

$$(3.2.3)\qquad \Gamma^*(t_1, t_2) = \int_0^{t'} F(t, u)\, F(t', u)\, du$$

as a continuation of $\Gamma(t_1, t_2)$ when the sign of the difference $t_1 - t_2$ changes. The difference

$$(3.2.4)\qquad \gamma(t_1, t_2) = \tfrac{1}{2}\left[\Gamma(t_1, t_2) - \Gamma^*(t_1, t_2)\right] = -\int_t^{t'} F(t, u)\, F(t', u)\, du$$

will be considered as the *singular part* of $\Gamma(t_1, t_2)$ for small values of $t_2 - t_1$.

LEMMA 3. *If* $n = 2p + 1$ *and*

$$(3.2.5)\qquad F(t, u) = (t-u)^p f(t) + o\left[(t-u)^p\right], \qquad u \to t\,,$$

where $f(t)$ *is a continuous function of* t, *then*

$$(3.2.6)\qquad \gamma(t_1, t_2) = (-1)^{p-1} \frac{(p!)^2}{2n!} f(t)\, f(t')\, (t'-t)^n + o\left[(t'-t)^n\right], \qquad t' - t \to 0\,.$$

This is an immediate consequence of formulas (3.2.4) and

$$(3.2.7)\qquad \int_t^{t'} (u-t)^p (t'-u)^p du = (t'-t)^n B(p+1, p+1) = \frac{(p!)^2}{n!} (t'-t)^{2p+1}.$$

When p and n are not integers, the lemma holds if $p!$ and $n!$ are replaced by $\Gamma(p+1)$ and $\Gamma(n+1)$.

In other words, if p is an integer, lemma 3 may be stated as follows: *if* $F(t, u)$ *has continuous derivatives of orders* $\leqq p$, in a neighborhood of every point $t = u > 0$, *and if* (3.2.5) *holds, then the difference*

$$(3.2.8)\qquad \Gamma(t_1, t_2) - (-1)^{p-1} \frac{(p!)^2}{2n!} f(t_1)\, f(t_2)\, |t_1 - t_2|^n$$

is continuous and has continuous derivatives up to order n.

As a particular case: *if* $P(u)$ *is a polynomial or an analytic function, if*

$$(3.2.9)\qquad P^2(u) = \frac{(2p)!}{2(p!)^2} (1-u)^{2p} + o\left[(1-u)^{2p}\right], \qquad u \to 1\,,$$

and if $F(t, u) = P(u/t)$, *then* $\Gamma(t_1, t_2)$ *has the property that is given for* $\Gamma_n(t_1, t_2)$ *by the first continuity theorem.*

3.3. *The first analytic expression of* $M(t)$. If

$$(3.3.1)\qquad F(t, u) = P\left(\frac{u}{t}\right) = b_0 + \sum_1^p b_h \left(\frac{u}{t}\right)^{2h-1},$$

it follows from lemma 2 that $\Gamma(t_1, t_2)$ has the form

$$(3.3.2)\qquad \Gamma(t_1, t_2) = g_0(t) + \sum_1^p \frac{g_h(t)}{t'^{2h-1}}\,,$$

and from lemma 1 that $\Gamma(t_1, t_2)$ is a homogeneous function of degree 1. Hence it has the form

(3.3.3) $$a_0 \frac{t}{2} - \frac{1}{2} \sum_1^p a_h \frac{t^{2h}}{t'^{2h-1}}$$

obtained for $\Gamma_n(t_1, t_2)$ [see formula (2.5.10)].

Since the continuity theorem that was proved in 2.5 is sufficient to define $a_0, a_1, \cdots, a_p$,[5] we see that if $P(u)$ fulfills the condition (3.2.9), then $\Gamma(t_1, t_2) = \Gamma_n(t_1, t_2)$, and $\phi(t)$ is an expression of $M(t)$, in the case $n = 2p + 1$.

We have now only to calculate the polynomial $P(u)$ that fulfills both conditions (3.2.9) and (3.3.1). Obviously it is

(3.3.4) $$P(u) = c_p \int_u^1 (1 - x^2)^{p-1} dx = c_p \left[I_{2p-1} - \int_0^u (1 - x^2)^{p-1} dx \right],$$

where

(3.3.5) $$I_{2p-1} = \int_0^1 (1 - x^2)^{p-1} dx = \int_0^{\pi/2} \sin^{2p-1} \theta \, d\theta$$

is the same integral as in formula (2.2.3), and

(3.3.6) $$\lim_{u \to 1} \frac{P^2(u)}{(1-u)^{2p}} = \frac{c_p^2 2^{2p-2}}{p^2} = \frac{(2p)!}{2 (p!)^2}.$$

Hence

(3.3.7) $$c_p^2 = 2p^2 \frac{(2p)!}{2^{2p} (p!)^2} = 2p^2 \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2p-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2p)}$$
$$= \frac{4p^2}{\pi} I_{2p}.$$

Therefore, the constant term in $P(u)$ is given by

(3.3.8) $$b_0^2 = c_p^2 I_{2p-1}^2 = \frac{4p^2}{\pi} I_{2p-1}^2 I_{2p} = p I_{2p-1}.$$

Since ξ_u may be replaced in formula (3.1.1) by $-\xi_u$, the sign of c_p (or of $b_0 = c_p I_{2p-1}$) is not essential, and we have proved

THEOREM 3.3. *If $n = 2p + 1$, and if $P(u)$ is the polynomial defined by* (3.3.4) *and* (3.3.7), *then $M(t)$ may be represented by the formula*

(3.3.9) $$M(t) = \int_0^t P\left(\frac{u}{t}\right) \xi_u \sqrt{du}.$$

3.4. *Particular cases.* If $n = 1$ ($p = 0$), one has obviously

(3.4.1) $$M_1(t) = \int_0^t \xi_u \sqrt{\frac{du}{2}} = \frac{1}{\sqrt{2}} X(t).$$

If $n = 3$ ($p = 1$),

(3.4.2) $$M_3(t) = \int_0^t \left(1 - \frac{u}{t}\right) \xi_u \sqrt{du} = \frac{1}{t} \int_0^t X(u)\, du.$$

[5] In section 2.6, a_0 was known, and we used only p among the $p + 1$ equations given by the continuity theorem; these equations define here $\Gamma(t_1, t_2)$ up to a constant factor. This factor is now defined by the last equation.

If $n = 5$ ($p = 2$),

$$(3.4.3)\quad M_5(t) = \int_0^t \left[\left(1-\frac{u}{t}\right)^2 - \frac{1}{3}\left(1-\frac{u}{t}\right)^3\right] \xi_u \sqrt{3du} = \int_0^t \left(\frac{2}{3} - \frac{u}{t} + \frac{u^3}{3t^3}\right) \xi_u \sqrt{3du}.$$

If $n = 7$ ($p = 3$),

$$(3.4.4)\quad M_7(t) = \int_0^t \left[\left(1-\frac{u}{t}\right)^3 - \frac{3}{4}\left(1-\frac{u}{t}\right)^4 + \frac{3}{20}\left(1-\frac{u}{t}\right)^5\right] \xi_u \sqrt{10du}$$

$$= \int_0^t \left(\frac{2}{5} - \frac{3}{4}\frac{u}{t} + \frac{1}{2}\frac{u^3}{t^3} - \frac{3}{20}\frac{u^5}{t^5}\right) \xi_u \sqrt{10du}.$$

3.5. *The derivatives of $M(t)$ and the second continuity theorem.* If $F(t, u)$ in formula (3.1.1) is a continuous function of t and u, and is differentiable in t, and if $dt > 0$, then

$$(3.5.1)\quad \delta\phi(t) = F(t, t)\, \xi_t \sqrt{dt} + dt \int_0^t \frac{\partial F(t, u)}{\partial t} \xi_u \sqrt{du},$$

and $\phi(t)$ is a.s. differentiable if and only if $F(t, t) = 0$. Its derivative is

$$(3.5.2)\quad \phi'(t) = \int_0^t F_1(t, u)\, \xi_u \sqrt{du}, \qquad F_1(t, u) = \frac{\partial F(t, u)}{\partial t}.$$

If $F_1(t, t) = 0$, then $\phi'(t)$ is also differentiable, and so on.

If one applies this remark to $M(t)$, one sees that $M(t)$ has a.s. p continuous derivatives. The first derivative is

$$(3.5.3)\quad M'(t) = c_p \int_0^t \left(1 - \frac{u^2}{t^2}\right)^{p-1} \frac{u}{t^2}\, \xi_u \sqrt{du},$$

the pth derivative is

$$(3.5.4)\quad M^{(p)}(t) = c_p \int_0^t \frac{\partial^{p-1}}{\partial t^{p-1}} \left[\frac{u}{t^2}\left(1 - \frac{u^2}{t^2}\right)^{p-1}\right] \xi_u \sqrt{du},$$

and this function is not differentiable. Applying again formula (3.5.1) to $M^{(p)}(t)$, one has

$$(3.5.5)\quad \delta M^{(p)}(t) = c'_p t^{-p} \xi_t \sqrt{dt} + dt \int_0^t F_{p+1}(t, u)\, \xi_u \sqrt{du}$$

where

$$(3.5.6)\quad c'_p = 2^{p-1}(p-1)!\, c_p,$$

$$(3.5.7)\quad F_{p+1}(t, u) = \frac{\partial^p F_1(t, u)}{\partial t^p} = c_p \frac{\partial^p}{\partial t^p}\left[\frac{u}{t^2}\left(1 - \frac{u^2}{t^2}\right)^{p-1}\right].$$

The first term in the right side of (3.5.5) may be called the singular part of $\delta M^{(p)}(t)$. In the same sense, the singular part of $\delta M(t)$ is

$$(3.5.8)\quad c'_p t^{-p} \int_0^\tau \frac{(\tau - u)^p}{p!} \xi_u \sqrt{du} = \frac{c'_p}{p!}\left(\frac{\tau}{t}\right)^p \xi'_t \sqrt{\frac{\tau}{n}},$$

where $\tau = dt$ and ξ'_t is a reduced Gaussian variable.

Thus we have proved the following theorem.

SECOND CONTINUITY THEOREM. *$M(t)$ is a.s. continuous and has continuous derivatives of orders 1, 2, $\cdots$, p. The pth derivative $M^{(p)}(t)$ is not differentiable. One has*

$$(3.5.9)\quad \delta M^{(p)}(t) = c'_p t^{-p} \xi_t \sqrt{dt} + O(dt),$$

$$(3.5.10)\quad \delta M(t) = \sum_1^p \frac{M^{(h)}(t)(dt)^h}{h!} + \frac{c'_p}{p!\sqrt{n}} \frac{(dt)^{p+1/2}}{t^p} \xi'_t + O[(dt)^{p+1}].$$

Formula (3.5.5) gives a more precise result for $M^{(p)}(t)$. We shall consider in section 3.8 and in part 4 the analogous formulas $M(t)$ and the other derivatives of this function. In all these formulas, dt is always positive.

3.6. *The functions $\phi_p(t)$ and $\Psi_p(t)$. Special reference to the case $n = 5$.* We shall consider here, not the general function $\phi(t)$ defined by (3.1.1), but the special functions

$$\phi_p(t) = t^{2p-1}M(t), \qquad \Psi_p(t) = \frac{d^p\phi_p(t)}{dt^p}. \tag{3.6.1}$$

The covariance of these functions are

$$t_1^{n-2}t_2^{n-2}\Gamma_n(t_1, t_2) \quad \text{and} \quad \frac{\partial^{2p}[t_1^{n-2}t_2^{n-2}\Gamma_n(t_1, t_2)]}{\partial t_1^p \partial t_2^p}. \tag{3.6.2}$$

They are polynomials in t and t', and the second is the simpler. If $n = 3$ ($p = 1$), the covariance of $\Psi_1(t)$ is t, and one deduces that $\Psi_1(t)$ is the Brownian function $X(t)$. Then

$$tM(t) = \int_0^t X(u)\, du = \int_0^t (t-u)\, \xi_u \sqrt{du}, \tag{3.6.3}$$

and we find again the result given in 3.4.

As p increases, the calculation becomes more and more complicated, and the method used in 3.3 is better. However, this second method gives a result that was not given by the first method. To show it, it is sufficient to consider the case $n = 5$ ($p = 2$). It was fully discussed in a previous paper, that was presented to the recent Symposium on Mathematical Probability in Brooklyn (April, 1955). We shall briefly summarize it.

If $n = 5$, the covariance $\Psi_2(t)$ is

$$36t^2t' - 8t^3, \tag{3.6.4}$$

and this leads us to consider the more general Gaussian function

$$\Psi(t) = \int_0^t (\lambda t + \mu u)\, \xi_u \sqrt{du}. \tag{3.6.5}$$

It has the covariance

$$\alpha t^2 t' + \beta t^3, \qquad \alpha = \lambda^2 + \frac{\lambda\mu}{2}, \quad \beta = \frac{\lambda\mu}{2} + \frac{\mu^2}{3}, \tag{3.6.6}$$

and it follows that, independently of the trivial remark that λ and μ may be replaced by $-\lambda$ and $-\mu$, the covariance (3.6.4) is obtained for the two functions

$$\Psi_{2,1}(t) = 2\int_0^t (2t-u)\, \xi_u \sqrt{3du}, \quad \Psi_{2,2}(t) = 2\int_0^t (-3t+4u)\, \xi_u \sqrt{3du}. \tag{3.6.7}$$

Thus, one has two different analytic expressions of the same random function. Coming back to $M(t)$, one has for $M_5(t)$ (the subscript is here the value of n) the two expressions

$$\begin{aligned} M_{5,1}(t) &= \int_0^t \left[\left(1-\frac{u}{t}\right)^2 - \frac{1}{3}\left(1-\frac{u}{t}\right)^3\right] \xi_u \sqrt{3du} \\ &= \int_0^t \left(\frac{2}{3} - \frac{u}{t} + \frac{u^3}{3t^3}\right) \xi_u \sqrt{3du}, \end{aligned} \tag{3.6.8}$$

$$\begin{aligned} M_{5,2}(t) &= \int_0^t \left[\left(1-\frac{u}{t}\right)^2 - 2\left(1-\frac{u}{t}\right)^3\right] \xi_u \sqrt{3du} \\ &= \int_0^t \left(-1 + 4\frac{u}{t} - 5\frac{u^2}{t^2} + 2\frac{u^3}{t^3}\right) \xi_u \sqrt{3du}. \end{aligned} \tag{3.6.9}$$

The first formula is the same as in section 3.4. However, $M_{5,2}(t)$ is a new expression of $M_5(t)$. We call special attention to the fact that in this expression the integral has a term in u^2/t^2 that does not exist in $M_{5,1}(t)$.

By a quite analogous method, one finds for $M_7(t)$ an expression $M_{7,1}(t)$ that is the same as in section 3.4, and another expression

$$(3.6.10)\quad M_{7,2}(t) = \int_0^t \left[\left(1-\frac{u}{t}\right)^3 - 3\left(1-\frac{u}{t}\right)^4 + \frac{3}{2}\left(1-\frac{u}{t}\right)^5\right] \xi_u \sqrt{10du}$$
$$= \int_0^t \left(-\frac{1}{2}+\frac{3}{2}\frac{u}{t} - 4\frac{u^3}{t^3} + \frac{9}{2}\frac{u^4}{t^4} - \frac{3}{2}\frac{u^5}{t^5}\right) \xi_u \sqrt{10du},$$

with a term in u^4/t^4 that does not exist in $M_{7,1}(t)$.

In the general case, the algebraic equations that define the unknown coefficients in

$$(3.6.11)\qquad \Psi_p(t) = \int_0^t (\lambda_0 t^{p-1} + \lambda_1 t^{p-2}u + \cdots + \lambda_{p-1}u^{p-1})\, \xi_u \sqrt{du}$$

seem to have 2^p solutions. Since nothing essential is changed if all the λ_h are replaced by $-\lambda_h$, one has at most 2^{p-1} distinct solutions. However, if $p = 3$, one finds only two real and distinct solutions, instead of $2^2 = 4$. This circumstance seems to be general. However, this is not proved, and in any case the general formulas for the second two expressions $\Psi_{p,2}(t)$ and $M_{2p+1,2}(t)$ are not as simple as those obtained above for the first two expressions of $\Psi_p(t)$ and $M_{2p+1}(t)$.

3.7. *Special properties of $M_{5,1}(t)$ and $M_{5,2}(t)$.* One deduces from the definition (3.6.5) of $\Psi(t)$

$$(3.7.1)\qquad \delta\Psi(t) = st\delta X(t) + \lambda X(t)\, dt, \qquad s = \lambda + \mu,\ X(t) = \int_0^t \xi_u \sqrt{du}.$$

Obviously, if $X(u)$ is given in $(0, t)$, $\Psi(t)$ is known; it may be deduced from formula (3.6.5), or from (3.7.1) with the Cauchy condition $\Psi(0) = 0$. To see whether the converse statement is true, let us set $\lambda = ks$, and write (3.7.1) in the form[6]

$$(3.7.2)\qquad s\,\delta\,[t^k X(t)] = t^{k-1}\delta\Psi(t).$$

It shows that, if a solution $X^*(t)$ of the equation (3.7.1) is known, the general solution is

$$(3.7.3)\qquad X(t) = X^*(t) + ct^{-k},$$

and we have to look whether we have a Cauchy condition to define c.

If $k > -1/2$, then $-k < 1/2$, and the a.s. condition

$$(3.7.4)\qquad X(t) = O(t^{-k}), \qquad t \downarrow 0,$$

is fulfilled only for one value of c. If $k = -1/2$, c may be deduced from a Cauchy condition in a generalized sense. On the contrary, if $2k + 1 < 0$, and if $X^*(t)$ is a possible function $X(t)$, all functions given by formula (3.7.3) are possible functions $X(t)$. Then we have no reason to say that one of these functions is the right $X(t)$, and c is a Gaussian random variable with positive standard deviation σ. In his previous paper the author has proved that

$$(3.7.5)\qquad \sigma^2 = -\frac{1+2k}{k^2}\, t^{1+2k}.$$

[6] Here, we suppose $s = 0$; if $s = 0$, $\Psi(t)$ is a.s. differentiable, and $X(t)$ is its derivative.

One has exactly opposite results if $\Psi(u)$ is given in (t, ∞). Then $X(u)$ is known in (t, ∞) if $k \leqq -1/2$, and depends on an unknown variable c, which is a Gaussian variable, if $k > -1/2$. Let us notice that, if $X(u)$ is given in (t, ∞), $\Psi(t)$ is not known in this interval but depends on the unknown constant

$$(3.7.6) \qquad \int_0^t u\,\xi_u \sqrt{du}\,.$$

Let us now apply these results to the function $M_5(t)$. Obviously, if $M_5(u)$ is given in $(0, t)$, then the corresponding function $\Psi_2(t)$ is known, and, since it has two expressions of the form (3.6.5), with the same $s = \lambda + \mu = \sqrt{12}$, and two different values of k,

$$(3.7.7) \qquad k_1 = 2 > -\tfrac{1}{2}\,, \qquad k_2 = -3 < -\tfrac{1}{2}\,,$$

it corresponds to two different Brownian functions $X_1(t)$ and $X_2(t)$. One deduces from (3.7.1)

$$(3.7.8) \qquad \frac{\delta\Psi_2(t)}{\sqrt{12}} = t\,\delta X_1(t) + k_1 X_1(t)\,dt = t\,\delta X_2(t) + k_2 X_2(t)\,dt\,,$$

and the property of $\Psi(t)$ that was proved above gives the following theorem.

THEOREM 3.7. *If $M_5(u)$ is given in* $(0, t)$, *then $\Psi_2(u)$ is known in the same interval, and the function $X_1(u)$ that is connected with the first expression $M_{5,1}(t)$ of $M_5(t)$ is known; however, $X_2(u)$ depends on an unknown constant, and if $X_2(u)$ is given in* $(0, t)$, *one has more information than if $X_1(u)$ is given.*

On the contrary, if $X_1(u)$ is given in (t, ∞), *one has more information than if $X_2(u)$ is given.*

3.8. *Markovian systems and stochastic differential equations.* Obviously, with the exception of the trivial cases $k = 0$ and $k = 1$, $\Psi(t)$ is not a Markovian function, but $X(t)$ and $\Psi(t)$ are together a Markovian system, and this is true for $\Psi_{2,1}(t)$ and $X_1(t)$ as well as for $\Psi_{2,2}(t)$ and $X_2(t)$.

Let us consider first the system

$$(3.8.1) \quad X_1(t) = \int_0^t \xi_u \sqrt{du}\,, \quad \Psi_2(t) = \Psi_{2,1}(t) = 2\int_0^t (2t-u)\,\xi_u \sqrt{3du}\,,$$

which is equivalent to the stochastic differential equations

$$(3.8.2) \quad \delta X_1(t) = \xi_t \sqrt{dt}\,, \quad \Psi_2(t) = 2\sqrt{3}\,\xi_t\sqrt{dt} + 4\sqrt{3}\,X_1(t)\,dt\,, \qquad dt > 0\,,$$

with the Cauchy conditions $X_1(0) = \Psi_2(0) = 0$. Since $\Psi_2(t) = \phi_2''(t)$, one deduces from (3.7.2), where $k = 2$, $s = 2\sqrt{3}$,

$$(3.8.3) \qquad 2\sqrt{3}\,t^2 X_1(t) = \int_0^t u\,d\Psi_2(u) = t\Psi_2(t) - \phi_2'(t)\,,$$

and the system

$$(3.8.4) \quad \frac{d}{dt}\phi_2'(t) = \Psi_2(t)\,, \quad \delta\Psi_2(t) = 2\sqrt{3}\,\xi_t\,t\sqrt{dt} + \frac{2dt}{t^2}\left[t\Psi_2(t) - \phi_2'(t)\right]$$

is an equation of order two in $\phi_2'(t)$. By elimination of $\Psi_2(t)$, one has

$$(3.8.5) \qquad \delta\phi_2'(t) = \phi_2''(t)\,dt + \frac{dt^2}{t^2}\left[t\phi_2''(t) - \phi_2'(t)\right] + 2\,\xi_t\,t\sqrt{dt^3}\,,$$

and equations of order three may easily be obtained for $\phi_2(t)$ and $M_5(t) = t^{-3}\phi_2(t)$. This last equation was given in the author's previous paper.

Let us now present some important remarks. We may consider $\phi_2'(t)$ as a *Markovian function of order two*, since, if the *two numbers* $\phi_2'(t)$ and $\phi_2''(t)$ are known, then the continuation of $\phi_2'(t)$ to the right does not depend on the values of $\phi_1'(u)$ in $(0, t)$. In equation (3.8.5),

$$\frac{2}{t^2}\left[t\phi_2''(t) - \phi_2'(t)\right] \tag{3.8.6}$$

may be considered as *an expected second derivative* of $\phi_2'(t)$, or a third derivative of $\phi_2(t)$. More precisely, if $\phi_2'(t)$ is given in $(0, t)$, and if $m'(t'\,|\,t)$ is the conditional expectation of $\phi_2'(t)$, then

$$\left[\frac{\partial^2 m'(t'\,|\,t)}{\partial t'^2}\right]_{t'=t} = \frac{2}{t^2}\left[t\phi_2''(t) - \phi_2'(t)\right]. \tag{3.8.7}$$

It is easy to deduce this formula from

$$\begin{aligned} E\{\phi_2'(t_1)\,\phi_2'(t_2)\} = \gamma(t_1, t_2) &= \frac{\partial^2}{\partial t_1 \partial t_2}\left[t_1^3 t_2^3 \Gamma_5(t_1, t_2)\right] \\ &= 6t^3 t'^2 - 2t^4 t'. \end{aligned} \tag{3.8.8}$$

One has

$$t'^2 \frac{\partial^2 \gamma}{\partial t'^2} = 2\left(t' \frac{\partial \gamma}{\partial t'} - \gamma\right). \tag{3.8.9}$$

If $E_t(Y)$ is the conditional expectation of Y when $\phi_2'(u)$ is known in $(0, t)$, and if $u < t < t'$, one has $\gamma(u, t') = E\{\phi_2'(u) E_t[\phi_2'(t')]\}$. Then the last equation may be written in the form

$$E\left\{\phi_2'(u)\left[t'^2 \frac{\partial^2 m'(t'\,|\,t)}{\partial t'^2} - 2t' \frac{\partial m'(t'\,|\,t)}{\partial t'} + 2m(t'\,|\,t)\right]\right\} = 0, \tag{3.8.10}$$

which holds for every $u \leqq t$, and, since $m(t'\,|\,t)$ is necessarily a linear function of the known values of $\phi_2'(u)$, one deduces

$$t'^2 \frac{\partial^2 m(t'\,|\,t)}{\partial t'^2} - 2t' \frac{\partial m(t'\,|\,t)}{\partial t'} + 2m(t'\,|\,t) = 0. \tag{3.8.11}$$

If $t' = t$, one finds again formula (3.8.7).

It is easy to integrate equations like (3.8.5) [or the analogous equations with $\phi_2(t)$ or $M_5(t)$]. We speak here of an integration to the right: $\phi_2'(t)$ and $\phi_2''(t)$ are known for the considered value t, and we want to write $\phi_2'(t')$ and $\phi_2''(t')$, $t' > t$, in the *canonical forms*

$$\phi_2'(t') = m' + \sigma'\xi', \qquad \phi_2''(t') = m'' + \sigma''\xi'', \tag{3.8.12}$$

where we have to find, not only m', m'', σ', σ'', but also the covariance $\rho = E(\xi'\xi'')$. The solution of this problem is easily deduced from the formulas

$$\begin{cases} \phi_2''(t') = \Psi_{2,1}(t') = 2\displaystyle\int_0^{t'} (2t' - u)\,\xi_u \sqrt{3du}, \\ \phi_2'(t') = \displaystyle\int_0^{t'} \Psi_{2,1}(t)\,dt = 2\int_0^{t'} (t'^2 - ut')\,\xi_u \sqrt{3du}, \end{cases} \tag{3.8.13}$$

and from theorem 3.7. From this theorem it follows at once that the canonical forms of $\phi_2'(t')$ and $\phi_2''(t')$ are

$$\begin{cases} \phi_2'(t') = 2\displaystyle\int_0^{t} (t'^2 - ut')\,\xi_u \sqrt{3du} + 2\int_t^{t'} (t'^2 - ut')\,\xi_u \sqrt{3du}, \\ \phi_2''(t') = 2\displaystyle\int_0^{t} (2t' - u)\,\xi_u \sqrt{3du} + 2\int_t^{t'} (2t' - u)\,\xi_u \sqrt{3du}, \end{cases} \tag{3.8.14}$$

and, since the analogous formula holds for $M_{5,1}(t)$, this enables us to say that $M_{5,1}(t)$ and formulas (3.8.13) give the *canonical form* of $M_5(t)$, $\phi_2'(t)$ and $\phi_2''(t)$. Indeed the variables ξ_u' in $(0, t)$ and the ξ_u' in (t, t') are independent, and in both formulas the first integral is a linear function of the given numbers $\phi_2'(t)$ and $\phi_2''(t)$. Then one has

$$(3.8.15)\quad m' = 2\int_0^t (t'^2 - ut')\, \xi_u \sqrt{3du} = 2t'\int_0^t [\,(t'-2t) + (2t-u)\,]\, \xi_u \sqrt{3du}$$
$$= 2\sqrt{3}\,t'(t'-2t)\, X_1(t) + t'\phi_2''(t)\,,$$

or, taking into account (3.8.3),

$$(3.8.16)\quad m' - \phi_2'(t) = \frac{t'(t'-2t)}{t^2}\,[t\phi_2''(t) - \phi_2'(t)\,] + t'\phi_2''(t) - \phi_2'(t)$$
$$= \frac{t'(t'-t)}{t}\,\phi_2''(t) - \frac{(t'-t)^2}{t^2}\,\phi_2'(t)\,.$$

If $t' - t = dt$, we find again formula (3.8.11). By the same method, one obtains

$$(3.8.17)\quad m'' - \phi_2''(t) = 4\sqrt{3}\,(t'-t)\, X_1(t) = 2\,\frac{t'-t}{t^2}\,[t\phi_2''(t) - \phi_2'(t)\,]\,.$$

Now, since the second terms in formula (3.8.14) are $\sigma'\xi'$ and $\sigma^2\xi^2$, one has

$$(3.8.18)\quad \begin{cases} \sigma'^2 = 12t'^2\displaystyle\int_t^{t'} (t'-u)^2\,du = 12t'^2\int_0^{t'-t} v^2\,dv = 4t'^2\,(t'-t)^3 \\ \sigma''^2 = 12\displaystyle\int_t^{t'} (2t'-u)^2\,du = 12\int_0^{t'-t} (t'+v)^2\,dv \\ \qquad\qquad = 4\,(t'-t)\,(7t'^2 - 5tt' + t^2) \\ \rho\sigma'\sigma'' = 12\displaystyle\int_t^{t'} t'\,(t'-u)\,(2t'-u)\,du = 12t'\int_0^{t'-t} v\,(t'+v)\,dv \\ \qquad\qquad = 2\,(t'-t)^2\,(5t'-2t)\,. \end{cases}$$

Formulas (3.8.15) to (3.8.18) give the complete solution of the continuation of $\phi_2'(t)$ to the right. It is easy to write analogous formulas for $\phi_2(t)$ or $M(t)$.

3.9. *The backward stochastic differential equation.* Let us now consider $\Psi_2(t)$ and $X(t)$ as defined by

$$(3.9.1)\quad X_2(t) = \int_0^t \xi_u \sqrt{du}\,, \qquad \Psi_2(t) = \Psi_{2,2}(t) = 2\int_0^t (-3t + 4u)\, \xi_u \sqrt{3du}\,,$$

or, what is the same, by the differential equations

$$(3.9.2)\quad \delta X_2(t) = \xi_t \sqrt{dt}\,, \qquad \delta\Psi_2(t) = 2\sqrt{3}\,t\,\delta X_2(t) - 6\sqrt{3}\,X_2(t)\,dt\,,\ dt > 0\,,$$

with the Cauchy conditions $X_2(0) = \Psi_2(0) = 0$. Each of the functions $X(t)$ and $\Psi(t)$ has the same distribution as in section 3.8, but the joint distribution is not the same. Instead of (3.8.3), we have

$$(3.9.3)\qquad 2\sqrt{3}\,\frac{X_2(t)}{t^3} = \int_\infty^t \frac{d\Psi_2(u)}{u^4}\,,$$

and as was already shown, $X_2(t)$ is known if $\Psi_2(u)$ is given in (t, ∞) but it is not a function of $\phi_2'(t)$ and $\phi_2''(t) = \Psi_2(t)$. Thus we have a Markovian system with $\phi_2'(t)$ and $\Psi_2(t)$, and another with $\Psi_2(t)$ and $X_2(t)$. Since, if one sets

$$(3.9.4)\qquad J_h = \int_0^t u^h \xi_u \sqrt{du}\,,$$

one has

$$X_2(t) = J_0, \qquad \Psi_2(t) = 2\sqrt{3}\,(-3J_0 t + 4J_1), \tag{3.9.5}$$

$$\begin{aligned} \phi_2'(t) &= \int_0^t \Psi_2(t)\,dt = \int_0^t (-3t^2 + 8tu - 5u^2)\,\xi_u\sqrt{3du} \\ &= \sqrt{3}\,(-3t^2 J_0 + 8tJ_1 - 5J_2), \end{aligned} \tag{3.9.6}$$

one sees clearly that $X_2(t)$ and $\Psi_2(t)$, which depend only on J_0 and J_1, do not give the same information as $\Psi_2(t)$ and $\phi_2'(t)$, as this latter function depends also on J_2.

Since functions Ψ and ϕ are defined by integrals taken from 0 to t, the continuation to the left and the continuation to the right are quite different problems. We shall speak of both in part 5. However, we shall prove at once that the continuation of $\phi_2'(t)$ to the left is connected with a stochastic differential equation which is not equation (3.8.5), where dt is essentially positive.

Starting again from (3.8.8), we obtain

$$t^2\frac{\partial^2\gamma}{\partial t^2} - 6t\frac{\partial\gamma}{\partial t} + 12\gamma = 0, \qquad \gamma = \gamma(t, u),\ u \geqq t, \tag{3.9.7}$$

and, if $t \leqq t' \leqq u$, we deduce, by the same reasoning as in section 3.8,

$$E\left\{\phi_2'(u)\left[t^2\frac{\partial^2 m'(t\,|\,t')}{\partial t^2} - 6t\frac{\partial m'(t\,|\,t')}{\partial t} + 12m'(t\,|\,t')\right]\right\} = 0, \tag{3.9.8}$$

$$t^2\left[\frac{\partial^2 m'(t\,|\,t')}{\partial t^2}\right]_{t'=t} = 6t\phi_2''(t) - 12\phi_2'(t), \tag{3.9.9}$$

and finally, for negative dt, one has

$$\delta\phi_2'(t) = \phi_2''(t)\,dt + 3\,[t\phi_2''(t) - 2\phi_2'(t)\,]\frac{dt^2}{t^2} + 2\,\xi_t\,t\,|\,dt\,|^{3/2}. \tag{3.9.10}$$

This equation is analogous to (3.8.5), but holds for $dt < 0$. The coefficient of dt^2/t^2 is not the same in both equations.

The continuation of $\phi_2'(t)$ to the left is defined by this equation, and the same method may be applied to $M'(t)$ or $\bar{M}(t)$. However, it seems preferable to deduce these continuations from the formulas given in sections 3.3 and 3.4, as we shall do in part 5.

4. A general theory of the Gaussian random functions

4.1. *Gaussian sequences.* The canonical form of a Gaussian r.v. is $X = \mu + \sigma\xi$, where μ is its expectation and σ is its standard deviation. Then ξ is a reduced Gaussian r.v.

Consider a sequence $\{\xi_n\}$, $n = 1, 2, \cdots$, of such independent reduced variables. Joint Gaussian variables, X_n, with $E(X_n) = 0$, may be defined as functions of the ξ_ν by the recurrence formula

$$X_n = \sum_1^{n-1} b_{n,\nu}X_\nu + \sigma_n'\xi_n \tag{4.1.1}$$

where σ_n' is a conditional standard deviation, or by the explicit formula

$$X_n = \sum_1^{n-1} a_{n,\nu}\,\xi_\nu + s_n\xi_n. \tag{4.1.2}$$

Let $\mu_{n,\nu}$, $n > \nu$, denote the conditional expectation of X_n when $X_1, X_2, \cdots, X_\nu$ are given. If

$$(4.1.3) \qquad m_{n,\nu} = \sum_1^\nu a_{n,\rho}\,\xi_\rho \qquad m'_{n,\nu} = \sum_1^\nu b_{n,\rho}\,X_\rho\,,$$

we may consider the form (4.1.2) as *canonical* if $m_{n,\nu} = \mu_{n,\nu}$, and the form (4.1.1) as *canonical* if $m'_{n,\nu} = \mu_{n,\nu}$, for every ν and $n > \nu$. Then, the canonical definitions of the sequence give at once the conditional canonical form of every X_n if $X_1, X_2, \cdots, X_\nu$, $\nu < n$, are known.

If the numbers σ'_n are all positive, one has a one-one correspondence between ξ_n and X_n, for every n, and obviously each of the formulas (4.1.1) and (4.1.2) may be deduced from the other; both are canonical and $s_n = \sigma'_n$. Things are quite different if the set $\mathcal{E}_0$ of the subscripts of vanishing σ'_n is not empty ($\mathcal{E}$ will denote the complementary set). Then

$$(4.1.4) \qquad X_\nu = \sum_1^{\nu-1} b_{\nu,\rho}\,X_\rho\,, \qquad \nu \in \mathcal{E}_0\,,$$

and, if $X_1, X_2, \cdots, X_{\nu-1}$ are known, X_ν gives no new information. We have then $\mu_{n,\nu} = \mu_{n,\nu-1}$, and, since ξ_ν is not a.s. 0, the condition $m'_{n,\nu} = \mu_{n,\nu}$ (that has to be fulfilled as well for $\nu - 1$ as for ν) implies $b_{n,\nu} = 0$. Then *the recurrent form* (4.1.1) *is canonical if and only if* $b_{n,\nu} = 0$ for every $\nu \in \mathcal{E}_0$ and every $n > \nu$.

If this condition is not fulfilled, the considered form is not canonical, but formula (4.1.4) may be used to eliminate the terms in X_ν with $\nu \in \mathcal{E}_0$, and one obtains the canonical recurrent definition of the same sequence.

From this definition, it is easy to deduce an explicit definition, by successive eliminations of $X_{n-1}, X_{n-2}, \cdots$ in formula (4.1.1). Since the variables ξ_ν with $\nu \in \mathcal{E}_0$ are not present in these formulas, and since, if $n \in \mathcal{E}$, one has a one-one correspondence between ξ_n and X_n, the obtained explicit formulas have the following properties: one has always $s_n = \sigma_n$, and $\mathcal{E}_0$ may be defined as the set of the subscripts ν of vanishing s_ν; for those ν, and $n > \nu$, $a_{n,\nu} = 0$; the ξ_ρ with $\rho \leqq \nu$ and $\rho \in \mathcal{E}$ give exactly the same information as the corresponding X_ρ; then $m_{n,\nu} = m'_{n,\nu}$, and the representation is canonical.

On the contrary, when one starts from formula (4.1.2) with given coefficients $a_{n,\nu}$ and s_n, it may happen that $\sigma_\nu = 0$, and $a_{n,\nu} \neq 0$. Then, if n is the smallest integer $n > \nu$ with $a_{n,\nu} \neq 0$, ξ_ν gives information on X_n that $X_1, X_2, \cdots, X_{n-1}$ do not give, and $s_n < \sigma'_n$. One always has $s_n \leqq \sigma'_n$.

Let us summarize the results:

Definition. *The formulas* (4.1.2) *give the canonical form of the sequence if* $m_{n,\nu} = \mu_{n,\nu}$, *for every* ν *and* $n > \nu$.

Theorem 4.1. (a) *The form* (4.1.2), *with* $s_n \geqq 0$ *for all* n, *is canonical if and only if, whenever* $s_\nu = 0$ *and* $n > \nu$, *then* $a_{n,\nu} = 0$. (b) *It is canonical if and only if* $s_n = \sigma'_n$ *for all* n. *One always has* $s_n \leqq \sigma'_n$.

It is easy to get rid of the restriction $s_n \geqq 0$. One may replace ξ_ν, σ_ν and $a_{n,\nu}$ by ϵ_ν, $\epsilon_\nu\sigma_\nu$, and $\epsilon_\nu a_{n,\nu}$, where $\epsilon_\nu = \pm 1$; nothing is changed in the formulas. Then one has always *one and only one class of equivalent canonical forms* for a Gaussian sequence. If $E\{X_n\} = \mu_n \neq 0$, one has only to add μ_n to the right side of formula (4.1.2).

4.2. *The general Gaussian functions of a continuous parameter. Their canonical form.*

Let C_G denote the class of all Gaussian r.f. that are defined in $(0, \infty)$ and have identically zero expectations; these restrictions are not essential. If $\phi \in C_G$ and if we apply the results of section 4.1 to the sequence $\{\phi(n\tau)\}$, and let τ tend to zero, we are led to theorems that are a quite natural generalization of those obtained for the sequences.

Let us first notice that, in formula (4.1.1), the last term is essential, and one obtains at the limit, not an explicit expression of $\phi(t)$, but a stochastic differential equation

$$\delta\phi(t) = \int_0^t G(t, dt, u)\, \phi(u)\, du + \sigma(t, dt)\, \xi_t, \qquad dt > 0\,. \tag{4.2.1}$$

Such equations, with $G(t, dt, u) = dt\,G(t, u)$ and $\sigma(t, dt) = \sigma(t)\sqrt{dt}$, have been considered in previous papers of the author, [4] and [5]. Other forms are obtained in this paper for the functions $M_{2p+1}(t)$ and $\phi_p(t)$; then the first term of $\delta\phi(t)$ is a known polynomial in dt and the form of $\sigma(t, dt)$ is $c_p(dt/t)^p\sqrt{dt}$. This function is identically zero only if the process is deterministic.

On the contrary, formula (4.1.2) leads to an explicit expression of $\phi(t)$

$$\phi(t) = \int_0^t F(t, u)\, \xi_u \sqrt{du} = \int_0^t F(t, u)\, dX(u)\,, \tag{4.2.2}$$

where $X(u)$ is the Wiener r.f. Since we have the same r.f. $\phi(t)$ if $F(t, u)$ is replaced by $\epsilon(u)\, F(t, u)$, where $\epsilon(u) = \pm 1$, we have to speak, not of an expression of $\phi(t)$, but of a class of equivalent forms. If possible, we shall choose the particular $F(t, u)$ that is positive for sufficiently small $t - u$, but this is not essential.

We know, from part 3, that a well-defined function $\phi(t)$ may have two different classes of representations of the form (4.2.2). More precisely than in part 3, we shall say that this form gives a canonical representation of $\phi(t)$ if, for every positive t, one has exactly the same information by giving $\phi(u)$ in $(0, t)$ as by giving the value of ξ_u in the essential part of this interval. We consider an interval (t', t'') as unessential if no more information is obtained by giving $\phi(t)$ in $(0, t'')$ than by giving this function in $(0, t']$. The essential part of $(0, t)$ is the set of all points in $(0, t)$ that belong to no unessential interval. Obviously, if u_0 belongs to an unessential interval, the *canonical kernel* $F(t, u)$ is identically zero for $u = u_0$, $t \in (u_0, \infty)$. Since $\phi(u)$ is obviously known in $(0, t)$ when $X(u)$ is known, if a representation is not canonical, then $X(u)$ gives more information than $\phi(u)$. For given t and $t' > t$, the canonical form of the r.v. $\phi(t')$, when $X(u)$ is given in $(0, t)$, is obviously

$$\phi(t') = m(t' \mid t) + s(t' \mid t)\, \xi' \tag{4.2.3}$$

with

$$m(t' \mid t) = \int_0^t F(t', u)\, \xi_u \sqrt{du}\,, \tag{4.2.4}$$

$$s(t' \mid t)\, \xi' = \int_t^{t'} F(t', u)\, \xi_u \sqrt{du}\,, \tag{4.2.5}$$

and

$$s^2(t' \mid t) = \int_t^{t'} F^2(t', u)\, du\,. \tag{4.2.6}$$

Now if

$$\phi(t') = \mu(t' \mid t) + \sigma(t' \mid t)\, \xi \tag{4.2.7}$$

is the canonical form of the r.v. $\phi(t')$ when $\phi(u)$ is given in (0, 5), one has the following theorem.

THEOREM 4.2. *Formula* (4.2.2) *gives the canonical form of the r.f.* $\phi(t)$ *if and only if*

$$\mu(t' \mid t) = \int_0^t F(t', u)\, \xi_u \sqrt{du}, \qquad 0 < t < t'. \tag{4.2.8}$$

Then $\sigma = s$. *If this condition is not identically fulfilled, then*

$$E\{ m^2(t' \mid t) - \mu^2(t' \mid t) \} \tag{4.2.9}$$

is nonnegative and is positive at least for some values of t and t'.

More briefly, *the canonical form minimizes* $E(m^2)$ *and maximizes* s^2, *and, for this form,* $m = \mu$, $s = \sigma$.

Since $m = \mu$ means that $X(u)$ and $\phi(u)$, if given in $(0, t)$, give exactly the same information on $\phi(t')$, the first part of the theorem is obvious and gives a new definition of the canonical representation.

Since

$$\begin{aligned} E\{\phi^2(t')\} &= E\{ m^2(t' \mid t)\} + s^2(t' \mid t) \\ &= E\{ \mu^2(t' \mid t)\} + \sigma^2(t' \mid t), \end{aligned} \tag{4.2.10}$$

the expectation (4.2.9) is $\sigma^2 - s^2$. Further, since $X(u)$ gives at least the same information as $\phi(u)$, and gives more information about $\phi(t')$ (at least for some values of t and t'), if the representation is not canonical, the second part of the theorem is also obvious.

An important problem which is not solved by theorem 4.2 is to recognize, when the kernel $F(t, u)$ is given, whether it gives the canonical form of $\phi(t)$. It was solved in section 3.7 in the very particular case when $F(t, u) = \lambda t + \mu u$. A more general case will be considered in section 4.6(b).

4.3. *The uniqueness theorem.*

THEOREM 4.3. *A well-defined function* $\phi(t)$ *has at most one class of equivalent canonical representations.*

PROOF. If $\phi(t)$ is well defined, $\mu(t' \mid t)$ is a well-defined r.f., and, if (4.2.2) is the canonical form of $\phi(t)$, formula (4.2.8) holds, and we have

$$E\{ \mu^2(t' \mid t)\} = \int_0^t F^2(t', u)\, du, \tag{4.3.1}$$

where $F^2(t', t)$ is the derivative of a known function. Then

$$F(t, u) = \epsilon(t, u)\, F^*(t, u), \tag{4.3.2}$$

where $F^*(t, u)$ is known, and $\epsilon(t, u) = \pm 1$.

The sign $\epsilon(t, u)$ may be arbitrarily chosen for a particular value t_0 of t. Then the connection between $X(u)$ and $\phi(u)$ [in other words, between the numbers ξ_u and $\phi(u)$] is well defined in $(0, t_0)$, and $F(t, u)\sqrt{du} = d\phi(t)/d\xi_u$ is also well defined in this interval. Since we may apply successively this result for a sequence of indefinitely increasing values of t_0, we have no other choice than a sign depending only on u, and the theorem is proved.

We have now to speak of the existence theorem. To do this, we shall use the Schwartz theory of distributions.

4.4. *Application of the Schwartz theory of distributions.* Let us first recall the main idea of this theory. Let C denote the class of all real functions $\varphi(t)$ that are continuous for every real t, have continuous derivatives of all orders, and each of which is different from zero only in a finite interval (which is not necessarily the same for two different functions).

If $f(t)$ is a function that is integrable in every finite interval, then the inner product

$$\int_{-\infty}^{+\infty} f(t)\,\varphi(t)\,dt = f\cdot\varphi = U(\varphi) \tag{4.4.1}$$

is a linear functional of $\varphi(t)$, well defined and finite for every function of class C. The converse does not hold; $\varphi'(t_0)$, for instance, does not have the form (4.4.1). The idea of Schwartz is to associate to every linear functional $U(\varphi)$ that is well defined in C a symbolic function $f(t)$ for which (4.4.1) holds. Unless it is not an ordinary function, its definition in an open interval depends only on the values of $U(\varphi)$ for functions which vanish outside of this interval. For instance, if $U(\varphi) = \varphi(t_0)$, $f(t)$ is the Dirac symbolic function $\Delta(t - t_0)$, and is 0 for $t \neq t_0$.

Now, if $f(t)$ is differentiable, one has

$$f'\cdot\varphi = \int_{-\infty}^{+\infty} f'(t)\,\varphi(t)\,dt = -\int_{-\infty}^{+\infty} f(t)\,\varphi'(t)\,dt = -U(\varphi')\,, \tag{4.4.2}$$

and the derivative is the symbolic function associated with $-U(\varphi')$. Also the pth derivative is the function associated with $(-1)^p U[\varphi^{(p)}]$. Then we may define $f^{(p)}$ by

$$f^{(p)}\cdot\varphi = (-1)^p U[\varphi^{(p)}]\,, \tag{4.4.3}$$

and, with this definition, every symbolic function $f(t)$ has derivatives of all orders, which we shall denote $D^{(p)}[f(t)]$. In this section, we shall use only the first derivative $D[f(t)]$ of a nondecreasing function $f(t)$, and, setting $D[f(t)] = \sigma^2(t)$, we shall say that $\sigma(t)$ is a *σ-function.*

Suppose now that the considered function $\phi(t)$ has independent increments. Then it is well defined by its variance $S^2(t)$, which is a nondecreasing function of t, and, if one defines $F(t)$ by formula

$$F^2(t) = DS^2(t)\,, \tag{4.4.4}$$

then one has

$$\phi(t) = \int_0^t F(u)\,\xi_u\sqrt{du}\,. \tag{4.4.5}$$

Obviously, $F^2(u)$ is a Lebesgue integrable function if and only if $S(0) = 0$ and $S(t)$ is absolutely continuous. However, with the Schwartz derivative, these restrictions are not necessary; $S^2(t)$ may have jumps, and $S(0) = 0$ does not exclude $S(+0) > 0$.

Let us notice that, if $F(u)$ is not an ordinary function, it is not possible to replace in formula (4.4.5) $\xi_u\sqrt{du}$ by $dX(u)$, and consider $\phi(t)$ as a linear functional of $X(u)$. But it may be represented by $X[S^2(t)]$.

Quite analogous remarks hold for the Maruyama integral [13]

$$\phi(t) = \int_0^1 F(t, u)\,\xi_u\,du\,. \tag{4.4.6}$$

If $F(t, u)$ is an ordinary function, $\phi(t)$ may be written as a functional

$$\phi(t) = \int_0^1 F(t, u)\,dX(u)\,, \tag{4.4.7}$$

of $X(u)$, and depends only on an enumerable set of r.v. ξ. If we introduce σ-functions, if for instance $F^2(t, u) = \Delta[u - t/(1 + t)]$ [in that case, $\phi(t)$ is the value of ξ_u for $u = t/(1 + t)$], formula (4.4.6) may represent r.f. that depend on a nonenumerable set of r.v. ξ_u. Thus formula (4.4.6) is able to represent r.f. that have not the form 4.4.7.

Let us now introduce a definition of separability that is not exactly the definition due to Doob. We shall say that a r.f. $U(t)$ is *separable* if we may find a sequence $\{t_n\}$ such that, if all the values $U(t_n)$ are known, then $U(t)$ is known. Unless $U(t)$ is a very special r.f., the set $\{t_n\}$ is necessarily everywhere dense. Then

THEOREM 4.4. *$\phi(t)$ may be represented by formula* (4.4.7) *if and only if it is separable and belongs to C_G.*

PROOF. Let us first recall that it is easy to define a one-one correspondence between $X(t)$, if defined in $(0, T)$, and an infinite sequence of reduced Gaussian r.v. ξ_n that preserves the probability.

Now, if $\phi(t)$ is a separable function, and belongs to the class C_G, then we may choose a sequence t_n satisfying the separability definition. If we apply to the sequence $\{X(t_n)\}$ the results of section (4.1), each $\phi(t_n)$, and also $\phi(t)$ for every t, is a linear function of the variables ξ_n, hence also of the function $X(t)$ that is associated with the sequence $\{\xi_n\}$.

Now, let $m^*(t|u_1)$ be the conditional expectation of $\phi(t)$ when $X(u)$ is given in $(0, u_1)$. It is, for fixed t, a Gaussian r.f. of u_1 with independent increments, to which we may apply formula (4.4.5). Then

$$(4.4.8) \qquad m^*(t|u_1) = \int_0^{u_1} F(t, u)\, \xi_u \sqrt{du} = \int_0^{u_1} F(t, u)\, dX(u),$$

and, if $u_1 = 1$, we obtain formula (4.4.7). Thus the first part of the theorem is proved. Moreover, since this formula depends on the choice of the correspondence between $X(u)$ and the sequence $\{\xi_n\}$ (on which every orthogonal substitution is possible), we see that $\phi(t)$ has an infinity of representations of the form (4.4.7), which is not a canonical form.

Conversely, let $\phi(t)$ be defined by formula (4.4.7); it is a known linear function of the variables ξ_n that are associated with $X(t)$. Since obviously $\phi \in C_G$, we have only to prove that this function is separable. To do it, let us consider a transfinite sequence $\{t_n\}$ such that, if all the $X(t_\nu)$ with ν preceding n are known, $X(t_n)$ is unknown. When it is no longer possible to continue this sequence, $X(t)$ is a known function of the $X(t_n)$. We have to prove that the set of all the t_n is then at most an enumerable set.

This is obvious. Each of the $X(t_n)$ gives new information on the sequence $\{\xi_\nu\}$. Thus, at least one of the conditional standard deviations $\sigma_n(\xi_\nu)$ is decreasing. For each ξ_ν, and therefore also for the set of all ξ_ν, this may happen at most an enumerable infinity of times. Q.E.D.

It would be easy to give a modified proof that avoids the use of a transfinite sequence, but the use of the Zermelo axiom seems to be essential.

4.5. *The existence theorem.* Roughly speaking, this theorem may be stated as follows: *every r.f. of class C that fulfills a suitable continuity condition has one* (and then only one) *class of equivalent canonical representations.*

It is the main theorem of this theory. Unfortunately, we did not succeed in obtaining a necessary and sufficient continuity condition. We shall only present some remarks that show what kind of conditions may be considered, and that they are not very restrictive.

Let us suppose that $\phi(u)$ is given in $(0, t)$, and let

$$(4.5.1) \qquad \phi(t+dt) = \mu(t+dt|t) + \sigma(t, dt)\, \xi_t, \qquad dt > 0,$$

be the canonical form of $\phi(t+dt)$. The condition is that, *for sufficiently small dt, the value of ξ_t gives sufficient information on the behavior of $\phi(u)$ in $(t, t+dt)$, and that we shall nevermore need supplementary information.*[7] Then, for every $t' > t$,

$$\mu(t' \mid t+dt) - \mu(t' \mid t) = F(t', t)\, \xi_t \sqrt{dt}\,, \tag{4.5.2}$$

where $F(t', t)$ is a σ-function of t, and, since $\mu(t' \mid t') = \phi(t')$, formula (4.2.2) results from an integration with respect to t.

Let us first notice that this reasoning holds even if $\sigma(t, dt)$ is identically 0, as happens when $\phi(t)$ is an analytic r.f. Then ξ_t gives no new information; but we need no such information. In this case $F(t, u)$ is, for given t, a σ-function of a quite special kind, and the integral (4.2.2) depends only on the behavior of $X(u)$ in an arbitrarily small neighborhood of the origin.

Now, if we suppose that $\phi(t)$ has a.s. continuous derivatives up to order p, and that $\phi^{(p)}(t)$ is not differentiable, one has, for every given t,

$$\sigma(t, dt) = o\,[dt^p]\,, \tag{4.5.3}$$

and ξ_t gives information on $\delta\phi^{(p)}(t)$ as well as on $\delta\phi(t)$. If for instance, as happens in many applications, $\sigma(t, dt)$ has the form $\sigma(t)(dt)^{p+1/2}$, $\delta\phi^{(p)}(t)$ is known up to an error that is a.s. $o(dt)$, and we do not need more.

Then, to conceive an example where the necessary continuity condition is not fulfilled, we may suppose that the t-axis is divided into two complementary sets $\mathcal{E}_0$ and $\mathcal{E}_1$, and that $\phi(t)$ has quite different definitions in $\mathcal{E}_0$ and in $\mathcal{E}_1$. For instance, let $X_0(t)$ and $X_1(t)$ be two independent Wiener functions, and $\phi(t)$ be $X_0(t)$ or $X_1(t)$ according as $t \in \mathcal{E}_0$ or $\mathcal{E}_1$. In this case, the existence of a canonical representation of $\phi(t)$ depends on the definition of $\mathcal{E}_0$ and $\mathcal{E}_1$.

Let us first suppose that $\{t_n\}$, with $t_0 = 0$, is a sequence of indefinitely increasing numbers, and that $t \in \mathcal{E}_0$ or $\mathcal{E}_1$ according as the largest t_n that is $\leqq t$ has an even or an odd subscript. Then we want to know $X_0(t)$ in the intervals $[t_{2p}, t_{2p+1})$, $p = 0, 1, \cdots$, and $X_1(t)$ in the complementary intervals. Both may easily be deduced from one function $X(t)$ that varies as $X_i(t)$, $i = 0, 1$, in (t_{2p+i}, t_{2p+i+1}), and has at the point t_{2p+i} a jump defined by $X(t_{2p+i}+0) - X(t_{2p+i}-0) = X_i(t_{2p+i}) - X_i(t_{2p-1+i})$. Then $\phi(t)$ is a function of $X(t)$ that may be easily written in the form (4.2.2).

On the contrary, if $\mathcal{E}_0$ is the set of all rational numbers, to define $\phi(t)$, we need to know both functions $X_i(t)$ in every small interval $(t, t+dt)$, and one number ξ_t is not able to give the necessary information. Then $\phi(t)$ has no canonical representation.

In spite of such examples, the necessary continuity condition is not very restrictive. In this last example, we may replace $X_0(t)$ by a Gaussian additive function that has a jump $a_\nu \xi'_\nu$ at every rational point t_ν, with $\sum a_\nu^2$ (with $t_\nu < t$) $< \infty$ for every finite t; then we have only to set $X(t) = X_0(t) + X_1(t)$, and we obtain a canonical representation of $\phi(t)$.

Let us finally remark that the measurability of $\phi(t)$ is not a necessary condition. Indeed, if $\phi(t) = f(t)\phi_0(t)$, where $\phi_0(t)$ has a canonical representation, then $\phi(t)$ also has one, even if the known function $f(t)$ is not measurable.

[7] Although this is not very precise, the reader will understand here why we consider the necessary and sufficient condition for the existence of the canonical representation as a continuity condition.

4.6. *The derivatives of $\phi(t)$.* (a) Let us now suppose that $\phi(t)$ is given by formula (4.2.2), where $F(t, u)$ is a differentiable function of t. Then for small and positive dt, one has

$$\delta\phi(t) = dt \int_0^t \frac{\partial F(t, u)}{\partial t} \xi_u \sqrt{du} + F(t, t)\, \xi_t \sqrt{dt}, \tag{4.6.1}$$

and we see that $\phi(t)$ *is a.s. differentiable if and only if* $F(t, t) = 0$; *the derivative is then*

$$\phi'(t) = \int_0^t F_1(t, u)\, \xi_u \sqrt{du}, \qquad F_1(t, u) = \frac{\partial F(t, u)}{\partial t}. \tag{4.6.2}$$

If $F_1(t, t) = 0$, and if $F_1(t, u)$ is also differentiable, we obtain the second derivative $\phi''(t)$, and so on. If for instance

$$F(t, u) = \frac{(t-u)^p}{p!}, \tag{4.6.3}$$

then the pth derivative of $\phi(t)$ is the Brownian function $X(t)$, and every term $\xi_u\sqrt{du}$ may be considered as a random impulsion of order p (speed if $p = 1$, acceleration if $p = 2$). If

$$F(t, u) = \exp \frac{-1}{t-u}, \tag{4.6.4}$$

then $\phi(t)$ is a.s. infinitely differentiable, but not analytic.

(b) Let us now return to the problem that was stated at the end of section 4.2: *if $F(t, u)$ is given, recognize whether it is a canonical kernel.*

Obviously, this problem depends on the Volterra equation

$$\phi(t) = \int_0^t F(t, u)\, dX(u) \equiv F(t, t)\, X(t) - \int_0^t \frac{\partial F(t, u)}{\partial u} X(u)\, du, \tag{4.6.5}$$

and the kernel $F(t, u)$ is canonical if and only if, for given $\phi(t)$, this equation has at most one solution $X(t)$ that is a possible value for the Wiener function. Let us consider the case when

$$F(t, t) = F_1(t, t) = \cdots = F_{p-1}(t, t) = 0, \tag{4.6.6}$$

and $F_p(t, u)$ exists, is continuous, and is $\neq 0$ if $t = u > 0$; $F_p(0, 0)$ may be 0. Then $\phi(t)$ has a.s. derivatives up to order p, that vanish for $t = 0$, and (4.6.5) is equivalent to

$$\frac{d^p\phi(t)}{dt^p} = \int_0^t F_p(t, u)\, dX(u) \equiv F_p(t, t)\, X(t) - \int_0^t \frac{\partial F_p(t, u)}{\partial u} X(u)\, du. \tag{4.6.7}$$

Then, if

$$K(t, u) = \frac{1}{F_p(t, t)} \frac{\partial F_p(t, u)}{\partial u} \tag{4.6.8}$$

is continuous in the area $0 \leqq u \leqq t < \infty$, the Volterra equation

$$X(t) = \int_0^t K(t, u)\, X(u)\, du \tag{4.6.9}$$

has no other solution than $X(t) \equiv 0$, and (4.6.7) has at most one solution. Then $F(t, u)$ is canonical.

On the contrary, if $K(t, u)$ is not continuous, (4.6.7) is a singular Volterra equation, and the considered problem is more difficult. This happens for all the kernels considered in part 3, to define the functions $M_{2p+1}(t)$, $\phi_p(t)$, $\Psi_p(t)$ when $p > 1$. With these kernels, equation (4.6.9) has solutions that are nonidentically zero, and one has to ask whether these solutions are suitable values for the difference of two Wiener functions. The answer was given in section 3.7 for $p = 2$, $n = 5$. In the general case, the problem is more difficult.[8]

4.7. *The Goursat kernels.* We shall say that $F(t, u)$ is a *Goursat kernel of order* $p + 1$ if it may be written in the form

$$F(t, u) = \sum_0^p f_h(t)\varphi_h(u), \tag{4.7.1}$$

and not in an analogous form with less than $p + 1$ terms. Then the functions $f_h(t)$ on one hand, and the functions $\varphi_h(u)$ on the other hand, are linearly independent.

If we use the Schwartz derivatives, each of the functions $f_h(t)$ has derivatives of all orders, and $\sum c_h f_h(t)$ is the general solution of a linear differential equation of order $p + 1$

$$\mathcal{E}_t[f(t)] \equiv [D^{p+1} - A_1(t)D^p - \cdots - A_p(t)]f(t) = 0, \tag{4.7.2}$$

and one has

$$\mathcal{E}_t[F(t\ u)] = 0. \tag{4.7.3}$$

Conversely, if (4.7.3) holds, $F(t, u)$ has for every given u the form $\sum c_h f_h(t)$, and is a Goursat kernel of order $\leqq p + 1$. Thus the definition of the Goursat kernels may be given in the following form: $F(t, u)$ is a Goursat kernel of order $\leqq p + 1$ if it satisfies a differential equation of the form (4.7.2).

Lemma 2 in section 3.2 may now be stated as follows: *if $F(t, u)$ is a solution of equation* (4.7.3), *then the covariance $\Gamma(t_1, t_2)$ is a solution of*

$$\mathcal{E}_{t'}[\Gamma(t_1, t_2)] = 0, \qquad t' = \max(t_1, t_2). \tag{4.7.4}$$

The converse statement is not always correct, as shown by the following example. The covariance $\Gamma_5(t_1, t_2)$ of $M_5(t)$, considered as a function of t and t', is a Goursat kernel of order three, and, considered for fixed t as a function of t', is a solution of an equation of order three and of the form (4.7.2). We have seen in section 3.6 that two different kernels $F_{5,1}(t, u)$ and $F_{5,2}(t, u)$ give the same r.f. $M_5(t)$. The first kernel $F_{5,1}(t, u)$, which gives the canonical form of $M_5(t)$, verifies the same equation, but $F_{5,2}(t)$ is a Goursat kernel of order four, and is not a solution of this equation.

This remark leads us to the following theorem.

THEOREM 4.7. *If $F(t, u)$ is the kernel of the canonical representation of $\phi(t)$, equations* (4.7.3) *and* (4.7.4) *are equivalent.*

We have already proved that (4.7.3) implies (4.7.4). To prove the converse statement, let us start from (4.7.4) and write it in the form

$$E\{\phi(t)\,\mathcal{E}_{t'}[\mu(t'\,|\,t_1)]\} = 0, \qquad t \leqq t_1 \leqq t'. \tag{4.7.5}$$

Since this formula holds for every $t \leqq t_1$, and $\mathcal{E}_{t'}[\mu(t'\,|\,t_1)]$ is a linear functional of $\phi(t)$, $0 < t \leqq t_1$, it means that a.s.

$$\mathcal{E}_{t'}[\mu(t'\,|\,t_1)] = 0, \tag{4.7.6}$$

[8] Let us notice here that theorem 4.8 will give a necessary condition for the canonical character of a given kernel. It follows from this theorem that the kernels in $M_{5,2}(t)$ and $M_{7,2}(t)$ in section 3.6 are not canonical. A necessary and sufficient condition will be given in another paper.

that is, if t_1 is replaced by t,

$$\int_0^t \mathcal{E}_{t'} [F(t', u)] \, \xi_u \sqrt{du} = 0 , \qquad u \leqq t \leqq t' . \tag{4.7.7}$$

Since such an integral is a.s. zero if and only if the integrand is identically zero, (4.7.3) is proved.

4.8. *Markovian functions of order* $p+1$.

1°. We shall say that a r.f. $\phi(t)$ *is a Markovian r.f. of order* $\leqq p+1$ *in the wide sense* if, for every t, p numbers $\varphi_h(t)$, $h = 1, 2, \cdots, p$, may be found such that, if $\phi(t)$ and these numbers are known, then the behavior of $\phi(u)$ in (t, ∞) is independent of all other information on the values of this function in $(0, t)$. The order is $p+1$ if it is $\leqq p+1$ and not $\leqq p$. If $V(t)$, a vector r.f., with values in a $(p+1)$ dimensional space, is Markovian, then each of its components is a Markovian r.f. of order $\leqq p+1$ in the wide sense.

THEOREM 4.8. *If a r.f.* $\phi \in C_G$ *is defined by its canonical representation and if its variance is positive for every positive* t, *then it is a Markovian r.f. of order* $p+1$ *in the wide sense if and only if the kernel* $F(t, u)$ *is a Goursat kernel of order* $p+1$.

PROOF. If $F(t, u)$ has the form (4.7.1), and if we set

$$J_h(t) = \int_0^t \varphi_h(u) \, \xi_u \sqrt{du} , \tag{4.8.1}$$

then

$$\mu(t' \mid t) = \sum_0^p f_h(t') J_h(t) \tag{4.8.2}$$

depends only on the $p+1$ r.v. $J_h(t)$. Since $\phi(t)$, for every positive t, depends at least on one of these r.v., $\mu(t' \mid t)$ is also, for every $t' < t$, a known function of $\phi(t)$ and the p other variables $J_h(t)$. Since $\sigma(t', t)$ is not a r.v., the definition of $\phi(t')$ in (t, ∞) depends also only on these $p+1$ variables. One notices that this first part of the present theorem holds even if $F(t, u)$ is not a canonical kernel.

Conversely, if the numbers $\varphi_h(t)$ considered in the definition of a Markovian r.f. of order $p+1$ may be found, then since $\mu(t' \mid t)$ is a linear function of $\phi(u)$, $0 < u \leqq t$, it verifies, for every $t' > t = t_1$, an equation of form (4.7.5). Then (4.7.3) follows as in the proof of theorem 4.7. Q.E.D.

2°. We shall say that $\phi(t)$ is a *Markovian r.f. of order* $p+1$ *in the restricted sense* if (a) it is a Markovian r.f. of order $p+1$ in the wide sense; (b) the derivatives $\phi^{(k)}(t)$ up to order p exist a.s. [this implies the a.s. continuity of $\phi(t)$ and its derivatives up to order $p-1$]; (c) these p derivatives may be chosen as functions $\varphi_h(t)$.

It is easy to give a constructive definition of the most general r.f. of class C_G that has these properties. From (a) and theorem 4.8, we deduce that we have to start from a kernel $F(t, u)$ of the form (4.7.1). We deduce from section 4.6 that (b) is then equivalent to the following conditions:

(b_1) the derivatives $f_h^{(k)}(t)$, $k = 1, 2, \cdots, p$, exist. This implies that the functions $f_h(t)$ and the derivatives $f_h^{(k)}(t)$, $k = 1, 2, \cdots, p-1$, are continuous.

(b_2) $$F_k(t, t) = \sum_0^p f_h^{(k)}(t) \, \phi_h(t) = 0 , \qquad k = 0, 1, \cdots, p-1 .$$

Then

$$\phi^{(k)}(t) = \sum_0^p f_h^{(k)}(t) \, J_h(t) \qquad k = 0, 1, 2, \cdots, p , \tag{4.8.3}$$

and (c) may be stated as follows: these $p+1$ linear functions of the $J_h(t)$ are independent; this happens if and only if the Wronsky determinant

$$\Delta(t) = \| f_h^{(k)}(t) \| , \qquad h, k = 0, 1, \cdots, p , \tag{4.8.4}$$

does not vanish. This condition and condition (b_1) may be replaced by the following: the functions $f_h(t)$ are $p+1$ independent solutions of an equation of the form (4.7.2) that has only regular points (this means that the Cauchy problem has one and only one solution, for every $t > 0$).

Now, we have to find functions $\varphi_h(t)$ that fulfill condition (b_2). Let us first notice that, since $\Delta(t) \neq 0$, $F_k(t, t)$ cannot be 0 for $k = 0, 1, \cdots, p$, then if (b_2) holds, $\phi(t)$ will have a.s. derivatives up to order p, but not of order $p+1$, unless all the $\varphi_h(t)$ vanish for the same value of t. If $\Delta(t)$ is written in the form

$$\Delta(t) = \sum_0^p f_h^{(p)}(t)\, \Delta_h(t) , \tag{4.8.5}$$

then, for every $t > 0$, at least one of the $\Delta_h(t)$ is $\neq 0$. Suppose $\Delta_0(t) \neq 0$.[9] Then equations (b_2) define all the ratios $\varphi_h(t)/\varphi_0(t)$, and these ratios are continuous functions of t. Then we have only to choose a σ-function $\varphi_0(t)$ and we have the most general solution of our problem. Since we keep the same r.f. when $\varphi_0(t)$ is replaced by $\epsilon(t)\varphi_0(t)$, where $\epsilon(t) = \pm 1$, it is well defined when the functions $f_h(t)$ and $\varphi_0^2(t)$ are chosen.

Now, we have still to prove that the constructed $F(t, u)$ is a canonical kernel. That is obvious. Since the functions $\phi^{(k)}(t)$, $k = 0, 1, \cdots, p$, are independent linear functions of the $J_h(t)$, $\mu(t' \mid t)$, which is a known function of the $J_h(t)$, is also a known function of the $\phi^{(k)}(t)$, and is known when $\phi(u)$ is given in $(0, t)$.

These results are a generalization of those of part 3. The function $M_{2p+1}(t)$ is a Markovian function of order $p+1$, in the restricted sense, and the condition that we have used to find the coefficients a_h is equivalent to condition (b) of this section. If n is an even number $> 2p$, $M_n(t)$ has a.s. derivatives up to order p, but does not seem to be a Markovian r.f. of every finite order.

3°. One may wonder if it is possible to generalize this theory by using Schwartz derivatives $D^k\phi(t)$ instead of the ordinary derivatives. The answer is no. If $\phi(u)$ is known in $(t - dt, t)$, we may consider $D^k\phi(t-0)$ as known. However, only in the case of continuous derivatives is $D^k\phi(t-0) = D^k\phi(t+0)$, and these numbers may replace the numbers $J_h(t)$ and give information that is sufficient to define $\mu(t' \mid t)$.

In general, as we have seen in the case of the function $\Psi(t)$ in part 3, such information is given only by $\phi(t)$ and $X(t)$, where $X(t)$ is given by an integral in $(0, t)$, and is unknown when $\phi(t)$ is given only in a small interval $(t, t - dt)$.

4°. If $\phi(t)$ is continuous, its values in $(0, t)$ depend at most on an enumerable infinity of parameters and if $p \to \infty$, the wide definition of Markovian r.f. has no meaning. On the contrary, however, the restricted definition leads to a new kind of Markovian r.f. Then $\phi(t)$ has a.s. derivatives of all positive orders, and $\mu(t' \mid t)$ depends only on the values of $\phi(t)$ and these derivatives at the time t. A slightly different definition is the following one: if $\phi(t - dt)$ is known, up to an error that is $O[(dt)^p]$ for every p (when $dt \to 0$), then $\mu(t' \mid t)$ is known. Since the existence of the derivatives is not necessary, this definition is slightly more general than the preceding one.

[9] If none of the functions $\Delta_h(t)$ is $\neq 0$ for every positive t, then we may replace $\varphi_0(t)$ by $\varphi(t) = \sum |\varphi_h(t)|$.

4.9. *Stochastic differential equations.* If, in the formula

$$\phi(t') - \phi(t) = \mu(t' \mid t) - \phi(t) + \sigma(t', t), \tag{4.9.1}$$

we suppose $t' - t = dt > 0$, we have an equation that may be called a stochastic differential equation. In general, as we have just seen, $\mu(t' \mid t)$ depends on all the values of $\phi(u)$ in $(0, t)$ and this equation has also the character of an integral equation. Only if $\phi(t)$ is a Markovian r.f. of finite or infinite order in the restricted sense is it a purely differential stochastic equation. If the order is $p + 1$, one has

$$\frac{\partial^k \mu(t' \mid t)}{\partial t'^k} = \phi^{(k)}(t), \qquad k = 1, 2, \cdots, p, \tag{4.9.2}$$

and the derivative of order $p + 1$ is given by formula (4.7.5). Then we obtain

$$\delta\phi(t) = \sum_1^p \phi^{(k)}(t) \frac{dt^k}{k!} + \frac{dt^{p+1}}{(p+1)!} \sum_0^p A_{p-k}(t)\, \phi^{(k)}(t) + \sigma(t + dt, t)\, \xi_t, \tag{4.9.3}$$

where

$$\sigma^2(t', t) = \int_t^{t'} F^2(t', u)\, du. \tag{4.9.4}$$

In (4.9.3), the last term is obviously $O[(dt)^{p+1/2}]$, but not $O[(dt)^{p+1}]$.

Let us apply this result to the function $M_{2p+1}(t)$ considered in part 3. Then equation (4.7.2) is the Euler equation satisfied by

$$1, \frac{1}{t}, \frac{1}{t^3}, \cdots, \frac{1}{t^{2p-1}}, \tag{4.9.5}$$

and may be written

$$f^{(p+1)}(t) = \sum_1^p a_k t^{k-p-1} f^{(k)}(t) \tag{4.9.6}$$

where the numbers a_k are, for every given p, defined by

$$\begin{aligned} s(s-1)\cdots(s-p+1) - \sum_2^p a_k s(s-1)\cdots(s-k+2) - a_1 \\ = (s+2)(s+4)\cdots(s+2p). \end{aligned} \tag{4.9.7}$$

For small values of p, it is easy to obtain their numerical values. It is also easy to deduce σ from formula (4.9.4) and from the expression of $F(t, u)$ in section 3.3. Then one obtains for positive dt,

$$M(t) = \sum_1^p M^{(k)}(t) \frac{dt^k}{k!} + \frac{dt^{p+1}}{t^{p+1}(p+1)!} \sum_1^p a_k t^k M^{(k)}(t) + \sigma_p \frac{dt^{p+1/2}}{t^p}\, \xi_t, \tag{4.9.8}$$

$$M'(t) = \sum_1^p M^{(k)}(t) \frac{dt^{k-1}}{(k-1)!} + \frac{dt^p}{t^{p+1} p!} \sum_1^p a_k t^k M^{(k)}(t) + \sigma'_p \frac{dt^{p-1/2}}{t^p}\, \xi'_t, \tag{4.9.9}$$

where

$$(2p+1)\,\sigma_p^2 = (2p-1)\,\sigma_p'^2 = \frac{(2p)!}{2\,(p!)^2}. \tag{4.9.10}$$

One also obtains easily the equations that $\phi(t) = t^{n-2} M(t)$ and $\phi'(t)$ satisfy. For this $\phi(t)$ as well as for $M(t)$, we encounter a special case. One of the functions $f_h(t)$ is a constant; hence it disappears in the canonical form of the derivative, and this derivative is a Markovian r.f. of order p in the restricted sense. When this does not happen, it is a Markovian r.f. of the same order as the integral, in the wide sense. This circumstance is not repeated at the second derivation, and the equations verified by $M''(t)$ or $\phi''(t)$ are not purely differential stochastic equations.

Coming back to general equation (4.9.3), it is easy to integrate it. We have to choose arbitrarily $p+1$ independent solutions of equation (4.7.2), and take them as $f_h(t)$. Then the $\varphi_h(u)$ are known up to a common factor $\varphi(u)$, as we have proved in section 4.8, 2°, and $\varphi(u)$ is then given by formula (4.9.4), up to the sign, which is of no consequence. Then $F(t, u)$ is known, and the Cauchy problem is easy to solve. The formulas

$$(4.9.11)\quad \phi^{(k)}(t') = \int_0^t F_k(t', u)\,\xi_u\sqrt{du} + \int_t^{t'} F_k(t', u)\,\xi_u\sqrt{du}\,, \quad k=0, 1, \cdots, p\,,$$

where the first terms are known functions of the given $\phi^{(k)}(t)$, give not only the conditional canonical form of the unknown $\phi^{(k)}(t')$, but their conditional joint distribution. This method was already applied in section 3.8 to obtain the joint distribution of $\phi_2'(t)$ and $\phi_2''(t)$.

4.10. *Final remarks.*

1°. In this part, we have considered only processes that start from time $t=0$. However, nothing essential is changed if t is replaced by a continuously increasing function of another parameter. Then we may suppose that the initial time t_0 is an arbitrary time that may be $-\infty$.

We may also choose $-t$ as a new parameter. Then we obtain a quite different definition of the same process. It has a backward canonical form

$$(4.10.1)\qquad \phi(t) = \int_t^{\infty} G(t, u)\,\xi_u\sqrt{du}\,,$$

and a backward stochastic differential equation, which is different from the forward equation.

In the case of a Goursat kernel, these two equations have a simple connection. The covariance $\Gamma(t_1, t_2)$ may be written in the form (3.2.2) if and only if the canonical kernel $F(t, u)$ is a Goursat kernel of order $p+1$. Then $G(t, u)$ is a Goursat kernel of the same order, and may be deduced from the functions $g_h(t)$ in the same way that $F(t, u)$ is deduced from the functions $f_h(t)$. Equation (4.7.2) is replaced by another of the same form. The Markovian character of every order, in the wide sense as well as in the restricted sense, is unchanged. In this last case, equation (4.9.3) is replaced by another of the same form, which holds for negative dt, and has another term in dt^{p+1}.

In the case of the r.f. $M_{2p+1}(t)$, one has $f_h(t)g_h(t) = t$. Then it is easy to deduce the Euler equation satisfied by $g_h(t)$ from the equation satisfied by $f_h(t)$. The term in $\xi_t'(dt/t)^p\sqrt{dt}$ has been written in section 3.5 for the forward differential equation; it is the same for the backward equation, $\sqrt{dt}$ being replaced by $\sqrt{|dt|}$.

2°. A generalization of the preceding remark may be applied to the Gaussian functions of two (or more than two) variables. To apply our theory and obtain a canonical form for such a function $\phi(t_1, t_2)$, we have to define the set of points (u_1, u_2) that will be considered as preceding (t_1, t_2). If we define this set by the conditions

$$(4.10.2)\qquad u_i \leq t_i\,, \qquad \sum u_i < \sum t_i\,, \qquad i = 1, 2$$

then the generalization of formula (4.2.2) is

$$\phi(t_1, t_2) = \int_0^{t_1}\int_0^{t_2} F(t_1, t_2, u_1, u_2)\,\xi_{u_1, u_2}\sqrt{du_1 du_2}. \tag{4.10.3}$$

Let us notice that these conditions (4.10.2) are already used in the definition of the distribution function of joint r.v. X and Y; this function is the probability that the point (X, Y) precedes a given point. This definition depends on the axis. It is easy to obtain intrinsic definitions, by introducing the Schwartz density of probability, or considering the probability as a function of a set.

On the contrary, for our problem, these methods are complicated or even impossible, and it seems not possible to avoid the introduction of conditions (4.10.2) or other conditions that would also be arbitrary and not intrinsic. For this reason, the theory stated in this part is especially interesting in the case of Gaussian r.f. of a time-parameter.

5. The continuation of $M(t)$

5.1. *The continuation to the right.* The tool for this problem is given by formula

$$M(t) = c_p\int_0^t \xi_u\sqrt{du}\left[I_{2p-1} - \int_0^{u/t}(1-x^2)^{p-1}\,dx\right], \tag{5.1.1}$$

where $M(t) = M_n(t)$, with $n = 2p+1$, and

$$c_p^2 = \frac{4p^2}{\pi} I_{2p}, \qquad b_0^2 = c_p^2 I_{2p-1}^2 = I_{2p-1}, \tag{5.1.2}$$

which was proved in section 3.3. We have already deduced

$$M'(t) = c_p\int_0^t \left(1-\frac{u^2}{t^2}\right)^{p-1}\frac{u}{t^2}\,\xi_u\sqrt{du}. \tag{5.1.3}$$

If $\sigma_p(t', t)$ is the standard deviation of $M'(t')$, $t' > t$, when $M'(u)$ is given in $(0, t)$ [and then $M(u)$ is known, since $M(0) = 0$], then since formula (5.1.3) is the canonical form of $M'(t)$, one has

$$\begin{aligned} t'\sigma_p^2(t', t) &= c_p^2\int_t^{t'}\left(1-\frac{u^2}{t'^2}\right)^{2p-2}\frac{u^2du}{t'^3} \\ &< \frac{c_p^2}{2}\int_{t^2}^{t'^2}\left(1-\frac{v}{t'^2}\right)^{2p-2}\frac{dv}{t'^2} = \frac{c_p^2}{2(2p-1)}\left(1-\frac{t^2}{t'^2}\right)^{2p-1} \end{aligned} \tag{5.1.4}$$

and, since $c_p^2 = O(p^{3/2})$, $p \to \infty$, $\sigma_p^2(t', t)$ is, for fixed $t > 0$ and $t' > t$, a rapidly decreasing function of p. For $t = 0$, one has

$$t'E\{M'^2(t')\} = c_p^2\int_0^1 (1-u^2)^{2p-2}u^2du = \frac{c_p^2}{4p-1} I_{4p-3}, \tag{5.1.5}$$

and

$$\lim_{p\to\infty} E\{M'^2(t')\} = \frac{1}{4\sqrt{2t'}}. \tag{5.1.6}$$

Thus, when $n = 2p+1 \to \infty$, $M'(t')$ is not a very small Gaussian r.v., and it follows from formula (5.1.4) that, when $M'(u)$ is given in $(0, t)$, it is known with an indefinitely decreasing error. Thus the process tends to a deterministic process.

Let us notice that this is independent of the difficulties that arise when, in the definition of the Brownian function $X(A)$, n tends to infinity, and which will be considered in part 6. In this part, we consider only a function $M_n(t)$ of a real variable t. Its covariance $\Gamma_n(t_1, t_2)$ has a limit $\Gamma_\omega(t_1, t_2)$ given by formula (2.3.2). Then $M_n(t)$ has a limit $M_\omega(t)$ in the Bernoulli sense. This means that the probability distribution in a functional space has a limit. We knew already that $M_\omega(t)$ is a.s. indefinitely differentiable. Now we know something more: the stochastic process $M_\omega(t)$ is not really a stochastic process, and $M_\omega(u)$ is known in $(0, \infty)$ when it is given in an arbitrary small interval $(0, 5)$. It is also known if all its derivatives are given at a point t. The question then arises: is $M_\omega(t)$ an a.s. analytic function?

5.2. *The analytic character of $M_\omega(t)$.*

THEOREM 5.2. *$M_\omega(t)$ is a.s. an analytic function.*

PROOF. If $n = 2p + 1$ and $h \leqq p$, one deduces from formula (5.1.3)

$$(5.2.1) \qquad \sigma^2_{n,h} = E\{[M_n^{(h)}(t)]^2\} = c_p^2 \int_0^t \left\{ \frac{d^{h-1}}{dt^{h-1}} \left[\frac{u}{t^2} \left(1 - \frac{u^2}{t^2}\right)^{p-1} \right] \right\}^2 du .$$

We first have to find the limit $\sigma^2_{\omega,h}$ of this expression, for fixed h, when $p \to \infty$. For this calculation

$$(5.2.2) \qquad \frac{d}{dt} \frac{(t^2 - u^2)^{p+\alpha}}{t^{2(p+\beta)}} = \frac{(t^2 - u^2)^{p+\alpha}}{t^{2(p+\beta)}} \left[\frac{2(p+\alpha)u^2}{t(t^2 - u^2)} + \frac{2(\alpha - \beta)}{t} \right]$$

may be replaced by

$$(5.2.3) \qquad \frac{(t^2 - u^2)^{p+\alpha}}{t^{2(p+\beta)}} \frac{2pu^2}{t(t^2 - u^2)} ,$$

and consequently

$$(5.2.4) \qquad \frac{d^{h-1}}{dt^{h-1}} \left[\frac{u}{t^2} \left(1 - \frac{u^2}{t^2}\right)^{p+h-2} \right]$$

may be replaced by

$$(5.2.5) \qquad \left[\frac{2pu^2}{t(t^2 - u^2)} \right]^{h-1} \frac{u}{t^2} \left(\frac{t^2 - u^2}{t^2} \right)^{p-1} = (2p)^{h-1} \frac{u^{2h-1}(t^2 - u^2)^{p-h}}{t^{2p+h-1}} ,$$

and, setting $u^2 = t^2 v$, we obtain

$$(5.2.6) \qquad \begin{aligned} \sigma^2_{\omega,h} &= \lim_{p\to\infty} c_p^2 \frac{(2p)^{2h-2}}{2t^{2h-1}} \int_0^1 v^{2h-3/2} (1 - v)^{2p-2h} \\ &= \lim_{p\to\infty} c_p^2 \frac{(2p)^{2h-2}}{2t^{2h-1}} B(2h - \tfrac{1}{2}, 2p - 2h + 1) \\ &= \lim_{p\to\infty} c_p^2 \frac{(2p)^{2h-2}}{2t^{2h-1}} \frac{\Gamma(2h - \frac{1}{2})\Gamma(2p - 2h + 1)}{\Gamma(2p + \frac{1}{2})} \\ &= 2^{-5/2} t^{1-2h} \Gamma(2h - \tfrac{1}{2}) \lim_{p\to\infty} p^{-3/2} c_p^2 . \end{aligned}$$

Finally, taking account of formula (5.1.2), we have

$$(5.2.7) \qquad \sigma^2_{\omega,h} = \frac{1}{4\sqrt{2\pi}} \Gamma(2h - \tfrac{1}{2}) t^{1-2h} .$$

Since, for large h, $\Gamma(2h - 1/2)/(h!)^2$ is approximately $2^{2h-1}/h^2\sqrt{\pi}$ the general term of the Taylor series

$$(5.2.8) \qquad M_\omega(t + \tau) = M_\omega(t) + \sum_1^\infty \frac{\tau^h}{h!} \sigma_{\omega,h} \xi_h ,$$

where ξ_h is a reduced Gaussian r.v., does not tend to zero if $\tau > t/2$, and this series is a.s. divergent. Although the ξ_h are not independent, since

$$a_h = Pr\{\,|\,\xi_h\,|\geqq c\sqrt{2\log h}\,\} < \frac{c'h^{-c^2}}{\sqrt{\log h}}, \qquad c' = \frac{1}{c\sqrt{\pi}}, \tag{5.2.9}$$

and $\sum a_h < \infty$ if $c > 1$, one deduces from the Cantelli lemma that the contrary inequality

$$|\,\xi_h\,| < c\sqrt{2\log h} \tag{5.2.10}$$

holds a.s. for sufficiently large h. Hence the considered Taylor series is a.s. convergent if $2|\tau| < t$, and the theorem is proved.

More precisely, we know that the radius of convergence of this series is $t/2$. Thus, a.s., if $t = re^{i\theta}$, $M_\omega(t)$ is *a regular analytic function for* $|\theta| < \pi/6$, *and all the points of the half straight lines* $\theta = \pm\pi/6$ ($\sin\theta = \pm 1/2$) *are singular points.*

5.3. *The continuation to the left.* Many different problems may be considered. For instance, according to whether $M(u)$ or $\bar{M}(u)$ is given in (t, ∞), the problem is not the same. The most interesting problem arises when $X(A)$ is given in the entire region outside of a sphere Ω_t. Then $\bar{M}(u)$ is known in (t, ∞), and, since $M(t) = \bar{M}(t) - X(0)$, if $\mu + \sigma\xi$ is the conditional canonical form of $M(t)$, the canonical form of $X(0)$ is $\bar{M}(t) - \mu - \sigma\xi$. The problem is to find μ and σ (σ obviously has the form $k_p\sqrt{t}$). Since $M(t)$ is invariant under the addition of a constant to $X(A)$, $M'(u)$ and $\bar{M}(u)$, if given in (t, ∞), give exactly the same information on $M(t)$, and since $M'(t)$ is a Markovian r.f. of order p in the restricted sense, one has exactly the same information if $M'(t)$, $M''(t)$, $\cdots$, $M^{(p)}(t)$ are given, or also if one knows the integrals $J_h(t)$ [see formula (4.8.1)], which we shall write in the form

$$J_{2h-1}(t) = \int_0^t u^{2h-1}\xi_u\sqrt{du}, \qquad h = 1, 2, \cdots, p\,. \tag{5.3.1}$$

Since the only unknown term in $M(t)$ is now

$$b_0 J_0(t) = b_0\int_0^t \xi_u\sqrt{du}, \tag{5.3.2}$$

and since we may suppose $t = 1$ (after that the generalization is easy), we are reduced to the following problem. *Setting*

$$J_k = \int_0^1 u^k\xi_u\sqrt{du}, \tag{5.3.3}$$

one supposes that $J_1, J_3, \cdots, J_{2p-1}$ *are given, and one seeks the conditional canonical form of* J_0.

It obviously has the form

$$J_0 = a_1'J_1 + a_2'J_3 + \cdots + a_p'J_{2p-1} + c_p\xi\,, \tag{5.3.4}$$

and, to find the numbers a_h' and c_p, one has to minimize

$$c_p^2 = E\Big\{\Big[J_0 - \sum_1^p a_h J_{2h-1}\Big]^2\Big\} = \int_0^t\Big(1 - \sum_1^p a_h' u^{2h-1}\Big)^2\,du\,. \tag{5.3.5}$$

The minimizing polynomial may easily be deduced from the theory of the Legendre polynomials. If we set

$$\frac{d^n}{dx^n}\,(1 - x^2)^n = \lambda_n P_n(x)\,, \tag{5.3.6}$$

$$\lambda_n^2 = \int_0^1 \left[\frac{d^n}{dx^n}(1-x^2)^n\right]^2 dx = (2n)!\int_0^1 (1-x^2)^n dx \tag{5.3.7}$$

$$= (2n)!\, I_{2n+1} = \frac{2^{2n}(n!)^2}{2n+1};$$

$P_n(x)$ is the Legendre polynomial, normalized in (0, 1), and if we represent 1 in the interval (0, 1) by the series

$$1 = \sum_0^\infty a_h P_{2h+1}(x)\,, \qquad 0 < x < 1\,, \tag{5.3.8}$$

our problem is solved by the formulas

$$\sum_0^{p-1} a_h' u^{2h+1} = \sum_0^{p-1} a_h P_{2h+1}(x)\,, \tag{5.3.9}$$

$$c_p^2 = \sum_p^\infty a_h^2\,. \tag{5.3.10}$$

The coefficients a_p are given by the Fourier-Legendre formula

$$a_p = \frac{1}{\lambda_n}\int_0^1 \frac{d^n}{dx^n}(1-x^2)^n dx = \frac{1}{\lambda_n}\left[\frac{d^{2p}}{dx^{2p}}(x^2-1)^{2p+1}\right]_{x=0}\,, \qquad n = 2p+1\,, \tag{5.3.11}$$

$$= \frac{(-1)^p}{\lambda_{2p+1}}(2p)!\,\frac{(2p+1)!}{p!\,(p+1)!} = \frac{(-1)^p}{\lambda_{2p+1}}\,\frac{2p+1}{p+1}\left[\frac{(2p)!}{p!}\right]^2.$$

Then, we deduce from formula (5.3.7)

$$a_p^2 = \frac{4p+3}{2^{4p+2}(p+1)^2}\,\frac{(2p!)^2}{(p!)^4} = \left[\frac{1\cdot 3\cdot\cdots\cdot(2p-1)}{2\cdot 4\cdot\cdots\cdot(2p)}\right]^2 \frac{4p+3}{4(p+1)^2} \tag{5.3.12}$$

$$= \frac{4}{\pi^2} I_{2p}^2 \frac{4p+3}{4(p+1)^2} = \frac{4}{\pi^2}(I_{2p}^2 - I_{2p+2}^2)\,,$$

and from formula (5.3.10),

$$c_p^2 = \frac{4}{\pi^2} I_{2p}^2\,. \tag{5.3.13}$$

This is the conditional variance of $J_0(t)$. That of $b_0 J_0(t)$ is then

$$k_p^2 = \frac{4}{\pi^2} b_0^2 I_{2p}^2 = \frac{4}{\pi^2} p I_{2p}^2 I_{2p-1} = \frac{1}{\pi} I_{2p} = \frac{1}{2}\,\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2p-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2p)}\,. \tag{5.3.14}$$

The values $k_0^2 = 1/2$, $k_1^2 = 1/4$ are easy to find by elementary calculations. By the same method, the author had found

$$k_2^2 = \tfrac{3}{16}\,, \qquad k_3^2 = \tfrac{5}{32}\,. \tag{5.3.15}$$

However, the calculations become more and more complicated, and only the Legendre polynomials seem able to lead to the general formula.

The same method may be used to find the conditional canonical forms of $M(t_0)$ and $M'(t_0)$, $t_0 < t$, when $M(t)$ and its p derivatives (or only the derivatives) are given at the point t. In this case the calculation is more complicated.

When $p \to \infty$, $c_k \to 0$. This was already known, since $M_\omega(t)$ is analytic, but it is a particularly important theorem, because it gives, not only properties of $\bar{M}_\omega(t)$, but also of the Brownian function $X(A)$ itself. When this function is given in the neighborhood of a sphere, it is known exactly at the center; on concentric spheres, one knows only averages; but, for increasing p, the convergence to this known average is faster than at the center.

Consequences of these remarks will be stated in part 6.

6. The function $X(A)$ in Euclidean space and in Hilbert space

6.1. *The function $X(A)$ in Euclidean space $E_n(n = 2p + 1)$.* One of the most important of our theorems on $M(t)$ is that it is a Markovian r.f. of order $p + 1$ in the restricted sense. This property of $M(t)$ is connected with properties of $X(A)$. The author did not succeed in finding satisfactory proofs of all these properties and will only present in this section a brief sketch of a theory that merits further development.

Let us first speak of two particular problems which can be solved by the methods considered in parts 2 and 3. The first concerns the average $\bar{M}(r, t)$ of $X(A)$ on the circle

$$x_0 = t, \quad x_1^2 + x_2^2 + \cdots + x_n^2 = r^2, \tag{6.1.1}$$

of the $(n + 1)$-dimensional Euclidean space. If $n = 2p + 1$, the covariance of the r.f.

$$M(r, t) = \bar{M}(r, t) - \bar{M}(0, t) = \bar{M}(r, t) - X(t), \tag{6.1.2}$$

where $X(t)$ is a Wiener r.f., is given by an elementary integral and it is easy to extend our continuity theorem to this case. $\bar{M}(r, t)$ has a.s. continuous derivatives up to order p, whereas $M(r, t)$ is not differentiable with respect to t.

Let $\mu(r, t)$ denote the conditional expectation of $X(t)$ when $\bar{M}(\rho, u)$ is given for $\rho = r$ and every value of u. This function is a weighted average of the given values of $\bar{M}(r, u)$, and has a.s. continuous derivatives up to order p. Since $\bar{M}(r, t)$ and $\mu(r, t)$ tend a.s. to $X(t)$ as r tends to zero, the irregular Wiener function $X(t)$ is then a limit of differentiable functions of t, with a number of continuous derivatives that increases with n. In the Hilbert space, $\bar{M}(r, t)$ and $\mu(r, t)$ are analytic, and $X(t)$ appears as a limit of analytic functions.

In the second particular problem, the spheres are replaced by parallel planes. Then it is not possible to speak of an average in the whole plane; but we may consider a weighted average, for instance,

$$m_a^*(t) = \frac{1}{\sqrt{2\pi a}} \int_0^\infty e^{-r^2/2a} M(r, t)\, dr, \tag{6.1.3}$$

or

$$m_a(t) = \frac{2}{a^2} \int_0^a M(r, t)\, r\, dr, \tag{6.1.4}$$

which is an average in the finite circle $r < a$. Both have continuous derivatives up to order p.

This has important consequences. The r.f. $X(A)$ is not differentiable, and one cannot speak of its normal derivative. However, $m_a'(t)$ is, for every fixed a, a mean derivative, normal to the plane $x_0 = t$, and in the $(n + 1)$-dimensional space, if $n \geqq 2p + 1$, the mean derivatives of orders $1, 2, \cdots, p$ exist. The form of the derivative of order $q \leqq p$ is obviously $c_{p,q}\, \xi_q a^{1/2-q}$, and it is probably very large if a is small, and has no limit (either finite or infinite) when a tends to zero.

This is connected with a well-known property of $X(A)$. If this function is given on a surface Ω, which has a tangent plane that varies continuously, and if in a sufficiently small neighborhood of a point H of Ω and on the normal straight line one chooses two symmetric points A and B, then there exists a negative correlation between $X(A) - X(H)$ and $X(B) - X(H)$. This is easily understood. If A is on the positive side of the surface, there is a positive correlation between $X(A) - X(H)$ and the mean normal derivative at the point H, and a negative correlation between $X(B) - X(H)$ and this derivative.

Now let Ω denote a regular surface that divides the space into two regions, and let us suppose that the following information on $X(A)$ is given: a function $x(A)$ is given such that $X(A) - x(A) = o(\delta^p)$, where δ is the absolute distance between A and the surface Ω. Then the following statement is probably right: *if such information, which we shall call the Cauchy condition, is given, then the values of the r.f. $X(A)$ on one side of Ω are independent of its values on the other side. If this statement is right,* one can say that $X(A)$ *also has a Markovian character* (to define a Markovian r.f. of order $p + 1$ in an n-dimensional space, one has to speak, not of $p + 1$ numbers, but of $p + 1$ arbitrary functions depending on a point of a surface). However, this is not at all obvious. Even if Ω is a sphere, this is not a consequence of the known properties of $M(t)$. If t is the radius of the sphere, if $OA_0 = t_0$, $OA_1 = t_1$, where $t_0 < t < t_1$, and if θ is the angle A_0OA_1, one has to prove that the conditional covariance $\gamma(t_0, t_1, \theta)$ of $X(A_0)$ and $X(A)$ when $x(A)$ is given in a neighborhood of Ω is zero. We have only proved that

$$\int_0^\pi \sin^{n-2}\theta\, \gamma(t_0, t_1, \theta)\, d\theta = 0\,. \tag{6.1.5}$$

Preliminary steps to the proof of the stated theorem would be to prove it in the case of the sphere, and also to consider other simple families of surfaces depending on one parameter, and suitable weighted averages on these surfaces. The existence of a Markovian character for such averages is a necessary, but not sufficient, condition for the correctness of the stated theorem. The author calls attention to these unsolved problems.

6.2. *The Hilbert space. Preliminary remarks.* In this section, we shall summarize known results, some of which are proved in the author's books [2] and [3], the other in his recent paper [12]. In the last sections, we shall give again a complete statement of the most important results of that paper.

The Hilbert space E_ω, as well as the Euclidean n-space E_n, is separable. Hence we may choose a sequence $\{A_n\}$ that is everywhere dense in this space, and define the Gaussian sequence $\{X(A_n)\}$. The difference is that, in the Euclidean n-spaces, this sequence defines a.s. a continuous function $X(A)$, but not in the Hilbert space. However, it defines a.s. a function $X(A)$ that has the following properties: for every integer n, it is continuous, not only in the $x_1x_2\cdots x_n$ plane, but in every Euclidean space $E_n \subset E_\omega$, and also a.s. in every n-dimensional surface with a tangent plane that varies continuously

The following remark shows how such a function may be discontinuous in E_ω. Let A_n be the point $x_n = a\sqrt{2}$ of the x_n axis, and $X_n = X(A_n)$. It is easy to find the canonical definition of the sequence $\{X_n\}$, and to deduce the following theorem: *as $n \to \infty$, the average*

$$m_n = \frac{1}{n}(X_1 + X_2 + \cdots + X_n) \tag{6.2.1}$$

has a.s. a limit m, and the conditional distribution of X_{n+1} when $X_1, X_2, \cdots, X_n$ are given is $X_n = m_{n-1} + \sigma_n\xi_n$, where σ_n tends to a. Then the sequence $\{X_n\}$ is a.s. not bounded, and since this result holds for arbitrarily small a, one can a.s. find a sequence $\{A'_n\}$ such that A'_n tends to O, and $X(A'_n)$ has every given (finite or infinite) limit. Thus $X(A)$ is neither continuous nor finite in a neighborhood of every point O. However, such sequences cannot be found in an n-dimensional plane, and are quite exceptional. Probably, one cannot find a continuous path such that A tends to O and $X(A)$ to a limit different from $X(O)$.

We may conclude that the Brownian function $X(A)$ is intrinsically defined in E_ω. Although it is often convenient to choose a sequence of axes Ox_n, the r.f. $X(A)$ does not depend on these axes.

Now let Ω_n denote the sphere $r = t$ in the Euclidean n space $x_1x_2 \cdots x_n$, and $M_n(t)$ the average of $X(A) - X(O)$ on the surface of this sphere. It is easy to write the sequence $M_n(t)$ in the canonical form, and to deduce that it has a.s. a limit $M(t)$. Then, in the Hilbert space, $M(t)$ is not only a r.f. defined by its covariance, but has the same geometrical meaning as in the n space.

6.3. *The determinism of $X(A)$ in the Hilbert space. The case of the sphere.* We shall say that the *Cauchy conditions of infinite order* for the function $X(A)$ are given on a surface if a function $x(A)$ is given such that, for every p, one has uniformly

$$X(A) - x(A) = o(\delta^p) \tag{6.3.1}$$

when the distance δ between A and Ω tends to zero. Then the following question arises: *when the Cauchy conditions of infinite order for $X(A)$ are given on a surface that divides the space in two regions R_1 and R_2, are the values of $X(A)$ in R_1 independent of its values in R_2?*

1°. If Ω is a sphere (with center O), the answer is yes. This is an obvious consequence of the following theorem.

THEOREM 6.3. *If the Cauchy conditions of infinite order are given for $X(A)$ on a sphere Ω of the Hilbert space, then $X(A)$ is known in the region inside this sphere.*

In such cases, it is more suggestive to consider the Cauchy conditions of infinite order as *Dirichlet conditions;* we shall do so.

PROOF. Since $\bar{M}(t)$ and all its derivatives are known when t is the radius t_0 of Ω, this analytic function is known, and $X(O) = M(O)$ is known.

Now let A_0 be a point different from O in the inside of Ω, and P the plane that contains A_0 and is perpendicular to OA_0. P is another Hilbert space in which we may apply the preceding result. If $M^*(t)$ is the function that replaces $M(t)$ when the initial space and O are replaced by P and A_0, the Dirichlet conditions of infinite order for $X(A)$ are known in P on the sphere which is the intersection of P and Ω; then $X(A)$ is known at the point A_0. Q.E.D.

Then it follows from the known properties of $M(t)$, applied to a family of spheres that have as center a given point A_0 inside of Ω, that the average of $X(A)$ is known on every sphere of this family, even if it is entirely outside of Ω. However, one has no reason to think that $X(A)$ is known outside of Ω. Although the author has not computed the conditional expectation of $X(A)$ outside of Ω, he is convinced that it is positive.

2°. Let us now consider E_ω as the product space of two spaces P and E', where P is a Hilbert space and E' may be a Euclidean space or a Hilbert space. Let C denote the intersection $\Omega \cap P$ (the center of Ω is not necessarily in P). We shall say that P is a plane in E_ω, and C a circle of the sphere Ω. When the Dirichlet (or Cauchy) conditions are given on C, considered as a circle of Ω, they are known on C, considered as a sphere of the plane P, and, since theorem 6.3 may be applied when E_ω and Ω are replaced by P and C, we see that *if the Dirichlet conditions* (of infinite order) *are given, not on the whole sphere* Ω, *but only on the circle* C, $X(A)$ *is known in the smallest convex set containing* C.

For this theorem, it is an essential restriction that P have an infinite number of dimensions.

3°. Up to the end of this paper, we shall call *closed curve* the boundary of a part of a surface, and shall consider only circles that are curves; then, in E_n(or E_ω), a circle contains points depending on $n - 2$ (or $\omega - 2$) parameters.

Now let us suppose that the Cauchy conditions are given, not on a whole sphere Ω of E_ω, but on a part Ω_1 of Ω, bounded by a curve C. Let Ω_1' denote a part of Ω_1 bounded by a circle C', the intersection of Ω and a plane P. There results immediately from the theorem stated in 2° of this section that $X(A)$ is known in the volume between Ω_1' and P. Then, obviously, it is known in the volume bounded by Ω_1 and a surface S, and C is the boundary of S. If C is a circle, S is the inside of this circle. Thus the following problem arises: What happens in the general case? What are the properties of S?

To give an answer, it is better to replace the sphere by any closed surface. We shall do it in the last section. We shall first give some lemmas concerning the sphere, in E_n and in E_ω.

6.4. *Properties of the sphere.*

1°. We shall term a curve C that divides the sphere Ω_n of space E_n into two parts of equal area, a *median curve* of Ω_n. It is known that the median curve that has the smallest measure is the circle.

Let $N(\epsilon, C)$ denote the ϵ-neighborhood of C on Ω_n, that is, the locus of all points of Ω_n the distance of which to C is less than ϵ. Then

THEOREM 6.4.1. *On a given sphere of* E_n *and for given* ϵ, *the median curve for which* $N(\epsilon, C)$ *has the smallest area is a circle.*

On this subject, see chapter 1, section 3 in the author's book [3]. An immediate consequence is

THEOREM 6.4.2. *When* $n \to \infty$, *for fixed* ϵ *and spheres of fixed radius, and for every sequence of median curves* C_n *of these spheres, one has*

$$\lim \frac{m\,[N(\epsilon, C_n)]}{m\,(\Omega_n)} = 1 \tag{6.4.1}$$

[$m(\cdot)$ is the measure of the considered area].

2°. Another immediate consequence is the following. Let Ω be a sphere of E_ω with center O and radius R, Ω_n its intersection with an n-dimensional plane P_n containing O, and $U(A)$ a uniformly continuous function of A, defined on Ω. Then

THEOREM 6.4.3. *If* $U(A)$ *is known on a median curve of* Ω_n, *its average on* Ω_n *is known, up to an error that has an upper bound tending to zero when* $n \to \infty$.

We may consider that $X(A)$ has an average on Ω if, and only if, the averages on Ω_n have a limit that does not depend on the choice of the plane P_n. It may happen that this limit does not exist. For instance, the function

$$\sum_1^\infty x_{2\nu}^2 \tag{6.4.2}$$

has on Ω_n an average that is 0 or R^2 according as P_n is $Ox_2x_4\cdots x_{2n}$ or $Ox_1x_3\cdots x_{2n-1}$, and one cannot speak of its average on Ω, unless one gives a precise definition, which depends necessarily on the order of the axis.

3°. Since it is impossible to give a satisfactory definition of the measure on Ω, one cannot retain the definition of median curve that was given for Ω_n. Hence we shall give the following definition: a curve C on Ω is a *median curve* if, for arbitrary small ϵ, and arbitrarily large n_0, one can find $n > n_0$, and corresponding planes P_n, spheres Ω_n, and median curves C_n, such that

$$C_n \subset N(\epsilon, C) . \tag{6.4.3}$$

Then, if $U(A)$ is known on C, it is known on C_n with an arbitrarily small error, and the conclusion of theorem 6.4.3 holds. However, it is still necessary to be very careful, before speaking of the average of $X(A)$ on Ω.

In the following section, when we shall speak of median curves, to define an average in Hilbert space as the limit of an average in E_n, we shall only consider n, P_n and C_n such that $C_n \subset N(\epsilon_n, C)$, where $\epsilon_n \to 0$.

6.5. *Application to* $X(A)$. Since $X(A)$ is not continuous in E_ω, one may wonder whether the conclusion of theorem 6.4.3 may be applied to this function.

If the considered median curve C_n is a circle, the answer is given by the following theorem.

THEOREM 6.5.1. *The average of $X(A)$ is a.s. the same on a sphere Ω with radius r and on every given median circle C of this sphere.*

PROOF. The function $\bar{M}(r, h)$ considered in section 6.1 is a.s. continuous, for every positive r. Then, if the considered circle is in the plane $x_0 = 0$, the average $\bar{M}(r \cos \theta, r \sin \theta)$ is a.s. a continuous function of θ, and, when $\epsilon \to 0$, the average of $X(A)$ in $N(\epsilon, C)$, which is $\bar{M}(r)$, tends to $M(r, 0)$, which is its average in C. Hence the theorem is proved.

Since this theorem holds for every positive r, all the derivatives of $M(r)$ are a.s. the same if $\bar{M}(r)$ is defined as the average of $X(A)$ on Ω, or on the intersection of this sphere and of the plane $x_0 = 0$. An obvious consequence is that the Dirichlet conditions for $X(A)$ give exactly the same information on the values of $X(A)$ in the plane $x_0 = 0$ if given on the whole sphere or only on its intersection with this plane.

We knew this already. It is an obvious consequence of the theorem proved in 2° of section 6.3. However, the new proof points the way to an important generalization. It is difficult to prove that this generalization is right; the author may only say that he is convinced it is right.

The idea is that the new proof starts from a continuity theorem. If the value of $X(A)$ or the Dirichlet conditions are given on a curve C, and if C' is deduced from C by a slight modification, a randomization occurs, and, if we compare the conditional standard deviations of an average on C' and of $X(A')$ for a particular point of C', the first is much smaller than the second. Thus the new information obtained on $X(O)$ when $X(A)$ or the Dirichlet conditions are given on C' amounts to almost nothing.

Now, let us consider a median curve C on the sphere Ω. Its measure and the measure of its ϵ-neighborhood, in the sense deduced from theorems 6.4.1 and 6.4.2 by a passage to the limit, are as small as possible if C is a circle. Thus one has at least as much information when $X(A)$ or the Dirichlet conditions are given on C as when they are given on a median circle. Moreover, all the examples that were considered in this paper lead to the idea that the randomization that was just considered is better for an $(n + 1)$-dimensional set than for an n-dimensional set, and better for a large surface than for a small one. That is the best reason to think that, when the Dirichlet conditions are given on a median curve C, the values of $X(A)$ in $N(\epsilon, C)$ have a.s. the following property: if they are known, the new information on $X(A)$ may be neglected if ϵ is small. Since $N(\epsilon, C)$ is almost all the sphere Ω, it follows that $\bar{M}(t)$ and all its derivatives are known for the considered value of t, $\bar{M}(t)$ is known for every t, and $X(O)$ is known.

Although these remarks are not a proof, the following theorem is very likely.

THEOREM 6.5.2. *If $X(A)$ is given on a median curve of the sphere Ω_t, then $\bar{M}(t)$ is a.s. equal to a well-defined linear function of the given values* [to avoid any confusion, let us say more precisely that, in this theorem, the given values are not an arbitrary function of A on Ω, but some information on a function that is supposed to be a possible function $X(A)$, and has a.s. some restrictive properties].

Although this theorem is not actually proved, it is easy to prove the following one.

THEOREM 6.5.3. *If the Dirichlet conditions of infinite order are given for $X(A)$ on a median curve Γ of the sphere Ω_t, then $X(O)$ is known.*

PROOF. One has only to prove that $\bar{M}(t)$ and all its derivatives $M^{(p)}(t)$ are known for the value t considered. Since $\bar{M}(t)$ is a.s. an analytic function, it will follow that $\bar{M}(0) = X(O)$ is known.

Let $N(\epsilon, \tau, \Gamma)$, where $\tau < \epsilon$, be the locus of points of $\Omega_{t+\tau}$ whose distance to Γ is less than ϵ. The average of $X(A)$ on this area is $\bar{M}(t + \tau)$, and, if for instance $\epsilon = 2\tau$, it is known up to an error that is $O(\tau^p)$, for every fixed p and arbitrarily small τ. Then $\bar{M}(t)$ and $M^{(p)}(t)$ are known. Q.E.D.

6.6. *Generalization.*

1°. *The case of a closed surface.*

THEOREM 6.6.1. *If the Dirichlet conditions for $X(A)$ are given on a convex and closed surface S, then $X(A)$ is known at every interior point O.*

PROOF. Let Ω denote a sphere with center O, A a point of Ω, r the distance OA, and a the intersection of the ray OA and Ω. Then r is a continuous function of a, and S may be defined by this function.

Let Ω_n be the intersection of Ω and the plane $Ox_1x_2 \cdots x_n$ (or every n-dimensional plane containing O); let ρ_n be the median value of r on Ω_n. Then $r = \rho_n$ defines a median curve on this sphere. When n tends to infinity, since the numbers ρ_n are bounded, there exists at least a number ρ that is a limit of ρ_n for a partial sequence of subscripts n. We may suppose that Ω has the radius ρ. Then $r = \rho$ is a median curve of Ω, and, since the Dirichlet conditions are given on S, they are known on this curve, and it follows from theorem 6.5.3 that $X(O)$ is known. Q.E.D.

2°. The condition that S is convex is not essential. If several points of this surface are on the same ray, one has only to choose one of them, and the proof holds. The following remarks, however, are necessary to avoid possible confusion.

One may wonder whether one obtains the same value of $X(O)$ if, for instance, the chosen value of r is, on every ray, the largest or the smallest. If the given values of $X(A)$ are arbitrary, the answer is no; the given values are superabundant. However, the existence of $X(A)$ has been proved, and no contradiction is possible. Thus, if the given conditions are superabundant, one may avoid every difficulty if one supposes that they are deduced from a sample function. Then they are fulfilled by at least one function belonging to a set that may be called the set of possible functions, and, if one chooses any one of the several methods to obtain $X(O)$, one is sure that the result does not depend on this choice.

Now, if S is not a convex surface, it may happen that the median value ρ_n will be replaced by an interval (ρ_n', ρ_n''). Even if ρ_n is well defined, it may happen that, for indefinitely increasing n, one has, not a well-defined limit ρ, but all the points of an interval. Then ρ may be arbitrarily chosen in this interval, and we have an infinity of methods to obtain $X(C)$. However, there is no need to consider the case when all do not give the same value. If the given values give some information on a sample function, one is a.s. that no contradiction is possible.

3°. *The case of a curve.* The condition that S be a closed surface is no longer essential. We have already proved it in the case of the sphere. To know what happens when the Dirichlet conditions are given on an open part S' of a closed surface S, the best method is to consider first the case when they are given only on the boundary C of S'.

We shall use the following definitions. A half cone with vertex O is called a ***median cone of space*** if its intersection with a sphere with center O is a median curve of this sphere. If a point a describes this curve, and if A is the point of the ray Oa defined by $OA = f(a)$, where $f(a)$ is a given function, we shall say that the curve C that A describes, *seen from O*, is a ***median curve of the space***, or that O is a ***median point of view of C***.

Let us now consider a line L and a curve C surrounding L. This means that every continuous surface bounded by C has at least a common point with L. If, for instance, L is an infinite line that is asymptotic to the two sides of a straight line, it is obvious that it contains at least one median point of view of C. Then the locus of these points of view is at least a surface bounded by C. It is also obvious that, in an n-dimensional space, it is in general a surface; however, things are more complicated in the Hilbert space. Then, to see what happens, if for instance the x_1 axis is surrounded by C, we shall apply first the preceding result in the $x_1x_2\cdots x_n$ plane, and suppose that, for sufficiently large n, L is in this plane. Then we have in this plane a curve C_n which is the intersection of C and the plane, and the locus of the median points of view of C_n is a surface S_n, that has at least a common point A_n with L. It may happen that, when n tends to infinity, A_n and S_n have limits A and S, and, in this case, $X(A)$ is known on S. However, it may also happen that A_n have no limit. In this case one may conceive of S_n as a vibrating membrane, and every point of a volume V is a limit of points of a partial sequence of these surfaces. Then $X(A)$ is known in the whole volume V.

It is easy to show by an example that this second case is possible. Let us set

$$\sum_1^\infty x_{2n}^2 = r^2, \qquad \sum_1^\infty x_{2n-1}^2 = r'^2, \tag{6.6.1}$$

and let us consider, on the cylinder

$$r^2 + r'^2 = R^2, \tag{6.6.2}$$

of the $Ox_0x_1\cdots$ space, the curve

$$(C) \qquad x_0 = r. \tag{6.6.3}$$

If $0 < \rho < R$, in the plane $x_0 = a$ which is a Hilbert space, the curve

$$(C') \qquad r^2 = a^2, \qquad r'^2 = R^2 - a^2 \tag{6.6.4}$$

is a part of C, and $N(\epsilon, C')$ is almost the whole surface bounding the volume

$$r^2 < a^2, \qquad r'^2 < R^2 - a^2 . \tag{6.6.5}$$

Then, if the Cauchy conditions are given on C, they are known on C' and $X(A)$ is known in the whole volume

$$0 < a < R, \qquad r^2 < a^2, \qquad r'^2 < R^2 - a^2 . \tag{6.6.6}$$

If one changes suitably the order of the axis, every point of this volume is a limit of points of the surfaces S_n.

4°. *The case of an open surface.* Let us come back to the case of a closed surface S, divided by C into two surfaces S_1 and S_2. In the first case, when the Dirichlet conditions given on C define $X(A)$ only on a surface Σ, if they are given in S_i, $i = 1$ or 2, $X(A)$ is known in the volume V_i between S_i and Σ. In the second case, when the Dirichlet conditions, given on C, define $X(A)$ on a volume V, then, if given in S_i, $X(A)$ is known in a volume V_i, and U is contained in the intersection of V_1 and V_2.

In such cases, the remarks presented in the second part of this section are essential. The given conditions are superabundant, but one may assume that no contradiction is possible. If S_i' is $S_i - N(\epsilon, C)$, with very small S, and if the Dirichlet conditions are given on S_i', then $X(A)$ is known in a volume V_i' and $V_1' \cup V_2'$ is only a part of the volume bounded by S. However, the difference may be entirely in a neighborhood of C, and then the want of given conditions in $N(\epsilon, C)$ hinders us only in obtaining $X(A)$ in this neighborhood.

5°. Finally let us recall that the properties of the surfaces S or volumes V that are loci of median points of view for curves C are connected with the properties of the minimal surfaces in the Hilbert space.[10] If the locus is a surface, it is a minimal surface. However, it is generally not a surface, and in this case the definitions of the mean curvature of a surface, and of the minimal surfaces, are relative to particular sequences of secant planes P_n. It follows that the theory is rather complicated, and not quite satisfactory. On the contrary, in the present theory, the conditional standard deviation of $X(A)$ for every possible information is always well defined, and the conclusion of the theory appears quite satisfactory.

REFERENCES

[1] Paul Lévy, *Théorie de l'Addition des Variables Aléatoires*, Paris, Gauthier-Villars, 1937 (1st ed.); 1954 (2nd ed.).

[2] ———, *Processus Stochastiques et Mouvement Brownien*, Paris, Gauthier-Villars, 1948.

[3] ———, *Problèmes Concrets d'Analyse Fonctionnelle*, Paris, Gauthier-Villars, 1950.

[4] ———, *Le Mouvement Brownien*, Mémorial des Sciences Mathématiques, No. 126, Paris, Gauthier-Villars, 1954.

[5] ———, "Wiener's random function, and other Laplacian random functions," *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, Berkeley and Los Angeles, University of California Press, 1951, pp. 171–187.

[6] ———, "Random functions: general theory with special reference to Laplacian random functions," *Univ. of California Publ. in Stat.*, Vol. 1 (1953), pp. 331–390.

[7] ———, "Le mouvement brownien à $n = 2p + 1$ paramètres, I," *C. R. Acad. Sci., Paris*, Vol. 239 (1954), pp. 1181–1183.

[8] ———, "Le mouvement brownien à $n = 2p + 1$ paramètres, II," *C. R. Acad. Sci., Paris*, Vol. 239 (1954), pp. 1584–1585.

[10] See the third part of the author's book [3].

[9] ———, "Le mouvement brownien à $n = 2p + 1$ paramètres, III," *C. R. Acad. Sci., Paris*, Vol. 240 (1955), pp. 1043–1044.
[10] ———, "Sur une classe de fonctions aléatoires gaussiennes," *C. R. Acad. Sci., Paris*, Vol. 240 (1955), pp. 1308–1309.
[11] ———, "Brownian motion depending on five parameters," a paper presented to the Symposium on Mathematical Probability and Its Applications, Brooklyn, N.Y., April 14, 1955.
[12] ———, "A Gaussian random function depending on a point of the Hilbert space," to be published.
[13] G. MARUYAMA, "The harmonic analysis of stationary stochastic processes," *Memoirs of the Faculty of Sci., Kyûsyû Univ.*, Ser. A., Vol. 4 (1949), pp. 45–106.